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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED in 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

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ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

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(Devoted to the Interests of Collegiate Mathematics)

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AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR ALBERT W. TUCKER

It is now six years since the Mathematical Association of America first conferred an Award for Distinguished Service to Mathematics. By the governing regulations, the award is to be made for outstanding service to mathematics of such a character as to influence significantly the field of mathematics or mathematical education on a national scale. In the intervening years, the Association has honored brilliant teachers, statesmen of the mathematical community, creative scientists, and crusading reformers of the nation's mathematical curriculum. The man we honor today has played all of these rôles with dedicated enthusiasm. By his willing service to those who would learn, teach, advance, or use mathematics, Albert William Tucker has added measurably to the vigor and quality of mathematics today.

Born and reared a Canadian, Al Tucker came to Princeton for graduate study in 1929 after receiving his A.B. and A.M. from the University of Toronto. He obtained his Ph.D. in 1932 and, with the exception of short periods of leave, his colleagues at Princeton have had the pleasure of his company ever since.

To define the unique quality of a man in a short citation is an impossible task. The list of offices held, the honors conferred, the projects completed are far too cold and impersonal to convey much of the essential achievement. Although this impressive record must be cited, it is the human evidence that is the most real and important.

Consider Al Tucker's record as a teacher. He joined the Princeton Faculty in 1933. He was appointed Assistant Professor in 1934, Associate Professor in 1938, and Professor in 1946. He succeeded Emil Artin as Albert Baldwin Dod Professor in 1953. Princeton has shared his teaching talents on a number of occasions. In 1949–50 he was Visiting Professor at Stanford University and in the summer of 1952 a guest lecturer at the Massachusetts Institute of Technology. He was Philips Visitor at Haverford College in 1953–54 and 1958–59, Visiting Lecturer for the Mathematical Association in 1956–57, and since 1963 a guest lecturer at the Rockefeller Institute. He was Fulbright Lecturer at four Australian universities in 1956 and lecturer at several European universities in 1959 for the Organization for European Economic Cooperation.

Behind this formal record stands a veritable army of mathematicians now active who give evidence of a man who taught them with exquisite care and precision. Among them is a student who learned from Al Tucker the Borsuk conjecture on the total curvature of a knot in a course on geometric concepts and went on to solve the problem while still a freshman. His lectures sparkle with penetrating examples. Perhaps the most famous is an illustration of non-zero-sum game theory constructed for an expository seminar before a general audience at Stanford in 1949–50. This game, which goes by the name of the Prison-

ALBERT WILLIAM TUCKER

er's Dilemma, has inspired countless research papers and at least one entire book. He has been generous with his materials as well. With the procrastination of a confirmed perfectionist, his examples and teaching exercises have found their way to the mathematical public not through his own books but through the books of others.

His record as a statesman of the mathematical community is equally impressive. He has been a Council member and Trustee of the American Mathematical Society, President of the Mathematical Association of America, a Vice President of the American Association for the Advancement of Science (AAAS) and Chairman of the Conference Board of the Mathematical Sciences (CBMS). It is characteristic that his service extends beyond the confines of individual professional organizations to strengthen our ties with other sciences through the AAAS and to unite the manifold interests within the mathematical sciences through the CBMS. At the highest national level, his wisdom was sought as a member of the first President's Committee on the National Medal of Science and as a consultant to the President's Science Advisory Committee. Many young research mathematicians have benefitted from his representation of mathematics on the Basic Physical Sciences Program Committee of the Alfred P. Sloan Foundation. In all of these positions, his colleagues have learned to appreciate the wise deliberation with which he confronts any task, large or small.

Al Tucker's introduction to applied mathematics came during World War II when, in addition to teaching in the Army Specialized Training Program, and the Navy Pre-Radar Program, he served as associate director of the Fire Control Research Group at Princeton, working on target location and gunnery direction problems for the Office of Scientific Research and Development and for the Frankford Arsenal. Since 1948, he has directed a research project sponsored by the Office of Naval Research, which has centered on problems from the theory of games, mathematical programming and combinatorics. Here the special quality of Tucker's leadership has shown itself in the large number of young people who have produced basic results in these areas while participants in this project. Furthermore, his responsibility to the larger mathematical community has led to the propagation of these results through conferences and volumes of collected research.

From their inception, Al Tucker has been in the forefront of efforts to reform the mathematical curriculum at all levels. He has served on the Committee (then Commission) on the Undergraduate Program in Mathematics for most of the years since its establishment in 1953 and has contributed generously to the work of its panels. As Chairman of the Commission on Mathematics of the College Entrance Examination Board (CEEB), he laid firm foundations for the later work of the School Mathematics Study Group. Curriculum reform is no passing fancy with him. Many a less conscientious man would have left the work to others; Al Tucker continues to serve as Chairman of the Committee on Advanced Placement of the CEEB. Again, the measure of his service is not

the number of hours sitting with difficult colleagues who will not keep to the work at hand, but rather the quality of the final product that he has guided and, above all, the jewels of expository mathematics that appear anonymously through the reports and bear the imprint of his clarity and elegance.

One of the most fitting tributes to the qualities of Al Tucker's life in mathematics was composed by John Sloan Dickey, President of Dartmouth College, and read on the occasion of the award of an honorary degree of Doctor of Science in June, 1961. It captures such a large share of the debt we owe him that it seems proper to quote it directly:

"Nearly three decades ago you began an academic career at Princeton which became a mission to mathematics. In a field where scholarship scores only if the idea is both new and demonstrably true your ideas have won their way in topology, in the theory of games, and in linear programming. But even in mathematics a mission is more than ideas; it is also always a man, a man who cares to the point of dedication, whose concern is that others should care too, and who can minister to the other fellow, as the need may be, either help or forbearance. Because you, sir, embody in extraordinary measure both your profession's love of precision and man's need for conscientious leadership, mathematics in America at all levels is today higher than it was and tomorrow will be higher."

The Association is proud to enroll you among its honored own as a recipient of the Award for Distinguished Service to Mathematics.

H. W. KUHN

AWARD OF THE 1968 CHAUVENET PRIZE TO PROFESSOR MARK KAC

The Board of Governors of the Mathematical Association of America at its meeting on August 27, 1967, at the University of Toronto voted to award the 1968 Chauvenet Prize to Professor Mark Kac of the Rockefeller University for his paper "Can One Hear the Shape of a Drum?," published in this MONTHLY, 73 (1966), Part II (Slaught Paper No. 11), 1–23. Professor Kac is the first person to be awarded the Chauvenet Prize for a second time, having previously received it in 1950.

A certificate and monetary award in the amount of five hundred dollars was presented to Professor Kac at the time of the Business Meeting of the Association on January 26, 1968, in San Francisco.

The Chauvenet Prize is awarded for a noteworthy expository paper published in English, such as will come within the range of profitable reading for

members of the Association. The purpose of the prize is to stimulate the writing of expository works by American scholars. The 1968 Prize, awarded for a paper published in 1966 by a member of the Association, is the seventeenth award of the Chauvenet Prize since its institution by the MAA in 1925. For a list of the names of the previous winners, see this MONTHLY, 71 (1964), page 589, 73 (1965), pp. 2–3, and 75 (1967), p. 3.

Professor Kac was born on August 3, 1914 in Krzemieniec, Poland. He received the Ph.D. degree from the John Casimir University in Lwow in 1937, after which he worked as an actuary at the Phoenix Company, a Polish insurance firm. Professor Kac came to the United States in 1938, and after a year at The Johns Hopkins University, he joined the faculty of mathematics at Cornell University. He became a naturalized citizen in 1943, and from 1943 to 1945 served as a member of the Office of Scientific Research and Development. Professor Kac was a member of the Institute for Advanced Study at Princeton in 1951–52. He received the Parnas Foundation Fellowship in Poland in 1938–39 and held a Guggenheim Fellowship in 1946–47. He was elected to the American Academy of Arts and Sciences in 1959 and to the National Academy of Sciences in 1965. Professor Kac has been a member of the Council of the American Mathematical Society, its Vice-President from 1964 to 1966, and Editor of the Transactions of the American Mathematical Society from 1955 to 1958. He was appointed Professor and became a member of the Rockefeller Institute on July 1, 1961. He was Lorentz Visiting Professor of Theoretical Physics in Leiden, the Netherlands, in 1963, and has been Chairman of the Division of Mathematical Sciences of the National Research Council of the National Academy of Sciences in 1965–67.

Professor Kac's significant contributions to so many branches of mathematics and its applications, including the theory of probability, statistics, analysis, and number theory, are contained in his numerous papers—already more than 80 in number—which have appeared in many scientific publications throughout the world.

He is the author of Carus Monograph no. 12, *Statistical Independence in Probability, Analysis, and Number Theory*. The article for which he has received the Chauvenet Prize has been reprinted as a CEM Film Manual, since it is the script of his film with the same title.

In accepting the 1968 Chauvenet Prize, Professor Kac indicated that he felt both honored and flattered by having been voted for the second time the recipient of the Chauvenet Prize.

THE PROBLEM OF APOLLONIUS

H. S. M. COXETER, University of Toronto

1. Introduction. In the third century B.C., Apollonius of Perga wrote two books on Contacts (ἐπαφαι), in which he proposed and solved his famous problem: given three things, each of which may be a point, a line, or a circle, construct a circle which passes through each of the points and touches the given lines and circles. The easy cases are covered in the first book, leaving most of the second for the really interesting case, when all the three "things" are circles. As Sir Thomas Heath remarks [**10**, p. 182], this problem "has exercised the ingenuity of many distinguished geometers, including Vieta and Newton." I will not ask you to look at any of their methods for solving it. These are adequately treated in the standard textbooks [e.g. **11**, p. 118]. Nor will I inflict on you an enumeration of the possible relations of incidence of the three given circles. This was done with great skill in 1896 by Muirhead [**12**].

2. The Descartes Circle Theorem. I will confine the discussion to the two cases that were described by Descartes in letters of November 1643 to his favourite disciple, Princess Elisabeth, daughter of King Frederick of Bohemia [**8**, pp. 37–50]. The first letter deals with three nonintersecting circles, entirely outside one another. Descartes finds some relations between the radii and central distances. The details are clumsy but clear. The second letter deals with the limiting case when the three given circles are mutually tangent at three distinct points. He uses d, e, f to denote the radii of these three circles, and x for the radius of a fourth circle that touches them all externally. Thus d, e, f, x are the radii of four circles in mutual (external) contact. Unfortunately there is a gap in the argument (between pages 48 and 49) which precludes any clear understanding of the crucial steps leading to his conclusion:

$$\begin{aligned} &ddeeff + ddeexx + ddffxx + eeffxx \\ &\quad = 2deffxx + 2deeffx + 2deefxx + 2ddeffx + 2ddefxx + 2ddeefx. \end{aligned}$$

It seems strange today that he did not express this equation more concisely as

$$\frac{1}{dd} + \frac{1}{ee} + \frac{1}{ff} + \frac{1}{xx} = \frac{2}{ef} + \frac{2}{fd} + \frac{2}{de} + \frac{2}{dx} + \frac{2}{ex} + \frac{2}{fx}$$

or

$$2\left(\frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{x^2}\right) = \left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{x}\right)^2. \tag{2.1}$$

This beautiful result, which Pedoe [**13**, p. 634] very properly calls *The Descartes Circle Theorem*, was rediscovered almost exactly 200 years later by Mr. Philip Beecroft of Hyde, Cheshire. He published it in a journal not often consulted nowadays: "The Lady's and Gentleman's Diary for the year of our

Lord 1842, being the second after Bissextile, designed principally for the amusement and instruction of Students in Mathematics: comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit" [1].

Dr. Leon Bankoff of Los Angeles is a dentist who spends his spare time on the same delightful pursuit. When he saw the announced title of my Presidential Address, he kindly sent me a copy of this and two other papers by Beecroft. Although the Descartes circle theorem was rediscovered again in 1936 by Frederick Soddy [14], neither Soddy nor anyone else followed Beecroft in his brilliant idea of regarding the configuration of four circles in mutual contact as part of a configuration of eight circles, each passing through the three points of contact of three others, as in Figure 1.

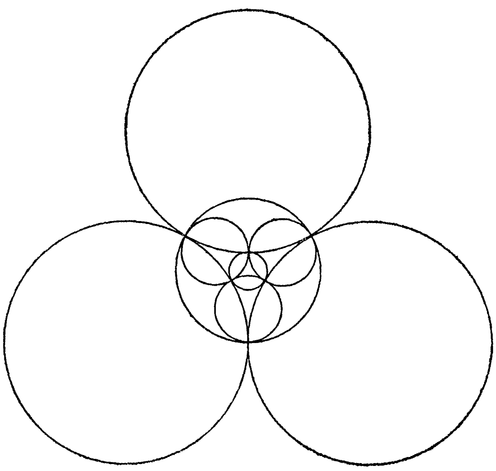

FIG. 1

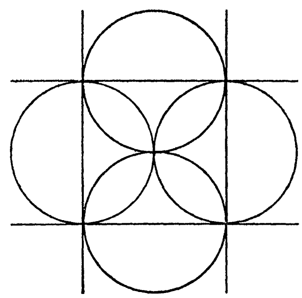

FIG. 2.

This configuration arises in its most symmetrical form when we consider a regular octahedron and its circumsphere. The eight face-planes of the octahedron cut the sphere in such a set of eight circles, and we can obtain the planar configuration by stereographic projection from an arbitrary point on the sphere. In particular, projection from a vertex of the octahedron yields four lines forming a square and four circles having the sides of this square as diameters, as in Figure 2. Any other case of Beecroft's configuration can be derived from this simple one by an inversion.

In the course of his rediscovery of the Descartes circle theorem, Beecroft noticed that equation 2.1 holds for *any* four circles in mutual contact, provided we make the convention that, when two circles have *internal* contact, we regard the larger circle as having a negative radius. Although he worked with radii, it is obviously more convenient to use curvatures (or, as Soddy would say,

"bends"), which are reciprocals of radii. Let

$$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \eta_1, \eta_2, \eta_3, \eta_4$$

be the curvatures of Beecroft's eight circles. Then the theorem says that

$$2\sum \epsilon^2 = \left(\sum \epsilon\right)^2, \tag{2.2}$$

and of course, we shall have also $2\sum\eta^2 = (\sum\eta)^2$. In the words of Soddy's poem,

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

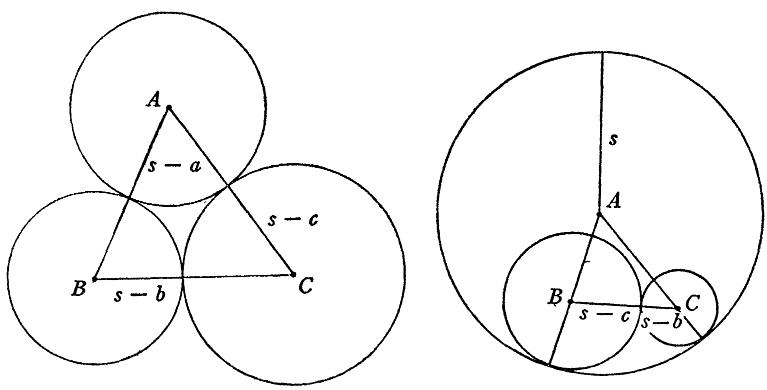

FIG. 3

Here is a simplified version of Beecroft's proof. Let a, b, c, s, r and r_a denote the sides, semiperimeter, inradius and first exradius of a triangle ABC, so that

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} \quad \text{and similarly} \quad r_a^2 = \frac{s(s-b)(s-c)}{s-a}$$

[7, pp. 60, 164 (Ex. 3)]. Any three mutually tangent circles can be regarded as having centres A, B, C, and radii $s-a$, $s-b$, $s-c$ or else s, $s-c$, $s-b$, as in Figure 3. Accordingly we write in the former case

$$\frac{1}{\eta_1} = r, \quad \frac{1}{\epsilon_2} = s - a, \quad \frac{1}{\epsilon_3} = s - b, \quad \frac{1}{\epsilon_4} = s - c,$$

and in the latter (with the minus sign for internal contact)

$$\frac{1}{\eta_1} = r_a, \quad \frac{1}{\epsilon_2} = -s, \quad \frac{1}{\epsilon_3} = s - c, \quad \frac{1}{\epsilon_4} = s - b.$$

It follows that

$$\epsilon_3\epsilon_4 + \epsilon_4\epsilon_2 + \epsilon_2\epsilon_3 = \left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4}\right)\epsilon_2\epsilon_3\epsilon_4$$

$$= \left\{\begin{array}{ccccc} \dfrac{(s-a)+(s-b)+(s-c)}{(s-a)(s-b)(s-c)} & = & \dfrac{s}{(s-a)(a-b)(s-c)} & = & \dfrac{1}{r^2} \\ \dfrac{s-b-c}{-s(s-c)(s-b)} & = & \dfrac{s-a}{s(s-b)(s-c)} & = & \dfrac{1}{r_a^2} \end{array}\right\} = \eta_1^2.$$

Similarly $\eta_3\eta_4+\eta_4\eta_2+\eta_2\eta_3=\epsilon_1^2$, and of course we can permute the subscripts 1, 2, 3, 4. Hence

$$\left(\sum \epsilon\right)^2 = \sum \epsilon^2 + 2\epsilon_1\epsilon_2 + \cdots = \sum \epsilon^2 + \sum \eta^2 = \left(\sum \eta\right)^2,$$

and $\sum\epsilon = \sum\eta$. Also

$$\begin{aligned} -\epsilon_1^2 + (\epsilon_2+\epsilon_3+\epsilon_4)^2 &= -\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 + 2\eta_1^2 \\ &= 2\eta_1(\eta_2+\eta_3+\eta_4) + 2\eta_1^2 \\ &= 2\eta_1 \sum \eta = 2\eta_1 \sum \epsilon, \end{aligned}$$

whence

$$-\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 2\eta_1. \tag{2.3}$$

Adding four such equations after squaring each side, we obtain

$$\sum \epsilon^2 = \sum \eta^2,$$

whence $2\sum\epsilon^2 = \sum\epsilon^2 + \sum\eta^2 = (\sum\epsilon)^2$. Thus 2.2 is proved.

This version of Beecroft's proof formally resembles one of Pedoe's proofs [**13**, p. 638], but the meaning is quite different.

One way of expressing the connection between the four ϵ's and the four η's is to remark that they are the roots of quartic equations

$$(\epsilon - 2u)^2\epsilon^2 + (v - w)\epsilon + 2uw = 0,$$
$$(\eta - 2u)^2\eta^2 - (v - w)\epsilon + 2uv = 0.$$

For instance, in Figure 1, where

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \sqrt{3} - 1, \qquad \epsilon_4 = \sqrt{3} + 3,$$
$$\eta_1 = \eta_2 = \eta_3 = \sqrt{3} + 1, \qquad \eta_4 = \sqrt{3} - 3,$$

we have $u=\sqrt{3}$, $v=-2(2+\sqrt{3})$, $w=2(2-\sqrt{3})$. Again, in Figure 2, where

$$\epsilon_1 = \epsilon_2 = \eta_3 = \eta_4 = 0 \quad \text{and} \quad \eta_1 = \eta_2 = \epsilon_3 = \epsilon_4,$$

we have $u=v=0$.

One pretty result which Beecroft seems to have missed is

$$\sum \epsilon\eta = 0. \tag{2.4}$$

This appeared, with a geometric proof, in the "Diary" for 1846 [1]. For an algebraic proof we can use (2.3) in the form

$$\epsilon_1 + \eta_1 = \tfrac{1}{2} \sum \epsilon,$$

whence $\epsilon_1+\eta_1=\epsilon_2+\eta_2=\epsilon_3+\eta_3=\epsilon_4+\eta_4$ and

$$\sum \epsilon\eta = \sum \epsilon(\epsilon+\eta) - \sum \epsilon^2 = \tfrac{1}{2}\left(\sum \epsilon\right)^2 - \sum \epsilon^2 = 0.$$

3. Triads of nonintersecting circles. Although radii and curvatures belong to Euclidean geometry, it should not be forgotten that the problem of Apollonius is still meaningful in the wider field of the *inversive* plane, which may be thought of as the surface of a sphere, or as the Euclidean plane completed by a single point at infinity. In this kind of geometry circles have no "centres," but two intersecting circles still determine an angle, and two nonintersecting or tangent circles have an *inversive distance* δ such that, if an inversion transforms one of the circles into a line and the other into a circle of radius b whose centre is at distance p from the line, $\cosh \delta = p/b$ [7, pp. 130, 176 (Ex. 4)].

In Beecroft's configuration, each of the eight circles is orthogonal to three, tangent to three, and at a certain inversive distance δ from the remaining one. Figure 2 shows that $\cosh \delta = 2$, whence $\delta = \log(2+\sqrt{3})$, the logarithm of the ratio of the radii of the two concentric circles in Figure 1.

In fact, any two nonintersecting circles can be inverted into concentric circles, and their inversive distance is equal to the logarithm of the ratio of the radii (the greater to the smaller) of these two concentric circles [7, pp. 121, 123].

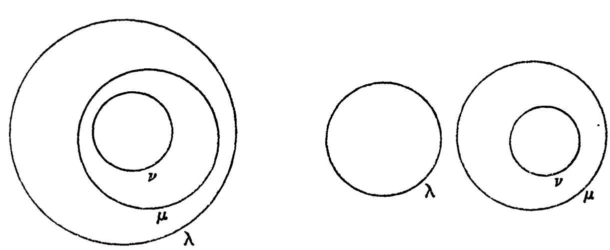

FIG. 4

By thinking of circles on a sphere (without any distinction between "great" and "small" circles), we see that, when the inversive problem of Apollonius is considered for three non-intersecting circles, the number of solutions can only have two possible values: zero or eight. The number is 0 if the circles are *nested*, as in Figure 4 (where every circle tangent to λ and ν intersects μ in two distinct points). It is 8 in the remaining case (Figure 5), where we naturally speak of the three nonintersecting circles as an *Apollonian triad*. In particular, any three circles that belong to a nonintersecting pencil of coaxal circles are nested.

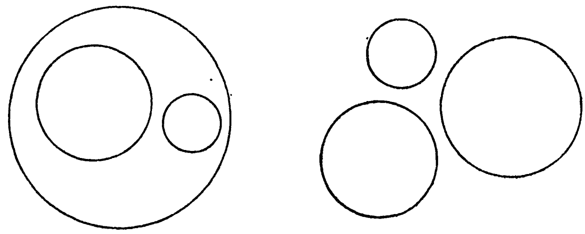

FIG. 5

4. A "Nontriangle Inequality" for nested circles. It is clear from considerations of continuity that, for any three positive numbers α, β, γ, there exists an Apollonian triad of circles whose inversive distances have these values. Accordingly, it is remarkable that the mutual inversive distances of three nested circles satisfy a "nontriangle inequality" (which thus serves as a necessary, but not sufficient, condition for three circles to be nested):

(4.1) *Among the mutual inversive distances between three nested circles, one is greater than or equal to the sum of the other two. Equality holds only when the three circles are coaxal.*

Although this is a theorem of inversive geometry, the simplest proof employs Euclidean ideas. Let λ, μ, ν be the nested circles, as in Figure 4, and let α, β, γ be their inversive distances: λ to μ, μ to ν, ν to λ. Since the circles are nonintersecting, there is at least one circle ρ orthogonal to all of them [11, p. 34]. Since either of the intersections of μ and ρ is the centre of a circle inverting these two circles into perpendicular lines, we lose no generality by taking μ to be a line parallel to the radical axis of λ and ν, as in Figure 6.

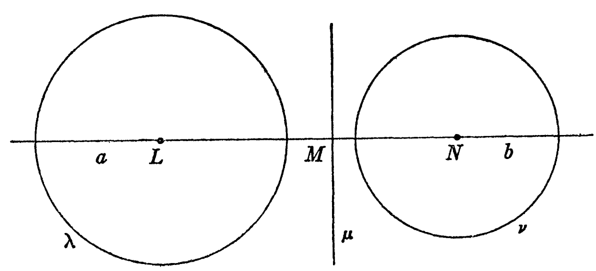

FIG. 6

Let λ and ν have centres L and N, radii a and b, and let μ meet LN in M, so that $LM = a \cosh \alpha$ and $MN = b \cosh \beta$. Since

$$(a \cosh \alpha + b \cosh \beta)^2 = LN^2 = a^2 + b^2 + 2ab \cosh \gamma$$

[6, p. 77], it follows that

$$(a \sinh \alpha - b \sinh \beta)^2 = 2ab \left\{\cosh \gamma - \cosh (\alpha + \beta)\right\},$$

whence $\gamma \geqq \alpha+\beta$, with equality only when $a \sinh \alpha = b \sinh \beta$. Since $(a \sinh \alpha)^2$ and $(b \sinh \beta)^2$ are the powers of M with respect to λ and ν, this exceptional case is when μ coincides with the radical axis of λ and ν. From the standpoint of inversive geometry, this means that the nested circles λ, μ, ν are coaxal. Thus (4.1) is proved.

By regarding the inversive plane as a sphere, we see that each circle determines an enveloping cone which can be regarded as the null cone at a point in an *exterior-hyperbolic* space [**5**, pp. 83–84]. The inversive distance between two nonintersecting circles now appears as the non-Euclidean distance between two points lying on a secant of the sphere, and the three nested circles are represented by the vertices of the kind of triangle for which the nontriangle inequality was observed as long ago as 1907 by E. Study [**16**, p. 108; see also **3**, p. 225].

Du Val [**9**] proved in 1924 that the events in de Sitter's space-time can be represented by the points of an exterior-hyperbolic 4-space: the part of real projective 4-space that lies outside a nonruled quadric 3-fold. Thus the exterior-hyperbolic 3-space that we have been discussing may be regarded as a 3-dimensional section of de Sitter's 4-dimensional world, and the terminology of space-time is appropriate. For instance, the light-cone at a given event is the enveloping cone from a given point to the absolute quadric surface Ω (the 3-dimensional section of the quadric 3-fold).

The "dictionary" relating the inversive plane to exterior-hyperbolic 3-space begins as follows:

Circle	Point (or "event")
Coaxal circles	Collinear points
Intersecting pencil	Spacelike line
Tangent pencil	Null line (tangent to Ω)
Non-intersecting pencil	Timelike line (secant to Ω)
Limiting points	The beginning and end of eternity
Orthogonal pencils	Polar lines
Angle of intersection of circles	Space interval
Inversive distance	Time interval
Homography	Lorentz transformation

To establish the connection, we regarded Ω as a sphere. Pedoe [**13**, p. 635] prefers a paraboloid of revolution.

5. Two special solutions of the problem. After that wild excursion, let us return to the Euclidean plane and consider two nonintersecting (or possibly tangent) circles of *equal* radius b. Since their inversive distance α is twice the inversive distance between either circle and their radical axis, the Euclidean distance between their centres in $2b \cosh \frac{1}{2}\alpha$. Hence three nonintersecting (or possibly tangent) circles can be inverted into *congruent* circles if and only if their three inversive distances α, β, γ are such that $\cosh \frac{1}{2}\alpha$, $\cosh \frac{1}{2}\beta$, $\cosh \frac{1}{2}\gamma$ are the

mutual (ordinary) distances of three points in the Euclidean plane. Since three congruent circles are tangent to two parallel lines, or to two concentric circles, according as their centres are or are not collinear, we can deduce the following theorem of inversive geometry:

(5.1) *Among the eight circles that touch an Apollonian triad with inversive distances* α, β, γ, *two are nonintersecting* (*or tangent*) *if and only if each of the three numbers* $\cosh \frac{1}{2}\alpha$, $\cosh \frac{1}{2}\beta$, $\cosh \frac{1}{2}\gamma$ *is less than the sum of the other two* (*or one of them is equal to the sum of the other two*).

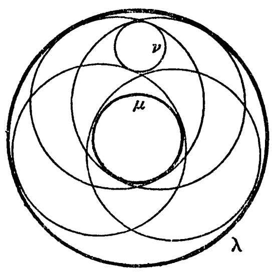

FIG. 7

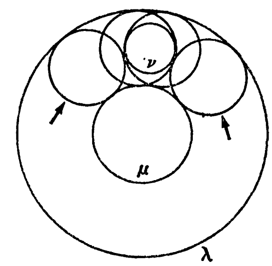

FIG. 8

After noticing that three circles cannot always be inverted into congruent circles, Roger Johnson [11, p. 97] says "This negative result is highly regrettable." In order to refute his pessimistic attitude, let us work again in the Euclidean plane and consider an Apollonian triad consisting of two concentric circles λ, μ, whose radii satisfy $a > b$, and a third circle ν. Since λ and μ are concentric, the circles that touch both consist of two one-parameter families of congruent circles in the closed annulus bounded by λ and μ: one family having radius $\frac{1}{2}(a+b)$ (Figure 7) and one having radius $\frac{1}{2}(a-b)$ (Figure 8). Since the triad $\lambda\mu\nu$ is Apollonian, ν must lie strictly within this annulus and have radius less than $\frac{1}{2}(a-b)$.

The eight solutions of the problem of Apollonius for $\lambda\mu\nu$ consist of four members of each family. We see at once from the figures that each of λ, μ, ν is separated from the other two by two of the eight. (Of the four circles in Figure 7, two

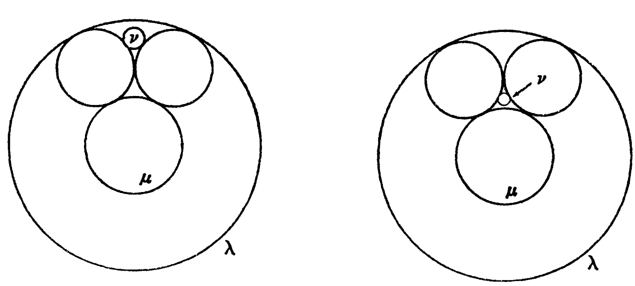

FIG. 9

surround μ and ν, separating them from λ, and two surround μ, separating it from λ and ν. Of the four in Figure 8, two surround ν, separating it from λ and μ.) The two circles emphasized in Figure 8 are special in that they do not separate λ, μ, ν at all. In this figure they happen to be nonintersecting (as in Theorem 5.1), but they could just as easily be tangent, as in Figure 9, or intersecting, as in Figure 10.

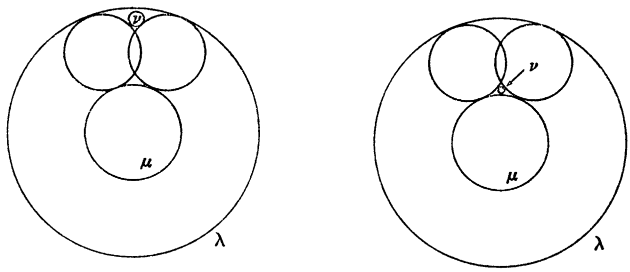

FIG. 10

It is this last possibility that Johnson considered "regrettable." Its redeeming feature is that, of the four "crescents" or *lunes* into which these two intersecting circles decompose the inversive plane, just one contains all three of the original circles λ, μ, ν. Consequently, when the intersecting circles are inverted into intersecting lines, which decompose the Euclidean plane into four angular regions, the new versions of λ, μ, ν are all inscribed in the same one of the four angles, that is, they are homothetic in pairs from the same centre of dilatation. Our conclusion may be summed up as follows:

(5.2) *Every Apollonian triad can be inverted into three circles which are either congruent or homothetic.*

6. Mid-circles. Any two nonintersecting circles have a unique *mid-circle* which inverts them into each other [7, pp. 121–122]. For instance, the mid-circle of two concentric circles is concentric with them, and its radius is the geometric mean of the two radii. Since mid-circles invert into mid-circles, we are now ready to prove the following nice theorem:

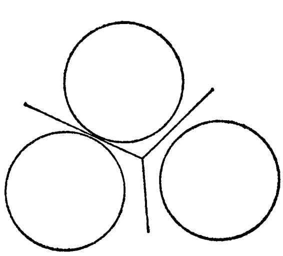

FIG. 11

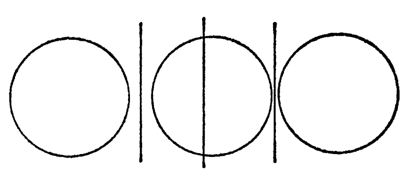

FIG. 12

(6.1) *The three mid-circles of an Apollonian triad are coaxal.*

If the Apollonian triad can be inverted into three congruent circles, their mid-circles become their radical axes [7, p. 35], which are either concurrent (Figure 11) or parallel (Figure 12), and of course three concurrent or parallel lines are a special case of three coaxal circles. If, on the other hand, the Apollonian triad can be inverted into three homothetic circles (inscribed in an angle), their mid-circles become concentric circles (Figure 13), which are another special case of coaxal circles.

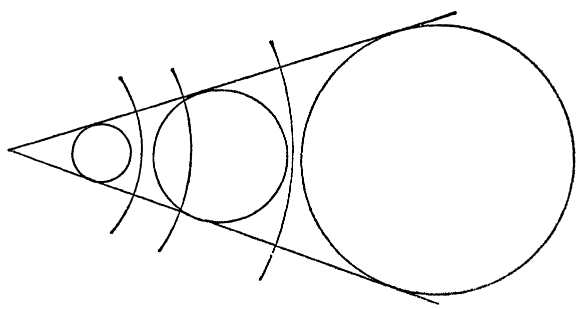

FIG. 13

The most familiar Euclidean example of an Apollonian triad consists of three circles all outside one another. In this case the centres of the three mid-circles are the external centres of similitude of the pairs of circles (Figure 14). Since coaxal circles have collinear centres, the inversive Theorem (6.1) has the Euclidean corollary

(6.2) *If three non-intersecting circles of different sizes are mutually external, so that every two of them have four common tangents, then the three points of intersection of the pairs of external common tangents are collinear.*

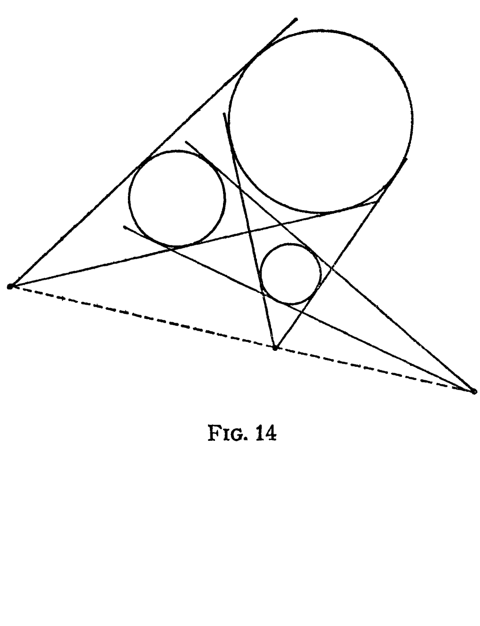

FIG. 14

This result is of special interest in the present context, because it was used as a lemma in Apollonius' own solution of his problem [10, p. 182]. Although it follows easily from Menelaus [2, p. 188], it pleased Herbert Spencer so much that he wrote of it as "a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe" [15, pp. 187–188; see also pp. 606–608].

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References

1. Philip Beecroft, Properties of circles in mutual contact, Lady's and Gentleman's Diary, 1842, pp. 91–96; 1846, p. 51.
2. N. A. Court, College Geometry, 2nd ed., Barnes and Noble, New York, 1952.
3. H. S. M. Coxeter, A geometrical background for de Sitter's World, this MONTHLY, 50 (1943) 217–228.
4. ———, Introduction to Geometry, Wiley, New York, 1961.
5. ———, Geometry, In vol. 3 of T. L. Saaty's *Lectures on Modern Mathematics*, Wiley, New York, 1965.
6. ———, Inversive distance, Ann. Mat. Pura Appl., (4), 71 (1966) 73–83.
7. ——— and S. L. Greitzer, Geometry Revisited, New Math. Library, Vol. 19, New York and Syracuse, 1967.
8. René Descartes, Œuvres, Publiées par C. Adam et P. Tannery, vol. 4, Paris, 1901.
9. Patrick Du Val, Geometrical note on de Sitter's World, Phil. Mag. (6), 47 (1924) 930–938.
10. Thomas Heath, A History of Greek Mathematics, Vol. 2, Oxford, 1921.
11. R. A. Johnson, Advanced Euclidean Geometry, Dover, New York, 1960.
12. R. F. Muirhead, On the number and nature of the solutions of the Apollonian contact problem, Proc. Edinburgh Math. Soc., 14 (1896) 135–147.
13. Daniel Pedoe, On a theorem in geometry, this MONTHLY, 74 (1967) 627–640.
14. Frederick Soddy, The kiss precise, Nature, 137 (1936) 1021.
15. Herbert Spencer, Autobiography, Vol. 1, New York, 1904.
16. Eduard Study, Beiträge zur nicht-euklidische Geometrie, Amer. J. Math., 29 (1907) 101–167.

ON UNIQUENESS OF OPTIMAL BASIC SOLUTIONS TO LINEAR PROGRAMS

C. B. MILLHAM, Iowa State University

We consider a linear programming problem of the form:

$$\text{maximize } cx \text{ subject to } Ax \leqq b,\ x \geqq 0,$$

where A is an $m\times n$ matrix, x, c are n-dimensional column and row vectors, respectively, and b is an m-dimensional column vector, together with the dual,

$$\text{minimize } ub \text{ subject to } uA \geqq c,\ u \geqq 0,$$

where u is an m-dimensional row vector.

We wish to consider, under the further assumption of nondegeneracy in basic solutions, some results published by Goldman and Tucker [1], in order to derive a result (Theorem 3) on uniqueness of optimal bases to the dual programs.

Goldman and Tucker make the following definition:

Let M_1 and M_2 be arbitrary subsets of the set M of indices $\{1, \cdots, m\}$, while N_1 and N_2 are arbitrary subsets of the set N of indices $\{1, \cdots, n\}$. Then each of

$$\text{I:}\quad \begin{aligned} (Ax)_i &= b_i, &&\text{all } i \text{ in } M_1\\ x_j &= 0, &&\text{all } j \text{ in } N_2\end{aligned}$$

and

$$\text{II:}\quad \begin{aligned} (uA)_j &= c_j, &&\text{all } j \text{ in } N_1\\ u_i &= 0, &&\text{all } i \text{ in } M_2\end{aligned}$$

is a system of equated constraints. Systems (I) and (II) are *dual* if $M_2 = M - M_1$ and $N_2 = N - N_1$. In this case we renumber so that $M_1 = \{1, \cdots, p\}$, $N_1 = \{1, \cdots, q\}$, and the systems may be written as

$$\begin{array}{ll} \text{I:}\ a_{11}x_1 + \cdots + a_{1q}x_q = b_1 & \text{II:}\ u_1a_{11} + \cdots + u_pa_{p1} = c_1,\\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\\ a_{p1}x_1 + \cdots + a_{pq}x_q = b_p & u_1a_{1q} + \cdots + u_pa_{pq} = c_q\\ x_{q+1} = \cdots = x_n = 0 & u_{p+1} = \cdots = u_m = 0. \end{array}$$

If the matrix

$$A_1 = \begin{Bmatrix} a_{11} & \cdots & a_{1q}\\ \cdot & \cdot\ \cdot\ \cdot & \cdot\\ a_{p1} & \cdots & a_{pq}\end{Bmatrix}$$

is nonsingular, then a system of equated constraints written in one of the above forms is a *nonsingular square system.*

A *feasible* vector x is one satisfying the constraints $Ax \leqq b$, $x \geqq 0$. A vector x is an *extreme* feasible vector if x cannot be written as the mean of two other feasible vectors x', x''.

Goldman and Tucker then prove the following results. The numbering is theirs.

THEOREM 7. *Feasible vectors x and u are optimal if and only if they satisfy dual systems of equated constraints.*

THEOREM 8. *There is a unique pair of dual systems of equated constraints such that each is the maximal system of equated constraints satisfied by all optimal vectors.*

THEOREM 9. *A feasible nonzero x or u is an extreme feasible vector if and only if it satisfies a nonsingular square system of equated constraints.*

THEOREM 10. *Feasible nonzero x and u are extreme optimal vectors if and only if they satisfy dual nonsingular square systems of equated constraints.*

The authors also make the following geometric observations: since the set of points satisfying a set of linear inequalities is by definition a convex polyhedral set, the set of feasible vectors for a linear programming problem is such a convex polyhedral set. The set of feasible vectors x satisfying (I) forms, if nonempty, a *face* of the convex polyhedral set of constraints for the maximization problem, and the set of feasible u satisfying (II) forms, if nonempty, a face of the convex polyhedral set of constraints for the minimization problem. If (I) and (II) are dual, the faces they determine are dual. Theorem 7, then, says that feasible x and u are optimal if and only if they lie in dual faces. Theorem 8 states that the optimal vectors for each problem form a face of the corresponding convex polyhedral set of feasible vectors, that these two "optimal" faces are dual, and that they are the only pair of dual faces consisting entirely of optimal vectors. A nonsingular system of equated constraints leads to 0-dimensional faces, or "vertices," and Theorem 9 states that the notion of "extreme vector" is essentially that of "vertex." Theorem 10 states that feasible nonzero x or u are optimal if and only if they are dual vertices of the set of feasible vectors.

Normal solution procedures for solving a maximization problem call for augmenting the coefficient matrix A with an $m\times m$ identity matrix I to get the system $Ax+I\bar{x}=b$, $x\geqq 0$, $\bar{x}\geqq 0$. The column vectors of I are called *slack* vectors, and the components of $\bar{x}$ are *slack* variables. We will say that a vector $\operatorname{col}(x, \bar{x})$ is *feasible* if it satisfies $Ax+I\bar{x}=b$, $x\geqq 0$, $\bar{x}\geqq 0$. A feasible vector $\operatorname{col}(x, \bar{x})$ is a *basic* feasible vector if the column vectors of (A, I) associated with the positive components of $\operatorname{col}(x, \bar{x})$ form a basis for R^m or if, in case b is expressed by $\operatorname{col}(x, \bar{x})$ as a positive combination of $h<m$ column vectors from (A, I), there are $m-h$ other vectors in (A, I) that can be adjoined to those in the positive combination to form a basis for R^m. In the latter case, $\operatorname{col}(x, \bar{x})$ is said to be a *degenerate* basic feasible vector. We will refer to a basis associated with a basic feasible vector as a feasible basis, and to any basis associated with a degenerate basic feasible vector as a degenerate feasible basis.

It is easy to show that if $\operatorname{col}(x, \bar{x})$ is a basic feasible vector, then x is extreme, and conversely.

In solving, a sequence of basic feasible solutions together with concurrent evaluations of the objective function cx is determined, until an optimal feasible solution is found. We now reexamine some of the results of Goldman and Tucker. Our results have the following geometric interpretations: Theorem 1 states that if the vector $\operatorname{col}(x, \bar{x})$ associated with a basic feasible solution is nondegenerate, then x defines a unique 0-dimensional face of the convex polyhedral set of feasible vectors for the maximization problem. Theorem 2 states that, in the absence of degeneracy, there is a 1-1 correspondence between feasible bases and 0-dimensional faces of the convex polyhedral set of constraints. Theorem 3 states that if the optimal face for the maximization problem is 0-dimensional and

there is a vector in the optimal face for the minimization problem that satisfies only one nonsingular system of equated constraints, then the optimal face for the minimization problem is also 0-dimensional.

THEOREM 1. *If the vector* $\operatorname{col}(x, \bar{x})$ *associated with a basic feasible solution is nondegenerate, then* x *satisfies a unique nonsingular (square) system of equated constraints.*

Proof. Suppose the basis consists of k slack vectors and $m-k$ columns of A, and suppose further (with no loss in generality) that any basis slack vector is nonzero only in one of the last k components, so that the system of associated equations can be written

$$\text{(A)}\quad \begin{pmatrix} a_{11} & a_{12} \cdots a_{1m-k} & 0 \cdots 0 \\ \cdots & \cdots & \cdots \\ a_{m-k1} & a_{m-k2} \cdots a_{m-k,m-k} & 0 \cdots 0 \\ a_{m-k+1,1} & \cdots \quad a_{m-k+1,m-k} & 1 \cdots 0 \\ \cdots & \cdots & \cdots \\ a_{m\,1} & \cdots \quad a_{m\,m-k} & 0 \cdots 1 \end{pmatrix} \operatorname{col}(x_1, \cdots, x_{m-k}, \bar{x}_{m-k+1}, \cdots, \bar{x}_m) = \operatorname{col}(b_1, \cdots, b_m).$$

We note that $x_{m-k+1} = \cdots = x_n = 0$, and that the first $m-k$ equations are

$$\text{(B)}\quad \begin{pmatrix} a_{11} & \cdots & a_{1m-k} \\ \cdot & \cdots & \cdot \\ a_{m-k1} & \cdots & a_{m-km-k} \end{pmatrix} \operatorname{col}(x_1, \cdots, x_{m-k}) = \operatorname{col}(b_1, \cdots, b_{m-k}).$$

Now the columns of the matrix in (B) are linearly independent, for if not, there would be a vector $x' \neq 0$ such that

$$\begin{pmatrix} a_{11} & \cdots & a_{1m-k} \\ \cdot & \cdots & \cdot \\ a_{m-k1} & \cdots & a_{m-km-k} \end{pmatrix} \operatorname{col}(x_1', \cdots, x_{m-k}') = \operatorname{col}(0, \cdots, 0)$$

and since $x_i > 0$, $i = 1, \cdots, m-k$, there is $\epsilon > 0$ such that $\operatorname{col}(x_1, \cdots, x_{m-k}) + \epsilon \operatorname{col}(x_1', \cdots, x_{m-k}')$, $\operatorname{col}(x_1, \cdots, x_{m-k}) - \epsilon \operatorname{col}(x_1', \cdots, x_{m-k}')$ are both $\geqq 0$, and by adjoining $n-(m-k)$ 0-components to each we produce two n-dimensional column vectors having $\operatorname{col}(x_1, \cdots, x_{m-k}, 0, \cdots, 0)$ as their mean, contradicting the assumption that x is extreme (because basic). Hence the rows of this matrix are linearly independent also, and (B) is one nonsingular square system of equated constraints satisfied by x.

By the nondegeneracy assumption, $\bar{x}_i > 0$ for $i = m-k+1, \cdots, m$, and $x_j > 0$ for $j = 1, \cdots, m-k$. Thus, if x satisfies any other system of equated constraints defined by a set $\overline{M}_1$ and a set $\overline{N}_2$ of indices, then $\overline{M}_1$ must be a subset of $M_1 = \{1, 2, \cdots, m-k\}$ and $\overline{N}_2$ must be a subset of $N_2 = \{m-k+1, \cdots, n\}$. Thus, if $\overline{M}_1 \neq M_1$ or $\overline{N}_2 \neq N_2$, the resultant matrix determined by $\overline{M}_1$ and $\overline{N}_1$

$=N-\overline{N}_2$ must have fewer than $m-k$ rows or more than $m-k$ columns. In neither case can the matrix be square. This proves the theorem.

COROLLARY. *If* col$(x, \bar{x})$ *is nondegenerate, feasible, and basic, the rank of the matrix associated with the nonsingular system of equated constraints is given by* $m-k$, *where* m *is the number of constraint inequalities in the problem and* k *is the number of slack vectors in the basis associated with* col$(x, \bar{x})$.

THEOREM 2. *In the absence of degeneracy, distinct feasible bases determine distinct nonsingular systems of equated constraints.*

Proof. Let col$(x, \bar{x})$ and col$(x', \bar{x}')$ be vectors generated by two distinct feasible and nondegenerate bases from the augmented system

$$Ax + I\bar{x} = b$$
$$x \geqq 0, \qquad \bar{x} \geqq 0.$$

Since both bases are nondegenerate, each determines a pair (M_1, N_2),(M_1', N_2') of indices i, j, where, as before, M_1 and M_1' are subsets of $\{1, \cdots, m\}$ and N_2, N_2' are subsets of $\{1, \cdots, n\}$. Suppose $M_1 = M_1'$, $N_2 = N_2'$. Then the matrices determined by the indices $i \in M_1$, $j \in N_2$ and by the indices $i \in M_1'$, $j \in N_2'$, respectively, are equal, and we may renumber so as to assume $M_1 = M_1' = \{1, \cdots, p\}$, $N_2 = N_2' = \{1, \cdots, p\}$, and the matrix is

$$\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ a_{p1} & \cdots & a_{pp} \end{pmatrix}.$$

Since the matrix is nonsingular, we have $x_j = x_j'$, $j = 1, \cdots, p$, and $x_j = x_j' = 0$, $j = p+1, \cdots, n$. For $i \notin M_1 = M_1'$, we have

$$a_{i1}x_1 + \cdots + a_{ip}x_p = a_{i1}x_1' + \cdots + a_{ip}x_p' < b_i.$$

For both bases, b is expressed as a (unique) linear combination of the vectors in the particular basis, and the ith slack vector is a member of both bases. Furthermore, its coefficient in each of the two linear combinations is

$$\bar{x}_i = b_i - a_{i1}x_1' - \cdots - a_{ip}x_p' = b_i - a_{i1}x_1 - \cdots - a_{ip}x_p > 0.$$

This completes the proof.

THEOREM 3. *Suppose that the augmented maximization problem has a unique optimal feasible basis and that the augmented dual problem has an optimal feasible basis that is nondegenerate. Then the augmented dual problem has a unique optimal feasible basis.*

Proof. Let col$(x, \bar{x})$ be the optimal solution vector to the augmented problem

$$\text{maximize } c'x \text{ subject to } Ax + I\bar{x} = b, \qquad x \geqq 0, \quad \bar{x} \geqq 0.$$

By Theorem 1, x satisfies a unique nonsingular system of equated constraints, and by Theorem 10 of Goldman and Tucker, feasible x, u are extreme optimal

vectors for the dual problems if and only if they satisfy dual nonsingular systems of equated constraints.

Let the nonsingular system of equated constraints satisfied by x be designated by $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$, $i \in M_1$, $x_j = 0$, $j \in N_2$, where the cardinality of M_1 is equal to the cardinality of $N - N_2$. It is possible to renumber so that the system can be written

$$\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ a_{p1} & \cdots & a_{pp} \end{pmatrix} \operatorname{col}(x_1, \cdots, x_p) = \operatorname{col}(b_1, \cdots, b_p), \quad x_{p+1} = \cdots = x_n = 0.$$

Here $M_1 = \{1, \cdots, p\}$, $N - N_2 = \{1, \cdots, p\}$. If u is any optimal extreme vector for the dual, we see that u must then satisfy the equated constraints dual to this set, specifically,

$$(u_1, \cdots, u_p) \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{p1} & \cdots & a_{pp} \end{bmatrix} = (c_1, \cdots, c_p),\ u_{p+1} = \cdots = u_n = 0.$$

Since

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{p1} & \cdots & a_{pp} \end{bmatrix}$$

is nonsingular, $(u_1, \cdots, u_p)$ is uniquely determined, hence so is $(u_1, \cdots, u_p, 0 \cdots 0)$. We wish to find a basic solution for the augmented dual problem

$$\begin{gathered} \text{minimize } ub \text{ subject to} \\ A^T u^T - I\bar{u}^T = c, \quad u \geqq 0, \quad \bar{u} \geqq 0, \quad \text{or} \\ uA - \bar{u}I = c, \quad u \geqq 0, \quad \bar{u} \geqq 0. \end{gathered}$$

There are n constraint inequalities in the system $A^T u^T \geqq c$; since $u_{p+1} = \cdots = u_m = 0$ and we are assuming nondegeneracy, none of the last $m - p$ columns of A^T can appear in the basic solution, hence $n - p$ vectors from $(-I)$ must appear in the basis.

By nondegeneracy, they correspond to $n - p$ nonzero components of $\bar{u}$. Since

$$\bar{u}_j = (uA)_j - c_j = 0 \qquad \text{for } j = 1, \cdots, p,$$

these components must be the $\bar{u}_j$ for $j = p+1, \cdots, n$. Thus the optimal basis for the dual problem is uniquely determined (with our renumbering) as consisting of the first p columns of A^T and the last $n - p$ columns of $(-I)$, and this completes the proof.

Reference

1. A. J. Goldman and A. W. Tucker, Theory of Linear Programming, editors: H. W. Kuhn, and A. W. Tucker, Linear Inequalities and Related Systems, Princeton University Press, 1956 pp. 53–97.

MERSENNE-FORM AND FERMAT-FORM NUMBER CONGRUENCES

R. R. SEEBER, IBM, Poughkeepsie, N. Y.

This note gives congruences for Mersenne-form and Fermat-form numbers, the principal results being given by Theorem 2 and by (6) and (10) of Table 1; both (6) and (10) are similar in form to Wilson's Theorem.

All results after Theorem 2 are given in tabular form. Tables 2 and 3 give subsidiary statements, not all new, some of which are required for Table 1 proofs.

We define Mersenne-form and Fermat-form numbers by

$$M(n) = 2^n - 1, \tag{1}$$

and

$$F(n) = 2^n + 1. \tag{2}$$

Here and in what follows the letters denote nonnegative integers unless further restricted.

The proofs depend on a theorem given by an anonymous writer [1]. He employed nth roots of unity and the irreducibility of the associated cyclotomic equation to prove:

THEOREM 1. *If n is a prime, then the sums of the numbers $1, 2, 3, \cdots, n-1$ taken t at a time, for a fixed t, $0 \leqq t < n$, when divided by n give each of the residues $1, 2, 3, \cdots, n-1$ an equal number of times, $D(n-1, t)$, and the residue zero one more time or one less time according as t is even or odd.*

REMARK. $D(n-1, t)$ is given by $(\binom{n-1}{t} - (-1)^t)/n$.

Also since $2^n \equiv 1 \pmod{M(n)}$, we have

$$2^{kn+m} \equiv 2^m \pmod{M(n)}. \tag{3}$$

Expanding the products gives, for $n > 1$,

$$M(1)M(2) \cdots M(n-1) = \sum_{t=0}^{n-1} (-1)^{n-1-t} A(n-1, t) \tag{4}$$

and

$$F(1)F(2) \cdots F(n-1) = \sum_{t=0}^{n-1} A(n-1, t), \tag{5}$$

where $A(n-1, t)$ is the sum of all those terms of the complete product in (5) that can be formed by selecting t of the powers of 2 and $n-1-t$ of the 1's.

We see that the terms in the sum $A(n-1, t)$ are powers of 2, some of which may equal or exceed 2^n. By virtue of (3) we may reduce these to powers less than n but greater than or equal to zero, thus forming the new set of coefficients

$B(n-1, t)$ where

$$B(n-1, t) \equiv A(n-1, t) (\bmod M(n)).$$

But the exponents in the powers of 2 in the sums $B(n-1, t)$ were thus derived exactly in conformity with Theorem 1. Hence, if n is prime, we have:

$$\begin{aligned} B(n-1, t) &= 2^0(D(n-1, t) + (-1)^t) + D(n-1, t)(2^1 + 2^2 + \cdots + 2^{n-1}) \\ &= D(n-1, t)(2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}) + (-1)^t \equiv (-1)^t (\bmod M(n)). \end{aligned}$$

This gives:

THEOREM 2. *$A(n-1, t) \equiv (-1)^t$ (mod $M(n)$) if n is prime.*

This theorem is somewhat more than we need to prove (6) and (10), which now follow from (4) and (5), respectively.

Also, since $A(n, t) = 2^n A(n-1, t-1) + A(n-1, t)$ for $0 < t < n$, we have by Theorem 2 and (3):

COROLLARY. *$A(n, t) \equiv 0$ (mod $M(n)$) for $0 < t < n$ if n is prime.*

TABLE 1. Congruences

(1) $M(n) = 2^n - 1$; (2) $F(n) = 2^n + 1$.

Ref. No.	*Congruences*	*Conditions*	*Refs. for Proofs*
(6)	$M(1)M(2) \cdots M(n-1) \equiv (-1)^{n-1} n$ (mod $M(n)$)	if n is a prime.	Th. 2
(7)	$M(1)M(2) \cdots M(n-1) \equiv \pm n$ (mod $M(n)$)	only if n is a prime.	(15), (20)
(8)	$M(1)M(2) \cdots M(n-1) \equiv 0$ (mod $M(n)$)	iff $n = 6$.	Ref. [3]
(9)	$M(1)M(2) \cdots M(n-1) \equiv (n/2)M(n/2)$ (mod $M(n)$)	if $n = 2^{k+1}$.	(17)
(10)	$F(1)F(2) \cdots F(n-1) \equiv 1$ (mod $M(n)$)	if n is an odd prime.	Th. 2
(11)	$F(1)F(2) \cdots F(n-1) \equiv 0$ (mod $M(n)$)	iff $n = 2^{k+1}$.	(10), (15), (21)
(12)	$F(1)F(2) \cdots F(n-1) \equiv F(n/2)$ (mod $M(n)$)	if $n/2$ is an odd prime.	(10), (16)
(13)	$M(1)M(2) \cdots M(n-1) \equiv \pm F(1)F(2) \cdots F(n-1)$ (mod $F(n)$)	if $n = 4(k+1) \pm 1$.	(4), (5)

The converse of (6) is included in (7), and (8) was given by Zsigmondy [3]. We return later for the proofs of (7), (9), (11), (12), and (13).

In Table 2 we give quotients and remainders for divisions of M's and F's. We need only the remainders but also give the quotients for the proofs. Let $Q(x, y)$ be the quotient and $R(x, y)$ be the remainder on dividing x into y in the usual way, i.e.,

$$y = xQ(x, y) + R(x, y), \qquad 0 \leqq R(x, y) < x. \tag{14}$$

By dividing $M(5)$ into $M(12)$ in binary, it is easy to "see" the derivation of (15). (Compare [2].)

TABLE 2. Quotients and Remainders

(1) $M(n) = 2^n - 1$; (2) $F(n) = 2^n + 1$; (14) $y = xQ(x, y) + R(x, y)$, $0 \leqq R(x, y) < x$.

Ref. No.	x	y	$Q(x, y)$	$R(x, y)$	Conditions
(15)	$M(b)$	$M(a)$	$\left(\sum_{i=0}^{Q(b,a)-1} 2^{bi}\right) 2^{R(b,a)}$	$M(R(b, a))$	$b > 0$
(16a)	$M(b)$	$F(a)$	$\left(\sum_{i=0}^{Q(b,a)-1} 2^{bi}\right) 2^{R(b,a)}$	$F(R(b, a))$	$\begin{cases} b > 2 \text{ or} \\ b = 2 \text{ and } R(b, a) = 0 \end{cases}$
(16b)	$M(b)$	$F(a)$	$\left(\sum_{i=0}^{Q(2,a)-1} 2^{2i}\right) 2 + 1$	0	$b = 2$ and $R(b, a) = 1$
(16c)	$M(b)$	$F(a)$	$F(a)$	0	$b = 1$
(17a)	$F(b)$	$M(a)$	$\left(\sum_{i=0}^{Q(2b,a)-1} 2^{2bi}\right) 2^{R(2b,a)}M(b)$	$M(R(b, a))$	$R(2b, a) < b,\ b > 0$
(17b)	$F(b)$	$M(a)$	$\left(\sum_{i=0}^{Q(2b,a)-1} 2^{2bi}\right) 2^{R(b,a)+b}M(b) + M(R(b,a))$	$M(b - R(b, a))2^{R(b,a)} = M(b) - M(R(b, a)) > 0$	$0 < b \leqq R(2b, a)$
(17c)	$F(b)$	$M(a)$	$M(a - 1)$	1	$a > 0,\ b = 0$
(17d)	$F(b)$	$M(a)$	0	0	$a = b = 0$
(18a)	$F(b)$	$F(a)$	$\left(\sum_{i=0}^{Q(2b,a)-1} 2^{2bi}\right) 2^{R(2b,a)}M(b)$	$F(R(b, a))$	$R(2b, a) < b,\ b > 0$
(18b)	$F(b)$	$F(a)$	$\left(\sum_{i=0}^{Q(2b,a)-1} 2^{2bi}\right) 2^{b}M(b) + 1$	0	$R(2b, a) = b,\ b > 0$
(18c)	$F(b)$	$F(a)$	$\left(\sum_{i=0}^{Q(2b,a)-1} 2^{2bi}\right) 2^{R(b,a)+b}M(b) + M(R(b,a))$	$M(b - R(b, a))2^{R(b,a)} + 2 = F(b) - M(R(b, a)) > 0$	$0 < b < R(2b, a)$
(18d)	$F(b)$	$F(a)$	2^{a-1}	1	$a > 0,\ b = 0$
(18e)	$F(b)$	$F(a)$	1	0	$a = b = 0$
(19)	$F(b) - M(R(b, a))$	$F(b)$	1	$M(R(b, a)) > 0$	$0 < b < R(2b, a)$

$$10000100 = (2^5 + 1)2^2 = \left(\sum_{i=0}^{Q(5,12)-1} 2^{5i}\right) 2^{R(5,12)}$$

```
       10000100
11111)111111111111
       11111
          11111
          11111
             11 = M(2) = M(R(5, 12)).
```

In similar fashion we may derive (16), (17), and (18); however, the proofs in each case, including (19), follow directly by seeing that (14) is satisfied for the given conditions.

TABLE 3. Highest Common Divisors

(1) $M(n)=2^n-1$; (2) $F(n)=2^n+1$; (14) $y=xQ(x, y)+R(x, y)$, $0\leqq R(x, y)<x$.

Ref. No.	x	y	(x, y)	Conditions	Refs. for Proofs
(20)	$M(b)$	$M(a)$	$M((b, a))$	$b>0$.	(15)
(21a)	$F(b)$	$M(a)$	$F(b)$	$a=0$.	(15), (16)
(21b)	$F(b)$	$M(a)$	$F((b, a))$	$a>0$, $b>0$, $R(2(b, a), a)=0$.	(17), (18)
(21c)	$F(b)$	$M(a)$	1	Otherwise than for (21a) and (21b).	
(22a)	$F(b)$	$F(a)$	$F(0)$	$a=b=0$.	(16), (18)
(22b)	$F(b)$	$F(a)$	$F((b, a))$	$a>0$, $b>0$, $R(2(b, a), a)>0$, $R(2(b, a), b)>0$.	(19)
(22c)	$F(b)$	$F(a)$	1	Otherwise than for (22a) and (22b).	

In (20), (21), and (22) of Table 3 we give highest common divisors for M's and F's. By considering the remainders in successive steps of Euclid's algorism, we see that their possible forms are given in the remainders column of Table 2. A little consideration shows that divisors in all cases will have to be of form $F(u)$ or $M(v)$. Then Table 2 is used to identify the divisors that will leave a zero remainder for both arguments of (x, y).

Returning to Table 1, consider the proof of (7). Assume n is composite with

$$n = \prod_{i=1}^{k} p_i^{s_i} \tag{23}$$

where the p_i are distinct primes and $q_i=p_i^{s_i}$. Since $q_i | n$, $M(q_i) | M(n)$ by (15); and $M(q_i) | M(1)M(2) \cdots M(n-1) \pm n$ for $i=1, 2, \cdots, k$. If $k>1$, then $q_i<n$ and $M(q_i)$ is among the factors $M(1), M(2), \cdots, M(n-1)$; hence, $M(q_i) | n$ also. Since n is a common multiple of all the $M(q_i)$, it is a multiple of their least common multiple. Since the q_i are coprime, the $M(q_i)$ are also coprime by (15). Hence the least common multiple of the $M(q_i)$ is their product, and we have $M(q_1)M(q_2) \cdots M(q_k) | q_1q_2 \cdots q_k$. But this is impossible since $M(q_i)>q_i$ for $q_i>1$. Thus for $k>1$, n cannot be composite.

If $k=1$, then $n=q_1=p_1^{s_1}$ and $s_1>1$. Since $p_1 | n$, $M(p_1) | M(n)$ by (15); and $M(p_1) | M(1)M(2) \cdots M(n-1) \pm n$. Now $p_1<n$ and $M(p_1)$ is among the factors $M(1), M(2), \cdots, M(n-1)$; hence, $M(p_1) | n$ also. Since $M(p_1)>1$, we must have $M(p_1)=p_1^t$ where $0<t\leqq s_1$ and $p_1 | M(p_1)$. This is false for $p_1=2$. If $p_1>2$, then $p_1 | M(p_1-1)$ by Fermat's Theorem. But p_1 cannot divide both $M(p_1)$ and $M(p_1-1)$ since $(M(p_1), M(p_1-1))=M((p_1, p_1-1))=1$ by (20). Hence n is not composite, which completes the proof of (7).

Next consider the proof of (9). Since

$$M(n) = M(2^{k+1}) = M(2^k)F(2^k) = M(n/2)F(n/2),$$

we must show that

$$C = M(1)M(2) \cdots M(2^k - 1)M(2^k + 1)M(2^k + 2) \cdots M(2^k + 2^k - 1) - 2^k \equiv 0 \pmod{F(2^k)}.$$

Now by (17b) we have $M(2^k + x) \equiv M(2^k) - M(x) \pmod{F(2^k)}$ if $0 < 2^k \leqq R(2^{k+1}, 2^k+x)$, i.e., if $0 \leqq x < 2^k$. But $M(2^k) - M(x) = F(2^k) - 2 - (F(x) - 2) = F(2^k) - F(x)$. Thus $M(2^k+x) \equiv -F(x)$ (mod $F(2^k)$ if $0 \leqq x < 2^k$ and

$$C \equiv M(1)M(2) \cdots M(2^k - 1)(-1)F(1)F(2) \cdots F(2^k - 1) - 2^k \pmod{F(2^k)}$$

or $C \equiv (-1)^1 M(2)M(4) \cdots M(2^{k+1}-2) - 2^k \pmod{F(2^k)}$. This telescoping is repeated, giving $C \equiv (-1)^2 (M(2^k))^1 M(4)M(8) \cdots M(2^{k+1}-4) - 2^k$ and finally $C \equiv (-1)^k (M(2^k))^k - 2^k \pmod{F(2^k)}$. Since this expression is divisible by $M(2^k)+2 = F(2^k)$, thus $C \equiv 0 \pmod{F(2^k)}$, thereby completing the proof of (9).

Now consider the proof of (11). The first part follows immediately since $M(2^{k+1}) = F(2^0)F(2^1) \cdots F(2^k)$. If n is not a power of 2, it is either an odd prime or has an odd prime divisor p; both of these cases lead to contradictions. In the former case, $F(1)F(2) \cdots F(n-1) \equiv 1 \pmod{M(n)}$ by (10). In the latter case with $n = pm$, we have $M(p) \mid M(n)$ by (15) and thus $M(p) \mid F(1)F(2) \cdots F(n-1)$. But by (21c), $(F(x), M(p)) = 1$ for $0 < x < n$, since $p > 0$ and $R(2(x, p), p) > 0$. Thus $M(p) \nmid F(1)F(2) \cdots F(n-1)$, completing the proof of (11).

To prove (12), it is only necessary to show that

$$F(1)F(2) \cdots F(n/2 - 1)F(n/2 + 1)F(n/2 + 2) \cdots F(n - 1) \equiv 1 \pmod{M(n/2)}, \tag{24}$$

since $M(n) = F(n/2)M(n/2)$ with $n/2$ an odd prime. By (16a), $F(n/2+x) \equiv F(x) \pmod{M(n/2)}$, since $n/2 > 2$ for $0 < x < n/2$. Also $F(1)F(2) \cdots F(n/2-1) \equiv 1 \pmod{M(n/2)}$ by (10). Hence (24) is satisfied and (12) is proved.

Finally, to prove (13), we observe, first, that $2^{kn+m} \equiv (-1)^k 2^m \pmod{F(n)}$, which is similar to (3); and, second, that $A(x, x-t) = 2^{x(x+1)/2-(x+1)t} A(x, t)$, which follows from symmetry considerations in the definition of $A(x, t)$. Now applying (4) and (5), we have (13) directly.

As to the converses of (9), (10), (12), and (13), we can offer them only as conjectures; the conjectures have been verified by computation for $n < 35$ for the converse of (13) and for $n < 71$ for the others. For the converse (10), we see by (21) that there is no immediate proof in the manner of that for (7); obviously n must be odd. Also it appears likely that there are additional relationships similar to (9) and (12).

The author is indebted to L. Hellerman for a suggestion used in the proofs and to the referee for directing attention to reference 3.

References

1. Anonymous, Théorèmes et problèmes sur les nombres, Jour. für Math. (Crelle), 6 (1830) 100–106. See also L. E. Dickson, History of the Theory of Numbers, Vol. 1, p. 327, note 1.

2. D. Shanks, Solved and Unsolved Problems in Number Theory, Spartan, Washington, 1962, 17–19.

3. K. Zsigmondy, Zur Theorie der Potenzreste, Monatshefte Math. Phys., 3 (1892) 265–284. See also Dickson, History, p. 386, note 44.

A SIMPLE PROOF OF THE DENJOY-CARLEMAN THEOREM

P. J. COHEN, Stanford University

It is a well-known theorem that if f is an analytic function and $f^{(n)}(a)=0$ for all $n\geqq 0$, then f is identically zero. On the other hand there are infinitely differentiable functions defined on the real line such as $f(x)\equiv\exp(-1/x^2)$ which are not identically zero and $f^{(n)}(0)=0$ for all n. An analytic function may be defined as one whose Taylor series converges to it, which by the usual formula for the remainder term, will happen if the derivatives do not grow too rapidly in a given interval. More exactly, if for x in $[-a, a]$, $|f^{(n)}(x)|\leqq An!b^n$ where $b<1/a$, then the Taylor series for f converges uniformly to f in $[-a, a]$. Conversely, if f is analytic at $x=0$, the same inequality holds for some interval $[-a, a]$. This implies that a function such as $\exp(-1/x^2)$ must have its derivatives growing rapidly in an interval around zero. The theorem of Denjoy-Carleman is concerned with the rate of growth of the derivatives of such functions.

Let M_n be a sequence of positive numbers. We denote by $C\{M_n\}$ the class of infinitely differentiable functions on a given interval I satisfying $|f^{(n)}(x)|\leqq b^nM_n$ for some b, where b depends on f. By the above remarks the class $C\{n!\}$ is exactly the class of analytic functions. In 1912 Hadamard posed the question of determining those sequences M_n such that if $f\in C\{M_n\}$ and f and all its derivatives vanish at a point of I, then f is identically zero. Such a class we call a quasi-analytic class. Partial results were obtained by Denjoy and the answer in a definitive form was given by Carleman.

THEOREM. *Let $\beta_n=\inf_{k\geq n} M_k^{1/k}$. Then $C\{M_n\}$ is quasi-analytic if and only if $\sum 1/\beta_n=\infty$.*

The condition of Carleman can be put in a somewhat more convenient form. In the definition of β_n, the inf is necessary because if M_k is small for some large value of k, this may restrict the function more drastically than do the M_k for smaller values of k. In some sense, we would like to eliminate any abnormally low values of M_k for large k. First note that if $\liminf M_k^{1/k}<\infty$, then $C\{M_k\}$ is trivially quasi-analytic, since if $f^{(n)}(0)=0$ for all n, then repeated integration yields

$$|f(x)|\leqq M_n\frac{|x|^nb^n}{n!}.$$

If we let n tend to infinity, this would imply $f\equiv 0$. Assume then that $\liminf M_k^{1/k}=\infty$. If we plot $\log M_k$ versus k, this means that the lines connecting $(0, 0)$ to $(k, \log M_k)$ have slopes tending to $+\infty$. In turn, this implies the existence of a convex minorant, $\log M_k'$, that is, a sequence M_k' such that

(i) $\log M_k'$ is convex, i.e. $\log M_k'\leqq\frac{1}{2}(\log M_{k-1}'+\log M_{k+1}')$.

(ii) $M_k'\leqq M_k$.

(iii) There is a sequence $0=n_0<n_1<n_2<\cdots$ such that $M_{n_i}=M_{n_i}'$ and $\log M_k$ is a linear function of k for $n_i\leqq k\leqq n_{i+1}$.

It is possible then to show that the above condition on M_k can be rephrased as follows.

THEOREM. *Either of the following two conditions*

$$\text{(i)} \quad \sum (M_n')^{-1/n} = \infty \qquad \text{(ii)} \quad \sum M_n'/M_{n+1}' = \infty$$

is a necessary and sufficient condition for the class $C\{M_n\}$ to be quasi-analytic.

The proof that each of the above conditions is equivalent to the previous one, although elementary, is not entirely obvious. A proof can be found in [1], p. 23, and uses the inequality

$$\sum_{n=1}^{\infty} (a_1 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n.$$

We shall here concern ourselves only with condition (ii). Observe that if we set $g(x)=f(bx)$ for any b, then $g^{(n)}(x)=b^n f^{(n)}(bx)$. This shows that the role of b in the definition of $C\{M_n\}$ is not very significant.

We begin by showing, ([1] p. 103), that if $\sum M_k/M_{k+1}<\infty$, there is a function with compact support such that $|f^{(k)}(x)| \leqq M_k$. Since f has compact support, there will be a point x_0 such that $f^{(n)}(x_0)=0$ for all n. Let χ_i be defined by

$$\chi_i(x) = \frac{1}{2\mu_i} \quad \text{for } |x| \leqq \mu_i,$$

$\chi_i(x)=0$ if $|x|>\mu_i$, where $\mu_i=M_{i-1}/M_i$, $\mu_0=1/M_0$. Then the integral of χ_i is 1. We define f by the infinite convolution

$$f = \chi_0 * \chi_1 * \chi_2 * \cdots.$$

Since $\sum \mu_i$ converges, clearly f has compact support. Also, differentiating yields

$$f^{(k)}(x) = m_0 * \cdots * m_{k-1} * \chi_k * \chi_{k+1} * \cdots,$$

where m_i is the measure whose support is the two points $-\mu_i$, μ_i and which assigns to the first $1/2\mu_i$ and to the second $-(1/2\mu_i)$. Thus

$$|f^{(k)}(x)| \leqq \frac{1}{\mu_0}\cdot\frac{1}{\mu_1}\cdots\frac{1}{\mu_{k-1}} \sup |g_k|,$$

where $g_k=\chi_k * \cdots$. Since all the χ have integral one, $\sup |g_k| \leqq \sup |\chi_k| = 1/2\mu_k$. Thus

$$|f^{(k)}(x)| \leqq \frac{1}{\mu_0}\cdot\frac{1}{\mu_1}\cdots\frac{1}{\mu_k} = M_k.$$

Actually, the above inequalities can at first be applied only to finite products but the uniform bound on the $(k+1)$th derivatives of these products easily implies the convergence of the kth derivatives. Our main concern is the proof of

the converse portion of the theorem. The original proof of Carleman used Fourier transforms and arguments concerning analytic functions of a complex variable. Our proof will be entirely elementary. An elementary proof has previously been given by Bang [1], p. 107. The present proof is very similar to that of Bang but we feel it is more straightforward and makes the essential point clearer.

For the sake of exposition, we first assume that $M_k' = M_k$, i.e. that $\log M_k$ is a convex sequence. Let $f^{(k)}(0) = 0$ for all k and $|f^{(k)}(x)| \leqq M_k$ for all x in the interval $[0, a]$. If $\sum M_k/M_{k+1} = \infty$, we shall show f is identically zero in $[0, a]$. The fact that $\log M_k$ is a convex sequence means that $M_0/M_1 \geqq M_1/M_2 \geqq \cdots$. Let $0 < \alpha < 1$ be fixed, and fix an integer $n > 0$. Define

$$x_0 = 0, \quad x_1 = \alpha \frac{M_{n-1}}{M_n}, \quad x_2 = x_1 + \alpha \frac{M_{n-2}}{M_{n-1}}, \cdots, x_n = x_{n-1} + \alpha \frac{M_0}{M_1}$$

and assume x_n is in $[0, a]$. Clearly $|f^{(k)}(x)| \leqq \alpha^{n-k} M_k$ for x in $[0, x_1]$ if $k \leqq n$. This is seen by induction since if $|f^{(k)}(x)| \leqq \alpha^{n-k} M_k$ for x in $[0, x_1]$,

$$|f^{(k-1)}(x)| \leqq \alpha \frac{M_{n-1}}{M_n} \alpha^{n-k} M_k \leqq \alpha^{n-k+1} M_{k-1}$$

since $M_{n-1}/M_n \leqq M_{k-1}/M_k$.

In general, we shall estimate $|f^{(j)}(x)|$ for x in $[0, x_i]$ if $j \leqq n - i + 1$. To do this we need a definition.

Definition. *For* $0 < \alpha < 1$, *define* $B_{i,j}$ *for* $j \geqq i \geqq 0$ *by* $B_{0,j} = 0$, $B_{i,i} = 1$ *for* $i > 0$, *and* $B_{i+1,j+1} = B_{i,j+1} + \alpha B_{i+1,j}$ *if* $j > i$.

We now assert that

$$|f^{(j)}(x)| \leqq B_{i,n-j+1} M_j \qquad \text{for } x \text{ in } [0, x_i], \quad j \leqq n - i + 1. \tag{1}$$

This is trivial for $i = 0$ and for $j = n - i + 1$ and arbitrary i. If we assume (1) for previous values of i and all greater values of $j \leqq n - i + 1$, then for x in $[x_{i-1}, x_i]$ and $j \leqq n - i$

$$|f^{(j)}(x)| \leqq |f^{(j)}(x_{i-1})| + \alpha \frac{M_{n-i}}{M_{n-i+1}} \cdot B_{i,n-j} M_{j+1}. \tag{2}$$

Now, since $j \leqq n - i$, $M_{n-i}/M_{n-i+1} \leqq M_j/M_{j+1}$ and we have

$$|f^{(j)}(x)| \leqq (B_{i-1,n-j+1} + \alpha B_{i,n-j}) M_j \tag{3}$$

so $|f^{(j)}(x)| \leqq B_{i,n-j+1} M_j$ as asserted.

Lemma 1. *There is a constant* A, *such that for sufficiently small* α,

$$B_{i,j} < A\alpha \quad \text{if} \quad j > i > 0.$$

The theorem is an immediate consequence of the lemma. We first obtain $|f(x)| \leqq A\alpha M_0$ for x in $[0, x_n]$. Now

$$x_n = \alpha\left(\frac{M_0}{M_1} + \cdots + \frac{M_{n-1}}{M_n}\right)$$

so by choosing n large enough our hypothesis implies $|f(x)| \leqq \alpha A M_0$ for all x in $[0, a]$. Since α can be chosen arbitrarily small, $f(x) \equiv 0$.

The proof of the lemma is a simple calculation. If we define $\phi_0(x) \equiv 1$, and $\phi_i(x) = (Cx/i)^i$ for $i \geqq 1$ and C a suitable constant, using the fact that $j^i \geqq (j-1)^i + i(j-1)^{i-1}$ we verify that

$$\phi_{i-1}(j) + \phi_i(j-1) \leqq \phi_i(j) \quad \text{if} \quad i > 0, \quad j \geqq i.$$

From this we deduce by induction, for $i, j \geqq 0$,

$$B_{i+1,j+1} \leqq \phi_i(j)\alpha^j + \phi_{i-1}(j-1)\alpha^{j-1} + \cdots + \phi_0(j-i)\alpha^{j-i}. \tag{4}$$

For, assuming (4) for $B_{i,j+1}$ and $B_{i+1,j}$, we get

$$\begin{aligned} B_{i+1,j+1} &= B_{i,j+1} + \alpha B_{i+1,j} \\ &\leqq \phi_{i-1}(j)\alpha^j + \cdots + \phi_0(j-i+1)\alpha^{j-i+1} \\ &\quad + \phi_i(j-1)\alpha^j + \cdots + \phi_1(j-i)\alpha^{j-i+1} + \phi_0(j-i-1)\alpha^{j-i} \\ &\leqq \phi_i(j)\alpha^j + \cdots + \phi_1(j-i+1)\alpha^{j-i+1} + \phi_0(j-i)\alpha^{j-i}. \end{aligned}$$

Thus $B_{i+1,i+2} \leqq \phi_i(i+1)\alpha^{i+1} + \cdots + \alpha$. Since

$$\phi_n(n+1) = \left(\frac{C(n+1)}{n}\right)^n \leqq eC^n$$

we obtain that for $\alpha < 1/(2C)$

$$B_{i+1,i+2} < \frac{e\alpha}{1 - C\alpha} < 2e\alpha.$$

If $j > i$, $B_{i,j} \leqq B_{j-1,j}$ and the lemma and hence the theorem for $\log M_k$ convex follows.

We now turn to the general case in which the M_k are not logarithmically convex.

LEMMA 2. *Let $\xi_0 < \xi_1 < \cdots < \xi_{n-1} < x$ and assume $|f^{(n)}(t)| \leqq M$ for t in $[\xi_0, x]$. Then*

$$\begin{aligned} |f(x)| \leqq |f(\xi_{n-1})| &+ |f'(\xi_{n-2})|(x - \xi_{n-2}) + \cdots \\ &+ \frac{|f^{(n-1)}(\xi_0)|}{(n-1)!}(x-\xi_0)^{n-1} + \frac{M}{n!}(x-\xi_0)^n. \end{aligned}$$

Proof. By induction on n. The case $n=1$ is trivial. Assume the lemma for $n-1$. Applying it to $f'(x)$ we obtain

$$|f'(u)| \leqq |f'(\xi_{n-2})| + \cdots + \frac{|f^{(n-1)}(\xi_0)|}{(n-2)!}(u-\xi_0)^{n-2} + \frac{M}{(n-1)!}(u-\xi_0)^{n-1}.$$

If we integrate both sides from ξ_{n-1} to x, using the fact that for $i \leqq n-2$

$$\int_{\xi_{n-1}}^{x} (u-\xi_i)^{n-2-i} du \leqq \int_{\xi_i}^{x} (u-\xi_i)^{n-2-i} du = \frac{(x-\xi_i)^{n-1-i}}{(n-1-i)}$$

and

$$|f(x)| - |f(\xi_{n-1})| \leqq \int_{\xi_{n-1}}^{x} |f'(u)| du,$$

we obtain the statement of the lemma.

From the definition of M_k' we know that there is a sequence $0=n_0<n_1<n_2< \cdots$ such that $M_{n_i}=M'_{n_i}$ and M_k'/M_{k+1}' is constant for $n_i \leqq k < n_{i+1}$. Let n be such that $M_n = M_n'$ and define

$$x_0 = 0, \cdots, x_i = x_{i-1} + \alpha \frac{M'_{n-i}}{M'_{n-i+1}}, \cdots, x_n = x_{n-1} + \alpha \frac{M_0'}{M_1'}.$$

We claim, in analogy with (1), that for x in $[0, x_i]$ and $j \leqq n-i+1$

$$|f^{(j)}(x)| \leqq B_{i,n-j+1} M_j'. \tag{5}$$

The proof proceeds by induction on i as before, with some slight modification. Assume (5) holds for all smaller values of i. We first show that

$$|f^{(n-i+1)}(x)| \leqq M'_{n-i+1} \qquad \text{for } x \text{ in } [0, x_i]. \tag{6}$$

This step was obvious in the previous case since $M_k' = M_k$ were given as bounds for $f^{(k)}$. Now it requires proof. If $n-i+1$ belongs to the sequence $n_0<n_1< \cdots$ it is of course clear. If not, let $p<n-i+1<q$ where p, q are consecutive numbers of that sequence. Put $\lambda = q-(n-i+1)$ and

$$R = \frac{M_p'}{M_{p+1}'} = \cdots = \frac{M_{q-1}'}{M_q'}.$$

Assume α so small that Lemma 1 gives $B_{i,j}<\frac{1}{2}$ for $j>i>0$. Assuming that (5) holds for smaller values of i, we can write, by Lemma 2, for $x_{i-1} \leqq x \leqq x_i$

$$\begin{aligned}
|f^{(n-i+1)}(x)| &\leqq |f^{(n-i+1)}(x_{i-1})| + 2\alpha R |f^{(n-i+2)}(x_{i-2})| + \cdots + \frac{((\lambda+1)\alpha R)^{\lambda}}{\lambda!} M_q' \\
&\leqq \frac{1}{2} M'_{n-i+1} + 2\alpha R M'_{n-i+2} + \frac{(3\alpha R)^2}{2!} M'_{n-i+3} + \cdots + \frac{((\lambda+1)\alpha R)^{\lambda}}{\lambda!} M_q' \\
&= M'_{n-i+1}\left(\frac{1}{2} + 2\alpha + \frac{(3\alpha)^2}{2!} + \cdots + \frac{((\lambda+1)\alpha)^{\lambda}}{\lambda!}\right) \\
&< M'_{n-i+1}
\end{aligned}$$

which proves (6). Now we can prove (5) for $j<n-i+1$, exactly as before.

Namely, for x in $[x_{i-1}, x_i]$ and $j \leqq n-i$ if we assume (5) holds for $j+1$, we obtain

$$\begin{aligned} |f^{(j)}(x)| &\leqq |f^{(j)}(x_{i-1})| + \alpha \frac{M'_{n-i}}{M'_{n-i+1}} \cdot B_{i,n-j} M'_{j+1} \\ &\leqq B_{i-1,n-j+1} M'_j + \alpha B_{i,n-j} M'_j \end{aligned}$$

since $M'_{n-i}/M'_{n-i+1} \leqq M'_j/M'_{j+1}$. Thus $|f^{(j)}(x)| \leqq B_{i,n-j+1} M'_j$ and the theorem is completely proved.

Reference

1. S. Mandelbrojt, Séries adhérentes régularisation des suites applications, Gauthier-Villars, Paris, 1952.

MATHEMATICAL NOTES

SIMILAR CONFIGURATIONS IN MEASURABLE SETS

T. A. Bick, Union College, Schenectady, N. Y.

1. Introduction. The purpose of this note is to extend the known result which states that every set of positive two-dimensional Lebesgue measure in R^2 contains the vertices of an equilateral triangle to the same statement for any polygon. Extension to higher dimensions is possible with a modification of the proof given here. The author is pleased to acknowledge the many helpful suggestions made by Professor John F. Randolph.

2. Notation and preliminaries. We restrict ourselves to Euclidean 2-space R^2, with Lebesgue measure m^2 and the usual topology. For $p \in R^2$ and $r > 0$, $C(p, r) = \{q \in R^2 : |p-q| \leqq r\}$; if $A \subset R^2$ is bounded and measurable, the *upper (metric) density of A at p* is defined as

$$\limsup_{r \to 0} \frac{m^2(A \cap C(p, r))}{m^2(C(p, r))}$$

and is denoted by $\overline{D}(A, p)$.

The *lower density of A at p*, denoted by $\underline{D}(A, p)$, is the corresponding lim inf; if $\overline{D}(A, p) = \underline{D}(A, p)$, the *density of A at p*, denoted by $D(A, p)$, is their common value. Clearly, $0 \leqq \underline{D}(A, p) \leqq \overline{D}(A, p) \leqq 1$; if p is such that $D(A, p)$ exists and equals 1, p is called a *density point* of A.

THEOREM 1. (Lebesgue Metric Density Theorem, [1], p. 129): *If $A \subset R^2$ is measurable, then almost every point of A is a density point of A.*

3. Similar configurations.

LEMMA. *Let* $0<k_i\leqq 1$, $i=1, 2, \cdots m$, *and set*

$$\lambda_1 = 1/(1+k_1^2) + k_1^2/(1+k_1^2)(1+k_2^2) + \cdots + k_1^2 \cdots k_{m-1}^2(1+\tfrac{1}{2}k_m^2)/(1+k_1^2)\cdots(1+k_m^2)$$

$$\lambda_2 = 1/(1+k_2^2) + k_2^2/(1+k_2^2)(1+k_3^2) + \cdots + k_2^2 \cdots k_{m-1}^2(1+\tfrac{1}{2}k_m^2)/(1+k_2^2)\cdots(1+k_m^2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_m = (1+\tfrac{1}{2}k_m^2)/(1+k_m^2).$$

Then $\frac{1}{2}<\lambda_m<\lambda_{m-1}< \cdots <\lambda_1<1$.

Proof. Induction on m.

THEOREM 2. *Let* $A_1\subset R^2$ *be measurable with* $m^2A_1>0$. *Let* $q_1, \cdots, q_n$ *be a finite number of distinct points in* R^2 *with any configuration. Then* A_1 *contains* n *points with a similar configuration.*

Proof. Select one of the q_i, call it p_1. We consider now the distances from p_1 to the other $n-1$ points; let p_2 be the point in $\{q_i\}-\{p_1\}$ furthest from p_1 (if more than one satisfies, the choice is arbitrary). Next let p_3 be the next furthest point from p_1, and so on.

Let

$$\begin{aligned} d_1 &= |p_1-p_2| & \alpha_1 &= \text{angle from } \overline{p_1p_2} \text{ to } \overline{p_1p_3} \\ d_2 &= |p_1-p_3| & \alpha_2 &= \text{angle from } \overline{p_1p_3} \text{ to } \overline{p_1p_4} \\ &\cdots\cdots\cdots\cdots \\ d_{n-1} &= |p_1-p_n| & \alpha_{n-2} &= \text{angle from } \overline{p_1p_{n-1}} \text{ to } \overline{p_1p_n} \end{aligned}$$

We observe that $d_i>0$, $i=1, 2, \cdots n-1$, the points being distinct. Next let $k_i=d_{i+1}/d_i$, $i=1, 2, \cdots, n-2$, so $0<k_i\leqq 1$. Finally, let λ_i, $i=1, a, \cdots, n-2$ be defined as in the lemma, so that $\frac{1}{2}<\lambda_{n-2}<\lambda_{n-3}\cdots<\lambda_1<1$. Since $m^2A_1>0$, we may select, in view of Theorem 1, a density point $p_0\in A_1$. Since $\lambda_1<1$, we may select a disc $C_1(p_0, r_1)\equiv C_1$ such that $m^2(A_1\cap C_1)>\lambda_1 m^2C_1$. The remainder of the proof consists of $n-2$ stages.

Stage 1. We first rotate R^2 about p_0 through α_1. Denote the image of $A_1\cap C_1$ by $B_1\cap C_1$ (clearly the image of C_1 is C_1). It is clear that $m^2(A_1\cap C_1)=m^2(B_1\cap C_1)$. Next we shrink R^2 by a factor of k_1 about p_0. Denote the image of $B_1\cap C_1$ by A_2 and the image of C_1 by C_2. Then $C_2=C(p_0, k_1r_1)$, and it is readily seen that $m^2C_2=k_1^2m^2C_1$: Also $A_2\subset C_1$, so $A_2\cap C_2=A_2\cap C_1$. Hence $m^2(A_2\cap C_2)=k_1^2m^2(B_1\cap C_1)=k_1^2m^2(A_1\cap C_1)$. Since $m^2(A_1\cap C_1)>\lambda_1m^2C_1$, we have $m^2(A_1'\cap C_1)<(1-\lambda_1)m^2C_1$,

and $m^2(A_2'C_1) < (1-k_1^2\lambda_1)m^2C_1$. We now assert that $A_1 \cap A_2 \cap C_1 \neq \emptyset$. Indeed, suppose not, then $C_1 = (A_1 \cap A_2)' \cap C_1$. Hence

$$m^2C_1 \leqq m^2(A_1' \cap C_1) + m^2(A_2' \cap C_1) < (1 - \lambda_1 + 1 - \lambda_1 k_1^2)m^2C_1$$
$$= (2 - \lambda_1(1 + k_1^2))m^2C_1. \quad \text{Now } \lambda_1 > 1/(1 + k_1^2) \Rightarrow 2 - \lambda_1(1 + k_1^2) < 1.$$

Hence $m^2C_1 < m^2C_1$, a contradiction establishing that $A_1 \cap A_2 \cap C_1 \neq \emptyset$. Note that at this point we may conclude that A_1 contains the vertices of a triangle similar to that formed by p_1, p_2 and p_3.

We noticed above that $m^2((A_1 \cap A_2)' \cap C_1) < (2-\lambda_1(1+k_1^2))m^2C_1$. It follows that

$$m^2((A_1 \cap A_2) \cap C_1) > (1 - (2 - \lambda_1(1 + k_1^2)))m^2C_1 = (\lambda_1(1 + k_1^2) - 1)m^2C_1.$$

But $\lambda_1(1+k_1^2) - 1 = k_1^2\lambda_2$, so

$$m^2((A_1 \cap A_2) \cap C_1) > k_1^2\lambda_2 m^2C_1 = \lambda_2 m^2C_2.$$

Moreover, since $A_1 \cap A_2 \cap C_1 = A_1 \cap A_2 \cap C_2$, we have $m^2(A_1 \cap A_2 \cap C_2) > \lambda_2 m^2 C_2$. This last inequality is meaningful, since $\lambda_2 < \lambda_1 < 1$.

The procedure for Stages 2 through $n-3$ is similar to that for Stage 1. At the completion of Stage $n-3$, we have the following information: (1) The set A_1 contains $n-1$ points (one of them p_0) which have configuration similar to that formed by $p_1, \cdots p_{n-1}$;

(2) $$m^2((A_1 \cap A_2 \cap \cdots \cap A_{n-2}) \cap C_{n-2}) > \lambda_{n-2} m^2 C_{n-2}.$$

To complete the proof, we proceed with

Stage $n-2$. We first rotate R^2 about p_0 through α_{n-2} and shrink R^2 by k_{n-2} about p_0. The image of $A_1 \cap A_2 \cap \cdots \cap A_{n-2} \cap C_{n-2}$ under this double transformation we denote by $A_{n-1} \cap C_{n-2}$. We see that $m^2(A_{n-1} \cap C_{n-2}) = k_{n-2}^2 m^2(A_1 \cap A_2 \cap \cdots \cap A_{n-2} \cap C_{n-2}) > k_{n-2}^2 \lambda_{n-2} m^2 C_{n-2}$.
Hence $m^2((A_1 \cap \cdots \cap A_{n-2})' \cap C_{n-2}) < (1-\lambda_{n-2})m^2C_{n-2}$, and $m^2(A_{n-1}' \cap C_{n-2}) < (1-k_{n-2}^2\lambda_{n-2})m^2C_{n-2}$. Suppose now that $A_1 \cap \cdots \cap A_{n-2} \cap A_{n-1} \cap C_{n-2} = \emptyset$. Then,

$$\begin{aligned} m^2C_{n-2} &= m^2((A_1 \cap \cdots \cap A_{n-1})' \cap C_{n-2}) \\ &\leqq m^2((A_1 \cap \cdots \cap A_{n-2})' \cap C_{n-2}) + m^2(A_{n-1}' \cap C_{n-2}) \\ &< (1 - \lambda_{n-2} + 1 - \lambda_{n-2}k_{n-2}^2)m^2C_{n-2} \\ &= (2 - \lambda_{n-2}(1 + k_{n-2}^2))m^2C_{n-2}. \end{aligned}$$

We will have our contradiction ($m^2C_{n-2} < m^2C_{n-2}$) if we can show that $2-\lambda_{n-2} \cdot (1+k_{n-2}^2) < 1$. But $\lambda_{n-2} = (1+\frac{1}{2}k_{n-2}^2)/(1+k_{n-2}^2)$, so the desired inequality is immediate. We then conclude that $A_1 \cap A_2 \cap \cdots \cap A_{n-1} \neq \emptyset$, which means that A_1 contains n points—one of them p_0—with configuration similar to that

formed by $p_1, \cdots, p_n$, and hence also similar to that formed by $q_1, \cdots, q_n$.

In view of Theorems 1 and 2, the following corollary is evident:

COROLLARY. *Let $A \subset R^2$ be measurable with $m^2 A > 0$, and let $q_1, \cdots, q_n$ be a finite number of points in R^2 with any configuration, Then almost any point of A may be chosen as any of the vertices of a figure with all vertices in A, and with configuration similar to that formed by $q_1, \cdots, q_n$.*

Extension to dimension $t > 2$ would involve proving a lemma like the one above, with k_i^t replacing k_i^2. Extension to even bounded countable sets is impossible in general, since the countable set may be chosen to be dense, and the set of positive measure chosen to be nowhere dense.

Reference

1. S. Saks, Theory of the Integral, 2nd rev. ed., Dover, New York, 1964.

A SIMPLEX WITH A LARGE CROSS SECTION

D. W. WALKUP, Boeing Scientific Research Laboratories, Seattle, Washington

The longest line segment contained in a plane triangular region is one of the sides. Similarly, as Brands and Laman [1] and Eggleston [2] have shown, the plane section of a tetrahedron with the largest area is one of the faces. A limited generalization of these results to higher dimensions has been proven by Eggleston, Grünbaum, and Klee in Section 7 of [3]. There they show that the volume of a polyhedron P of dimension $d-1$ with at most $d+1$ vertices contained in a simplex S of dimension d is less than or equal to the volume of some face of S of dimension $d-1$. It is the purpose of this note to show that the natural generalization of the results in 2 and 3 dimensions does not hold—that there is a 5-dimensional simplex S with a 4-dimensional cross section T of greater volume than any of the 4-dimensional faces of S.

Repeated use of the Pythagorean theorem and induction on d will show that

$$\begin{aligned} \mu_d(S^d) &= 2^{-(1/2)d}(d+1)^{1/2}(d!)^{-1} \\ \alpha(S^d) &= 2^{-1/2}(d+1)^{1/2}d^{-1/2} \\ R(S^d) &= 2^{-1/2}(d+1)^{-1/2}d^{1/2} \\ r(S^d) &= 2^{-1/2}(d+1)^{-1/2}d^{-1/2}, \end{aligned}$$

where S^d is a unit d-simplex, i.e. a simplex of dimension d whose edges are of unit length, and μ_d, α, R, and r are respectively the d-dimensional volume, the altitude, and the radii of the circumscribed and inscribed spheres.

Consider the special case when d is odd (and greater than 1 to avoid trivial cases), so that $d = 2t+1$. Let H be a hyperplane through the origin in Euclidean

d-space and let J' and J'' be complementary orthogonal t-dimensional subspaces of H. Erect a line M from the origin perpendicular to H and lay off points p' and p'' on M a distance

$$b = \tfrac{1}{2}[1 - 2R(S^t)^2]^{1/2} \tag{1}$$

on either side of H. Lay out unit t-simplices S' and S'' with centroids at p' and p'' so that the flats which they span affinely are parallel translates of J' and J'' respectively. It is comparatively easy to see that the $2(t+1)$ vertices of S' and S'' taken together are the vertices of a unit d-simplex S^d, and the intersection T of H with S^d is the product of a pair of orthogonal simplices similar to but half the edge lengths of S' and S''. Thus

$$\mu_{2t}(T) = [2^{-t}\mu_t(S^t)]^2 = 2^{-3t}(t+1)(t!)^{-2},$$

and, of course, if F is any $(d-1)$-face (i.e. face of dimension $d-1$) of S^d,

$$\mu_{2t}(F) = \mu_{2t}(S^{2t}) = 2^{-t}(2t+1)^{1/2}[(2t)!]^{-1}.$$

Next consider a family of figures $S^d(\delta)$, $1 \geqq \delta > 0$, obtained from $S^d = S^d(1)$ by the contractions which carry every point p of S^d into a point $p(\delta)$ lying on the line through p perpendicular to H but δ times as far from H as p. Every such contraction leaves T fixed, so that $\mu_{2t}[T(\delta)] = \mu_{2t}(T)$. The volume of any $(d-1)$-face $F(\delta)$ of $S^d(\delta)$ is proportional to the length of a line segment $L(\delta)$ in $F(\delta)$ whose projection in H is orthogonal to the intersection of H and the hyperplane determining F. The $(d-1)$-face F of S^d is either the convex hull of S' and a $(t-1)$-face G of S'' or the convex hull of S'' and a $(t-1)$-face of S'. Without loss of generality we may suppose the former, in which case a suitable line segment $L(\delta)$ is the one connecting the centroid of $S'(\delta)$ and the centroid of $G(\delta)$. This segment has length $[(2\delta b)^2 + 0 + r(S^t)^2]^{1/2}$, where b is the distance given in (1) above. From a straightforward computation it follows that

$$\lim_{\delta \to 0} \mu_{2t}[F(\delta)] = \frac{\mu_{2t}(F)}{(2t+1)^{1/2}} = 2^{-t}[(2t)!]^{-1}$$

and

$$\lim_{\delta \to 0} \frac{\mu_{2t}[T(\delta)]}{\mu_{2t}[F(\delta)]} = \frac{t+1}{t^{1/2}} \cdot \frac{t^{1/2}(2t)!}{2^{2t}(t!)^2}. \tag{2}$$

Now consider the right side of (2) as a function of t. The first value strictly greater than unity occurs when $t=2$. Thus there is a 5-dimensional simplex with a 4-dimensional cross section of greater volume than any of its 4-dimensional faces. But more than this, from an application of Stirling's formula it follows that the right side of (2) is asymptotically equal to $(t/\pi)^{1/2}$. Thus no upper bound independent of d can be given for the ratio of the volume of a cross section to the volume of the largest $(d-1)$-dimensional face of a d-dimensional simplex.

References

1. J. J. A. M. Brands and G. Laman, Plane section of a tetrahedron, this MONTHLY, 70 (1963) 338–339.
2. H. G. Eggleston, Plane section of a tetrahedron, this MONTHLY, 70 (1963) 1108.
3. H. G. Eggleston, B. Grünbaum, and V. Klee, Some semicontinuity theorems for convex polytopes and cell-complexes, Comment. Math. Helv., 39 (1964) 165–188.

DIRECTED GRAPH REALIZATION OF DEGREE PAIRS

ANNE BROWNLEE, Florida Presbyterian College

This paper, which is an extension of work done by Hakimi [1], is concerned with the realizability of a finite set of nonnegative integer pairs as the degrees of the vertices of a directed graph. The degree pair (d_i^+, d_i^-) associated with vertex v_i of a graph represents the number of arcs with which v_i is incident; d_i^+ is the number of arcs for which v_i is the terminal vertex and d_i^- is the number for which v_i is the initial vertex. The directed graphs considered here may have parallel arcs (distinct arcs with the same initial and terminal vertices), but they may not have loops (arcs of the type $a(v_i, v_i)$).

In every set of n integer pairs, it is assumed that the pairs are ordered such that $d_i^+ + d_i^- \leqq d_{i+1}^+ + d_{i+1}^-$ for every $i = 1, \cdots, n-1$. Furthermore, for every pair, $d_i^+ + d_i^- > 0$. A vertex which corresponds to a pair which has the property $\min(d_i^+, d_i^-) = 0$ is called a *compact* vertex.

In [1] the following question was considered: Given a set of n nonnegative integer pairs, how can we tell whether or not there exists a directed graph of n vertices with the given set as its set of degree pairs? If such a graph G exists, the set is said to be realizable, and G is called a realization of the set. We now consider the question: Given a set of integer pairs with at least one realization G, is it possible to tell whether or not any realization contains a cycle?

In [1], the following theorems are proved:

THEOREM 1 (Hakimi). *Given a set of pairs of nonnegative integers* $\{(d_i^+, d_i^-) \mid i = 1, \cdots, n\}$, $n \geqq 2$, *the set is realizable as the degree pairs of the vertices of an n-vertex directed graph if, and only if*

(a) $\sum_{i=1}^n d_i^+ = \sum_{i=1}^n d_i^-$ *and* (b) $\sum_{i=1}^{n-1} (d_i^+ + d_i^-) \geqq d_n^+ + d_n^-$.

THEOREM 2 (Hakimi). *A directed graph G contains a cycle if G contains at most one compact vertex.*

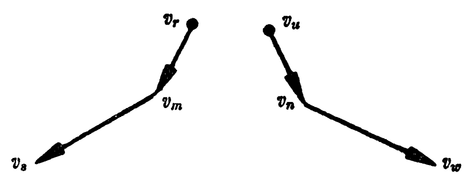
Theorem 2 implies a necessary condition that a set S have a cycleless realization; namely, S contains at least two pairs which define compact vertices. This condition, however, is not sufficient; for instance, the set $\{(0, 2), (2, 0), (2, 2), (3, 3)\}$ has no cycleless realization. The following theorem, our chief result, gives a necessary and sufficient condition that all realizations of a set be cycleless.

THEOREM 3. *Given a realizable set S, a necessary and sufficient condition for all realizations of S to be cycleless is:*

$$\min(d_i^+, d_i^-) > 0 \quad \textit{for at most one } i.$$

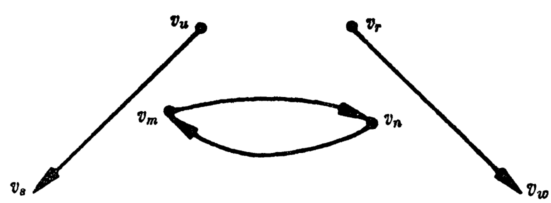
Proof. (a) Assume the condition holds. A cycle must contain at least two vertices, and these can not be compact. Hence the condition is sufficient.

(b) Conversely, assume S has only cycleless realizations, and suppose $\min(d_i^+, d_i^-) > 0$ for at least two i's. Let G be an arbitrary realization of S, and denote two noncompact vertices of G by v_m and v_n. By assumption G has no cycles, so the arcs $a(v_m, v_n)$ and $a(v_n, v_m)$ cannot both be in G. Assume neither of these arcs exists in G. Since v_m is noncompact, there must exist vertices v_r, v_s such that the arcs $a(v_r, v_m)$ and $a(v_m, v_s)$ are in G. Similarly, there must exist two vertices v_u, v_w such that the arcs $a(v_u, v_n)$ and $a(v_n, v_w)$ are in G.



Partial Subgraph of G

The vertices v_r, v_s, v_u, v_w do not all have to be distinct, but there must be at least two distinct vertices, for if, say, $v_r = v_s$, we have a cycle in G, contrary to our assumption. Now we replace the arcs $a(v_r, v_m)$ and $a(v_n, v_w)$ by the arcs $a(v_r, v_w)$ and $a(v_n, v_m)$, and the arcs $a(v_m, v_s)$ and $a(v_u, v_n)$ by the arcs $a(v_m, v_n)$ and $a(v_u, v_s)$. The resulting graph G' is clearly another realization of S. (If some of the vertices v_r, v_u, v_w coincide, it may be necessary to perform another transformation to eliminate a loop.) The resulting graph G' contains the cycle $a(v_m, v_n)$, $a(v_n, v_m)$; hence we have a contradiction.



Partial Subgraph of G'

If we assume one of the arcs $a(v_m, v_n)$ or $a(v_n, v_m)$ exists, say, $a(v_m, v_n)$, then we must have vertices v_r and v_s such that $a(v_r, v_m)$ and $a(v_n, v_s)$ are in G. By replacing these arcs by $a(v_r, v_s)$ and $a(v_n, v_m)$ we again arrive at a graph G'' which contains the cycle $a(v_m, v_n)$, $a(v_n, v_m)$, contrary to assumption. This completes the proof.

The problem remaining to be solved in this immediate field is that of finding conditions which guarantee the existence of at least one cycleless realization for a given set.

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Reference

1. S. L. Hakimi, On the degrees of the vertices of a directed graph, J. Franklin Inst., No. 4, 279 (1965) 290–308.

SOME SEPARABLE FORMS OF THE RICCATI EQUATION

P. R. P. RAO AND V. H. UKIDAVE, Indian School of Mines, Dhanbad, India.

We present below some separable forms of the Riccati equation

$$y'(x) = f(x) + g(x)y(x) + h(x)y^2(x), \tag{1}$$

and to that end state and prove two theorems.

THEOREM 1. *If there exist constants k, c and a function $v(x)$ such that $f=chv^2$ and $v'-gv=kf$, the substitution $y(x)=v(x)u(x)$ reduces* (1) *to the form*

$$u' = (cfh)^{1/2}(1 - ku + u^2/c), \tag{2}$$

if g is continuous, f, h are differentiable, and are both positive or both negative on the interval of integration, provided a certain condition holds (*see* (3), (4), *or* (5)).

Proof. Substituting $y=vu$, we obtain from (1)

$$vu' = f - (v' - gv)u + hv^2u^2.$$

Hence, it is obvious that, if $v'-gv=kf$ and $f=chv^2$, (1) reduces to (2).

Now, one can take c or k to be unity, determine v, and then utilize the other of the equations involving k or c to impose the condition that is sufficient (but not necessary) to obtain (2). Thus taking $k=1$, under the assumption that

$$f^{-1}e^G \frac{d}{dx}\left(\left(\frac{f}{h}\right)^{1/2}e^{-G}\right) = c^{1/2}; \quad G = \int^x g\,dx \tag{3}$$

the form (2) is obtainable. Again taking $c=1$, under the assumption that

$$(hf' - h'f - 2fgh)/2(fh)^{3/2} = k, \tag{4}$$

the form (2) is obtainable. A. K. Rajgopal [1] obtained a result, which can be obtained by taking $k=0$ under the assumption that

$$g^{-1}\frac{d}{dx}\left(\log\frac{f}{h}\right) = 2. \tag{5}$$

The result obtained by Allen and Stein [2] corresponds to $c=1$.

We note that the transformation $y=vu$ effects a change in the scale factor, and that if this transformation reduces (1) to the form (2), the functions f, h need to be just differentiable.

THEOREM 2. *If there exist constants a, b and a function $v(x)$, such that $H=ahv^2$ and $v'+gv=bH$, where $H=f+(d/dx)(g/h)$, the substitution $y=vu-(g/h)$ reduces equation* (1) *to the form*

$$u'=hv(a-abu+u^2), \tag{6}$$

if f is differentiable, g and h are twice-differentiable, and H and h are both positive or both negative on the interval of integration, provided a certain condition holds (see (7), (8), *or* (9)).

Proof. Substituting $y=vu-(g/h)$, we obtain from (1)

$$vu' = H - (v' + gv)u + hv^2u^2.$$

Hence, if $v'+gv=bH$, and $H=ahv^2$, (1) reduces to (6).

Now taking a or b to be unity, v can be determined, and then, the other equation involving b or a can be utilised to impose the condition sufficient (but not necessary) to obtain (6). Thus, taking $b=1$, under the assumption that

$$\frac{d}{dx}((H/h)^{1/2}e^G)/He^G = a^{1/2}; \quad G = \int^x g dx, \tag{7}$$

equation (1) reduces to (6). Again taking $a=1$, under the assumption that

$$(hH' - h'H + 2ghH)/2(Hh)^{3/2} = b, \tag{8}$$

the form (6) is obtainable from (1).

If we take $b=0$, $v=e^{-G}$, and under the assumption that

$$g^{-1}\frac{d}{dx}\left(\log\frac{H}{h}\right) = -2, \tag{9}$$

equation (6) takes the form $u'=he^{-G}(k+u^2)$, where $k=(H\ e^{2G})/h$.

The result obtained in [3] corresponds to $a=1$.

Note that the transformation $y=vu-(g/h)$ effects a change in the origin, as well as the scale factor and that when this transformation reduces (1) to (6), the functions g, h are required to be twice-differentiable.

References

1. A. K. Rajgopal, On the generalized Riccati equation, this MONTHLY, 68 (1961) 777–779.

2. J. L. Allen and F. M. Stein, On solutions of certain Riccati differential equations, this MONTHLY, 71 (1964) 1113–1115.

3. P. R. P. Rao, The Riccati differential equation, this MONTHLY, 69 (1962) 995.

ON FINITE ABELIAN GROUPS

A. H. M. HOARE, The University of Birmingham, U. K.

We give in Theorem 1 a shorter proof of a result on finite abelian groups equivalent to the result in [1]. The method may be used to give short proofs of other theorems on finite abelian groups, of which we give two examples. The results extend easily to finitely generated abelian groups.

Throughout this paper we use group to mean finite additive abelian group. For any such group G a set of "invariants," $[e_1, e_2, \cdots, e_r]$, e_i dividing e_{i+1}, are defined as in [2]. G is then isomorphic to

$$Z_{e_1} \oplus Z_{e_2} \oplus \cdots \oplus Z_{e_r}.$$

If G can be generated by n elements then a set of r invariants can be found with $r \leqq n$. A set of invariants can also be found none of which is equal to one. We prove the uniqueness of this set in the corollary below.

We define ϕ_d, for any positive integer d, to be the endomorphism $x \to dx$ for all x in a group G. K_d, the kernel of ϕ_d, consists of all elements with order dividing d and has a set of invariants $[(d, e_1), (d, e_2), \cdots, (d, e_r)]$, where (d, e_i) is the highest common factor of d and e_i. The factor group G/K_d, and the subgroup G_d (the image of G under ϕ_d) have $[e_1/(d, e_1), e_2/(d, e_2), \cdots, e_r/(d, e_r)]$ as a set of invariants.

Let L_d denote the kernel of the group H under ϕ_d, H_d its image and $|L_d|$ the order of L_d.

THEOREM 1. *The groups G and H are isomorphic if and only if $|K_d| = |L_d|$ for all d.*

If G and H are isomorphic then clearly $|K_d| = |L_d|$. Conversely suppose $|K_d| = |L_d|$ for all d. Let $[e_1, e_2, \cdots, e_r]$ and $[f_1, f_2, \cdots, f_s]$ be sets of invariants for G and H respectively, in which none of the invariants is equal to one. Then putting $d = e_1$

$$|K_d| = e_1^r = |L_d| = \prod_{i=1}^{s} (e_1, f_i) \leqq e_1^s,$$

and putting $d = f_1$

$$|L_d| = f_1^s = |K_d| = \prod_{i=1}^{r} (f_1, e_i) \leqq f_1^r.$$

Therefore $r = s$ and $e_1 = f_1$.

Now let t be the least integer, if any, for which $e_t \neq f_t$, we can suppose $e_t > f_t$. Then putting $d = e_t$,

$$|K_d| = \left(\prod_{i=1}^{t-1} e_i\right) e_t^{r-t+1},$$

and

$$| L_d | = \left(\prod_{i=1}^{t-1} f_i\right)\left(\prod_{j=t}^{r} (e_t, f_j)\right) = \prod_{i=1}^{t-1} e_i \prod_{j=t}^{s} (e_t, f_j) < | K_d | .$$

Therefore $e_t = f_t$ for all t.

THEOREM 2 (cf. [2] vol. I, 148–151). *If G is a group with a set of invariants $[m_1, m_2, \cdots, m_r]$ where m_{i+1} divides m_i, and $m_r \neq 1$, and if H (with a set of invariants $[n_1, n_2, \cdots, n_s]$, where n_{i+1} divides n_i and $n_s \neq 1$) is isomorphic to a subgroup of G, then $r \geqq s$ and for all i, n_i divides m_i.*

Note that we have reversed the order of the invariants.

It is clear that $|L_d| \leqq |K_d|$ for all d. Let t be the least integer, if any, such that n_k does not divide m_k. Put $d = n_k/(m_k, n_k)$, then $d \neq 1$, $(d, n_k) = d$, and $(d, m_k) = 1$. Therefore

$$\begin{aligned} (d, n_i) &= d & i &= 1, \cdots, k \\ (d, m_i) &= d & i &= 1, \cdots, k-1 \\ &= 1 & i &= k, \cdots, r. \end{aligned}$$

Thus $|K_d| = d^{k-1}$ and $|L_d| \geqq d^k$, which gives a contradiction. Therefore n_k divides m_k. If $s > r$, then putting $d = n_{r+1} \neq 1$, gives the same contradiction.

COROLLARY. *G is isomorphic to H if and only if $r = s$ and $n_i = m_i$ for all i.*

For G is isomorphic to H if and only if G is isomorphic to a subgroup of H and vice-versa.

THEOREM 3 (cf. [3] Theorem 3). *The same result holds as in Theorem 2, if H is a homomorphic image of G.*

G, and therefore H, can be generated by r elements. Hence a set of not more than r invariants can be found for H; so applying the corollary above we get $s \leqq r$.

Let ϕ be the homomorphism from G onto H, then for a d, $\phi(dx) = d\phi(x)$, i.e., $\phi\phi_d G = \phi_d \phi G = H_d$.

Thus ϕ restricted to G_d is a homomorphism onto H_d. Let k be the least integer, if any, such that n_k does not divide m_k. Put $d = m_k$, then $n_k/(d, n_k) \neq 1$, and $[m_1/d, m_2/d, \cdots, m_{k-1}/d]$ and $[n_1/(d, n_1), n_2/(d, n_2), \cdots, n_k/(d, n_k), \cdots]$ are sets of invariants for G_d and H_d respectively.

But we have shown above that a homomorphic image of a group has fewer invariants not equal to one than the original group. This contradiction completes the proof.

References

1. R. McHaffey, Isomorphism of finite Abelian groups, this MONTHLY, 72 (1965) 48–50.

2. A. G. Kurosh, The Theory of Groups, Chelsea, New York, 1960.

3. C. MacLachlan, Abelian group of automorphisms of compact Riemann surfaces, Proc. London Math. Soc., (3) 15 (1965) 699–712.

adjacent in T. If they are not antipodal there is an irreducible path in G from $\mathbf{a}$ to $\mathbf{b}$ of length less than k. By the lemma, this path can be continued beyond $\mathbf{b}$ in an irreducible manner until it attains a length k at some vertex $\mathbf{c}$ which is at distance k from $\mathbf{a}$ because of the girth of G. Since $d_c \geqq 2$ there is a circuit of length $2k+1$ on $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. In this odd circuit there is a sequence of distinct vertices $(\mathbf{a}, \mathbf{c}, \mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{b})$ each antipodal to its predecessor and therefore constituting the vertices of a path in T joining $\mathbf{a}$ and $\mathbf{b}$.

Theorem 2 and the corollary to Theorem 1 show that G is regular.

Reference

1. A. J. Hoffman and R. R. Singleton, On Moore graphs of diameters 2 and 3, IBM J. Res. Develop., No. 5, 4 (Nov. 1960) 497–504.

DIVISIBILITY OF INFINITE COPRIME SEQUENCES

K. SELUCKÝ, V.U.T. of Brno, Czechoslovakia

A. W. F. Edwards [1] defined an "infinite coprime sequence" as an infinite class of positive integers arranged in order of ascending magnitude, in which the highest common divisor of any two members is unity. The first of the sequences presented by him is

$$u_n - p = u_{n-1}\cdot(u_{n-1} - p), \tag{1}$$

where $(u_1, p) = 1$, $u_1 > p$, u_1 and p positive integers. Let $p = 1$. Then the sequences (1) may be written in the form

$$u_n = u_{n-1}^2 - u_{n-1} + 1, \qquad u_1 > 1. \tag{2}$$

THEOREM 1. *For all infinite coprime sequences given by* (2), *and for any odd prime number* q, *if*

$$q \mid u_n, \quad \textit{then} \quad n \leqq (q+1)/2. \tag{3}$$

Proof. Arrange all the sequences (2) in columns of an infinite matrix. Then there are in the rows of the matrix the values of the following functions

Row 1	$u_1 = x$, x all integers, notation	$f_1(x)$
Row 2	$f(x) = x^2 - x + 1$	$f_2(x)$
Row 3	$f[f(x)] = f^2(x) - f(x) + 1$	$f_3(x)$
⋮	⋮	⋮
Row i	$f[f[f \cdots]] = (i-1) \times$ the composite function of the independent variable x	$f_i(x)$
⋮	⋮	⋮

All values of an $(i+1)$-th row are selected from the values of the previous ith row and hence it is true that

$$q \nmid f_i(x) \Rightarrow q \nmid f_j(x), \qquad j \geqq i. \tag{4}$$

Let us consider also that

$$q \mid f_i(x_1) \Rightarrow q \mid f_i(x_1 + kq) \tag{5}$$

$$f_{i+1}(x) = f_{i+1}(-x + 1)$$

$$q \mid f_{i+1}(x_1) \Rightarrow q \mid f_{i+1}[-(x_1 - q) + 1] \tag{6}$$

and let us determine the number of independent solutions of the congruence

$$f_i(x) \equiv 0 \pmod q, \qquad 1 \leqq x \leqq q \quad [\text{see } (5)].$$

(a) Generally besides a solution x_1 there also exists a solution x_2, $x_2 \neq x_1$, $x_2 = -(x_1 - q) + 1$ [see (6)], $1 \leqq x_2 \leqq q$, except the cases (b) and (c).

(b) In the first row $f_1(x)$ there always exists only the trivial solution $x_0 = q$.

(c) In another further row there can also exist but one solution if $x_1 = x_2 = (q+1)/2$. This case may occur at most once, since in accordance with the definition, the other values of the relevant column are not divisible by the prime number q.

(d) For $x = 1$ no solution can occur because the sequence of the values of the first column is formed always by unities and $q > 2$.

From the fact that the rows, divisible by the prime number q, are ordered in an array one below the other (4) and every solution x_1 or x_2 makes impossible the repeating of the same solution in other rows, it follows that

$$n \leqq 1 + 1 + \frac{q - 3}{2} = \frac{q + 1}{2} \cdot$$

(7) *Note.* By using the same method a similar result can be proved for certain other sequences of [1].

Reference

1. A. W. F. Edwards, Infinite coprime sequences, Math. Gaz., No. 366, 48 (1964) 416–422.

VISUALIZING UNIFORM CONTINUITY

DWIGHT PAINE, Wells College

How can one visualize uniform continuity? A continuous function defined on an interval has a connected graph, and although this fact is seldom used in proofs it is helpful in thinking about continuity. Is there likewise a helpful way to visualize uniform continuity?

One may feel that the graph of a uniformly continuous function must not become too steep; but as R. C. Buck points out [1, p. 68], this is too strong a condition. The square-root function, for example, is uniformly continuous even

though it has arbitrarily steep chords near the origin. In the theorem below, however, we find a similar property, easy to visualize, which does characterize the graph of a uniformly continuous function on an interval. The theorem says, essentially, that uniform continuity on an interval means that *steep chords are short.*

THEOREM. *For a real function f defined on a real interval, the following two conditions are equivalent*:

$$\text{(i)} \qquad \forall \epsilon > 0 \quad \exists\, \delta > 0 \quad \ni: |x - y| < \delta \qquad \Rightarrow |f(x) - f(y)| < \epsilon;$$

$$\text{(ii)} \qquad \forall \epsilon > 0 \quad \exists\, N > 0 \quad \ni: \left|\frac{f(x) - f(y)}{x - y}\right| > N \Rightarrow |f(x) - f(y)| < \epsilon.$$

Proof. I. Assume (i), and let $N = 2\epsilon/\delta$. We shall prove the contrapositive of (ii). Note that if $|f(x) - f(y)|$ is greater than or equal to ϵ it is in fact an integral multiple $k\eta$ of some number η in the interval $[\epsilon, 2\epsilon]$.

We can assume without loss of generality that $f(x) < f(y)$. If $x < y$, the intermediate value theorem assures us that there are numbers $x = x_0 < \cdots < x_k = y$ such that each $f(x_i) = f(x) + i\eta$. If $x > y$, then the numbers are $x = x_0 > \cdots > x_k = y$. In either case, each $|f(x_i) - f(x_{i-1})| = \eta \geqq \epsilon$, so by our hypothesis each $|x_i - x_{i-1}| \geqq \delta$. Consequently $|x - y| \geqq k\delta$, and

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leqq \frac{k\eta}{k\delta} = \frac{\eta}{\delta} \leqq \frac{2\epsilon}{\delta} = N.$$

II. Assume (ii), and let $\delta = \epsilon/N$. To prove the contrapositive of (i), we observe that if $|f(x) - f(y)| \geqq \epsilon$, then

$$|x - y| = \left|\frac{x - y}{f(x) - f(y)}\right| \cdot |f(x) - f(y)| \geqq \frac{1}{N} \cdot \epsilon = \delta.$$

If the domain is not an interval, (ii) still implies (i) but the two conditions need not be equivalent. For example, any function restricted to the integers satisfies the definition of uniform continuity automatically, because distinct points in the domain are never less than one unit apart. But it may still (like the function n^2) have arbitrarily steep and long chords.

Nevertheless, no matter what the domain, the two conditions are sure to be equivalent if the range is bounded, for then N may be taken to be the diameter of the range divided by δ. And of course the same is true if the domain is bounded, because then the range is bounded too.

Reference

1. R. Creighton Buck, Advanced Calculus, 2nd ed., McGraw-Hill, New York, 1965.

THE FACTORIZATION OF THE CYCLOTOMIC POLYNOMIALS MOD p

W. J. GUERRIER, University of Hawaii

Let n be a positive integer and denote by $F_n(X)$ the cyclotomic polynomial of order n. In teaching courses in algebraic number theory, I have found the theorem below on the factorization of $F_n(X)$ mod p very useful. I do not know, however, of any simple reference for this theorem. The object of this note is to provide such a reference.

THEOREM. *Let p be a prime and suppose that $p\nmid n$. Denote by ϕ the Euler ϕ-function.*

(i) *Set $f=$ the (multiplicative) order of p* mod n. *Then $F_n(X)$ factors* mod p *into a product of $\phi(n)/f$ distinct irreducible polynomials each of degree f.*

(ii) *For any positive integer, r, $F_{p^rn}(X)=F_n(X)^{\phi(p^r)}$* (mod p).

Proof. (i): Denote by Z_p the field of p elements and let K be the splitting field over Z_p of the polynomial $X^{p^f}-X$. Since $n\mid p^f-1$, K contains the nth roots of unity. Let ζ be a primitive nth root of unity. The map $x\to x^p$ is a generator for the Galois group of K/Z_p. Thus the minimal polynomial of ζ over Z_p is

$$(X-\zeta)(X-\zeta^p)\cdots(X-\zeta^{p^{f-1}})$$

and therefore $F_n(X)$ has an irreducible factor of degree f mod p.

Now choose another primitive nth root of unity η not among $\zeta,\zeta^p,\cdots,\zeta^{p^{f-1}}$. (Note that since $p\nmid n$, ξ^{p^i} is a primitive nth root of unity.) The polynomial

$$(X-\eta)(X-\eta^p)\cdots(X-\eta^{p^{f-1}})$$

is then a second irreducible factor of $F_n(X)$ of degree f. Continuing this process one arrives at the desired conclusion.

(ii): Let $\eta_1,\eta_2,\cdots,\eta_s$ $(s=\phi(n))$ be the primitive nth roots of unity and let ζ be a primitive $p^{r\text{th}}$ root of unity. Since $(n,p)=1$ each of the elements $(\eta_i\zeta^j)^{p^r}$ $i=1,\cdots,s$, $j=1,\cdots,p^r$ is a primitive nth root. On the other hand for $(j,p)=1$, $\eta_i\zeta^j$ is a primitive p^rnth root of unity and for $p\mid j$, $(\eta_i\zeta^j)^{p^{r-1}}$ is a primitive nth root. Thus one has

$$F_n(X^{p^r})=\prod_{i,j}(X-\eta_i\zeta^j)=\prod_{\substack{i,j\\(j,p)=1}}(X-\eta_i\zeta^j)\cdot\prod_{\substack{i,j\\p\mid j}}(X-\eta_i\zeta^j)$$
$$=F_{p^rn}(X)\cdot F_n(X^{p^{r-1}}).$$

Therefore,

$$F_{p^rn}(X)=F_n(X^{p^r})/F_n(X^{p^{r-1}})\underset{(\text{mod } p)}{\equiv}F_n(X)^{p^r}/F_n(X)^{p^{r-1}}$$
$$=F_n(X)^{p^{r-1}(p-1)}=F_n(X)^{\phi(p^r)}.$$

This completes the proof of the theorem.

ON THE PRODUCT OF EQUIVALENCE RELATIONS

M. G. STANLEY, Case Institute of Technology

An early pleasant surprise in the calculus of relations is the theorem that the product $P \circ Q$ of two equivalence relations is an equivalence relation if and only if they commute, $P \circ Q = Q \circ P$. Moreover, it can be quickly amplified and generalized so as to introduce substantial, but simple, related algebraic ideas. P. M. Cohn has indicated how to do this ([1] p. 98, Exercise 2), but overstated the generalization.

As in Cohn's exercise, we consider a (positively) ordered semigroup S, where $a \leqq b$ implies $xa \leqq xb$ and $ax \leqq bx$, and multiples satisfy $xa \geqq a$, $ax \geqq a$.

A product pq of two idempotents in a positively ordered semigroup is idempotent if and only if qp divides pq.

Proof. qp divides $pqpq$; conversely, if qp divides pq, $pq \leqq pqpq \leqq ppqq$.

The immediately relevant ordered semigroup S is the semigroup of reflexive relations on a set. The idempotents are the transitive relations. As for symmetry, conversion $R \rightarrow R^*$ is an anti-automorphism of S and the symmetric relations are the self-conjugate elements.

For two self-conjugate idempotents p, q in a symmetric positively ordered semigroup, pq is idempotent if and only if it is self-conjugate, and this is if and only if $pq = qp$.

Proof. The conjugate q^*p^* is precisely qp. If $pq = qp$, qp divides pq. If qp divides $pq = xqpy$, then $qp = y^*pqx^*$; since divisibility is anti-symmetric, $pq = qp$.

The proof actually showed that a product of two self-conjugate elements is self-conjugate if and only if they commute. However, it is not true (as asserted in Cohn's exercise) that a product of two idempotents is not idempotent unless they commute. To see this, it suffices to take a three-point set $\{a, b, c\}$, with P consisting of the diagonal and (a, b), Q of the diagonal and (b, c).

Reference

1. P. M. Cohn, Universal Algebra, Harper and Row, New York, 1965.

A THEOREM ON LINEAR OPERATORS AND THE TIETZE EXTENSION THEOREM

M. C. McCORD, University of Georgia

The following theorem on linear operators, which is similar to Proposition A of Kaufman [1], is an abstraction of part of Urysohn's proof [3], p. 293, of the Tietze Extension Theorem.

THEOREM (cf. [1]). *Suppose T is a bounded linear operator from a Banach space X to a normed linear space Y, for which there are real numbers α and β, $\beta < 1$, such that the following holds. For each y in Y of norm 1, there exists an x in X*

of norm $\leqq \alpha$ *such that* $\|y - Tx\| \leqq \beta$. *Then for each* y *in* Y, *there exists an* x *in* X *such that* $y = Tx$ *and* $\|x\| \leqq \alpha(1-\beta)^{-1}\|y\|$.

Since the theorem is slightly different from Proposition A of [1], we indicate the proof: Given y in Y, we construct recursively a sequence $(x_1, x_2, \cdots)$ in X such that for each $n \geqq 1$,

$$\left\| y - \sum_{k=1}^{n} Tx_k \right\| \leqq \beta^n \|y\| \quad \text{and} \tag{1}$$

$$\|x_n\| \leqq \alpha\beta^{n-1}\|y\|. \tag{2}$$

Condition (2) implies that the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent with a sum x of norm $\leqq \alpha(1-\beta)^{-1}\|y\|$. Condition (1) then implies that $y = Tx$.

Now we point out how this theorem can be used in the proof of the Tietze Extension Theorem. For any topological space E, let $C(E)$ be the Banach space of all bounded, continuous, real-valued functions on E, with the supremum norm.

Suppose E is a normal space with closed subspace A. Define a bounded linear operator T from $C(E)$ to $C(A)$ by $Tx = x|A$ (the restriction of the function x to A). The start on the proof of the Tietze Extension Theorem provided in the exercises of Kelley [2], p. 242, shows exactly that the operator T satisfies the hypothesis of the above Theorem, with $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$. Hence the Theorem implies that for each y in $C(A)$, there exists an x in $C(E)$ such that $x|A = y$ and $\|x\| \leqq (\frac{1}{3})(1-\frac{2}{3})^{-1}\|y\| = \|y\|$. This is the Tietze Extension Theorem.

References

1. R. Kaufman, Operators with closed range, Proc. Amer. Math. Soc., 17 (1966) 767–768.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.
3. P. Urysohn, Über die Mächtigkeit der zusammenhängenden Mengen, Math. Ann., 94 (1925) 262–295.

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the MONTHLY. *Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.*

ON COMPLETENESS OF PARTIALLY ORDERED SETS AND FIXPOINT THEOREMS FOR ISOTONE MAPPINGS

I. I. KOLODNER, Carnegie-Mellon University

Let P be a poset with smallest element a and largest element b. Let $f: P \to P$ be an isotone function and denote by F the set of fixpoints of f, and by $F_x = \{f^n(x): n = 1, 2, \cdots\}$—the orbit of x. If P is a complete lattice then $F \neq \emptyset$ and it is a complete lattice, though not necessarily a sublattice of P. See, e.g., Garrett Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. XXV, New York, 1948, pp. 53–54.

In applications the above requirement on P is frequently too restrictive. We show that:

THEOREM 1. *If every chain in P has a supremum (infimum) then $F \neq \emptyset$.*

However, the chain condition is not a necessary one for existence of fixpoints. Furthermore, F need not satisfy the chain condition. One notes also that if P is a lattice satisfying the chain condition, then P is already a complete lattice.

THEOREM 2. *Suppose that each enumerable chain $A \subset P$ has a supremum and an infimum, and that $f(\sup A) = \sup f(A)$, $f(\inf A) = \inf f(A)$. Then*: (i) $F \neq \emptyset$, (ii) *each enumerable chain in F has an infimum and a supremum in F* (iii) $\sup F = \inf F_b$, $\inf F = \sup F_a$. *Furthermore, if P is a lattice then F is also a lattice, but not necessarily a sublattice of P.*

In proving Theorem 1 one considers maximal chains in

$$\underline{F} = \{x \in P: x \leqq f(x)\} \quad \text{and in } \overline{F} = \{x \in P: x \geqq f(x)\}.$$

In proving Theorem 2 one considers enumerable chains in $\underline{F}$, $\overline{F}$ and in F.

Mail Address: Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213.

A NOTE ON CURVATURES ASSOCIATED WITH A VECTOR FIELD

R. N. KAUL, University of California, Berkeley

1. A curve on a surface such that its rectifying plane at each point contains a line of rectilinear congruence is called a hyperasymptotic curve [2, **3**]. The object of this note is to extend this notion for an arbitrary vector field on a given surface.

2. Let $u^\alpha = u^\alpha(s)$ $(\alpha = 1, 2)$ be the equations of a curve of class c^r $(r \geqq 3)$ on a surface S of class c^s $(s \geqq 2)$ $x^i = x^i(u^1, u^2)$ $(i = 1, 2, 3)$. Let $v^i = x^i{}_{,\alpha}\, p^\alpha$ denote a unit vector field in S. Let λ^i be the direction of a unique line of an arbitrary congruence at a point of S which may be expressed as

$$\lambda^i = x^i{}_{,\alpha}\, t^\alpha + rX^i. \tag{2.1}$$

Setting [**4**] $\delta v^i/\delta s = {}_vk\, w^i$, $\mu^i = v^i \times w^i$ it can be shown that

$$\mu^i = \frac{{}_vk_a}{{}_vk} X^i + \frac{e({}_vk_n)}{{}_vk} \epsilon^{\alpha\beta} p_\beta x^i{}_{,\alpha} \tag{2.2}$$

where X^i is a unit normal to the surface and ${}_vk_a$, ${}_vk_n$ [**4, 5**] are the associate and normal curvature of the vector field V respectively.

A curve on the surface shall be called a hyperasymptotic curve of the vector field V if at each point of the curve the direction λ^i lies in the plane spanned by v^i and μ^i.

Thus

$$\lambda^i = \alpha v^i + \beta \mu^i. \tag{2.3}$$

The elimination of the parameters α, β from (2.3) using (2.1) and (2.2) leads to the vanishing of the scalar function

$$_v k_\lambda = re({}_v k_n) + g_{\alpha\beta} t^\alpha q^\beta {}_v k_a \tag{2.4}$$

where $p^\delta{}_{,\sigma}\ (du^\sigma/ds) = {}_v k_a q^\delta$.

Thus vanishing of the scalar function $_v k_\lambda$ characterizes hyperasymptotic curves of the vector field V. Since for normal congruences $_v k_\lambda = e({}_v k_n)$, therefore $_v k_\lambda$ may be defined as the *relative normal curvature of the vector field V with respect to the curve on S. The principal directions (and lines of relative curvature) of V are given by $\partial({}_v k_\lambda)/\partial(du^\alpha) = 0$ i.e.,*

$$\epsilon^{\rho\sigma}[r d_{\sigma\delta} g_{\rho\mu} p^\delta - t^\alpha g_{\alpha\delta} g_{\mu\sigma} p^\delta_{,\rho}] du^\mu = 0, \tag{2.5}$$

and the orthogonal trajectories of the hyperasymptotic curves of V are [1]

$$\epsilon^{\sigma\delta} g_{\beta\delta}[r d_{\alpha\sigma} p^\alpha + g_{\alpha\rho} t^\alpha p^\rho_{,\sigma}] du^\beta = 0. \tag{2.6}$$

The first theorem, similar to Meusnier's theorem, stated below follows from (2.4) and the second from the fact that equation (2.5) is the same as equation (2.6).

THEOREM 2.1. *The relative normal curvature of the vector field V with respect to asymptotic directions of V at a point of the surface equals the product of the associate curvature by the cosine of the angle between the direction λ and u, the latter being a vector orthogonal to V.*

THEOREM 2.2. *The hyperasymptotic curves of the vector field V form an orthogonal net with the lines of relative curvature of V.*

The corresponding form of Euler's theorem, the torsion of a hyperasymptotic curve of the vector field V and its relation to the torsion of the union curve of V shall be considered elsewhere.

Mail address: Desh Bandhu College, University of Delhi, New Delhi, India.

References

1. L. P. Eisenhart, An introduction to differential geometry with use of tensor calculus, Princeton University Press, 1940.
2. R. S. Mishra, A problem in rectilinear congruence using tensor calculus, Bull. Calcutta Math. Soc., 42 (1950) 118–122.
3. ———, Union curves and hyperasymptotic curves on the surface of reference of a rectilinear congruence, Bull. Calcutta Math. Soc., 42 (1950) 213–216.
4. T. K. Pan, Normal curvature of a vector field, Amer. J. Math., 75 (1952) 955–966.
5. R. N. Kaul, Union curvature of a vector field, The Tensor (New Series), 7 (1957) 185–189.

LIMIT POINTS OF BOUNDED SEQUENCES

PAUL SCHAEFER, SUC at Geneseo, New York

Barone [1] showed that a sufficient condition for a bounded sequence $s=\{s_n\}$ to have a connected set of limit points is that $s_n-s_{n-1}\to 0$. His example, $s_n=e^{in}$, indicates that this condition is not necessary. However, it suggests the following theorem.

THEOREM. *Let $C(s)$ be the set of limit points of the bounded complex sequence s. Then $C(s)$ is connected if and only if there exists a subsequence y of s such that $C(y)=C(s)$ and $y_n-y_{n-1}\to 0$.*

Proof of sufficiency follows from Barone's result. To show necessity for bounded divergent real sequences, let $C(s)=[a, b]$. For each positive integer $n\geqq 2$, let $I_{nk}=[x_{k-1}, x_k]$, where $x_k=a+k(b-a)/n$, $k=1, \cdots, n$ and $x_0=a$. Then let J_{nk} be the open interval concentric with I_{nk} of length $2(b-a)/n$. We enumerate the set $\{J_{nk}\}$ as follows: $J_{21}, J_{22}, J_{33}, J_{32}, J_{31}, J_{41}, \cdots$, alternately from left to right and then right to left. Denote this enumeration by $J_1, J_2, \cdots$. Define a subsequence y of s by setting $y_1=s_n$ for $n=n_1$, where n_1 is the least subscript of all s_n in J_1. If $y_1, \cdots, y_{p-1}$ have been defined, let $y_p=s_n$ for $n=n_p$, where n_p is the least $n>n_{p-1}$ for which s_n is in J_p. Clearly, $y_n-y_{n-1}\to 0$ since the length of J_n tends to zero with increasing n. Also, $C(y)$ is seen to be equal to $C(s)$. The case for complex sequences can be handled in a similar, but slightly more complicated fashion.

Mail Address: 26 Lima Road, Geneseo, N. Y. 14454.

Reference

1. H. G. Barone, Limit points of sequences and their transforms by methods of summability, Duke Math. J., 5 (1939) 740–752.

PERIODIC SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS

LOUIS BRAND, University of Houston

In the linear difference equation

$$P(E)x(t) = \{f(t)\}_p, \qquad E = 1 + \Delta, \tag{1}$$

$P(E)$ is a polynomial of degree n and $\{f(t)\}_p$ is a sequence of period p on the set $\{0, 1, 2, \cdots\}$. In the operational calculus [1] in which $s=\{0, 1, 0, 0, \cdots\}$ is the shift operator and $r=1/s$, we have

$$\{E^k x(t)\} = r^k\{x(t)\} - x_0 r^k - x_1 r^{k-1} - \cdots - x_{k-1} r. \tag{2}$$

When (2) is applied to $\{f(t)\}_p$, $E^p f(t)=f(t)$ and

$$\{f(t)\}_p = (f_0 r^p + f_1 r^{p-1} + \cdots + f_{p-1} r)/(r^p - 1) = F(r). \tag{3}$$

On applying (2) and (3), equation (1) becomes

$$P(r)\{x(t)\} = Q(r) + F(r), \tag{4}$$

where $Q(r)$ is a polynomial of degree n involving the initial values $x_0, x_1, \cdots, x_{n-1}$ linearly. We now have the

THEOREM. *Equation* (1), *with the transform* (4), *has a unique solution of period p iff the zeros r_j of $P(r)$ are not poles of $F(r)$. Its n initial values are determined uniquely from the equations written for each r_j:*

$$Q^{(i)}(r_j) + F^{(i)}(r_j) = 0, \qquad i = 0, 1, \cdots, m_j - 1, \tag{5}$$

in which r_j is of multiplicity m_j and (i) denotes an i-th derivative.

The proof consists in showing that the solution having the initial values given by (5) may be put in the operational form (3) of a sequence of period p.

If the sums of the input and output sequences are S_f and S_x, $P(1)S_x = S_f$; and $P(1) = 0$ implies that $S_x = S_f = 0$.

Example. $(E^2 - 2E + 2)x(t) = \{0, 1, 0, -1\}_4$.

The zeros of $r^2 - 2r + 2$ are $1 \pm i$. The input sequence becomes $F(r) = r/(r^2+1)$ and $Q(r) = x_0r^2 + x_1r - 2x_0r$. $Q(1+i) + F(1+i) = 0$ yields two equations giving $x_0 = 2/5$, $x_1 = 1/5$; the given equation now completes the solution: $\{x\} = \{2/5, 1/5, -2/5, -1/5\}_4$. Here $P(1) = 1$ and $S_f = 0$, so that $S_x = P(1)\ S_f = 0$, a convenient check.

Mail Address: Dept. of Mathematics, University of Houston, Houston, TX 77004.

Reference

1. Louis Brand, A division algebra for sequences and its associated operational calculus, this MONTHLY, 71 (1964) 719–728.

ON CERTAIN CLASSES OF ASSOCIATIVE RINGS

A. R. BLASS AND C. V. STANOJEVIC, University of Detroit

Associative rings of characteristic p satisfying the identities $x^p + y^p = (x+y)^p$ or $xy^p = x^py$ were studied in [1] and [2] respectively. We shall consider rings satisfying analogous identities with an arbitrary function in place of the pth power. Let R be a ring with center Z, and let h be a map of R into itself. We define R to be an L_h-ring (P_h-, Z_h-ring) iff for all $x, y \in R$ $h(x+y) = h(x) + h(y)$, $(xh(y) = h(x)y$, $h(x) \in Z)$. If $h(x) = x^n$, we replace the subscript h by n. R has the n-Z property iff, whenever m is an integer with no prime factors $\geqq n$, $mx \in Z$ implies $x \in Z$.

THEOREM 1. *A P_h-ring containing a nondivisor of zero is an L_h-ring.*

THEOREM 2. *Let R be a P_h-ring such that $h(x)$ is a divisor of zero for at most one $x \in R$. R is commutative iff R is a Z_h-ring.*

THEOREM 3. *Let R be a P_h-ring containing a nondivisor of zero. If $h(x)=xg(x)$ for all $x\in R$, then there is a constant $t\in Z$ such that, for all $x\in R$, $h(x)=tx$.*

THEOREM 4. *For each $x\in R$, let $n-1$ integers $t_2(x), t_3(x), \cdots, t_n(x)$ be given, and let $h(x)=\sum_{j=2}^{n} t_j(x)x^j$. If R is an L_h-ring with the $n-Z$ property, then R is a Z_h-ring. In particular, an L_n-ring with the $n-Z$ property is a Z_n-ring.*

Mail Address: Dr. Caslav Stanojevic, Department of Mathematics, University of Detroit, Detroit, MI 48221.

References

1. Časlav Stanojević, A sufficient and necessary condition for commutativity of division rings with characteristic p, Bull. Soc. Math. Phys. Macedonie, 12 (1961) 25–27.

2. Alexander Abian and W. A. McWorter, On the structure of pre-p-rings, this MONTHLY, 71 (1964) 155–157.

ON FERMAT'S CONJECTURE

W. E. CHRISTILLES, St. Mary's University

In this paper Fermat's Conjecture, which states that there exist no nontrivial solution in integers α, β, and π of the equation

$$\alpha^n + \beta^n + \pi^n = 0$$

for n an integer greater than 2, is treated from an elementary standpoint. It is sufficient to demonstrate the impossibility of

$$\alpha^p + \beta^p + \pi^p = 0 \tag{1}$$

in integers α, β, and π which are pairwise coprime, for p an odd prime. The major result, which is proven only after the introduction of seventeen preliminary theorems, is the proof of the following theorem.

THEOREM. *Let α, β, π, and p be integers, with $p\neq 3$ an odd prime, such that α, β, and π are pairwise coprime, and $\pi<0<\alpha<\beta<|\pi|$. Moreover, let $\gamma>2$ be the largest integer, where $\gamma=jq$ and $j>2$, such that*

$$q\left|\left(\frac{\alpha^{2p}+\beta^{2p}+\pi^{2p}}{2}\right),\qquad j\right|\alpha\beta\pi, \text{ and } (p, \phi(\gamma)) = 1.$$

Equation (1) holds only if $\alpha>|\alpha+\beta+\pi|>\gamma$.

Mail Address: Mathematics Department, St. Mary's University, 2700 Cincinnati Avenue, San Antonio, TX 78228.

THE NUMBER OF FACTORS IN NONUNIQUE FACTORIZATIONS

W. D. BOUWSMA, Pennsylvania State University

Let $S_k=\{x\in J: x\equiv 1 \pmod{k}\}$, where $J=\{1, 2, 3, \cdots\}$ and the integer $k>2$. A number $x\in S_k$ is called prime in S_k provided it has no factors in S_k other than 1 and x. Factorization in such systems is often used to illustrate nonunique prime factorization. B. Jacobson [1] has shown that if $k>3$ and k is prime, the

number of prime factors in such a prime factorization is not unique. By a slight modification of Jacobson's procedure, one can show that for k composite and $k>6$, the number of prime factors in such a factorization is not uniquely determined by the number being factored.

On the other hand, for $k=3$, 4, or 6, the number of prime factors in the factorization of x in S_k is uniquely determined by x. This is proved by comparing with the prime factorization in J. For example, suppose $x\in S_3$, and the prime factorization of x in J is $x=y_1\cdots y_s z_1\cdots z_t$, where each $y_i\in S_3$ and each $z_j\equiv 2 \pmod 3$. Then the number of prime factors of x in any prime factorization in S_3 is $s+\frac{1}{2}t$. The proofs for S_4 and S_6 are similar.

Mail Address: Mathematics Department, Southern Illinois University, Carbondale, IL 62901.

Reference

1. Bernard Jacobson, Factorizations with unequal numbers of primes, this MONTHLY, 73 (1966) 1110.

THE DETERMINATION OF A GALOIS POLYNOMIAL FROM ITS ROOT POLYNOMIALS

HOWARD KLEIMAN, Research Student, University of London King's College

Let K be a field, $\overline{K}$ an algebraic closure of K, and $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ an irreducible monic polynomial in $K[x]$. If $f(x)=(x-\theta_1)\cdots(x-\theta_n)$ in $\overline{K}[x]$, we call f a *Galois polynomial* if the field $L=K(\theta_1)$ is a Galois (normal and separable) extension of K. This means that the θ_i are all distinct and can be written in the form $\theta_i=g_i(\theta_i)$, $(i=1, 2, \cdots, n)$, where $g_i(x)$ is a polynomial in $K[x]$ of degree $<n$. The polynomials $g_i(x)$ are uniquely determined by f; they are called the *root polynomials* of f. The Galois group G of L/K can be represented by the $g_i(x)$ where the group operation is composition mod $f(x)$.

In this note, I would like to give a condition under which a Galois polynomial f is uniquely determined by its root polynomials. Recall that a field K is *formally real* if $\sum_{i=1}^m c_i^2=0$ with $c_i\in K$ implies $c_1=\cdots=c_m=0$.

THEOREM 1. *Suppose that K is formally real, and that at least one of the polynomials g_i is of degree 2. Then f is uniquely determined by $g_1, \cdots, g_n$.*

Proof. If $P_j(x)=\sum_{r_1<\cdots<r_j} g_{r_1}\cdots g_{r_j}-(-1)^j a_j$, then it is divisible by $f(x)$. Since $\deg P_1<n=\deg f$, we have $P_1(x)=0$; thus $a_1=-\sum_{i=1}^n g_i(x)$. Suppose for definiteness that $g_2(x)=Ax^2+Bx+C$, $A\neq 0$. As g_i runs through the group G, so does $g_2(g_i)$ (after reduction mod f). In other words there is a permutation $i\to i'$ of $\{1, \cdots, n\}$ such that $g_2(g_i(x))\equiv g_{i'}(x) \mod f(x)$, $(i=1, 2, \cdots, n)$. Adding these congruences for all i, we obtain $\sum_{i=1}^n g_2(g_i(x))\equiv -a_1 \mod f(x)$, since $\sum_{i=1}^n g_{i'}(x)=\sum_{i=1}^n g_i(x)=-a_1$. Thus, putting $h(x)=\sum_{i=1}^n g_2(g_i(x))+a_1$, we have $h(x)\equiv 0 \mod f(x)$. Now $h(x)=\sum_{i=1}^n (Ag_i(x)^2+Bg_i(x)+C)+a_1$ $=\sum_{i=1}^n Ag_i(x)^2+(-Ba_1+nC+a_1)$. If $m=\max \deg g_i$, we can write $g_i(x)$ $=c_ix^m+d_ix^{m-1}+\cdots$ (where $c_i=0$ if $\deg g_i<m$). Thus, the leading term of

$h(x)$ is $A(c_1^2+\cdots+c_n^2)x^{2m}$. This term is not zero, since K is formally real. Thus, deg $h=2m$, and since $2\leqq m\leqq n-1$, we have $4\leqq\deg h\leqq 2n-2$. But $h(x)\equiv 0 \pmod{f(x)}$ means that $f(x)$ divides $h(x)$. Now a polynomial of degree $<2n$ can have at most one irreducible factor of degree n. Hence, f is uniquely determined by h, and so by $g_1,\cdots,g_n$.

The following theorems give further results in the general case:

THEOREM 2. *Suppose some g_i has degree $k>1$. Then there are at most $(n-1)(k-1)$ polynomials f with root polynomials $g_1,\cdots,g_n$.*

THEOREM 3. *Suppose $g_1,\cdots,g_n$ are all linear. Then they uniquely determine all the coefficients of f except for the constant term a_n, which cannot be determined from them.*

Acknowledgment. I would like to thank the referee for helpful suggestions, including stylistic improvements, and Theorem 2.

Mail Address: Queensborough Community College, Bayside, New York 11364.

ON EVALUATING A CERTAIN INTEGRAL

B. L. S. PRAKASA RAO, University of California, Berkeley

The aim of this note is to evaluate the integral

$$R(\tau_1,\tau_2)\equiv\int_{-\infty}^{\infty}[h(y-\tau_1)-h(y)][h(y-\tau_2)-h(y)]dy,$$

where $h(y)=\epsilon(y)|y|^{\lambda}$, $\epsilon(y)=\beta$ for $y\leqq 0$ and $\epsilon(y)=\gamma$ for $y>0$, and $0<\lambda<\frac{1}{2}$. We shall indicate the method here. Some of the details can be found in Prakasa Rao [1]. It is easily seen that $R(\tau_1,\tau_2)<\infty$ for any τ_1 and τ_2. Define $h_\alpha(y)=\epsilon(y)|y|^{\lambda}\exp\{-\alpha|y|\}$ and $\theta_\alpha(y)=[h_\alpha(y-\tau_1)-h_\alpha(y)]\,[h_\alpha(y-\tau_2)-h_\alpha(y)]$ for $\alpha\geqq 0$. It can be shown that $R(\tau_1,\tau_2)=\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\theta_\alpha(y)dy$ and then we have $R(\tau_1,\tau_2)=(2\pi)^{-1}\Gamma^2(1+\lambda)(\gamma^2+\beta^2-2\gamma\beta\operatorname{Cos}\pi\lambda)\{G(-1-2\lambda,0)\}$, where $G(a,\epsilon)=\int_{-\infty}^{\infty}|t|^{a-1}(e^{it\tau_3}-e^{it\tau_1}-e^{-it\tau_2}+1)e^{-\epsilon|t|}dt$ and $\tau_3=\tau_1-\tau_2$. It can now be shown that $G(z,\epsilon)$ is analytic in the region $\{z:\operatorname{Re}(z)>-2\}$ and $G(a,\epsilon)$ can be evaluated for $a>0$. In particular, we can evaluate $G(\eta,\epsilon)$ by analytic continuation for $\eta=-(1+2\lambda)$. Again by bounded convergence theorem we get that $R(\tau_1,\tau_2)=c[|\tau_1|^{2\lambda+1}+|\tau_2|^{2\lambda+1}-|\tau_1-\tau_2|^{2\lambda+1}]$ where the constant c can be evaluated explicitly. As a special case of this result, we have $\int_{-\infty}^{\infty}[h(y-\tau)-h(y)]^2dy=2c|\tau|^{2\lambda+1}$.

Mail Address: Dept. of Math., University of Illinois, Urbana, Ill. 61802.

Reference

1. B. L. S. Prakasa Rao, On evaluating a certain integral, Department of Statistics, University of California, Berkeley, 1966.

ON THE DIOPHANTINE EQUATION $y^2+p^2=x^3$

HYMIE LONDON, Pennsylvania State University

THEOREM. *In the equation*

$$y^2 + p^2 = x^3,\ p > 2 \text{ and prime, } x, y \text{ integers, } y \geqq 0 \tag{1}$$

(i) *if* $p=12a^2-1$, *a a positive integer, then* $(x, y)=(4a^2+1,\ 2a(4a^2-3))$ *is a solution*;

(ii) *if* $p^2=3a^2+1$, *a a positive integer, then* $(x, y)=(4a^2+1,\ a(8a^2+3))$ *is a solution*;

(iii) *if* $p\equiv 3 \pmod 4$, *then* (i) *and* (ii) *yield the only solutions, if any exist*;

(iv) *if* $p\equiv 1 \pmod 4$, *then* (i) *and* (ii) *yield at most two solutions. Since* $p=m^2+n^2$ *for some integers m and n, then by finding all the integer solutions,* (u, v) *of*

$$nu^3 + 3mu^2v - 3nuv^2 - mv^3 = 1, \tag{2}$$

we get the remaining solutions of (1) *if any exist, as*

$$(x, y) = (p(u^2 + v^2),\ p(mu^3 - 3nu^2v - 3muv^2 + nv^3)).$$

Proof. Suppose $(y, p)=1$. Then (1) implies $y+pi=(a+bi)^3$. From this $p=b(3a^2-b^2)$, which implies $b=\pm 1, \pm p$. Each of these can easily be settled.

Suppose $(y, p)=p$. Setting $y=py_1$, we get $p^2(y^2+1)=x^3$ which implies $x=px_1$. Thus $y_1^2+1=px_1^3$, which implies $p\equiv 1 \pmod 4$. Since $p=m^2+n^2$ for some integers m and n, then $y_1^2+1=px_1^3$ implies

$$y_1 + i = (m + ni)(u + vi)^3.$$

The rest follows by the usual techniques.

Mail Address: Department of Mathematics, McAllister Building, University Park, PA 16802.

PROPERTIES OF THE RADII OF SPHERES ASSOCIATED WITH REGULAR SIMPLEXES

A. C. WILDE, University of Michigan

An equilateral triangle has two interesting circles: one passing through the midpoints of the edges and one passing through the vertices. This idea can be extended to higher dimensional regular simplexes; an a-dimensional simplex has a interesting spheres. Let $r(a, b)$ denote the radius of the sphere in the a-dimensional simplex passing through the centroids of the b-dimensional simplexes $(0\leqq b<a)$. Then

$$r(a, b) = l\sqrt{\left[\frac{a-b}{2(a+1)(b+1)}\right]},$$

where l is the length of an edge. We shall consider the ratio

$$(1)\qquad t(a,b,n) = \frac{r(a,b)}{r(a,b+n-1)} = \sqrt{\left[\frac{(a-b)(b+n)}{(a-b-n+1)(b+1)}\right]}.$$

For each n, a symmetrical triangle can be formed that is related to the Pascal triangle by the relation

$$\prod_{b=0}^{a-n} t(a,b,n) = \binom{a}{n-1}.$$

Manipulating with (1), we can develop the infinite product,

$$n = \prod_{j=n}^{\infty} \frac{(j+2)(j+n)}{(j+1)(j+n+1)},$$

which in turn can be generalized to

$$\binom{n}{r} = \prod_{j=0}^{\infty} \frac{(j+r+1)(j+n-r+1)}{(j+1)(j+n+1)}.$$

An infinite number of radii are rational multiples of l. Also, there exist an infinite number of pairs of equal radii. If we let $f(r(a,b)) = r(a, a-b-1)$ and $g_k(r(a,b)) = r(k^2a+k^2-1, k^2b+k^2-1)$, then

$$(g_k \circ g_m)(r(a,b)) = g_{km}(r(a,b)) = (g_m \circ g_k)(r(a,b)),$$

and

$$(f \circ g_k)(r(a,b)) = (g_k \circ f)(r(a,b)).$$

Mail Address: 2675 Geddes Ave., Ann Arbor, Michigan 48104.

ON THE ARBITRARY POWER OF AN ARBITRARY (2×2)-MATRIX

P. J. BLATZ, North American Aviation Science Center

It is desired to determine a general expression for any power of a matrix

$$(1)\text{–}(2)\qquad M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \text{ in terms of } M = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

After writing recursion formulae of the form:

$$(3)\ \&\ (4)\qquad a_{n+1} = a_na_1 + b_nc_1 \quad\text{and}\quad b_{n+1} = a_nb_1 + b_nd_1$$

$$(5)\ \&\ (6)\qquad c_{n+1} = c_na_1 + d_nc_1 \quad\text{and}\quad d_{n+1} = c_nb_1 + d_nd_1$$

one can introduce an operator E in (3) and (4), or in (5) and (6):

$$(7)\ \&\ (8)\qquad (E - a_1)a_n = c_1b_n \quad\text{and}\quad (E - d_1)b_n = b_1a_n$$

and solve these simultaneous characteristic equations to obtain:

$$(M^2)_{ij} = \frac{\lambda_1^n - \lambda_j^n}{R} M_{ij} + \frac{II\,(\lambda_1^{n-1} - \lambda_j^{n-1})\delta_{ij}}{R} \tag{9}$$

where λ_1 and λ_2 are eigenvalues of M

$$I = \lambda_1 + \lambda_2 = a_1 + d_1 = \mathrm{Tr}(M) \tag{10}$$

$$II = \lambda_1\lambda_2 = a_1d_1 - b_1c_1 = \mathrm{Det}(M) \tag{11}$$

$$R = \lambda_1 - \lambda_2 = \sqrt{[I^2 - 4II]}. \tag{12}$$

This method may be readily generalized to obtain the jth power of an arbitrary $(n\times n)$-matrix, since the characteristic equation in n-space is merely

$$0 = \sum_{s=0}^{n} (-1)^s E^{n-s} I_s, \tag{13}$$

where I_s is the sth invariant, or

$$I_s = \mathrm{Tr}(M^s). \tag{14}$$

Thus the general solution will be a linear combination of terms which are the $(j-1)$-th power of each of the n roots of (13). In order to solve for the coefficients of this term, it is necessary to know the components of the first n matrices. So there is no advantage in using this method unless $j>n$.

Mail Address: 1049 Camino Dos Rios, Thousand Oaks, Calif. 91360.

GENERALIZATIONS OF A COMPLEX ANALOGUE OF THE REAL TCHEBICHEV POLYNOMIAL THEOREM

J. R. SLAGLE, National Institutes of Health, Division of Computer Research and Technology

The norm of $F(z)$ on a set D of complex numbers is, by definition,

$$\|F(z)\| = \sup_{z\in D} |F(z)|.$$

The following notational conventions hold throughout the paper: n is a nonnegative integer; $0<R<\infty$; and c's are complex constants.

This paper presents two generalizations of the following well-known complex analogue of the real Tchebichev polynomial theorem:

THEOREM 1. *Let D be an origin-centered disc of radius R. The n-th degree monic polynomial with the least norm on D is uniquely z^n and $\|z^n\| = R^n$.*

Proof. This is an immediate consequence of Theorem 2 below. It can also be proved directly *via* the Tchebichev Alternates Theorem for trigonometric polynomials.

Theorem 2 generalizes the "nth degree monic polynomial" of Theorem 1 to power series with one of the coefficients fixed.

THEOREM 2. *Let D be an origin centered disc of radius R. Let n be a non-negative integer and let c_n be a fixed constant. Of all functions of the form* $\sum_{j=0}^{\infty} c_j z^j$, *the (unique) one with the least norm on D is $c_n z^n$ and* $\|c_n z^n\| = |c_n| R^n$.

Theorem 3 generalizes the "disc" of Theorem 1 to a wide class of subsets of the disc.

THEOREM 3. *If D (i) is a subset of the disc $|z-c| \leqq R$ and (ii) contains $2n$ equally spaced points around the circle $|z-c| = R$, then the n-th degree monic polynomial with the least norm on D is uniquely $(z-c)^n$ and* $\|(z-c)^n\| = R^n$.

Mail Address: Div. of Computer Research and Technology, Bldg. 12A, National Institutes of Health, Bethesda, MD 20014.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT 06268.

LIMITS

LEONARD GILLMAN, University of Rochester

I propose a definition of limit for first-term calculus that is easier to understand than (ϵ, δ), intuitively suggestive, yet completely rigorous.

Proofs are qualitative rather than computational, and pictures are very informative. A good example is the theorem on the limit of the reciprocal, which becomes trivial. On the other hand, the details about composite functions are fussy.

These days we expect students to understand the definition of sup (and inf). The hurdle of "for all ϵ" is there, but mollified. The proposed definition of limit is based on sup. A student who understands sup (though lacking manipulative skill) can understand limit. Later on he can learn the (ϵ, δ) formulation with greater confidence.

It suffices to define limit from the left.

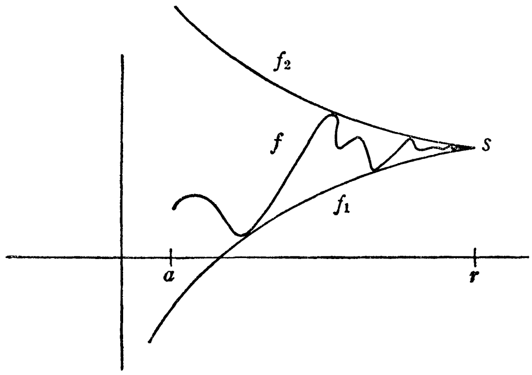
DEFINITION. $\lim_{x\to r^-} f(x) = s$ *if there exist an interval $[a, r)$ and functions f_1 and f_2 such that*

(a) $f_1 \leqq f \leqq f_2$ *on* $[a, r)$,

(b) f_1 *is increasing and* f_2 *is decreasing on* $[a, r)$,

(c) $\sup_{[a,r)} f_1 = \inf_{[a,r)} f_2 = s$.

Thus, f "zeros in" on the value s.



The definition is equivalent to (ϵ, δ). In fact, the conditions evidently imply (ϵ, δ)-$\lim_{r^-} f = s$. Conversely, if (ϵ, δ)-$\lim_{r^-} f = s$, choose $[a, r)$ on which f is bounded, and for $x \in [a, r)$ define $f_1(x) = \inf_{[x,r)} f$ and $f_2(x) = \sup_{[x,r)} f$; then f_1 and f_2 are as stated. (In terms of continuity, if f is defined at r and $f(r) = s$, the definition states that f is both upper semi-continuous and lower semi-continuous at r^-.)

Let us say in (b) and (c) that f_1 increases on $[a\ r)$ to the value s and f_2 decreases to s. Then the definition may be expressed:

$\lim_{x \to r^-} f(x) = s$ *if there is an interval* $[a, r)$ *on which* f *lies between two functions, one increasing to* s *and the other decreasing to* s.

(In the corresponding statement for the limit from the right, append "*as* $x \to r^+$.")

Some quick theorems: *If* f *has a limit at* r^- *then* f *is bounded on an interval* $[a, r)$. *If* f *is increasing on* $[a, r)$ *and* $\sup_{[a,r)} f = s$, *then* $\lim_{r^-} f = s$ (*take* $f_1 = f, f_2 = s$), *and correspondingly if* f *is decreasing. Also,* $\lim s = s$.

We now outline some additional proofs.

REMARK 1. In the definition, if $\sup_{[a,r)} f_1 > 0$, then $f_1(a_0) > 0$ for some $a_0 \in [a, r)$. Hence when $\sup f_1 > 0$ we may assume $\inf f_1 > 0$ (and correspondingly for f_2).

Let A be any set of numbers and let $-A = \{-\alpha : \alpha \in A\}$. Obviously, a number m is a lower bound of $-A$ if and only if $-m$ is an upper bound of A. Therefore $\inf(-A) = -\sup A$. More generally, any strictly increasing [decreasing] function from an interval onto an interval preserves [reverses] order and takes sups to sups [infs] and infs to infs [sups]. In the case discussed, $-f_2$ is increasing, $-f_1$ decreasing, and $-f_2 \leqq -f \leqq -f_1$, and so we have

(1) $$\lim(-f) = -\lim f.$$

Additional instances are its analogue:

(2) *If* $\lim f \neq 0$, *then* $\lim(1/f) = 1/(\lim f)$

(see REMARK 1), as well as

$$\lim(c + f) = c + \lim f \tag{3}$$

and $\lim(cf) = c \lim f$ (which includes (1)).

REMARK 2. Let $\lim_{r^-} f = s$ and $\lim_{r^-} g = t$, and let f_1, f_2, g_1, g_2 be as in the definition. Then on the smaller of the associated intervals, both sets of relations (a), (b), (c) are satisfied.

To prove the theorem on $\lim(f+g)$ we invoke the argument that if $\alpha \leqq m$ for all $\alpha \in A$, then $\sup A \leqq m$. This merely states that the least upper bound is the least of the upper bounds, but for some reason it baffles even graduate students. It should not be sprung on undergraduates without warning.

Evidently, f_1+g_1 is increasing. If $x \leqq y$ then

$$f_1(x) + g_1(y) \leqq f_1(y) + g_1(y) \leqq \sup(f_1 + g_1),$$

and two applications of (3) yield $s+t \leqq \sup(f_1+g_1)$. The reverse inequality is obvious. With the corresponding facts about f_2+g_2, we get the sum theorem (and then with (1), the difference theorem).

Similarly, if $s>0$ and $t>0$, then $\lim(fg) = st$. The case $s \geqq 0$, $t \geqq 0$ follows from this and the sum theorem by considering

$$\lim[(f + 1)(g + 1)].$$

The general product and quotient theorems then follow from (1) and (2).

The next two results use the fact that if $F \leqq G$, where F is increasing and G decreasing, then $\sup F \leqq \inf G$.

If $f \leqq g$ *then* $s \leqq t$. *Proof*: $f_1 \leqq g_2$.

Limits are unique. Proof: Assume that $\bar{f}_1 \leqq f \leqq \bar{f}_2$ on $[a, r)$, with $\bar{f}_1$ increasing, $\bar{f}_2$ decreasing, and $\sup \bar{f}_1 = \inf \bar{f}_2 = \bar{s}$. Then $f_1 \leqq \bar{f}_2$ and $\bar{f}_1 \leqq f_2$. Hence $s \leqq \bar{s}$ and $\bar{s} \leqq s$.

Next, *if* $f \leqq h \leqq g$ *and* $s = t$, *then* $\lim h = s$. *Proof*: $f_1 \leqq h \leqq g_2$. Note that to evaluate the limit of $(\sin x)/x$ by squeezing between cos and 1, one does not need this theorem but only the definition.

In the theorem on composite functions, we are given $\lim_{r^-} f = s$ and $\lim_{y \to s} g(y) = g(s)$ and are to prove $\lim_{r^-} (g \circ f) = g(s)$. Here f and f_i are defined on an interval $[a, r)$, and g and g_i are defined on an interval $[c, d]$ enclosing s. We may assume that $f_1(a) \geqq c$ and $f_2(a) \leqq d$. Now, either

$$g_1(f_1(x)) \leqq g(f(x)) \leqq g_2(f_1(x))$$

or

$$g_1(f_2(x)) \leqq g(f(x)) \leqq g_2(f_2(x)),$$

according as $f(x) \leqq s$ or $f(x) \geqq s$. (A picture makes this obvious.) Hence if h_1 is the pointwise minimum of $g_1 \circ f_1$ and $g_1 \circ f_2$, and h_2 is the pointwise maximum of $g_2 \circ f_1$ and $g_2 \circ f_2$, then

$$h_1 \leqq g \circ f \leqq h_2$$

on $[a, r)$, with h_1 increasing and h_2 decreasing. One then verifies that $\sup_{[a,r)} h_1 = \inf_{[a,r)} h_2 = g(s)$. (Alternatively, h_1 and h_2 can be defined by adding the functions in question and subtracting $g(s)$.)

The definition of limit is paraphrased for the limit of a sequence: $\lim x_n = s$ *if there exist an increasing sequence* (a_n) *and a decreasing sequence* (b_n) *such that* $a_n \leqq x_n \leqq b_n$ *for all* n *and*

$$\text{(d)} \qquad \sup a_n = \inf b_n = s.$$

In the same spirit we say that (x_n) *is a Cauchy sequence if the above hold with* (d) *replaced by*

$$\inf(b_n - a_n) = 0.$$

Convergence of Cauchy sequences becomes transparent.

Acknowledgment. I wish to thank R. H. McDowell for several helpful conversations.

METRIC COMPLETION SIMPLIFIED

OTOMAR HÁJEK, Caroline University, Prague, and Western Reserve University

The purpose of this note is to exhibit a construction of the completion of metric spaces which is considerably shorter and (in the author's opinion) conceptually simpler than that usually given. It appears that the considerable number of technical tricks of purely local significance can be replaced by references to previously established results on metric spaces which are important on their own—see next paragraph but one. (On the other hand, it is not apparent how, if at all, the construction below can be carried over to the problem of completion of uniform spaces.)

Assume given a metric space X, with $d(x, y)$ denoting the distance between points x, y in X. Let C be the linear space of all bounded continuous maps $\phi: X \to R^1$, provided with the uniform norm $\|\phi\| = \sup\{ |\phi(x)| \mid x \in X\}$.

It is assumed that the following two assertions have already been proved: C is always complete; a subset of a complete metric space Y is itself complete if and only if it is closed in Y.

Now fix a point $a \in X$, and define a map $h: X \to C$ by specifying, for any $x \in X$, the map $h(x) \in C$, i.e., $h(x): X \to R^1$, by $h(x)(u) = d(x, u) - d(a, u)$ for $u \in X$. Evidently indeed each $h(x) \in C$; e.g., boundedness follows from $\|h(x)\| \leqq d(x, a) < +\infty$. Similarly,

$$\|h(x) - h(y)\| = \sup\{ | d(x, u) - d(y, u) | \mid u \in X\} \leqq d(x, y),$$

and obviously ($u = y$, say) also $\|h(x) - h(y)\| \geqq d(x, y)$; thus h is an isometry into. Finally, $\overline{h[X]} = X^*$ is a closed subset of the complete space C, so that h maps X isometrically onto a dense subset of a complete space X^*.

In point of fact, this is the entire construction; however, one also has a further assertion on unicity of the completion X^*. Elementary category theory suggests inquiring whether the completion process can be interpreted as the action of a functor on appropriately defined categories, whereupon the unicity result appears as a functorial uniqueness formulated in a very natural manner.

Thus, let $\mathfrak{M}$ be the concrete category of all metric spaces and isometries into (more precisely, the objects X are to be provided with a distinguished point $a \in X$, but the morphisms need not preserve the distinguished points). Having a diagram $f\colon X \to Y$ in $\mathfrak{M}$, one forms the completions and isometric imbeddings as described above, obtaining

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ X^* & & Y^* \end{array}$$

Then one attempts to close this diagram commutatively by some isometry $f^*\colon X^* \to Y^*$. This is easily obtained by extending the isometry $h_Y \circ f \circ h_X^{-1}\colon h_X[X] \to Y^*$ over the set $\overline{h_X[X]} = X^*$ in the obvious manner (isometries carry Cauchy sequences into Cauchy sequences; Y^* is complete). It is then readily verified that $(X, f) \to (X^*, f^*)$ indeed defines a covariant functor T taking $\mathfrak{M}$ into its subcategory $\mathfrak{M}_c$ of complete spaces. Furthermore, the construction just performed shows that $h = \{h_X \mid X \text{ an object of } \mathfrak{M}\}$ is a natural transformation of the identity functor $\mathfrak{M} \to \mathfrak{M}$ into (a covariant functor) $T\colon \mathfrak{M} \to \mathfrak{M}_c$, with the property that $h_X[X]$ is dense in $T(X) = X^*$. Any natural transformation with these properties may be called a *completion*; the result on uniqueness may now be formulated thus: Any two completions are naturally equivalent, i.e., unique up to natural equivalences. The verification is both simple and natural. Given completions h, h' in

$$T \xleftarrow{h} \text{identity} \xrightarrow{h'} T',$$

for each $X \in \mathfrak{M}$ one constructs $e_X\colon T(X) \to T'(X)$ (and also $e'_X = (e_X)^{-1}$) as follows. Consider

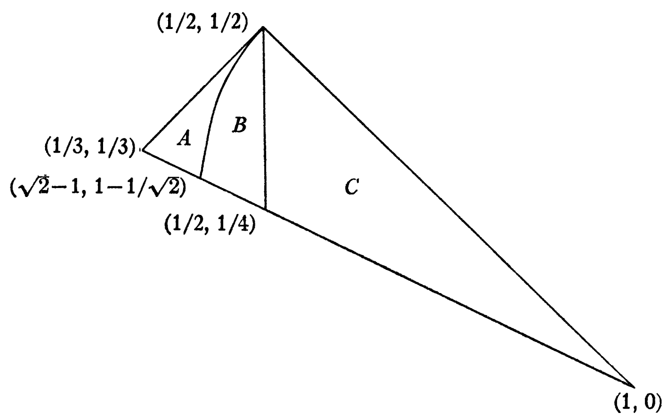
$$T(X) \xleftarrow{h_X} X \xrightarrow{h'_X} T'(X);$$

for any $x \in T(X)$ take $x_n \in X$ with $h_X(x_n) \to x$ (density of $h_X[X]$ in $T(X)$); then $h'_X(x_n)$ is Cauchy and converges (isometry, and completeness of $T'(X)$), so that one may set $e_X(x) = \lim h'_X(x_n)$. It is then easily verified that $e = \{e_X\}$ and $e' = \{e'_X\}$ are mutually inverse natural transformations $T \rightleftarrows T'$. (Apparently this construction also works for any subcategory $\mathfrak{M}'$ of $\mathfrak{M}$, in particular for $\mathfrak{M}'$ containing one object only.)

RANDOM OBTUSE TRIANGLES

N. T. GRIDGEMAN, National Research Council, Ottawa

Philipp Frank [1], referring to a discussion on the empirical probability that a "random triangle" is obtuse, points out that the answer depends on the specification of "the 'collective' in which the triangle is embedded," and he discusses two ways of characterizing a triangle: first, "giving one side a and the two adjacent angles α and β," and second, "giving the length of the three sides a, b, and c." Let us examine the corresponding problem in geometric probability. In this context the two angles in Frank's first case wholly characterize the triangle, and the side length a becomes an irrelevant scale factor. We have to define sample spaces of angles *or* sides. Dilatations, rotations, or reflections of the triangles will not be distinguished.



The two cases, *angle* and *side* sampling, can be dealt with jointly. For each a fixed total is needed. With angles this is given: 180°, which we here take as unity. With sides we shall analogously set the perimeter equal to unity. Thus we can designate three variables,

$$x \geqq y \geqq 1 - x - y \begin{cases} x < 1 \\ 1 - x - y > 0 \end{cases}$$

for both cases. It follows that, within certain limits, points (x, y) on the unit plane will specify triangles. Now two common limits for the sample spaces can immediately be written down in terms of Cartesian coordinates $(x, y) = (1/3, 1/3)$ and $(1/2, 1/2)$. A third limit will be $(1, 0)$ for the angles, and $(1/2, 1/4)$ for the sides. The spaces are shown in the Figure. The whole area, $A+B+C$, is the *angle* sampling space; that is to say, the co-ordinates x and y of any point within that area, multiplied by 180°, will give two angles characterizing a triangle. If the point lies in C, the resultant triangle will be obtuse; otherwise it will be acute—except that if the point lies on the line $x = 1/2$ the triangle will be right.

The *side* sampling space is the area $A+B$, for the coordinates x and y of any point in this space, together with the quantity $1-x-y$, can be formed into a triangle. If the point lies in B the triangle will be obtuse; otherwise it will be acute—except that if the point lies on the curve $x=1/2(1-y)-y$, the triangle will be right.

By inspection (for the angles) and by integration (for the sides) the probability that a random triangle is obtuse will be found to be

$$C/(A + B + C) = 3/4 \text{ for } \textit{angle} \text{ sampling}$$

and $B/(A+B)=9-12 \ln 2=0.68223 \cdots$ for *side* sampling.

If the sampling is of the set of points in the space the probability of a right triangle will of course be zero in both cases.

Populations of variables other than angles or sides can also be used to obtain random triangles. For instance the distribution of the vertices about the circumscribed circle may be considered. If three points on the circle are randomly located, it is easy to show that the probability that the resultant triangle is obtuse is 3/4, as for angle sampling. Another approach is to throw a circle, with one (circumferential) point identified, onto a plane so that it intersects a given straight line; then the triangle formed by the chord and the point has a probability of $1-1/\pi=0.68169 \cdots$ of being obtuse.

Reference

1. Philipp Frank, Philosophy of Science, Prentice-Hall, Englewood Cliffs, N. J., 1957.

SOME EXTENSIONS OF CANTOR'S POWER-SET THEOREM

JOHN HOOLEY, U.M.I.S.T., Manchester, England

To the more sophisticated members of the set-theoretical public, the results in this article would probably be regarded as simple special cases of more general theorems in cardinal arithmetic. These theorems however, need, for their proof, appeal to rather deeper principles (e.g. Zorn's Lemma) than we have in mind here where we use only arguments of an elementary, combinatorial character and one or two simple theorems on cardinality to squeeze a little more information out of Cantor's 'diagonal' proof. Our treatment might be appropriate in a course of informal set theory taking the subject about as far as is done in Chapter 2 of [1]. An application is to be found at the end of the article.

Each successive extension subsumes the previous ones, but although a proof of Extension (i) can be obtained by specializing the proof of Extension $(i+1)$, the proof actually given for Extension (i) is not *that* one but one slightly simpler in its details.

Assumptions. In Extensions (2), (3), we appeal repeatedly to the fact that removing one or two (in fact any finite number of) elements from an infinite set does not reduce the cardinal. In Extension (2), we also use the theorem (see

[2], p. 45) that removing a denumerable set from an infinite set, provided what is left is still infinite, does not reduce the cardinal. We also assume the content of Chapter 2 of [1], but our notation is different.

CANTOR'S THEOREM. *The cardinal, $\overline{\overline{S}}$, of any set S is less than that of its power-set, $\mathcal{P}(S)$, i.e. the set of all subsets of S.*

Proof. $\overline{\overline{S}} \leqq \overline{\overline{\mathcal{P}(S)}}$ is obvious, since we have the 1-1 correspondence $s \leftrightarrow \{s\}$ for each of $s \in S$. Now, suppose there were a 1-1 function f from S *onto* $\mathcal{P}(S)$ and consider the set D defined by

$$D = \{s \mid s \in S \text{ and } s \notin f(s)\}.$$

(N.B. This set is nonempty since $f^{-1}(\emptyset) \in D$.) $D \in \mathcal{P}(S)$ and so $D = f(t)$ for some $t \in S$; but it immediately follows that $t \in f(t)$ iff $t \notin f(t)$ and we conclude that the supposed f cannot exist. Therefore $\overline{\overline{S}} \neq \overline{\overline{\mathcal{P}(S)}}$ and by the Schroeder-Bernstein Theorem (knowledge of the Schroeder-Bernstein Theorem is not necessary, as will be indicated later), we cannot have $\overline{\overline{\mathcal{P}(S)}} \leqq \overline{\overline{S}}$. Therefore $\overline{\overline{S}} < \overline{\overline{\mathcal{P}(S)}}$.

The principle of the extensions. The aim is to show that what seem at first to be quite large parts of $\mathcal{P}(S)$ can be removed, while what remain are still cardinally larger than S. Suppose we remove from $\mathcal{P}(S)$ some class $\mathcal{R}$ of subsets of S so that what is left still has cardinal at least as great as $\overline{\overline{S}}$, then suppose there exists $f: S \to \mathcal{P}(S) - \mathcal{R}$ which is 1-1 and onto. Again we can define the 'diagonal' set D as above and apply the diagonal argument to it: this argument shows that $D \notin \mathcal{P}(S) - \mathcal{R}$ and therefore that $D \in \mathcal{R}$. The rest of the proof depends on the choice of $\mathcal{R}$ and in each case we obtain a contradiction by finding more subsets of S not containing their respective f^{-1} images than there are elements of D. We shall then conclude that $\overline{\overline{S}} < \overline{\overline{\mathcal{P}(S) - \mathcal{R}}}$.

EXTENSION (1). As $\mathcal{R}$, take the class of all nonempty finite subsets of S: except for the case $\overline{\overline{S}} = 1$, S must be infinite. The diagonal set D must be in $\mathcal{R}$, i.e. must be finite; suppose D has m members. S is an infinite subset of itself and so $S = f(s_0)$, say, and $s_0 \notin D$, since $s_0 \in f(s_0)$. $S - \{s_0\} = f(s_1)$, say, and $s_1 \neq s_0$ since $f(s_1) \neq f(s_0)$: also $s_1 \notin D$. $S - \{s_0, s_1\} = f(s_2)$, say, with $s_2 \neq s_0$, $s_2 \neq s_1$ and $s_2 \notin D$. By repeating this process, we define inductively a denumerable subset of S:

$$S^* = \{s_0, s_1, \cdots, s_n, \cdots\}.$$

Now $S^* = f(s)$ for some $s \in S$, but S^* is different from all the $f(s_n)$ since the latter include D whereas S^* is disjoint with D. Therefore $s \notin f(s)$ and so $s \in D$, say $s = d_1$.

Similarly $\{s_1, s_2, \cdots\}$ differs from all the $f(s_n)$ and from $f(d_1)$ and so must be $f(d_2)$, say, where $d_2 \in D$, $d_2 \neq d_1$. Continuing in this way, we have eventually, $\{s_{m-1}, s_m, \cdots\} = f(d_m)$, $d_m \in D$ and $d_1, d_2, \cdots, d_m$ are all different. However $\{s_m, s_{m+1}, \cdots\}$ is infinite and differs from all the $f(s_n)$ and from $f(d_1)$, $f(d_2)$, $\cdots, f(d_m)$, implying that D has at least $m+1$ elements, contrary to hypothesis. This contradiction shows that $\overline{\overline{S}} \neq \overline{\overline{\mathcal{P}(S) - \mathcal{R}}}$ and therefore, as before, that $\overline{\overline{S}} < \overline{\overline{\mathcal{P}(S) - \mathcal{R}}}$.

EXTENSION (2). Let S be nondenumerably infinite and take as $\mathcal{R}$ the class of all nonempty, finite or denumerably infinite subsets of S. Let

$$S^* = \{s \in S \mid f(s) = S - \{t\} \text{ for some } t \in S - D\};$$

then D is denumerable or finite but S^* is nondenumerable since there are as many sets $S - \{t\}$ $(t \in S - D)$ as there are elements of S (see Assumptions) and each is in the range of f. For the same reason as in the previous extension, S^* differs from $f(s)$ for all $s \in S^*$. Furthermore, every one of the nondenumerably many, nondenumerable subsets of S^* (e.g. all subsets of S^* of the form $S^* - \{s\}$ are nondenumerable) is the f-image of an element of S not in itself, implying that the set of $s \in S$ for which $s \notin f(s)$ is nondenumerably infinite; but this is the set D which we previously showed to be denumerable or finite. Therefore, again $\overline{\overline{S}} < \overline{\overline{\mathcal{P}(S) - \mathcal{R}}}$.

EXTENSION (3). Let $\mathcal{R}$ be the class of all nonempty subsets of S whose cardinal is less than $\overline{\overline{S}}$; (except for the cases $\overline{\overline{S}} = 0$, 1 or 2, S must here be infinite) then $\overline{\overline{D}} < \overline{\overline{S}}$. Let s_0 be any fixed element of $S - D$ and define

$$U = \{s \in S \mid f(s) = S - \{t\} \quad \text{for some } t \in S - D\},$$

$$V = \{s \in S \mid f(s) = S - \{t\} \quad \text{for some } t \in D \text{ such that } s \neq t\}$$

and $W = \{s \in S \mid f(s) = S - \{s_0, t\}$ for some $t \in D$ such that $S - \{t\} = f(t)\}$. If D is finite, we can take $S^* = U$ and the argument is just as for Extension (2), except that sets which are there described as 'nondenumerable,' should here be described as 'of cardinal greater than $\overline{\overline{D}}$.' Otherwise, let $S^* = U \cup V \cup W$.

The following properties are easily checked:

$$\overline{\overline{S^*}} = \overline{\overline{S}};$$

$$\text{for each } s \in S^*, f(s) \cap D \neq \varnothing;$$

$$S^* \cap D = \varnothing.$$

The last two properties show that, for any $s \in S^*$, $f(s)$ is not S^* nor a subset of S^* and so all subsets of S^* in the range of f are f-images of elements of D. However, there are as many sets of the form $S^* - \{s\}$ $(s \in S^*)$ as there are elements of S (since $\overline{\overline{S^*}} = \overline{\overline{S}}$) and each is in the range of f, implying that $\overline{\overline{D}} = \overline{\overline{S}}$, contrary to our previous conclusion that $\overline{\overline{D}} < \overline{\overline{S}}$. Therefore, again $\overline{\overline{S}} < \overline{\overline{\mathcal{P}(S) - \mathcal{R}}}$.

If, in any of the above arguments, any subset of S is used instead of S itself *as the domain of f*, the conclusion that $\overline{\overline{S}} \ngeq \overline{\overline{\mathcal{P}(S) - \mathcal{R}}}$ is established directly instead of indirectly *via* the Schroeder-Bernstein Theorem (see [2], p. 71).

Application. A common way of proving that the set of all real numbers on a unit interval is nondenumerable, uses Cantor's Theorem and the cardinality relations between the sets of the following sequence: **Z** (set of all natural numbers); $\mathcal{P}(\mathbf{Z})$; the set **B** of all infinite sequences of 0's and 1's; the reals on a unit interval (**I**). In the first step, we go from a denumerable to a nondenumerable set (**Z** to $\mathcal{P}(\mathbf{Z})$); then we pass, by an obvious one-one correspondence (see [1],

pp. 108–9), from $\mathcal{P}(\mathbf{Z})$ to $\mathbf{B}$; and finally we interpret the elements of $\mathbf{B}$ as binary representations of elements of $\mathbf{I}$. The irritating thing is that the last step is not one-one, some elements of $\mathbf{I}$ having two representations, one ending in $1\dot{0}$ and the other in $0\dot{1}$. However, the former representation, as an element of $\mathbf{B}$, corresponds to a *finite* subset of $\mathbf{Z}$ and so by consistently choosing the latter representation and rejecting all the sequences ending in $1\dot{0}$, we simply have to remove all the finite elements of $\mathcal{P}(\mathbf{Z})$ and the first step is provided by our Extension (1) instead of by the plain Cantor Theorem.

References

1. J. G. Kemeny, H. Mirkil, J. L. Snell and G. L. Thompson, Finite Mathematical Structures, Prentice-Hall, Englewood Cliffs, N. J., 1959.

2. A. A. Fraenkel, Abstract Set Theory, North Holland Publishing, Amsterdam, 1961.

NOTE ON THE ALTERNATING GROUP

E. L. Spitznagel, Jr., Northwestern University

The distinction between even and odd permutations is established in nearly every introductory modern algebra text. However, the most common proof leaves something to be desired, as Herstein remarks in *Topics in Algebra*, because it introduces a polynomial which seems completely extraneous to the subject. On the other hand, alternative proofs use permutation calculations which are fairly involved.

Following is a proof with more group-theoretic flavor than these other methods. Perhaps it will be found aesthetically more pleasing.

Define $\tau \in S_n$ to be even if τ can be represented as the product of an even number of transpositions. Let A_n be the set of all even $\tau \in S_n$. Then A_n is a subgroup of S_n, of index at most 2, and all we need do is show that some element of S_n is not contained in A_n.

Proposition. *$(1, 2) \notin A_n$, so $[S_n : A_n] = 2$.*

Proof. Suppose $(1, 2)$ can be written as a product of an even number of transpositions. Then the identity permutation ϵ can be written as a product of an odd number of transpositions. Let $\epsilon = (a, b) \cdots$ be a product of the smallest odd number k of transpositions which gives ϵ, and among all such products beginning with $(a, \ )$ let it be one containing the smallest number of a's.

Since $a\epsilon = a$, there must be another transposition in this product which moves a. Let (a, c) be the one closest to the left. Since $(d, e)(a, c) = (a, c)(d, e)$ and $(c, d)(a, c) = (a, d)(c, d)$, we have $\epsilon = (a, b)(a, f) \cdots$ is a product of length k containing the same minimum number of a's.

But then if $f \neq b$, $\epsilon = (a, f)(b, f) \cdots$ is a product of length k with fewer a's, while if $f = b$, then ϵ is a product of $k - 2$ transpositions, contradicting minimality of k.

COROLLARY. *If τ is even, then every time it is written as a product of transpositions, that product contains an even number of transpositions.*

Proof. If not, then we would have $(1, 2)\tau \in A_n$, and $\tau \in A_n$, so $(1, 2) \in A_n$.

A DYNAMICS APPROACH TO TRIGONOMETRIC DIFFERENTIATION

M. R. SPIEGEL, Rensselaer Polytechnic Institute, Hartford Graduate Center

In the usual textbooks on calculus, derivatives of trigonometric functions are obtained by first proving that

$$\frac{d}{d\theta}(\sin \theta) = \cos \theta \tag{1}$$

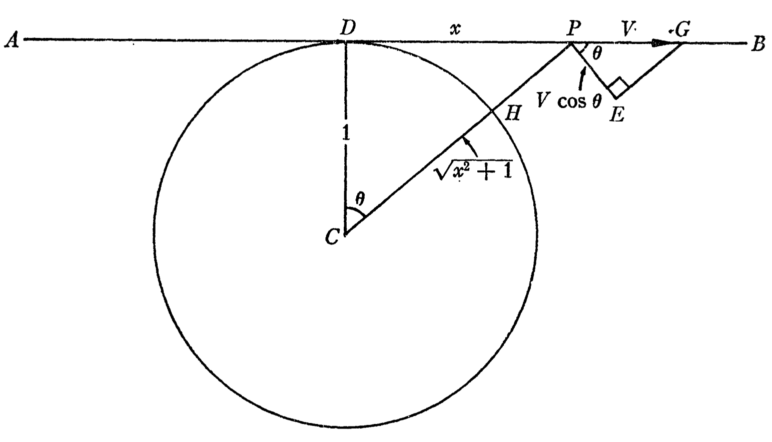
using

$$\lim_{\Delta\theta \to 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1. \tag{2}$$

Since a rigorous proof of (2) is difficult for the beginner, one often resorts to geometric demonstrations.

The purpose of this article is to show how derivatives of the trigonometric functions can be obtained by using elementary dynamics. To accomplish this let us refer to the figure below. In this figure we imagine that a particle P is moving with constant speed V along line AB and construct a unit circle with fixed center C and tangent to AB at D. Letting x denote the instantaneous distance of P from D and $\measuredangle PCD = \theta$, we see that the angular speed with which line segment CP rotates about C is given by

$$\frac{d\theta}{dt} = \frac{d}{dt}(\tan^{-1} x). \tag{3}$$



Now this angular speed can also be obtained in another way by first noting that the component PE of the speed in a direction parallel to the tangent to the circle at H is given by $V\cos\theta$. Then dividing by the distance $CP=\sqrt{(x^2+1)}$ we see that the angular speed is

$$\frac{V\cos\theta}{\sqrt{(x^2+1)}}=\frac{V}{x^2+1}=\frac{1}{x^2+1}\frac{dx}{dt}\cdot \tag{4}$$

using the fact that $V=dx/dt$. From the equality of (3) and (4) we find

$$\frac{d}{dt}(\tan^{-1}x)=\frac{1}{x^2+1}\frac{dx}{dt} \tag{5}$$

or equivalently,

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{x^2+1}\cdot \tag{6}$$

All other derivatives of trigonometric functions can now be found. For example to find the derivative of $\tan\theta$ we use the fact that $\tan\theta=x$ so that

$$\frac{d}{d\theta}(\tan\theta)=\frac{dx}{d\theta}=\frac{1}{d\theta/dx}=x^2+1=\sec^2\theta. \tag{7}$$

Similarly,

$$\frac{d}{d\theta}(\sin\theta)=\frac{d}{d\theta}\left(\frac{\tan\theta}{\sqrt{(\tan^2\theta+1)}}\right)=\cos\theta. \tag{8}$$

The physical approach used here proves to be especially interesting to the student of science or engineering since it provides an excellent example of the relationship of mathematical and physical concepts.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

Material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D.C. 20005.

UNDERGRADUATE RESEARCH: SOME CONCLUSIONS

KENNETH O. MAY, University of Toronto

Original contributions to mathematics by juveniles have a very long tradition. Indeed precocity and early involvement in creative work appear to be characteristic of practically all of the most productive mathematicians of the past

and there seems no reason to think that mathematics has changed enough to make this any less likely in the future.

The term "undergraduate research" has a somewhat different meaning. It refers to an educational activity designed to increase the output of creative mathematicians and scientists by giving young people the experience of original work in addition to the standard courses in which they "learn" what has been discovered by others. Undergraduate research activities may, and often do, lead to new publishable mathematical results, but this is not the goal. The activity may be considered successful if it enhances and accelerates the student's development as an original thinker. The important manifestations are an effort involving independent work and results that are original for the student, regardless of their newness to other mathematicians or significance for current mathematical research.

Undergraduate research as an educational activity, has a long history in the United States [**9**, **10**]. Part of the tradition stems from the senior thesis required at many institutions from the nineteenth century and earlier. At Reed College, under the loving guidance of Griffin, the senior thesis became an occasion for much original work [**2**, **3**]. Undoubtedly similar activities took place at other colleges, and this kind of activity shades off into similar pedagogical techniques not usually called undergraduate research. Examples are the "Moore method," contests, and the "discovery approach" at all levels. The inspiring teacher who stimulates his students to work on their own and solve challenging problems is promoting undergraduate research.

The recent development of undergraduate research programs is part of the general revival of interest in improving the teaching of mathematics at all levels. The new element has been a conscious organized effort to stimulate activity with or without financial support from outside. Having been involved in experimentation in this area for a decade at Carleton College, I here offer a few conclusions based on my own experiences and those of others whose work came to my attention as editor of Delta-Epsilon and a participant in many discussions [**1**, **5**, **6**].

I. Staff. It is essential to have a staff member A who is genuinely interested in undergraduates and enthusiastic about their work. Personal encouragement, interest, and enthusiasm are undoubtedly the most important means for motivating students. Mathematical originality and research activity are helpful but not essential, provided the staff member is intellectually alive and interested in mathematical ideas. It is also essential to have a staff member B who believes in the educational value of undergraduate research activities and who has sufficient influence to gain the necessary support from his colleagues and the administration. Supervision of undergraduate research takes time and energy. A small group of active students, possibly as few as three, is the equivalent of a course and should be so recognized. In the ideal department every member is both an A and a B, but a minimum is one of each with $A = B$.

II. Organization. Work has been carried out successfully with every imaginable organizational structure [4, 6]. Among these might be mentioned the following: A formal course for credit run like a seminar with students reporting on their individual or group research projects. Special honors courses and "problem" courses. A student-run colloquium at which problems are discussed and work reported, but with no credit and only very loose faculty involvement. An apparently anarchic "atmosphere" in which it "just happens" that students are inspired to do supplementary original work for their professors, some of which leads to publication. It would be impossible to list the variations, but it is essential that some conscious effort be extended by the faculty and students. "Anarchic" situations are usually a result of a carefully nurtured tradition, and definite arrangements for academic credit are helpful.

III. Publications. It frequently happens that undergraduate papers are published in the professional journals. This is more common than is generally realized, because the editors prefer not to call attention to the undergraduate origin of a paper. (I have been told that this preference arises from the fear of being deluged with undergraduate garbage.) However, such publication is not an important goal of an undergraduate research program. Student papers should not be judged in terms of their "publishability." On the other hand the very highest standards of clarity, exposition, and correctness should be demanded. Indeed one goal should be the eventual improvement of the presently very low standards of exposition in the mathematical community. A well written paper that reflects independent and original work on the part of the student, even if his results are not new, deserves praise and recognition. Moreover, possible publication provides incentive, and actual publication stimulates further effort. For these reasons it is desirable to have arrangements for local publication. This can easily be accomplished if staff and students keep in mind that they are not trying to emulate existing mathematical journals but instead are providing an opportunity for immediate publication and modest circulation of good student work. Publication dates need not be regular. Duplication of amateur typescript by ditto, mimeo, or multilith is adequate. The level can vary with the author's background. Reviews and news can be included. The work of such publication can be done largely by the students themselves and duplicating absorbed in the day-to-day operations of the department [1, 9].

IV. An ideal program? None exists, but a very good one would involve the following:

1. A student run colloquium.
2. A student run publication.
3. A student mathematics club that supports these activities and engages in others appropriate to the local situation.
4. A faculty member who keeps a friendly eye on things and helps as needed.
5. Some link of these activities with the curriculum.

Ideally, research-like activity should be part of every course (the quality and quantity varying with the context) and should be officially recognized by graduation honors, special courses, and the like. In short, undergraduate research should become a normal part of the curricular and extra-curricular educational process [8]. Teachers should be judged in part by the degree to which they stimulate independent work. But for many reasons this is not the case, and special programs and projects should be designed both for their immediate productivity and to assist in creating the environment in which they will no longer be necessary.

V. A(n) (inter)national student publication? College students sometimes read professional magazines, especially the *American Mathematical Monthly* and occasionally the more specialized research journals. But these periodicals are not really addressed to students. (Some of them do not seem to be addressed to anyone!) It is not just a matter of level, but more one of tone, style, and point of view. A magazine addressed to college mathematics students might make an important contribution to their education. It could, from a student point of view, review books of particular interest to undergraduates, present expositions that would inform and stimulate, include news of student mathematical organizations (Pi Mu Epsilon, Kappa Mu Epsilon, clubs, etc.) and activities, provide an outlet for the very best undergraduate writing and a forum for student discussion, contain a problem department designed for students,—and much more that cannot be anticipated. For example, it might print news stories (including pictures) of the Putnam winners, various solutions of Putnam questions, etc. It could encourage inter-college student meetings and competitions. Such a magazine might have its greatest individual impact where the student is "neglected" (at very small places with inadequate staff and at very large places with preoccupied staff). Its greatest value would be to reach and stimulate the undergraduate by putting him in communication with other students and mature mathematicians. The editorial staff should include students, though responsibility for continuity and management would best be assumed by the MAA or other continuing professional organization [7].

The work on which this paper is based was supported by grants from the National Science Foundation.

References

1. Delta-Epsilon, a journal for undergraduate work, published at Carleton College from January 1960, with NSF support 1961–1966. Volume 6 appeared in 1965–1966. Besides undergraduate papers, it contains reports on programs at various colleges. Backfiles are obtainable from the present editor, Prof. Roger Kirchner, Department of Mathematics, Carleton College, Northfield, Minnesota.

2. F. L. Griffin, Undergraduate mathematical research, this MONTHLY, 49 (1942) 379–385.

3. ———, Further experience with undergraduate mathematical research, this MONTHLY, 58 (1951) 322–325.

4. C. B. Lindquist, Honors programs for superior undergraduate mathematics students. OE-56015. Mathematics Programs Series, Divis. of Higher Ed., USDHEW, Office of Education, Washington, 1964.

5. Kenneth O. May, Undergraduate research in mathematics, this MONTHLY, 55 (1948) 241–246.

6. Kenneth O. May and Seymour Schuster, editors. Undergraduate Research in Mathematics, Report of a conference held at Carleton College, June 19–23, 1961, with support of the National Science Foundation. Northfield, 1962. Out of print, but copies were sent to all U. S. departments and libraries at the college level. Includes papers, examples, devices, case histories, topics, and bibliography.

7. Particle, a quarterly "by and for science students" published in Berkeley, California, has managed to keep going since 1960, but with considerable difficulty because of the lack of continued financial backing and scientific guidance.

8. H. E. Roscoe, Original research as a means of education, Nature, October 23 and 30, (1873), 538–9 and 559–561. Roscoe, a chemist, makes a plea for research as part of education, sometimes in a quite quaint way. Summing up, he says that his aim has been to show that "if freedom of inquiry, independence of thought, disinterested and steadfast labor, habits of exhaustive and truthful observation, and of clear perception, are things to be desired as tending to the higher intellectual development of mankind, then original research ought to be included as one of the most valuable means of education." Poor Roscoe! Such appeals have secured equipment and leisure for research by professors, but education remains largely the memorization and regurgitation of the discoveries of others.

9. L. G. Simons, Undergraduate publications in mathematics, Scripta Mathematica, 8 (1941) 165–175. Describes many local student mathematics journals, of which the oldest appears to be the proceedings of the University of Toronto Mathematical and Physical Society (1882!). Simons cites an example of a phenomenon that appears to be quite common: the editor of a student publication turns up a few years later as a faculty advisor. It appears that one result of undergraduate research activities is to help produce teachers interested in students!

10. E. R. Sleight, Undergraduate research in Michigan, this MONTHLY, 48 (1941) 696–697.

THE UNDERGRADUATE MATHEMATICS CLUB—A PRELIMINARY REPORT

D. S. MARTIN, State University of New York at Buffalo

During the Spring of 1966, the Upper New York State Section of the MAA surveyed its member schools to evaluate the effectiveness of and to compile suggestions for the organization or improvement of mathematics clubs. A report was given at the Sectional meeting and subsequently printed and sent to the member schools.

Of the 54 (out of 88) replies, 12 were from 2-year schools. They reported no mathematics clubs and indicated that clubs would not be feasible for them because of lack of student time. About $\frac{2}{3}$ of the reporting 4-year schools had mathematics clubs, of which about 60% were active. Five others expressed interest in starting clubs.

The essential ingredient for a mathematics club is student involvement. Unfortunately, this is also the ingredient over which the faculty has least control. Periodically, on almost every campus there is a small group of students who are interested in a mathematics club but who do not know what to do. Some procedures and ideas, found successful on various campuses, are contained in the report.

The "preliminary" in the title of this report indicates a lack of satisfaction with the present situation. Copies of the original report, "The Undergraduate Mathematics Club –Report and Recommendations," may be obtained, while the supply lasts, from Professor H. M. Gehman, Executive Director, MAA, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. The author would appreciate hearing from anyone with program suggestions or any comment as to what can be done with mathematics clubs for the future.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before May 31, 1968. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2045. *Proposed by Zalman Rubinstein, Clark University*

Given a set of n distinct complex numbers z_i, $i=1, 2, \cdots, n$ which satisfy $\min_{i\neq j} |z_i - z_j| \geqq \max_i |z_i|$. Find the upper bound for n and classify all the maximal sets.

What can be said in the case of arbitrary Euclidean spaces?

E 2046. *Proposed by L. P. Zukowski, University of Michigan*

Each element of the set $S=\{112, 518, 322\}$ is divisible by 14. The 3×3 determinant formed from S by associating each number with a distinct row and each digit with a definite column, viz.

$$\begin{vmatrix} 1 & 1 & 2 \\ 5 & 1 & 8 \\ 3 & 2 & 2 \end{vmatrix} = 14$$

is also divisible by 14. Show that this generalizes to the result that any common factor δ, of n arbitrary n-digit integers is also a factor of the analogously formed $n\times n$ determinant.

E 2047. *Proposed by Hans Liebeck, University of Keele, England*

x and y are elements of finite order m and n respectively of a group, and $xy=yx$. What can you say about the order of xy?

E 2048. *Proposed by J. Barlaz, Rutgers– The State University*

(A) For what functions f can both u and $f(u)$ be harmonic? ($u=u(x, y)$, f a function of a single variable.)

(B) Prove: If u and v are harmonic and $|u+iv|\equiv$constant then both u and v are constant. (This, of course, is well known when $u+iv$ is an analytic function of $x+iy$.)

E 2049. *Proposed by T. S. Frank, Le Moyne College, Syracuse, N. Y.*

Let $\langle S, \cdot\rangle$ be a finite semigroup with identity in which the cross-cancellation law ($a\cdot x=x\cdot b$ implies $a=b$) holds. Then $\langle S, \cdot\rangle$ is an abelian group.

E 2050. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let D be the maximum value of all determinants of order n whose entries are real numbers in the range $a\leqq x\leqq b$. Show that the value D is achieved by a determinant whose entries are exclusively a and b.

E 2051. *Proposed by P. T. Bateman, University of Illinois*

If n is a positive integer, let $r_s(n)$ denote the number of solutions of the equation

$$x_1^2 + x_2^2 + \cdots + x_s^2 = n$$

in integers $x_1, x_2, \cdots, x_s$ and let $f_s(n)=(2s)^{-1}r_s(n)$. If $s=1, 2, 4, 8$, it is known that f_s is multiplicative, that is, $f_s(mn)=f_s(m)f_s(n)$ for any pair of coprime positive integers m, n. Prove that f_s is not multiplicative for any other value of s. (Cf. J. M. Gandhi, Bull. Amer. Math. Soc., 72(1966) 220–221.)

E 2052. *Proposed by G. C. Berresford, Lawrence University*

Let A be any square matrix, and let B be a matrix formed as follows. B is identical to A except that the kth row and kth column of A are interchanged to become the kth column and kth row of B, k being arbitrary. Prove that multiplying the elements of row k by the corresponding cofactors of column k in matrix A and adding the products, gives the same result as multiplying the elements of column k by the corresponding cofactors of row k in matrix B, and adding the products.

E 2053. *Proposed by W. E. Buker, Pittsburgh, Pa. Public Schools*

If the five Platonic solids are inscribed in the unit sphere, the one having the greatest volume is the dodecahedron. Show this. Is this also true for surface area?

E 2054. *Proposed by L. Carlitz and R. A. Scoville, Duke University*

Find the number of sequences of positive integers $(a_1, a_2, \cdots, a_n)$ such that

$$1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n; \quad a_i \leqq i \qquad (i = 1, 2, \cdots, n).$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Number of Divisors and Greatest Integer Function

E 1906 [1966, 774]. *Proposed by Erwin Just, Bronx Community College*

Prove that

$$\sum_{k=1}^{2n} \tau(k) - \sum_{k=1}^{n} [2n/k] = n,$$

where $\tau(n)$ is the number of divisors of n and $[x]$ is the greatest integer not exceeding x.

I. *Solution by P. R. Chernoff, Harvard University.* Unless k is a divisor of $2n+1$ or of $2n+2$, we have

$$\left[\frac{2n+2}{k}\right] - \left[\frac{2n}{k}\right] = 0.$$

Therefore

$$\sum_{k=1}^{n+1}\left[\frac{2n+2}{k}\right] - \sum_{k=1}^{n}\left[\frac{2n}{k}\right] = \tau(2n+1) + \tau(2n+2) - 1.$$

Hence, if $f(n)$ is the required sum, we have $f(n+1) - f(n) = 1$, and the result follows inductively from $f(1) = 1$.

II. *Solution by W. H. Wong, Clarkson College of Technology.* The result is a special case of the more general theorem:

If $F(n) = \sum_{d|n} f(d)$, *then* $\sum_{k=1}^{n} F(k) = \sum_{k=1}^{n} [n/k] f(k)$.

We need only take $F = \tau$ and $f = 1$, then

$$(*) \qquad \begin{aligned} \sum_{k=1}^{2n} \tau(k) &= \sum_{k=1}^{2n}\left[\frac{2n}{k}\right] \\ &= \sum_{k=1}^{n}\left[\frac{2n}{k}\right] + \sum_{k=1}^{n}\left[\frac{2n}{n+k}\right] = \sum_{k=1}^{n}\left[\frac{2n}{k}\right] + n. \end{aligned}$$

For the cited theorem see William J. Leveque, *Topics in Number Theory*, vol. I, p. 117.

Also solved by J. C. Abad, I. K. Abroub, A. N. Aheart, R. G. Albert, P. N. Bajaj, Gabriel Bastien, C. T. Beers, Alfred Brousseau, L. Carlitz, Anton Chernoff, John Christopher, Raymond Chu, C. A. Church, Jr., Jim Clark, Mickey Dargitz, J. F. Dillon, G. C. Dodds, H. M. Edgar, R. B. Eggleton (Australia), P. G. Engstrom, P. K. Garlick, R. E. Giudici, Jerry Goodman, Simon Green, M. G. Greening (Australia). K. E. Hirst (England), J. E. Homer, Jr., D. G. Huffman, Donald Jeffords, Marg. Johns, Peter Kornya, C. P. Lawes, J. C. Lazzard, A. Makowski (Poland), D. C. B. Marsh, L. J. Marx, C. C. McBride, W. A. Miller, William Moser, Wanda Jane Mourant, J. B. Muskat, C. B. A. Peck, Alfred Pietrowski, Stanley Rabinowitz, Simeon Reich (Israel), Lois J. Reid, Judith Richman, Shmuel Rosset (Israel), Steven Russ, J. S. Shipman, R. Sivaramakrishnan (India), Al Somayajulu, Judith C. Soriano, Sidney Spital, D. R. Stark, D. L. Stenson & C. Q. Artino, H. H. Thoyre, A. M. Vaidya (India), C. R. Wall, W. W. Whitman, C. T. Whyburn, and the proposer.

The relation (*) is found in many texts on Theory of Numbers: Uspensky & Heaslet (p. 98), Nagell (p. 26), Stewart (p. 53), Sierpinski (pp. 159–160). See also Polya & Szegö, *Aufgaben* II, p. 131, and U.S.S.R. *Olympiad Problem Book*, p. 25, no. 103.

Convergence for Solutions of $x=f(x)$

E 1907 [1966, 775]. *Proposed by Sidney Spital, California State Polytechnic College*

Steffenson's method for solving the equation $x=f(x)$ consists of forming recursively a sequence $\{x_n\}$, using

$$x_{n+1} = x_n - \frac{[f(x_n) - x_n]^2}{f(f(x_n)) - 2f(x_n) + x_n},$$

which (hopefully) converges to a solution s. Show that if f is twice continuously differentiable and if $f'(s)\neq 1$, then the errors $e_n=x_n-s$ satisfy the following condition of quadratic convergence

$$e_{n+1} = \frac{f''(s)f'(s)}{2(f'(s)-1)}\, e_n^2 + O(e_n^3).$$

Solution by H. H. Wang, IBM, Poughkeepsie, N.Y. If we use the Taylor's series and if we adopt the notation $f'=f'(s), f''=f''(s)$, then we can write

$$\begin{aligned}
f(x_n) &= f(s+e_n) = f(s) + e_nf' + \tfrac{1}{2}e_n^2f'' + O(e_n^3) \equiv s + g(e_n),\\
f(f(x_n)) &= f(s+g(e_n)) = f(s) + g(e_n)f' + \tfrac{1}{2}(g(e_n))^2f'' + O(e_n^3)\\
&= s + e_n(f')^2 + \tfrac{1}{2}e_n^2f'f''(1+f') + O(e_n^3),\\
(f(x_n)-x_n)^2 &= (g(e_n)-e_n)^2 = e_n^2(f'-1)^2 + e_n^3f''(f'-1) + O(e_n^4),\\
f(f(x_n)) - 2f(x_n) + x_n &= e_n(f'-1)^2 + \tfrac{1}{2}e_n^2f''(f'+2)(f'-1) + O(e_n^3).
\end{aligned}$$

From $e_{n+1}-e_n=x_{n+1}-x_n$ there follows

$$e_{n+1} = e_n - \frac{(f(x_n) - x_n)^2}{f(f(x_n)) - 2f(x_n) + x_n} = \frac{f''f'e_n^2 + O(e_n^3)}{2(f' - 1)(1 + O(e_n))}$$
$$= \frac{f''f'}{2(f' - 1)} e_n^2 + O(e_n^3).$$

Also solved by Marcia Ascher, L. P. Bush (Nigeria), J. S. Shipman, J. F. Traub, P. H. Young, and the proposer.

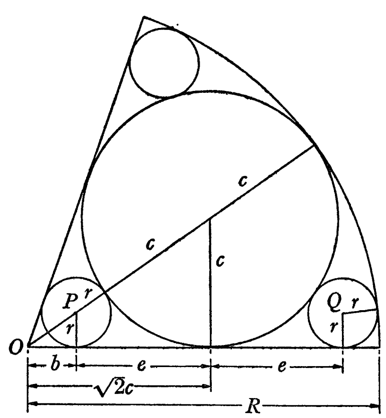
The solution may be found on p. 267 of J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, 1964. Although the result is stated in terms of Aitken's δ^2 process, the identification with Steffensen is made on the bottom of p. 268. Shipman finds the solution in Isaacson and Keller, *Analysis of Numerical Methods*, Wiley, New York, 1966, pp. 102–108. Further results in this direction are given in L. P. Bush's unpublished Master's Thesis, *Iterative Methods for Solving Equations* (Ohio University, June 9, 1963).

Tangent Spheres

E 1908 [1966, 775]. *Proposed by Edgar Karst, University of Oklahoma*

Four equal spheres are inscribed in a hemisphere of horizontal base such that each sphere touches two others and is tangent to both the hemisphere and its base. A radial section of the figure is now made by a vertical plane passing through the center of the hemisphere and the center of one of the inscribed spheres, and in this section a radius is drawn tangent to the circular section of the inscribed sphere. We now have a circular sector with its inscribed circle. Prove that the little circles inscribed in the corners of the sector and externally tangent to the inscribed circle are equal to one another.

Solution by Michael Goldberg, Washington, D. C. If four equal spheres of radius c, tangent to a horizontal plane, are arranged in the form of a square, each sphere touching its two neighbors, then the horizontal projection of a center of a sphere is at the distance $c\sqrt{2}$ from the center of the arrangement. The radius R of the circumscribed hemisphere is given by $R=(\sqrt{3}+1)c$. Let the center of this hemisphere be O, the center of the small inscribed sphere near O be P, and



the center of the small inscribed sphere in the far corner be Q. Then the radius r of the inscribed sphere at P satisfies the equation

$$\frac{r}{c} = \frac{R - 2c - r}{R - c}.$$

From this we obtain $r = (2-\sqrt{3})c$.

Let the horizontal projection of OP be b. Then, $b = r\sqrt{2}$. Let $e \equiv c\sqrt{2} - b = (\sqrt{6}-\sqrt{2})c$. If Q has the coordinates $(c\sqrt{2}+e,\ r)$ then $(c\sqrt{2}+e)^2 + r^2 = (13-4\sqrt{3})c^2$; and if Q is the center of the inscribed circle, then

$$\overline{OQ}^2 = (R - r)^2 = (13 - 4\sqrt{3})c^2.$$

This agreement verifies the fact that the small inscribed circles are of the same size.

Also solved by Jorge Dou (Spain), Ragnar Dybvik (Norway), Brother Thomas Flynn, Roman Kaluzniacki, Norman Miller, Barbara W. Nason, C. B. A. Peck, and J. M. Quoniam (France).

Center of a Group

E 1909 [1966, 775]. *Proposed by C. C. Lindner, Coker College, Hartsville, N.C.*

Prove that the center of a group is properly contained in every maximal subgroup having composite index.

Solution by Azriel Rosenfeld, University of Maryland. Let C be the center of G; then any subgroup H is normal in CH, the subgroup generated by C and H. If H is maximal, CH must be either H or G; and if it is the former, H contains C, as required. But if $CH = G$ we have H normal in G, and if H has composite index in G, the factor group G/H has composite order; but this is impossible since the maximality of H requires that G/H be a cyclic group of prime order.

Also solved by Marshall Atlas, P. R. Chernoff, John Christopher, D. Ž. Djoković, C. J. Duckenfield, P. K. Garlick, R. W. Gilmer, Jr., M. G. Greening (Australia), Colonel Johnson, Jr., Geoffrey Kandall, G. L. Loudner, Jr., D. C. B. Marsh, Ka Menehune, N. S. Natarajan (India), Stephen Rhodes & Gordon Woodward, Shmuel Rosset (Israel), W. R. Scott, Surjeet Singh & Qazi Zameeruddin (India), L. R. Vermani (India), and the proposer.

Set such that all Natural Numbers Occur Uniquely as Differences of Elements

E 1910 [1966, 775]. *Proposed by R. L. Graham, Bell Telephone Laboratories*

Show that there exists a sequence $a_1 < a_2 < \cdots$ of integers such that every positive integer occurs uniquely as a difference $a_i - a_j$ for some i and j. What can be said about the growth of such a sequence?

Remark by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa. The sequence $a_1 = 1$, $a_2 = 2$, $a_{2n+1} = 2a_{2n}$, and $a_{2n+2} = a_{2n+1} + r_n$, where r_n is the least natural number which cannot be represented in the form $a_j - a_i$ with $1 \leqq i < j \leqq 2n+1$, is proved to have the desired property in Sierpinski, *Elementary Theory of Numbers*, Warszawa, 1964, pp. 411–412. The first few terms are 1, 2, 4, 8, 16, 21, 42, 51, 102, 112, $\cdots$.

Also solved by R. B. Eggleton (Australia), Michael Goldberg, C. F. Marion, D. C. B. Marsh, P. G. Pantelidakis, Shmuel Rosset (Israel), W. W. Whitman, and the proposer.

Some of the solvers gave other valid sequences. However, the problem of finding which sequences lead to the smallest asymptotic growth is still unsolved.

Matrix with Prescribed First Row

E 1911 [1966, 775]. *Proposed by B. R. Toskey, Seattle University*

Suppose $a_{11}, \cdots, a_{1n}$ are given integers whose greatest common divisor is 1, and suppose $n \geqq 2$. Is it always possible to find a matrix (a_{ij}) with the given integers in the first row and all a_{ij} integers such that $\det(a_{ij}) = 1$?

Solution by D. C. B. Marsh, Colorado School of Mines. From the theory of numbers we know that if $\gcd(a_{11}, a_{12}) = g_2$ then $a_{11}x_1 + a_{12}x_2 = g_2$ has a solution in integers x_1, x_2. For $n = 2$ then, the integral matrix with first row a_{11}, a_{12} and determinant g_2 may be taken as

$$M_2 = \begin{bmatrix} a_{11} & a_{12} \\ -x_2 & x_1 \end{bmatrix}.$$

We proceed by induction, assuming the existence of an integral matrix M_{k-1} with first row $a_{11}, a_{12}, \cdots, a_{1,k-1}$ and determinant $g_{k-1} = \gcd(a_{11}, \cdots, a_{1,k-1})$. There exist integers y_k, x_k such that $g_{k-1}y_k + a_{1k}x_k = g_k = \gcd(a_{11}, \cdots, a_{1,k-1}, a_{1k})$. It is easy to construct

$$M_k = \begin{bmatrix} M_{k-1} & C \\ R & y_k \end{bmatrix},$$

where C is the $(k-1)\times 1$ matrix $[a_{1k}, 0, \cdots, 0]^T$ and R is the $1\times(k-1)$ matrix $[a'_{11}, a'_{12}, \cdots, a'_{1,k-1}]$, where $a'_{ij} = -a_{ij}x_k/g_{k-1}$. M_k is seen to have determinant $= g_k$ as required. Thus the induction is complete and the proposed question has an affirmative answer.

Also solved by P. R. Chernoff, J. F. Dillon, Philip Fung, K. E. Hirst (England), J. E. Homer, Jr., D. G. Huffman, R. S. Lee, L. J. Marx, N. S. Natarajan (India), Schmuel Rosset (Israel), Steven Russ, D. A. Smith, R. C. Thompson, and the proposer.

Smith and Thompson note that a solution is already given (except for a trivial adjustment to take care of ± 1) in connection with the equivalent problem E 1708 [1965, 671]. Besides further references given there, Thompson finds the problem in MacDuffee, *The Theory of Matrices*, p. 31.

Several other contributors made the unauthorized assumption that at least two of the a_{ij} were relatively prime.

Generalized Pythagorean Relation

E 1912 [1966, 775]. *Proposed by R. E. Williamson, Dartmouth College*

Prove the following generalization of the Pythagorean relation. Let P be a parallelotope spanned by k vectors in R^n, $k \leqq n$, that is, the set of combinations $a_1V_1 + \cdots + a_kV_k$, where $V_1, \cdots, V_k$ are the given vectors and the a_i satisfy $0 \leqq a_i \leqq 1$, $i = 1, \cdots, k$. Let $\{P_j\}$, $j = 1, 2, \cdots, \binom{n}{k}$, be the projections of P into the k-dimensional coordinate flats of R^n, $\binom{n}{k}$ in number. Then

$$V_k^2(P) = \sum_{j=1}^{\binom{n}{k}} V_k^2(P_j),$$

where V_k is k-dimensional volume in R^n.

Solution by Sidney Spital, California State Polytechnic College. It is known that the volume $V_k(P)$ of the parallelotope spanned by vectors $v_i (i=1, \cdots, k)$ is given by the square root of the Gramian, i.e.,

$$V_k^2(P) = \det\begin{bmatrix} (\mathrm{v}_1\cdot\mathrm{v}_1) & \cdots & (\mathrm{v}_1\cdot\mathrm{v}_k) \\ \vdots & & \vdots \\ (\mathrm{v}_k\cdot\mathrm{v}_1) & \cdots & (\mathrm{v}_k\cdot\mathrm{v}_k) \end{bmatrix}.$$

Now let the projection of v_i on the jth k-dimensional coordinate flat be called $v_{ij} (j=1, \cdots, \binom{n}{k})$. With respect to the orthonormal basis of R^n, v_{ij} forms a k component column vector and hence $M_j = (v_{1j}, \cdots, v_{kj})$ is a $k\times k$ matrix. G. E. Shilov (*Introduction to the Theory of Linear Spaces*, Prentice-Hall, 1961, p. 170) shows that

$$V_k^2(P) = \sum_{j=1}^{\binom{n}{k}} \det{}^2 (M_j). \tag{1}$$

However,

$$\det{}^2(M_j) = \det (M_j^T M_j) = \det\begin{bmatrix} (\mathrm{v}_{1j}\cdot\mathrm{v}_{1j}) & \cdots & (\mathrm{v}_{1j}\cdot\mathrm{v}_{kj}) \\ \vdots & & \vdots \\ (\mathrm{v}_{kj}\cdot\mathrm{v}_{1j}) & \cdots & (\mathrm{v}_{kj}\cdot\mathrm{v}_{kj}) \end{bmatrix}.$$

Since this is the Gramian for parallelotope P_j, $\det^2(M_j) = V_k^2(P_j)$. The Pythagorean generalization then follows from (1).

Also solved by P. R. Chernoff, and by the proposer.

Subset Sum of whose Elements is Divisible by m

E 1913 [1966, 776]. *Proposed by R. A. Jacobson, South Dakota State University*

Given the set of $2n$ integers $\{\pm a_1, \pm a_2, \cdots, \pm a_n\}$ and a positive integer $m < 2^n$. Show that it is always possible to select a subset S such that (i) $\pm a_i$ are not both contained in S; (ii) the sum of the elements of S is divisible by m.

Solution by Anton Chernoff, Student, Central High School, Philadelphia. Consider the $2^n - 1$ nonempty subsets $S_1, S_2, \cdots, S_{2^n-1}$ of $\{a_1, a_2, \cdots, a_n\}$ $(a_k \geqq 0)$, and let $F(S_i)$ denote the sum of the elements of S_i. If $F(S_i) \equiv 0 \pmod m$ for some $i(1 \leqq i \leqq 2^n - 1)$, then take $S = S_i$. Otherwise, since there are only m residue classes, it follows that there must exist $i, j, i \neq j, 1 \leqq i \leqq m, 1 \leqq j \leqq m$, such that

$F(S_i) \equiv F(S_j) \pmod{m}$. Now let S be chosen as follows: put in S the elements of S_i which are not in S_j and the negatives of the elements in S_j which are not in S_i. Now clearly $F(S) \equiv 0 \pmod{m}$ as required.

Also solved by J. C. Abad, J. L. Brown, Jr., J. F. Dillon, P. K. Garlick, Michael Goldberg, Jerry Goodman, D. G. Huffman, Donald Jeffords, D. C. B. Marsh, L. J. Marx, J. B. Muskat, Barbara W. Nason, W. W. Whitman, and the proposer.

Goodman notes the similarity of the present problem to E 1771 [1966, 543].

Group under Multiplication

E 1914 [1966, 776]. *Proposed by Don Redmont, Fremont, California*

Prove that the nth roots of unity constitute the only set of nth roots of a complex number which is a group under multiplication.

Solution by A. Makowski, Warsaw, Poland. Let z be a number such that some of the values of $z^{1/n}$ constitute a group. The unit of such a group may only be the number 1. Hence 1 is one of the values of $z^{1/n}$; therefore $z=1$.

Also solved by I. K. Abroub, A. N. Aheart, R. G. Albert, Larry Anderson, Marcia Ascher, Marshall Atlas, Anders Bager (Denmark), B. N. Bajaj, M. J. Balas, Merrill Barnebey, M. C. Bhandari (India), M. G. Brown, L. P. Bush, J. P. Celenza, P. R. Chernoff, John Dickinson & Dexter Strawther III, J. F. Dillon, G. C. Dodds, W. G. Dotson, Jr., E. S. Eby, W. O. Egerland, R. B. Eggleton (Australia), Larry Elewitz, M. A. Ettrick, Michael Goldberg, M. G. Greening (Australia), G. A. Heuer, K. E. Hirst (England), Stephen Hoffman, Bruce Hoyt, D. G. Huffman, Thomas Jefferson, Jr., J. A. Jensen, Erwin Just, Carolyn Kaluzniacki, Roman Kaluzniacki, Herbert Koller, J. A. Lambert (Australia), R. S. Lee, C. C. Lindner, L. J. Manx, D. C. B. Marsh, Norman Miller, W. A. Miller, P. L. Montgomery, Wanda J. Mourant, D. E. Moxness, Robert Patenaude, Alfred Petrowski, Stanley Rabinowitz, Simeon Reich (Israel), Stephen Rhodes, Shmuel Rosset (Israel), Steven Russ, David Sibley, D. L. Silverman, Indranand Sinha (India), Stephen Spindler, D. R. Stark, Preston Stein & Stan Dick, J. H. Tiner, Z. Z. Uoiea, A. M. Vaidya (India), Dimitrios Vathis (Greece), Kenneth Young, Qazi Zameeruddin (India), and the proposer.

Most of the solutions tacitly excluded the trivial case, $z=0$.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before July 31, 1968. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5550. *Proposed by C. C. Lindner, Coker College, Hartsville, S.C.*

Let $\mathfrak{A}$ be a complete set of $n-1$ orthogonal Latin squares of order n based on the symbols $1, 2, \cdots, n$. Denote by λ_{ij} the cell in the ith row and jth column of an $n \times n$ Latin square. Prove that if $i \neq p$ and $j \neq q$ then the cells λ_{ij} and λ_{pq} are occupied by the same symbol in exactly one of the Latin squares belonging to $\mathfrak{A}$.

5551. *Proposed by G. A. Heuer, University of California at Berkeley and Concordia College*

Find an ordered field in which every sequence is bounded, and every convergent sequence is ultimately constant.

5552. *Proposed by D. E. Crabtree, Amherst College*

It is well known that each characteristic vector for a nonsingular $n \times n$ matrix A is also a characteristic vector for the matrix adj A of (transposed) cofactors of elements of A. Prove that this is true for singular A also.

5553. *Proposed by P. R. Meyer, Hunter College*

It is well known that the topology of joint continuity is the product topology, but is there a topology of separate continuity? More precisely, if X, Y, and Z are topological spaces and f: $X \times Y \to Z$, is there a topology T for $X \times Y$ such that f is T-continuous if and only if f is separately continuous in each variable?

5554. *Proposed by Surjeet Singh, Kirori Mal College, Delhi, India*

Consider matrices with denumerably many rows and columns over a division ring D. Let R be the ring generated by all scalar matrices together with matrices having only a finite number of nonzero columns. Show that R is regular in the sense of von Neumann, and that R, considered as a right R-module, is not injective.

5555. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove or disprove the inequality $f(x)f(y) \geqq f(x)+f(y)-f(x+y)$, $(x,\ y \geqq 0)$, where

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

5556. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Every polynomial $f(x)$ irreducible over a perfect field R, and with an abelian Galois group, is normal. (An irreducible equation $f(x)=0$, $f(x) \in R[x]$, is called normal if $f(x)$ splits in the field obtained by the adjunction of an arbitrary root of $f(x)$ to R.)

5557. *Proposed by P. R. Chernoff, Harvard University*

Let E and F be measurable subsets of the unit interval having equal Lebesgue measures. Show that there is an essentially one-to-one measure preserving transformation T of the interval such that, up to sets of measure 0, $T(E)=F$.

5558. *Proposed by A. W. Goodman, University of South Florida*

The derivative of area is arc length. Let $\overline{R}(s)$ be the vector equation of a

smooth simple closed curve C that encloses a plane region of area A. Let $\overline{N}(s)$ be the unit outward normal. Define a parallel curve $C(r)$ at distance r from C by the vector equation $\overline{R}_r(s)=\overline{R}(s)+r\overline{N}(s)$. If $0\leqq r$ and r is sufficiently small then $C(r)$ will be a smooth simple closed curve that encloses a region with area $A(r)$. Prove that at $r=0$, $dA(r)/dr=L$ where L is the length of the curve C.

5559. *Proposed by Kwangil Koh, North Carolina State University, Raleigh*

A ring is said to be prime if and only if $aRb=0$, for a, b in R, implies that either $a=0$ or $b=0$. It is said to be semi-prime if and only if $aRa=0$ implies that $a=0$. Prove that a prime ring R with no nilpotent element except 0 is an integral domain (not necessarily commutative) and that there exists a semi-prime ring with no nilpotent element except 0, but with zero divisors.

SOLUTIONS OF ADVANCED PROBLEMS

An Inequality in the Complex Plane

5394 [1966, 547; 1967, 603]. *Proposed by P. J. O'Hara, Jr., University of Miami, Coral Gables, Florida*

Show that the functions, $(1-z)/(1-z^n)$, $n=1, 2, \cdots$ are uniformly bounded in modulus on the disk $\{z: |z-\frac{1}{2}| \leqq \frac{1}{2}\}$ in the complex plane; and that in fact for all n the bound is attained at $z=0$.

Editorial Note. The author of Solution I has discovered an error in her work and wishes to withdraw it. Solution II is complete. We include another solution.

III. *Solution by Robert Breusch, Amherst College.* With $f(z)=(1-z)/(1-z^n)$, f is regular on the given disk, thus $|f|$ takes its maximum on the circle $|z-\frac{1}{2}|=\frac{1}{2}$. There, $|z|=\cos\theta$, $z=e^{i\theta}\cos\theta$, $z^n=e^{in\theta}\cos^n\theta$. Since $|f(0)|=1$, $|f(1)|=1/n$, it suffices to prove: *If* $z=e^{i\theta}\cos\theta$, $0<\theta<\pi/2$, *then* $|1-z|<|1-z^n|$.

Proof. Refer to a drawing in which A is an arbitrary point on the upper half of the circle $|z-\frac{1}{2}|=\frac{1}{2}$ with affix z and argument θ. Let B be the point z^n, O the origin and C the point 1. Assume, contrary to the desired statement, that $BC\leqq AC$. Then, since AC is the shortest distance from C to the line OA, it follows that the measure of angle $COB\leqq\theta$. Also, either $n\theta\geqq 2\pi$, or $2\pi-n\theta\leqq\theta$; in either case, $\theta\geqq 2\pi/(n+1)$. Since $\theta<\pi/2$. we must have $n\geqq 4$ in order for our assumption to be possible. We also have

$$a \equiv \sin\theta > \frac{2\pi}{n+1}\left[1-\frac{4\pi^2}{6(n+1)^2}\right] > 4.6/(n+1) > 3.5/n.$$

Thus

$$(*) \qquad a \equiv AC > 3.5/n \quad \text{for } n \geqq 4.$$

Now $BC\geqq 1-OB$ so that our assumption implies

$$1 - \cos^n \theta \leqq a, \quad (1 - a) \leqq (1 - a^2)^{n/2}, \quad (1 - a)^2 \leqq (1 - a^2)^n,$$
$$1 \leqq (1 + a)^n (1 - a)^{n-2}.$$

Taking logarithms we have

$$0 \leqq n \cdot [a - a^2/2 + a^3/3] - (n - 2) \cdot [a + a^2/2 + a^3/3],$$
$$0 \leqq 2 - (n - 1) \cdot a + 2a^2/3 \equiv g(a).$$

Since g is decreasing for $0 \leqq a \leqq 1$, it follows that $a < 3.5/n$ for $n \geqq 4$, which contradicts $(*)$. The proof is now complete.

Sequence of Pseudo-norms on a Vector Space

5450 [1967, 89]. *Proposed by H. Kestelman, University College, London, England*

A real-valued function f defined on a vector space V is called a pseudo-norm if, for all x, y in V

$$f(x + y) \leqq f(x) + f(y) \quad \text{and} \quad f(-x) = f(x).$$

Suppose $f_1, f_2, \cdots$ are pseudo-norms in V, and E_0 is the set of those v in V with $\sum_n f_n(v) < \infty$; show that E_0 is either empty or is a group under addition.

If $\epsilon_1, \epsilon_2, \cdots$ is a sequence of elements of V, and S_ϵ is the set of all v with $\sum_n f_n(v + \epsilon_n) < \infty$, show that if S_ϵ includes at least two elements then E_0 is infinite and S_ϵ is a translate of E_0. Deduce when $V = R_n$, that if E_0 has positive interior Lebesgue measure then S_ϵ is either empty or else equal to R_n.

Solution by M. D. Mavinkurve, Siddharth College, Bombay, India. The hypothesis insures $f_n(x) \geqq 0$. If E_0 contains v_1 and v_2 it also contains $v_1 - v_2$, for $\sum f_n(v_1 - v_2) \leqq \sum f_n(v_1) + \sum f_n(v_2) < \infty$. Therefore E_0 (unless it is empty) is a subgroup of the additive group of V. That E_0 may actually be empty is seen by putting $f_n(x) = 1/n$ for all x in V.

Suppose S_ϵ contains two elements v_1 and v_2, so that $\sum f_n(v_1 + \epsilon_n) < \infty$ and $\sum f_n(v_2 + \epsilon_n) < \infty$. Then

$$\sum f_n(v_1 - v_2) = \sum f_n(v_1 + \epsilon_n - v_2 - \epsilon_n)$$
$$\leqq f_n(v_1 + \epsilon_n) + \sum f_n(v_2 + \epsilon_n) < \infty.$$

Therefore E_0 contains $v_1 - v_2 (\neq 0)$ and its integral multiples. E_0 is thus infinite. S_ϵ is then the coset $E_0 + v_1$.

When $V = R_n$ and E_0 has positive interior Lebesgue measure, E_0 contains a closed set of positive measure and, being a subgroup under addition, must coincide with R_n. In this case S_ϵ, if not empty, as a translate of R_n, coincides with it.

Also solved by D. A. Hejhal, G. A. Heuer, R. M. Rakestraw, Charles Riley, and the proposer.

In his solution Rakestraw shows that S_2 may be empty when $V = R_n$ and E_0 has positive interior measure.

Operators on a Hilbert Space

5451 [1967, 89]. *Proposed by J. P. Williams and Thomas Crimmins, University of Michigan*

It is well known that if T is a bounded linear operator from a Hilbert space H into itself, then T maps H *onto* itself if and only if its adjoint is bounded below: $\|T^*x\| \geqq C\|x\|$, $(x \in H)$ for some positive constant C (see Dunford and Schwartz, Vol. 1, p. 488, for example).

Show that the weaker condition $\|Tx\| + \|T^*x\| \geqq C\|x\| \, (x \in H)$ does not imply that T is one-to-one or that the range of T is dense. Does this condition imply that the range of T is closed?

Solution by P. R. Chernoff and W. C. Waterhouse, Harvard University. On the space of square-summable sequences, let

$$T(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \cdots) = (0, a_0, 0, a_2, a_3/2, a_4, a_5/3, a_6, a_7/4, \cdots).$$

Then $T^*(a_0, a_1, a_2, a_3, a_4, a_5, a_6, \cdots) = (a_1, 0, a_3, a_4/2, a_5, a_6/3, \cdots)$.
Hence the condition holds with $C = 1$. T is not one-to-one, and its range is neither closed nor dense.

Also solved by E. A. Nordgren and the proposers.

Sequences with an Odd Number of Real Elements

5452 [1967, 89]. *Proposed by E. S. Barnes, University of Malaya, Kuala Lumpur*

Let $n = 2m+1$, $n > 3$. Show that n real numbers $x_1, x_2, \cdots, x_n$ have the property that the sum of any m of them is less than or equal to the sum of the remaining $m+1$ if and only if $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is a nonnegative linear combination of the $2n$ vectors $\boldsymbol{E} \pm \mathbf{e}_i (i = 1, \cdots, n)$, where $\mathbf{e}_i$ are the unit vectors and $\boldsymbol{E} = \sum_1^n \mathbf{e}_i$.

Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands. Let S be the set of vectors $\mathbf{x} = (x_1, \cdots, x_n)$ satisfying the above condition. It is clear that for $i = 1, \cdots, n$ the vectors $\boldsymbol{E} + \mathbf{e}_i$ are in S and nonnegative linear combinations of vectors in S are in S. Hence it is sufficient to show that each $\mathbf{x}$ in S can be written as a nonnegative linear combination of the vectors $\boldsymbol{E} + \mathbf{e}_i$. Without loss of generality we may assume $x_1 \leqq x_2 \leqq \cdots \leqq x_{2m+1}$. Then we have

$$(x_1 + x_2 + \cdots + x_{m+1}) - (x_{m+2} + x_{m+3} + \cdots + x_{2m+1}) = \alpha \geqq 0.$$

Then

$$\mathbf{x} = \alpha \boldsymbol{E} + \sum_{i=1}^{m+1} (x_{m+2} - x_i)(\boldsymbol{E} - \mathbf{e}_i) + \sum_{i=m+3}^{2m+1} (x_i - x_{m+2})(\boldsymbol{E} + \mathbf{e}_i),$$

where all coefficients are nonnegative. As $\boldsymbol{E}$ is itself a nonnegative linear combination of the vectors $\boldsymbol{E} + \mathbf{e}_i$ (coefficients all equal to $1/n$) the proof is complete.

Also solved by M. G. Greening (Australia) and the proposer.

A Noncompact Space

5453 [1967, 89]. *Proposed by R. S. Doran, University of Washington*

Prove or disprove: If X is a countably compact, locally compact uniform space with unique uniform structure, then X is compact.

Solution by Howard Chitwood, Carson-Newman College, Jefferson City, Tennessee. A counterexample is suggested by Kelley, *General Topology*: Let X be the set of all ordinals less than the first uncountable ordinal, and let $\mathfrak{J}$ be the order topology on X.

(1) According to section E, pp. 162–163, X is locally compact and countably compact but not compact.

(2) According to section E, p. 204, there is a unique uniformity for X whose topology is $\mathfrak{J}$.

Also solved by M. A. Ettrick, T. E. Gantner, Jan Hejcman (Czechoslovakia), L. R. King, M. D. Mavinkurve (India), T. Memp, C. M. Pareek, W. J. Pervin, Francis Siwiec, D. H. Smith, Al Somayajulu, S. Swaminathan, H. H. Wicke, and the proposer.

An additional reference which provides the proof for the unique uniform structure of the example above is available in Gillman and Jerison, *Rings of Continuous Functions*, p. 238.

Swaminathan with his solution offers the result that X is compact if it is assumed additionally that X is complete relative to the unique uniform structure.

A Semigroup

5454 [1967, 90]. *Proposed by G. Szász, Nyiregyháza, Hungary*

What are the elements of the semigroup generated by the free generators a, b and subject to the generating relations $a^2b=a$, $b^2a=b$? Has this semigroup some idempotent elements?

Solution by D. Ž. Djoković, University of Waterloo, Canada. We have $aba = (a^2b)ba = a^2(b^2a) = a^2b = a$, $bab = b$. Any word in a, b by repeated application of $a^2b=a$, $b^2a=b$, $aba=a$, $bab=b$ can be reduced to one of the following forms:

$$(*) \qquad a^n,\ b^n,\ ab^n,\ ba^n,\ (n = 1, 2, \cdots).$$

Any word has the following invariants: (1) the first letter, (2) the difference between the total number of a's and the total number of b's. Hence all words (*) are different. The only idempotent elements are ab and ba; they are also right identities.

Also solved by D. T. Adams, W. J. Blundon, (England), J. P. Celenza, L. D. Crowson, L. J. Dixon, Morton Goldberg, M. G. Greening (Australia), G. A. Heuer, C. F. Hockett, R. R. Korfhage, S. Lajos (Hungary), C. W. Leininger, Jack Mettauer, Bohuslav Mišek (Czechoslovakia), Jagannath Mital (India), T. F. Mulcrone, F. D. Parker, D. E. Penney, Perry Scheinok, John Shafer, Philip Trauber, Z. Z. Uoiea, and the proposer.

Intra-Regular Semigroups

5455 [1967, 90]. *Proposed by G. Szász, Nyiregyháza, Hungary*

Let S be a semigroup having the following property: if a and b are any elements of S such that the principal ideal generated by a does not contain the

element b, then there exists a prime ideal P in S containing a but not containing b. Prove that S is intra-regular.

Solution by John Shafer, University of Massachusetts. Let s be any element of S and let T be the principal ideal generated by s^2. If s is not an element of T, then by the given property of S, there exists a prime ideal P containing s^2 but not s. But a prime ideal (or even a semi-prime ideal) cannot satisfy this condition. Hence s must be an element of T.

Now $T=Ss^2S\cup s^2S\cup Ss^2\cup\{s^2\}$. If $s\in\{s^2\}$, then $s=s^4\in Ss^2S$. If $s\in s^2S$, then $s=s^2x=s(s^2x)x=s(s^2)x^2$; so again $s\in Ss^2S$. Similarly if $s\in Ss^2$, then $s\in Ss^2S$. Therefore $s\in Ss^2S$ (or $s=xs^2y$ for some x, $y\in S$), and this is the requirement for S to be intra-regular.

Also solved by D. F. Dawson, C. W. Leininger, and the proposer.

Leininger and the proposer refer to Clifford and Preston, *The Algebraic Theory of Semigroups*, pp. 121–125, for a theory of semigroup decomposition which is pertinent to the solution of this problem.

Third Order Circulant Orthogonal Matrices

5456 [1967, 90]. *Proposed by F. D. Faulkner, U. S. Naval Postgraduate School*

Let $\mathfrak{A}=\{A\}$ be the set of all real, third order, circulant, orthogonal matrices:

$$A=\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}=A(a,b,c)\qquad\text{with } A^{-1}=A^T.$$

(1) Show that there is one A for each value of a, $1\geqq a\geqq\frac{2}{3}$, with $\frac{2}{3}\geqq b\geqq 0\geqq c\geqq -\frac{1}{3}$.

(2) Show that all matrices of $\mathfrak{A}$ are obtained by permuting the elements obtained in (1), and the negatives of such matrices.

(3) Show that there is an infinite number of matrices in $\mathfrak{A}$ whose elements are all rational.

(4) Show that if two adjacent elements of A are rational, then A has all rational elements.

Solution by Leonard Carlitz, Duke University. The condition $AA^T=I$ reduces to $a^2+b^2+c^2=1$, $ab+bc+ca=0$, which implies $a+b+c=\pm 1$.

(1) The assumption $1\geqq a\geqq\frac{2}{3}$ leads to the requirement that $a+b+c=1$ in order to obtain real solutions. Solving the equations

$$b^2+c^2=1-a^2,\qquad b+c=1-a$$

gives

$$b=\tfrac{1}{2}(1-a+\sqrt{(1+3a)(1-a)}),\qquad c=\tfrac{1}{2}(1-a-\sqrt{(1+3a)(1-a)}),$$

and $1\geqq a\geqq\frac{2}{3}$ implies $\frac{2}{3}\geqq b\geqq 0$, $0\geqq c\geqq -\frac{1}{3}$.

(2) It suffices to show that we may assume $1 \geqq a \geqq \frac{2}{3}$. Since $a^2+b^2+c^2=1$, the upper bound for a is immediate. Also we may assume that $b \leqq a$. Then

$$1 - a + \sqrt{(1+3a)(1-a)} \leqq 2a$$

which reduces to $12a^2 \geqq 8a$ or $a \geqq \frac{2}{3}$. The rest of (2) follows from symmetry considerations.

(3) The condition for rationality is $1+2a-3a^2=r^2$, where r is rational. This is equivalent to $(3a-1)^2+3r^2=4$, which is satisfied by

$$3a - 1 = \frac{u^2-3}{u^2+3}, \qquad r = \frac{2u}{u^2+3},$$

where u is an arbitrary rational number.

(4) follows immediately from $a+b+c=1$.

Also solved by J. P. Celenza, D. Ž. Djoković, M. G. Greening (Australia), H. E. Thomas, Jr., and the proposer.

Invertible Elements in a Ring

5457 [1967, 90]. *Proposed by T. J. Grilliot, Duke University*

Let R be a ring with identity. Prove that the sets

$$A = \{x \in R: x \text{ is not invertible}\},$$
$$B = \{x \in R: x \text{ is not left invertible}\},$$
$$C = \{x \in R: x \text{ is not right invertible}\},$$

are all equal if any one of them is closed under addition.

Solution by Kwangil Koh, North Carolina State University. Let $a \in A$. If $a \notin B \cap C$ then a is either left invertible or right invertible. If a is left invertible then there is $x \in R$ such that $1 = xa$ and $a = axa \neq 0$. Since $a \in A$, $1 - ax \neq 0$. Since $ax = axax$, $ax(1-ax)=0$ and $(1-ax)ax=0$. Hence ax, $(1-ax)$ are elements of $A \cap B \cap C$. Thus if any one of A, B, C is closed under addition, $ax+(1-ax)=1$ is an element of one of A, B, C. This is impossible. Similarly a cannot be right invertible. Thus $A \subseteq B \cap C$. Since $B \cup C \subseteq A$ always, it follows that $A = B = C$.

Also solved by C. B. Baytop & J. E. Joseph, H. R. Gage, E. R. Gentile (Argentina), M. G Greening (Australia), H. A. Guess, G. A. Heuer, T. P. Kezlan, M. D. Mavinkurve (India), Jagannath Mital (India), J. H. Oppenheim, L. J. Pratte, P. P. Sanchez, and the proposer.

Decomposition Theorems for Semigroups

5459 [1967, 90]. *Proposed by Thomas C. Brown, Reed College and Kiev State University, U.S.S.R.*

Let S be a semigroup with three generators g_1, g_2, g_3, in which $x^2=x^3$ and $x^2y=yx^2$ for all x, y in S. Show that S can be written as the union of seven pairwise disjoint semigroups and that seven is maximal.

Solution by David Ryeburn, Simon Fraser University, Burnaby, B. C., Canada. It is just as easy to generalize to the case where there are n generators, $g_1, \cdots, g_n$, and the same relations $x^2=x^3$, $x^2y=yx^2$ hold, and to show that S is the union of 2^n-1 pairwise disjoint semigroups, with 2^n-1 maximal.

Observe that $(xy)^2=(xy)^3=x(yx)^2y=xy(yx)^2=xy^2xyx=x^2y^3x=x^3y^3=x^2y^2$. It follows by induction that $(x_1x_2\cdots x_k)^2=x_1^2x_2^2\cdots x_k^2$ for each positive integer k.

Now for each of the 2^n-1 nonempty subsets, T, of $\{g_1, \cdots, g_n\}$, consider the class T^* of all members of S which can be written as a product of members of T but which cannot be written as a product of members of any proper subset of T. Clearly T^* is a semigroup. To see that it has no proper disjoint subsemigroups, note that if $T=\{g_{i_1}, \cdots, g_{i_k}\}$ and $x\in T^*$ then $x^2=g_{i_1}^2g_{i_2}^2\cdots g_{i_k}^2$, since the square of x will be the product of the squares of the exhibited factors of x, and since squares commute and are equal to higher powers, the squares in whatever order and with whatever repetitions may be rearranged into the form $g_{i_1}^2g_{i_2}^2\cdots g_{i_k}^2$. Thus all subsemigroups must share that element.

Clearly the semigroups T^* are disjoint and have S as their union. If we have m semigroups $S_1, \cdots, S_m$ which are disjoint and have S as their union then there must be at least m nonempty semigroups among the disjoint sets $T^*\cap S_i$, $i=1, \cdots, m$, $T\subset\{g_1, \cdots, g_n\}$, $T\neq\emptyset$. Since $T^*\cap S_i\subset T^*$ and all subsemigroups of T^* have an element in common, it follows that $m\leqq 2^n-1$.

We can generalize further. Suppose that the defining relations are $x^m=x^{m+1}$ and $x^my=yx^m$. Observe that

$$\begin{aligned}(xy)^m &= (xy)^{2m-1} = (xy)(xy)^{2m-2} = x(xy)^{2m-2}y = x^2y(xy)^{2m-3}y = x^2(xy)^{2m-3}y^2\\ &= x^3y(xy)^{2m-4}y^2 = \cdots = x^{m-1}y(xy)^my^{m-2} = x^{m-1}(xy)^my^{m-1}\\ &= x^my(xy)^{m-1}y^{m-1} = x^my^m,\end{aligned}$$

since the factor x^m commutes with x's and y's and absorbs x's. It follows that $(x_1x_2\cdots x_k)^m=x_1{}^mx_2{}^m\cdots x_k{}^m$. The above argument then applies, we have $x^m=g_{i_1}^mg_{i_2}^m\cdots g_{i_k}^m$ for all $x\in T^*$, and thus again the semigroup S is the union of 2^n-1 disjoint semigroups, with 2^n-1 maximal.

We can generalize further yet. Suppose the defining relations are $x^p=x^{p+1}$ and $x^qy=yx^q$. First suppose $q\geqq p$. Then $x^q=x^{q+1}$ so that the argument above applies with $m=q$. Suppose on the other hand that $q<p$. Choose any integer $l\geqq p/q$. Then $x^{lq}y=yx^{lq}$ since we may commute q x's at a time, l times. Since $lq\geqq p$, we return to the first supposition. Thus we have established that: *If S is a semigroup with n generators $g_1, \cdots, g_n$ in which $x^p=x^{p+1}$ for some positive integer p and all $x\in S$, and in which $x^qy=yx^q$ for some positive integer q and all $x\in S$, $y\in S$, then S can be written as a union of 2^n-1 pairwise disjoint semigroups, and 2^n-1 is maximal.*

Also solved by John Shafer, John Waddington, and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

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Functional Analysis, Theory and Applications. By R. E. Edwards. Holt, Rinehart and Winston, New York, 1965. xiii+781 pp. $25.00.

This is a welcome addition to the collection of books which have appeared on this subject in recent years. It is distinguished from many of the others by several features. First, a principal objective of the author is to devote equal attention to theory and to applications, and this is largely achieved. For example, Chapter 2 The *Hahn-Banach Theorem*, contains not only the usual geometric and analytic forms of this theorem, but also applications to potential theory, approximation theory, game theory, and other fields. The following chapter is concerned with three important fixed-point theorems (the Contraction Mapping Principle and the theorems of Markov-Kakutani and Schauder-Tychonoff) and several of their applications. The emphasis on providing illustrative examples is evident throughout the book. There is a large selection of excellent exercises.

Secondly, one is more impressed by the depth and detail of treatment than by the number of topics mentioned. Thus while Chapter 6 is devoted to the Open Mapping and Closed Graph Theorems in their usual settings, these are returned to in a later chapter on duality where their generalizations to B_r-complete spaces are presented. The development is from a very general point of view whenever possible.

The reviewer was occasionally puzzled by a rather unusual style of presentation. For example, Theorem 6.2.1 consists of two parts, one of which is actually a definition and the other a theorem. He also shares the fear expressed by the author as to the effectiveness of Chapter 5 which is 122 pages devoted to distributions and linear partial differential equations.

DUANE W. BAILEY, Amherst College

Functional Analysis. By George Bachman and Lawrence Narici. Academic Press. New York, 1966. xiv+530 pp. $14.50. (Telegraphic Review, April 1967.)

This is an introductory book on functional analysis intended for advanced undergraduates and beginning graduate students. The prerequisites are given in general terms as undergraduate mathematics through the junior year, and, specifically, as linear algebra and advanced calculus. The pace of the book is slow and its style is discursive. The exposition is broken down into twenty-nine short chapters, sometimes referred to as lectures, of varying lengths. Topics discussed in the early chapters include topology, metrics, norms, completeness, orthonormal systems and the traditional Banach space theorems (Hahn-Banach, closed graph, uniform boundedness). One of the largest chapters is an introduction to Banach algebras, in which analytic function theory (!) is applied to derive the spectral radius formula. The detailed development throughout the book is largely directed toward the spectral theorem for self-adjoint and normal operators on a Hilbert space, and culminates in a burst of proofs of this theorem. It is proved first for finite dimensional normal operators, for motivational purposes, and then for compact normal operators. Then come three different proofs for bounded self-adjoint operators, one for bounded normal operators, and, finally, two proofs for unbounded self-adjoint operators.

In view of the authors' aim to produce an in-depth, highly detailed, introductory treatment, there are some curious omissions. There is no complete proof that any of the standard norms are norms: the Cauchy-Schwarz inequality is used but not proved; the Hölder inequality is proved (for finite sums) but the Minkowski inequality is not derived from it. And although the completion process for a metric space is discussed in detail, no explicit infinite dimensional Banach space is proved to be complete except $C[a,b]$. The completeness of l_2, for example, is deduced from the general measure-theoretic proof of the completeness of $L_p(\mu)$, which of course can only be sketched in this setting.

This brings up what seems to the reviewer to be the unavoidable weakness of an early introduction to standard functional analysis, that in the absence of real variable theory, the most significant context for the subject being developed is missing. As suggested above, the authors attempt to meet this difficulty by sketching relevant contacts with real variables in a series of examples. However, these can't be of much help to the student who has not had the subject and who will end up with the spectral theorem in a vacuum, so to speak, whereas, the more advanced student is better served by a development making a more integrated and thorough going use of real variables. This is true all along the line, but is especially critical for the spectral theorem itself, the real meaning of which involves the general space $L_2(\mu)$ in an essential way.

LYNN H. LOOMIS, Harvard University

Integral Equations and their Applications, Vol. I. By W. Pogorzelski. Pergamon Press, New York and Polish Scientific Publishers, Warsaw, 1966. xi+714 pp. $18.50. (Telegraphic Review, April 1967.)

This is a readable and quite comprehensive treatise; together with the forthcoming second volume it promises to achieve distinction as a useful text and work of reference. It is also an honorable memorial to its late author's half century of research and teaching on integral equations and contiguous matters.

The three parts into which Volume I is divided are devoted respectively to (1) linear equations, (2) nonlinear problems and integral equations arising from differential problems, (3) singular equations. Most of Part 1 is taken up with Fredholm's determinantal approach to the equations of second kind and the Hilbert-Schmidt L^2 theory for symmetric kernels. This material is conventional but well organized. Part 2 opens with chapters on fixed-point theorems of Banach-Cacciopoli and Schauder and their applications to nonlinear equations (Schauder's theorem is proved in an appendix by Professor Roman Sikorski). In separate chapters on the elliptic, parabolic and hyperbolic cases, integral equation techniques are used to study partial differential problems of second order. Part 3 contains a good deal of fairly recent material developed by Russian and Polish mathematicians, much of it motivated by classical problems of Hilbert, Riemann and Poincaré in analytic function theory.

Favorable aspects of the work include: (1) Throughout, there is an engaging blend of classical analysis (inequalities, complex function theory) with topological and function-theoretic tools. (2) Nonlinear problems and techniques are introduced early and come up often. (3) Much material is to be found here which has previously existed only in scattered research papers. (4) The presentation is clear and well paced, the three translators have produced flowing, idiomatic English, and the Polish printers have done their job well.

It is with regret that I record the following: (1) A *gaffe* occurs in Chapter XII, where the Neuman Problem is assigned to von Neumann. (2) There are no exercises. (3) The physicist, though finding much that is useful, will not see physical problems treated directly. (4) Numerical methods are not even hinted at. (5) Multiple solutions of nonlinear equations are not discussed.

We may look forward to Volume II with interest.

A. T. LONSETH, Oregon State University

Lectures in General Algebra. By A. G. Kurosh, translated by Ann Swinfen. English translation edited by P. M. Cohn. Pergamon Press, Oxford, 1965. x+361 pp. $7.50.

This is the second translation of Kurosh's book (the first is reviewed in the July 1965 issue of this MONTHLY). The book is sufficiently interesting to merit a few more comments as well as a comparison of the two translations, for we have here an unusual collection of topics in modern algebra which can be read by persons with a relatively modest background. Novel features include material on non-associative rings, universal algebras, and groups with multi-operators. Kurosh has made no effort to encompass all areas of modern algebra, however. For example, there is no homological algebra.

Both translations are entirely adequate (this should come as no surprise since the previous Chelsea edition was translated by K. A. Hirsch). However, I find I prefer the Hirsch translation on several grounds. First of all, the Pergamon edition has no section numbers at the top of the page which largely negates Kurosh's excellent cross references. Secondly, the index is better in the Chelsea edition. Finally, the Chelsea edition is somewhat less expensive. There is a bibliography of important books on algebra published (in a dozen languages) during the last thirty years. Among these one finds a number of books in Russian which have yet to be translated into English. Surely the publishing houses would perform a more useful service in translating some of these books into English rather than duplicating what we already have in a completely satisfactory form.

Prospective authors of algebra books might note the preface where there is a discussion of writing a successor to van der Waerden's *Modern Algebra.* Kurosh makes no pretense of having done this but his Modern Algebra mosaic is an important contribution to mathematical literature.

E. A. KLOTZ, Swarthmore College

Real and Abstract Analysis. By Edwin Hewitt and Karl Stromberg. Springer-Verlag, New York, 1965. 476 pp. $9.50.

The theory of functions of a real variable is presented from the point of view that modern analysis, though largely abstract, has its roots in classical, concrete mathematics. In addition to a general and comprehensive development of the traditional tools of real function theory, topics of interest in contemporary research are included.

The book contains six chapters, the first two of which deal with the fundamentals of the subject. These begin with an informal review of the elements of set theory and with the construction of the real and the complex number fields, continue with the development of set theoretic topology as it applies to analysis and with the study of spaces of continuous functions, and conclude with a presentation of several versions of the Stone-Weierstrass theorem. In Chapter 3, the Riemann-Stieltjes integral is introduced, and by a very general extension, using the Darboux-Daniell method, the Lebesgue integral is derived and its properties explored, the chapter culminating with the Riesz representation theorem. Chapter 4 is devoted to a study of general Banach spaces, including the function spaces L^p and their conjugate spaces, and Hilbert spaces. The next chapter begins with a classical treatment of the theory of differentiation and continues with a characterization of those functions which are indefinite integrals. An extension of the concept of indefinite integral leads to the Lebesgue-Radon-Nikodým theorem which is proved under very general conditions. This generality is relaxed in the study of product spaces in Chapter 6, where Fubini's theorem, for example, is established on the assumption of σ-finiteness of the measure spaces.

The book gives a careful, thorough, and well organized presentation of differentiation and integration. It is replete with interesting and helpful remarks stressing the highlights of a theorem, emphasizing the importance of given hypotheses, indicating the place of a result in the general structure, or pointing out different directions in which the theory has developed. Although no bibliography is included, references and historical notes are

contained either in the body of the text or in the footnotes. There are numerous exercises of varying degree of difficulty and many provocative illustrative examples.

The book is an excellent source of reference for the professional, but as a text for first or second year graduate students, it offers far too much material, as the authors themselves indicate. In sketching an outline of an abbreviated course of study, however, even they find it difficult to make up their minds to limit the topics to be covered. For example, in the preface, the instructions given for the selection of sections to be taken up in Chapters 4 and 5 are at variance with the introductory remarks at the beginnings of these chapters. Further, for students without adequate mathematical maturity, the generality of the treatment may prove difficult, but the enthusiastic and challenging approach cannot fail to generate interest and curiosity.

DEBORAH TEPPER HAIMO, Southern Illinois University

Stochastic Processes. By N. U. Prabhu. Macmillan, New York, 1965. 233 pp. $7.95.

This is an intermediate level text in probability theory. Readers should have had a one-semester introduction to probability on the level of the first half of Volume I of Feller's "Introduction to Probability Theory," and a knowledge of advanced calculus. The book would be appropriate as a text for a second semester undergraduate probability course, or for a course in applied probability for students in the sciences, engineering, or operations research.

Stochastic processes being a vast subject, and the present volume being of modest size (233 pages), the selection of a limited number of topics is dictated. The author has concentrated most of his attention on Markov processes and renewal theory and on related applications to such areas as random walk, queueing theory, population growth and counter models.

Markov processes comprise roughly half the volume of the book. The first of three chapters on the subject, Chapter 2, starts with the theory of discrete parameter chains with a countable state space. The standard theorems on classification of states, limit probabilities, and stationary measures, are proved. Chains with the real line as their state space are defined but their properties are not developed in general, rather being illustrated by special examples.

Diffusion processes are studied in Chapter 3 from a "classical" standpoint, the Kolmogorov equations being derived by passage to the limit in the difference equations satisfied by the transition probabilities. Discontinuous processes are in the next chapter, their treatment being purely analytical, as in the diffusion case. (Sample path properties are not considered.) The Kolmogorov-Feller equations are derived and then a number of special processes are studied (e.g., Poisson, compound Poisson, birth-death).

Renewal theory is treated thoroughly in the last two chapters. The discrete renewal theorem and Blackwell's theorem are proved—first for positive random variables—and after the introduction of ladder indices for arbitrary variables. Age and residual life distributions are derived and stationary renewal processes discussed. Some aspects of fluctuation theory (distribution of maxima in particular) are developed.

In addition, there is an introduction summarizing definitions and prerequisites; and a chapter on second-moment processes, (Chapter I). Differentiation of processes, stochastic integrals, and spectral distributions are here introduced, but the treatment is much briefer than the parts on Markov processes and renewal theory, and does not go very far past the definition stage.

In addition to the numerous examples and applications in the body of the text, there is a substantial collection of complements and problems at the end of each chapter. The book is organized and written lucidly. It is a welcome addition to the textbook literature in applied probability.

P. E. NEY, University of Wisconsin

Scalar and Vector Fields: A Physical Interpretation. By R. B. McQuistan. Wiley, New York, 1965. 310 pp. $5.95.

The intention of the author is to "emphasize the physical meaning of the algebra, calculus, operators, and transformation theory associated with fields." He admits that "this emphasis on the physical interpretation has meant that certain aspects of the mathematical rigor have been neglected." In the first chapter, trouble arises more from failure to recognize that vector and field theory are mathematical models. This and careless definitions lead to unnecessary ambiguities such as *scalar* vs. *pseudoscalar;* the attempt to distinguish between *free vectors*, *sliding vectors*, and *point vectors;* and the "definition" of vector field as "when a vector quantity varies from point to point in a region of space." Many facts are stated without proof as if self-evident, some of which may be (the associative law for vector addition), but some of which are not (the distributive law for the cross product when the cross product is defined without using coordinates). Many arguments are heuristic and can lead to false implications (e.g., that f can be expanded in a Fourier series only if f^2 is integrable). Increments and differentials are not distinguished; "infinitesimal" covers both (e.g., "we can think of space curves with which we deal in the physical world as composed of many infinitesimal straight line segments . . . "). Starting slowly, the book accelerates rapidly, and includes a great many formal processes involving differentiation, integration, change of coordinates, and potential functions that are important to physicists and engineers. A good background in advanced calculus is needed. The book is not suitable for a mathematics course. For physicists and engineers, it seems most useful as a reference used in connection with other courses.

R. C. JAMES, Harvey Mudd College

Le Calcul Symbolique et ses Applications à la Physique Mathêmatique. By Pierre Humbert and Serge Colombo. Gauthier-Villars, Paris, 1965. 74 pp. Fr. 16.00.

This book is a revised and augmented version of an earlier volume by Humbert in the same series (Mêmorial des Sciences Mathêmatiques, edited by H. Villat). I was unable to locate a copy of the original edition, but gather from the preface that the revision modernized the notation and added material on special functions.

The book is a brief exposition of operator calculus using primarily the Laplace transform, but mentioning the transforms of Mellin, Weierstrass, Hankel, Stieltjes, Kantorovich-Lebedev and Fraser. The material is divided into three chapters: one which explains the operator calculus, a second which applies operational methods to problems in special functions, and a third which uses operational methods to solve equations for coupled electrical networks. There are no problems. The book has no index, but includes a bibliography of about 50 references of which about three quarters are earlier than 1950.

Because the book includes no problems, is written in French, and covers material readily available in English texts on operational methods, it would not be suitable as a text for U.S. colleges. For the same reasons it would not be very good for self-study by an English-speaking engineer or mathematician. I suspect that even a French-speaking engineer or mathematician could find better references for self-study.

B. A. TAGUE, Bell Telephone Laboratories

Concepts of Probability Theory. By Paul E. Pfeiffer. McGraw-Hill, New York, 1965. 399 pp. $10.50.

This is a book with a somewhat different approach in presenting the basic concepts of probability theory. Since it is written primarily for students of engineering and related physical sciences, the author utilizes several ideas and techniques appearing in the literature in these areas. The concept of probability as a mass, the use of minterm expansions and maps, binary designators and other notions from the theory of switching networks,

and the use of the indicator function for events to provide analytical expansions for discrete-valued random variables constitute some of the more unusual features.

The subject matter progresses logically starting with the classical probability model followed by the development of a mathematical model corresponding to "real world" concepts and relationships. The mathematical model is then extended by introducing the concept of a random variable and certain associated analytical means of describing the distribution of values of random variables. The concept of the integral is examined to allow a general formulation of the averaging operation. The integral on an abstract space, upon which a suitable measure is defined, is introduced, and the methods of Riemann and of Lebesgue are discussed. Mathematical expectation is treated as a probability-weighted average resulting from the application of abstract integrals. Classical theorems, such as the law of large numbers and the central limit theorem are developed, and the book ends with a treatment of random processes.

The textual treatment is illustrated by many representative examples and approximately 200 well-selected problems are included. Classified references appear at the end of each chapter, and useful supplementary material including a brief, but good, introduction to set theory is included in the appendices.

The treatment is lucid and, in general, rigorous. It provides for mature engineering students a good basic foundation in probability theory and should prove a valuable reference book for those engaged in engineering and scientific research and development.

E. B. ROESSLER, University of California, Davis

Representation Theory of Finite Groups. By Martin Burrow. Academic Press, New York, 1965. 185 pp. $3.45.

Except for the comprehensive (and voluminous) work by Curtis and Reiner on the representation theory of finite groups, this is the first textbook in English offering an introduction to the important theory of modular representations. For this reason alone, Burrow's brief and largely self contained book will be useful for many purposes. The account of the classical theory of representations over the field of complex numbers is developed from the point of view of the representation module. A certain amount of algebraic sophistication is expected of the reader, although many important algebraic tools are fully developed in the text. In particular, Wedderburn's Theorem on the structure of semisimple rings with minimal condition is proved in the text, and a glossary of definitions from the theory of groups, rings, ideals and fields is added as an appendix.

The applications include the famous theorems of Burnside and of Frobenius about the existence of proper normal divisors in certain groups and the construction of the irreducible representations of the symmetric group Σ_n. For $n \leq 5$, this construction is carried out in detail. All chapters contain good examples and all but the last one (on modular representations) have exercises. Some difficult theorems on induced and on modular representations are quoted without proof (and with a reference) to complete the picture of the general theory.

The style of the book is lucid and concise.

WILHELM MAGNUS, New York University

Theory of Retracts. By Sze-Tsen Hu. Wayne State University Press, Detroit, Mich., 1965. 234 pp. $13.50.

The theory of retracts is an important area connecting set-theoretic topology and algebraic topology. It has its beginning in Borsuk's pioneering work in the early 1930's, and has been gradually developed into a mature stage. Hu's book is the first one devoted to a thorough treatment of the subject. The book is a unique and valuable reference work for the topologists. But, it is also very readable for a graduate student having had a course in topology. Like the other books by the author, it is written in an extremely

clear and precise style, with great care and unusual completeness in detail.

The material is well organized. Chapter I deals with retracts and their properties. In Chapters II and III, absolute retracts (AR) and absolute neighborhood retracts (ANR) are first studied in a setting for certain general classes of topological spaces; then it is shown that the class of all metrizable spaces is the most natural setting for the theory. Chapter IV studies necessary and sufficient conditions for a metrizable space to be an ANR. Similar characterizations of locally n-connected spaces are given in Chapter V. The last two chapters are devoted to adjunction spaces and mapping spaces of AR's and ANR's, compact AR's and compact ANR's in Euclidean spaces and deformation retracts.

KY FAN, University of California at Santa Barbara

Geschichte und Theorie der Kegelschnitte und der Flächen zweiten Grades. By Kuno Fladt. Klett Verlag, Stuttgart, 1965. 311 pp. 48DM.

The author insists that this is not a book on the foundations of geometry and, in a sense, he is right. Yet it comes very close to being one. It is an attempt to give all the important properties of the conic sections and to show the consistency of the various definitions. He uses the Kleinian notion of the invariants under groups of transformations as a unifying principle.

He starts with the Euclidean transformations and generalizes them to the affine and finally the projective transformations. In the process, he develops those parts of the geometry of the real projective plane which have some bearing on conics. This, of course, includes a very substantial portion of classical projective geometry. He also indicates briefly how one might start with projective geometry and get Euclidean geometry from it.

The only background required is elementary analytic geometry. There does not seem to be any consensus in this country as to what geometry we should teach our prospective high school teachers. Perhaps we should consider Fladt's program as one possibility.

T. G. OSTROM, Washington State University

Differential Equations and Applications. By James B. Scarborough. Waverly Press, Baltimore, 1965. 479 pp. $8.25.

The novel feature of this book is the more or less self-contained treatment of a wide variety of interesting applications with which the author seeks to illustrate the fundamental role played by differential equations in mathematical models of nature. The first quarter of the book provides practical means for finding solutions of ordinary and partial differential equations which arise in the investigation of the physical phenomena. Since the author was guided by utility, the reader should not expect to achieve the same knowledge of the theory of differential equations as from a theoretically oriented book. In the remainder of the book, the author's expressed desire is to derive all the differential equations in the applications, with the exception of two or three *ab initio* from fundamental principles or from known physical laws. How well the author succeeds in this depends to a large extent on what "known physical principles" the reader is able to supply. The reviewer feels, however, that the author has achieved the goal outlined in his Preface: " . . . to show how differential equations arise, how to solve them in the most direct manner, and what the solutions . . . imply . . . (and) . . . to exhibit the power and utility of differential equations. . . . "

Prerequisite for this book is a first course in differential and integral calculus. Some knowledge of general physics is desirable especially if one is to follow the derivation and interpretation of the equations of motion.

R. L. BORRELLI, Harvey Mudd College

Einführung in die Funktionentheorie. By R. Nevanlinna and V. Paatero. Birkhäuser, Basel and Stuttgart, 1965. 388 pp. Swiss Francs 64.

This book represents a very complete and self-contained presentation of what the reviewer regards as a one year course covering almost all of the topics that properly belong to an elementary course in the theory of functions of a single complex variable. After fundamental definitions, the elementary analytic functions are studied to p. 111, where infinite series and power series are introduced. The Cauchy Theorem is then given, along with its usual applications including its relation to the fundamental group of the region. Following this, there is a chapter on harmonic functions, concluding with the Harnack Convergence Theorem, and a chapter on analytic continuation. After this, infinite product representations of entire functions and the Jensen formula are given. The authors then give us three very clear chapters on periodic functions, the Γ-function and the Riemann ζ-function. The book concludes with the Riemann Mapping Theorem, the construction of conformal maps and this problem's relation to the Dirichlet problem, and, finally, a proof of the Picard Theorem using the modular function. The presentation of this material is economical and coherent, and the format of the book has been very carefully considered to emphasize the orderly development of definitions and proofs. Each chapter has a large set of problems, many of them with hints.

R. T. HARRIS, University of Maryland

A Collection of Problems on Mathematical Physics. By B. M. Budak, A. A. Samarskii and A. N. Tikhonov. Pergamon Press, New York, 1964. xii+770 pp. $11.50.

This volume contains 835 problems, most of them stated in physical terms, but easily set up as boundary-value problems in second-order partial differential equations. There are references to the textbook by Tikhonov and Samarskii (also recently brought out by Pergamon Press in an English translation), but the problems should make a most valuable adjunct to any text or course on second-order partial differential equations.

The problems occupy pp. 1–160; the detailed hints, answers and solutions—almost a text in themselves—pp. 161–740. The remaining 30 pages contain formulas, tables, bibliographical references, and an index.

Chapters: One (short) on reduction to canonical form; two each (extensive) on equations of hyperbolic, parabolic and elliptic type, " . . . divided into paragraphs according to the method of solution . . . to give students the opportunity, by means of independent work, of gaining . . . skill." (Preface.) As the authors state, techniques such as operational, variational and differential methods, as well as integral equations, are not represented. To this, add first-order partial differential equations, a stepchild in this collection as also in many textbooks.

Translation, typography and general appearance are exemplary. The book is a most welcome contribution to an area where much of the learning must be by doing live problems.

FRITZ STEINHARDT, City College, CUNY

Diffusion Processes and their Sample Paths. By K. Ito and H. P. McKean, Jr. Springer-Verlag and Academic Press, Berlin and New York, 1965. xvi+321 pp. $14.50.

This book describes a measure-theoretic approach to Brownian motion with emphasis on properties of the sample paths. A Brownian motion is interpreted as a stochastic process with continuous paths and a Markov probability structure. The reader is assumed to have a strong background in probability theory. The book will be of interest primarily to other specialists in the field of stochastic processes. It is not a simple exposition for novices.

G. F. NEWELL, University of California, Berkeley

The Classical Moment Problem. By N. I. Akhiezer. Hafner, New York, 1965. x+253 pp. \$10.00.

The moment problem can be stated as follows: for a given sequence $\{s_k\}$ of real numbers, does there exist a nondecreasing function σ such that for

$$k = 1, 2, \cdots; \qquad s_k = \int_{-\infty}^{\infty} u^k d\sigma(u)?$$

This book examines the solvability of the problem. The chapter headings are: Infinite Jacobi matrices and their associated polynomials; the power moment problem; function theoretic methods in the moment problem; inclusion of the power moment problem in the spectral theory of operators; trigonometric and continuous analogues.

Each chapter is followed by a set of addenda and exercises which include many interesting results. The book should be of great interest to those concerned with the uses of classical and modern analysis. It pre-supposes a knowledge of elementary real and complex variable theory and some knowledge of operator theory.

STEPHEN HOFFMAN, SUNY at Cortland.

Regnekunsten i det gamle Norge, fra arilds tid til Abel. (Norwegian) (The Art of Calculating in Old Norway until the Time of Abel.) With an English Summary. By Viggo Brun. Universitetsforlaget, Oslo, 1962. 125 pp.

Alt er tall, matematikkens historie fra old tid til renessanse (Norwegian). (All is Number, a History of Mathematics from Antiquity to the Renaissance.) By Viggo Brun. Universitetsforlaget, Oslo, 1964. 239 pp.

Viggo Brun has in recent years written two books on the history of mathematics, one dealing with the early history in Norway, the other with the general story of the subject. Both are on a very elementary level, but with many interesting observations by the author.

The first book includes a variety of information which will probably be new to a non-Norwegian reader. There are accounts of early fraction computations from the legal codices of Viking times, astronomical considerations from Icelandic sources and the King's Mirror (about 1260), as well as the only medieval Norwegian manuscript on the rules of computations, the *Algorismus* by Hauk Erlandson (about 1300), a magistrate both in Iceland and Norway. It is strongly influenced by Sacrobosco. The first printed arithmetic did not appear until 1645. The author gives an account of problems and material taught in early schools as well as at the military academy and the academy for mining engineers at Kongsberg.

The book concludes with an account of Norwegian contributors to mathematics before 1800. Most are minor prophets, a couple of them were pupils of Tycho Brahe; F. C. Arentz (1736–1825) distinguished himself by an independent discovery of the solution of linear equations by determinants while Jens Kraft (1720–1765) wrote his main work on mechanics (1763), later translated into Latin. In the main stream of mathematics is Caspar Wessel (1745–1818), first to represent complex numbers geometrically in the plane. The author has succeeded in collecting a considerable number of new details concerning his life.

Viggo Brun's general history of mathematics is a pleasing presentation of some of the main facts of the subject, successfully avoiding the dreariness of the lists of many names. The choice of material from the various writers should be suitable at the high school or beginning college level. The last chapter brings us the priority quarrels concerning the solution of the cubic equation.

OYSTEIN ORE, Yale University

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses; T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year). A boldface star (★) marks a notable book that might be overlooked.

Algebra

Introduction to Modern Abstract Algebra. By David M. Burton (University of New Hampshire). Addison-Wesley, Reading, Mass., 1967. vii+310 pp. $8.95. Sets, functions, elementary number theory, groups, rings, vector spaces, with emphasis on group theory. The discussion includes the Sylow theorems, the Jordan-Hölder theorem, field extensions, prime and maximal ideals, Boolean rings and Boolean algebra. T (15–16).

Problems in Group Theory. By John D. Dixon (Univ. of New South Wales). Blaisdell, Waltham, Mass., 1967. xv+176 pp. $7.50. A collection of "challenging and yet accessible" problems together with their solutions, mostly taken from research papers published since 1950 and accompanied by many references to original sources. Topics include sub-groups, permutation groups, automorphisms, and finitely generated abelian groups, normal series, commutators and derived series, solvable and nilpotent groups, the group ring and monomial representations, Frattini subgroup, factorization, linear groups, representations and characters. Technical terms are defined. S, P, L.

Linear Algebra. By W. H. Greub (Univ. of Toronto). Third Edition. *Grundlehren der mathematischen Wissenschaften*, Vol. 97. Springer-Verlag, New York, 1967. xii+434 pp. $9.80. The essential character of this edition remains the same as that of the previous ones, namely a development based on an axiomatic treatment of vector spaces. The major change is the separation of multilinear algebra to appear in a later volume, *Grundlehren der mathematischen Wissenschaften*, Vol. 136. Other changes include the elimination wherever possible of the restriction to finite dimensionality, the use of a coefficient field of characteristic zero in place of a restriction to real and complex numbers, and the addition of many problems. T (15–17), S, P, L.

Combinatorial Theory. By Marshall Hall, Jr. (Cal. Inst. of Tech.). Blaisdell, Waltham, Mass., 1967. x+310 pp. $9.50. In response to the need for a systematic treatment of the discreet arising from a recent revival of interest in combinatorics, the author has dealt with problems of enumeration, choice, and the existence and construction of designs. Topics include inversion, generating functions, partitions, Ramsey's theorem, extrema, convex spaces and linear programming, Debruijn sequences, block designs, difference sets, finite geometries, Latin squares, Hadamard matrices, block designs, and completion and embedding. There is a six page bibliography. T(16–17), S, P, L.

Lie Groups for Pedestrians. By Harry J. Lipkin (Weizmann Inst. of Sci., Rehovoth, Israel). Second edition. North-Holland, Amsterdam, (Distributed by Interscience—Wiley, New York), 1966. ix+182 pp. $6.50. This introductory treatment for the nonspecialist is given a very favourable review by I. Kelson in *Science* 12 May, 1967. S, P, L.

Elements of Abstract Algebra, second edition. By John T. Moore (Univ. of Florida and Univ. of Western Ontario). Macmillan, New York, 1967. xv+349 pp. $8.95. The first edition was reviewed by Carruth in this MONTHLY, 70 (1963) 345. This second

edition differs primarily in the inclusion of additional material, problems, and illustrative examples in order to provide greater flexibility. T(15–16).

★*Algebra: Volume I.* By L. Redei (Math. Inst., Univ. of Szeged). Intntl. Ser. Pure and Applied Math. 91. Pergamon, New York, 1967. xviii+823 pp. $21.50. This impressive tome is the first volume of a treatise that is intended to contain an inclusive selection of classical and recent results. It is designed as an encyclopedic reference work on modern algebra rather than as a text. Volume I covers set-theoretical preliminaries, structures, operator structures, divisibility in rings, finite Abelian groups, operator modules, commutative polynomial rings, fields, ordered structures, fields with valuation, Galois theory, and finite one-step non-commutative structures. The translation from Hungarian includes some changes and revisions. P, S, L.

Analysis

Theoretical Analysis. By Lester J. Heider (Marquette Univ.) and James E. Simpson (Univ. of Kentucky). Saunders, Philadelphia, 1967. xii+379 pp. $8.50. Based on lecture notes for a two-semester course at Marquette, following the recommendations of the Committee on the Undergraduate Program for an elementary course on real variable theory, this text emphasizes definitions, theorems, and proofs and is not intended to replace the traditional advanced calculus course with its emphasis on techniques and applications. Among the topics are Hausdorff and metric spaces, the integrals of Riemann and Stieltjes, complete metric spaces, the Lebesgue integral, Hilbert spaces, multiple integrals and some applications of classical analysis. There is a list of symbols. T(15–16).

Geometry of Polynomials. By Morris Marden (Univ. of Wisconsin, Milwaukee). American Mathematical Society, Providence, Rhode Island, 1966. xiii+243 pp. $12.70. A revised edition of *The Geometry of the Zeros of a Polynomial in a Complex Variable (1949)*. Changes include rewriting, expansion, corrections, the inclusion of new material, and the addition of over 300 new titles to the bibliography. P, L.

Partial Differential Equations. By I. G. Petrovskii (Moscow Univ.). Saunders, Philadelphia, 1967. vii+410 pp. $9.00. Since there already exists a fine translation by A. Shenitzer of the first edition of Petrovsky's *Lectures on Partial Differential Equations* (Interscience, New York, 1954), one might expect some explanation for this translation of the third edition of 1961. The content of the two editions appears to be practically the same except for a supplement of about fifty pages in the later one. On the other hand the current translation by Scripta Technica appears to be inferior, and the book itself is produced cheaply, apparently by photo off-set from a machine-typed manuscript. There is no dust-jacket nor translator's preface, and the author's prefaces are omitted. A number of figures that appear in the earlier translation are also missing.

Methods of Contour Integration. By M. L. Rasulov(Baku). Translated by Scripta Technica. North-Holland, Amsterdam and Wiley, New York, 1967. xiv+439 pp. $19.00. A treatise containing primarily the results of the author relating to the residue method (including Tamarkin's results) and the contour-integral method. S, P.

Systems of Singular Integral Equations. By N. P. Vekua (Tiflis). Translated by A. G. Gibbs and G. M. Simmons. Noordhoff, Groningen, 1967. 216 pp. $9.90 (cloth), $9.20 (paper). A systematic presentation of results obtained by the author for systems of equations with Cauchy-type kernels. P.

New Methods for Solving Elliptic Equations. By I. N. Vekua (Tiflis). Translated by D. E. Brown and edited by A. B. Taylor. Wiley, New York, 1967. xii+358 pp. $16. A

treatise, including results here published for the first time, on this special branch of the theory of functions of a complex variable. S, P.

Computers and Computing

BASIC, An Introduction to Computer Programming Using the BASIC Language. By William F. Sharpe (Univ. of Washington). Free Press, New York, 1967. xi+137 pp. $6.75 (cloth), $3.95 (paper). BASIC was developed at Dartmouth College under the direction of J. G. Kemeny. T, S, P, L.

★*Mathematics and Computing: with FORTRAN Programming.* By William S. Dorn (IBM) and Herbert J. Greenberg (Univ. of Denver). Wiley, New York, 1967. xvi+595 pp. $8.95. Designed for a one year course in mathematics that takes account of the electronic computer, this book requires only high school mathematics but introduces topics in calculus, finite mathematics, linear algebra, logic and probability. It might be used as a transition course between high school and university mathematics, as a terminal course at the college level, or for the training of high school and elementary teachers. There are many historical remarks and biographical notes—including several on living mathematicians. There is a table of currently operative computing systems. This book pioneers in content, approach, and pedagogy, yet appears teachable. T(13), TT, S, L.

Advances in Computers. Edited by Franz L. Alt (National Bureau of Standards) and Morris Rubinoff (Univ. of Pennsylvania). Volume 8. Academic Press, New York, 1967. xii+345 pp. $14.50. Topics in this volume are time-shared computer systems, formula manipulation by computer, standards for computers and information processing, syntactic analysis of natural language, programming languages and computers, and incremental computation. A table of contents of the previous seven volumes is included. P, L.

Practical Five-Figure Mathematical Tables. By C. Attwood. St. Martin's Press, New York, 1967. viii+88 pp. $6.75. An unusually well-printed set of tables with mean proportional parts. This second edition includes several improvements. S.

Numerical Integration. By Philip J. Davis (Brown Univ.) and Philip Rabinowitz (Weizmann Institute). Blaisdell, Waltham, Mass., 1967. ix+230 pp. $7.50. One of a series intended for use in both classroom and computation laboratory. Topics include approximate integration over finite and infinite intervals, error analysis, approximate integration in two or more dimensions, automatic integration, and appendices, including a reprint of an article by Abramowitz, and bibliographies of FORTRAN and ALGOL Procedures, of tables, of books and articles. T(15–16), S, L.

Introductory Numerical Analysis. By Anthony J. Pettofrezzo (Florida State Univ.). Heath, Boston, 1967. 194 pp. $6.95. "... primarily for the undergraduate mathematics major, engineering student, or future high school mathematics teacher who needs some understanding of the underlying principles involved in numerical analysis." (A determinant is defined as a "square array of n^2 elements ..."). T(14).

Dictionary of Computer and Control Systems Abbreviations, Signs and Symbols. Edited by David D. Polon. Odyssey, New York, 1965. xvii+350 pp. $18.00. Abbreviations for text and drawings, signs, color codes, etc. for all areas of computer sciences and related fields. L.

★*Computation: Finite and Infinite Machines.* By Marvin L. Minsky (Electrical Engineering, M.I.T.). Prentice-Hall, Englewood Cliffs, N. J., 1967. xvii+317 pp. $12.00. Intended as an elementary but broad introduction to the field for mathematicians

and other professionals, this book assumes only a little algebra and develops its own necessary mathematics. Its three major divisions are entitled Finite-State Machines, Infinite Machines, and Symbol-Manipulation Systems and Computability. Two central ideas are "effective procedure" and "universal machine." There is a selected cross reference coded bibliography and a glossary-index. The style is thoughtful, exciting, and enjoyable. T(14–16), S, P, L.

Difference Methods for Initial-Value Problems. Second Edition. By Robert D. Richtmyer (Univ. of Colorado) and K. W. Morton (Culham Lab., Abingdon, England). Interscience-Wiley, New York, 1967. xiv+405 pp. $14.95. The first edition (1957) was favorably reviewed by R. B. Davis in Mathematical Reviews 20, No. 438. In this revision the authors have not attempted to cover all the new theoretical results of the last decade, but have continued to direct the book to users rather than theoretical numerical analysts. They comment that in spite of numerous theoretical results "most practical problems in physical science are too complex . . . to be yet covered by the theorems that have been published. . . It is just as necessary to utilize physical intuition, heuristic reasoning, and trial-and-error procedures as it was when von Neumann started the subject, nearly twenty-five years ago." Accordingly the authors have selected only a few theoretical advances for detailed description. T, P.

The Calculus of Observations, An Introduction to Numerical Analysis. By Sir Edmund Whittaker and G. Robinson. Fourth Edition. Dover, New York, 1967. xiv+397 pp. $2.75 (paper). Reprint of the fourth edition (1944) of the work originally published in 1924 under the title *The Calculus of Observations: A Treatise on Numerical Mathematics.* S, P, L.

Geometry

Modern Geometry, 2nd Edition. By Claire Fisher Adler (C. W. Post College of Long Island University). McGraw-Hill, New York, 1967. xi+302 pp. $8.75. This is an extensive revision of a book first published in 1958. Its three parts are entitled Foundations and an Introduction to Non-Euclidean Geometry; Pure, Nonmetric Projective Geometry; Algebraic Projective Geometry and Linear Algebra. The most important changes are an increased emphasis on logical rigor in the first part, the insertation of additional proofs in the second part, and a presentation using vectors and matrices in the third part so as to make it "a geometric approach to linear algebra and an algebraic approach to projective geometry." The book is intended both for future teachers and to round out geometric background of future mathematicians. T(13–14), TT, S.

Studies in Global Geometry and Analysis. Edited by S. S. Chern (Univ. of Cal., Berkeley). Studies in Mathematics 4. Mathematical Association of America, Distributed by Prentice-Hall, Englewood Cliffs, N. J., 1967. 197 pp. $6.00 ($3.00 for a single copy to members of the MAA). The latest in a series that is already justly known for its high quality. After the editor's introduction that locates the book historically and mathematically, there are six articles: What is analysis in the large? by Marston Morse, Curves and surfaces in Euclidean space by S. S. Chern, Differential forms by Harley Flanders, On conjugate and cut loci by Shoshichi Kobayashi, Surface Area by Lamberto Cesari, and Integral Geometry by L. A. Santalo. Like its predecessors, this volume will be extraordinarily useful as supplementary reading by students and faculty alike. S, P, L.

Linear Geometry. By K. W. Gruenberg (Univ. of London) and A. J. Weir (Univ. of Sussex). Van Nostrand, Princeton, N. J., 1967. viii+186 pp. $6.00. This book "aims to explain in suitably rigorous language the intuitively familiar concepts of euclidean,

affine and projective geometries and to study the relations between them. The method of exposition is purely algebraic. . . . " A first course in linear algebra is assumed. T(15–17), S, T, L.

Méthodes d'Algèbre Abstraite en Géométrie Algébrique. Ergebnisse der Mathematik und ihrer Grenzgebiete, 4. By Pierre Samuel. Springer-Verlag, New York, 1967. Second corrected edition. xii+133 pp. $6.50. This exposition of the "old" geometry of Weil-Zariski (old in the sense of being pre-Grothendieck) includes a review of algebraic definitions underlying the theory, a historical annex, and a table of comparative terminology in seven treatises. S, P, L.

Vorlesungen über differential Geometrie, und geometrische Grundlagen von Einsteins, Relativitätstheorie. By Wilhelm Blaschke. Chelsea, New York, 1967. Vol. I, x+311 pp. Vol. II, vii+259 pp., bound together. $10.95. A texturally unaltered reprint of the third edition of Vol. I on elementary differential geometry (1930) and Vol. II on affine differential geometry (1923). A third volume on the differential geometry of the circle and the sphere is not included here. S, P, L.

Logic and Foundations

Remarks on the Foundations of Mathematics. By Ludwig Wittgenstein. Edited by G. H. von Wright, R. Rhees and G. E. M. Anscombe. Translated by G. E. M. Anscombe. M.I.T. Press, Cambridge, Mass., 1967. xix+204 pp. $3.45 (paper). A reprint of the original translation published in 1956, this edition also gives German and English on facing pages. S, P.

Mathematical Logic. By Stephen Cole Kleene (Univ. of Wisconsin). Wiley, New York, 1967. xiii+398 pp. $10.95. Suitable for a rugged undergraduate or solid beginning graduate course in logic, this book consists of a first part that covers the essentials of propositional calculus, predicate calculus, and predicate calculus with an equality, followed by a second supplementary part on mathematical logic and the foundations of mathematics, dealing with computability, decidability, and additional topics on the predicate calculus. There is a good bibiliography. It is good that this important contributor to modern foundations has taken time to write an undergraduate text of this quality. T(15–17), S, P, L.

★*Problems in the Philosophy of Mathematics.* Edited by Imre Lakatos (Univ. of London). Proceedings of the International Colloquium in the Philosophy of Science, London, 1965, Volume I. North Holland, Amsterdam, 1967. xv+241 pp. $10.00. Nine papers, each followed by several discussions. Although based on the original presentations, the articles and discussions have been revised extensively. The papers are: Greek Dialectic and Euclid's Axiomatics by A. Szabo, The Metaphysics of the Calculus by A. Robinson, On a Fregean Dogma by F. Sommers, Recent Results in Set Theory by A. Mostowski, What Do Some Recent Results in Set Theory Suggest? by P. Bernays, On the Relevance of Post-Gödelian Mathematics to Philosophy by S. Korner, Informal Rigor and Completeness Proofs by G. Kreisel, Foundations of Mathematics: Whither Now? by L. Kalmar, and Logic and Heuristic in Mathematics Curriculum Reform by J. A. Easley, Jr. This fascinating volume should be widely read and placed in every mathematics library. The second and third volumes will be entitled The Problem of Inductive Logic, and Problems in the Philosophy of Science. S (History and Foundations), P, L.

Set Theory for the Mathematician. By Jean E. Rubin (Michigan State Univ.). Holden-Day, San Francisco, 1967. xi+387 pp. $10.75. Beginning with an introductory chapter including a brief, but good history, the book develops the von Neumann-

Bernays-Gödel set theory with the modifications due to A. T. Morse. This is a high quality book for serious students and for reference. It reaches to such topics as the generalized continuum hypothesis and the axiom of choice, but it can be adapted to courses as short as a single quarter. T(16–17), S, P, L.

Mathematical Logic. By Joseph R. Shoenfield (Duke Univ.). Addison-Wesley, Reading, Mass., 1967. vii+344 pp. $12.75. The author describes his intention as the collection of "the principal results in what seems to me the central topic of mathematical logic: proof theory, model theory, recursion theory, axiomatic number theory, and set theory". He has intentionally omitted "many interesting side topics" and has increased the content by exercises whose proofs often require considerable extension of the method of the text. Though intended for a first year graduate class, nothing more advanced than the simplest concepts of numbers and modern algebra are assumed. T(17), S.

Probability and Statistics

Handbook of Methods of Applied Statistics. Volume I. Techniques of Computation, Descriptive Methods, and Statistical Inference. Volume II. Planning of Surveys and Experiments. By I. M. Chakravarti (Univ. of North Carolina), R. G. Laha (Catholic Univ.), J. Roy (Indian Statistical Inst.). Wiley, New York, 1967. I. xiv +460 pp. $12.95. II. x+160 pp. $9.00. Each statistical method is described in detail with an explanation of its theoretical foundation but without proof. Computational procedures are discussed and illustrated. There are many examples and graded exercises for solution by the reader. The work is intended as a textbook in methods and as a reference for practicing statisticians and research workers in other fields. The lack of answers to the exercises will be regretted by nonstudent users. S, P, L.

Modern Factor Analysis. Second Edition, Revised. By Harry H. Harman (Educational Testing Service). University of Chicago, Chicago, 1967. xx+474 pp. $12.50. The first edition of this book (1960) was reviewed in Mathematical Reviews 28, #2610. Although the organization and length of the text remains unchanged, it has been totally revised on the basis of suggestions, criticisms, new results, and techniques based on advances in computing technology which have made some classical methods obsolete. Both the exercises and the substantial bibliography have been updated. T(16), S, P, L.

A Brief Introduction to Probability Theory. By John P. Hoyt (Indiana Univ. of Penn.). International Textbook, Scranton, Penn., 1967. viii+151 pp. $3.95 (paper). Originally written for a forty-five meeting course at the United States Naval Academy, this book presupposes elementary calculus. It stresses random variables, uses generating functions, and climaxes with a central limit theorem for sums of identically distributed independent random variables. The last two chapters are on estimation and hypothesis testing. T(14).

Elements of Nonparametric Statistics. By Gottfried E. Noether (Boston Univ.). Wiley, New York, 1967. x+104 pp. $7.95. Slanted toward statisticians "who have learned about nonparametric statistics from a user's manual and would like to penetrate more deeply", this book assumes a first course in statistics. S, P.

Stationary Random Processes. By Yu. A. Rozanov. Holden-Day, San Francisco, 1967. 211 pp. $10.95. The translation includes some improved results communicated by the author to the translator and the correction of minor errors. The four chapters are entitled harmonic analysis of stationary random processes, linear forecasting of continuous-parameter stationary processes, and random processes stationary in the strict sense. S, P.

Prediction Analysis. By John R. Wolberg (Israel Institute of Tech.). Van Nostrand, Princeton, N. J., 1967. xi+304 pp. $10.75. The author defines prediction analysis as the theory of predicting the standard deviations of results of experiments and he works in the context of least squares method of data analysis. S, P.

Topology

Homology of Cell Complexes. By George E. Cooke and Ross L. Finney (Univ. of Illinois). Mathematical Notes, Princeton University Press, 1967. vii+256 pp. $3.75 (paper). Based on introductory lectures in algebraic topology by Norman Steenrod in the Fall of 1963. The emphasis is on computing homology groups of complexes. T(17–18), S, P.

Elementary Topology. By Michael C. Gemignani (SUNY at Buffalo). Addison-Wesley, Reading, Mass., 1967. xi+258 pp. $9.75. The author has written for students with as little as three semesters of calculus and analytic geometry, but he recommends a course in real analysis to make "a deeper appreciation" possible. Topics are related to their historical origin in geometry and analysis. Metric spaces, topologies, derived topological spaces, continuity, the separation axioms, convergence, covering properties, compactness, connectedness, metrizability, complete metric spaces, homotopy theory. T(16–17).

A Seminar on Graph Theory. Edited by Frank Harary (Univ. of Michigan) and Lowell Beineke. Holt, Rinehart and Winston, New York, 1967. vii+116 pp. $5.95. This collection of fourteen papers, six by Harary and the rest by Beineke, Erdös, Paul Kelly, J. W. Moon, Nash-Williams, Rado, and C. A. B. Smith, and dedicated to George Polya, reports on a seminar at University College, London, during 1962–1963, which "was a good year for graph theory at U. C. L." The editor describes it as an "aperitif" and hopes it will encourage study of more systematic treatises. The level varies from elementary introductory to the presentation of recent results. S, P.

Fibre Bundles. By Dale Husemoller (Haverford College). McGraw-Hill, New York, 1966. xiv+300 pp. $14.50. This exposition of a subject that dates from the 1930's but got started in earnest only in the 1950's, includes quite recent material and is designed for a course presupposing general topology, basic homotopy and CW-complex theory (for parts I and II) and cohomology theory (for part III). There are exercises and a bibliography. T(17–18), S, P, L.

Topology: An Introduction with Application to Topological Groups. By George McCarty (Univ. of California, Irvine). McGraw-Hill, New York, 1967. xiii+270 pp. $8.95. An introduction to topology as a part of geometry. Topics include an introduction to sets, functions, groups, and metric spaces, topologies and topological groups, compactness and connectiveness, function spaces, the fundamental group, the fundamental group of the circle, the fundamental theorem of algebra, and locally isomorphic groups. T(after calculus and linear algebra).

Ordered Topological Vector Spaces. By Anthony L. Peressini (Univ. of Illinois). Harper and Row, New York, 1967. x+228 pp. $10.25. An ordered topological vector space is a topological vector space with an order that is compatible with its linear structure, i.e., a real vector space with a transitive, reflexive, anti-symmetric relation that is preserved under vector addition and multiplication by scalars. This book evolved from mimeographed lecture notes used in a seminar with prerequisite a course in functional analysis. It is not encyclopedic but moves slowly with many examples. There is a twelve page bibliography and a table of symbols. T(17–18), S, P, L.

Topological Vector Spaces, Distributions, and Kernels. By Francois Treves (Purdue Univ.). Academic Press, New York, 1967. xvi+565 pp. $18.50. " . . . not a treatise on functional analysis . . . lecture notes aimed at acquainting the graduate student with that section of functional analysis which reaches beyond the boundaries of Hilbert spaces and Banach spaces . . . " T(17–18), P.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Rufus Oldenburger, Director of the Automatic Control Center, School of Mechanical Engineering, Purdue University, has received the 1967 Donald P. Eckman Education Award from the Instrument Society of America.

Arkansas College: Mr. M. C. Kester, Tyler Junior College, has been appointed Assistant Professor; Associate Professor W. D. Moon has been promoted to Professor.

Brock University: Associate Professor H. E. Bell, Union College, has been appointed Associate Professor; Dr. E. R. Muller, University of Sheffield, has been appointed Assistant Professor; Assistant Professor A. S. B. Holland has been appointed Associate Professor and Assistant Dean of Arts and Science at the University of Calgary.

University of California, Irvine: Visiting Assistant Professor R. W. West, University of California, Los Angeles, has been appointed Assistant Professor; Assistant Professor R. J. Whitley, University of Maryland, has been appointed Associate Professor.

Carnegie-Mellon University: Assistant Professor S. P. Franklin has been promoted to Associate Professor; Associate Professors V. J. Mizel and R. A. Moore have been promoted to Professors; Associate Professor T. I. Seidman, Wayne State University, has been appointed Visiting Associate Professor; Assistant Professor H. L. Shapiro, Pennsylvania State University, has been appointed Visiting Assistant Professor.

Clemson University: Assistant Professors A. K. Bose, University of Alabama, and D. R. LaTorre, University of Tennessee, have been appointed Assistant Professors; Dr. J. D. Fulton, Oak Ridge National Laboratory, has been appointed Assistant Professor.

Cleveland State University: Assistant Professor R. L. Pruitt, Capital University, has been appointed Assistant Professor; Dr. C. C. Buck, Educational Research Council of Greater Cleveland, has been appointed Associate Professor; Associate Professor J. M. Egar, University of Akron, has been appointed Associate Professor; Associate Professor S. M. Robinson, Union College, has been appointed Adjunct Professor; Mr. J. N. Hanson has been promoted to Assistant Professor.

Colorado State College: Associate Professor W. D. Popejoy has been promoted to Professor; Assistant Professor Norman Eggert has been appointed Assistant Professor at Montana State University.

Colorado State University: Drs. R. E. Gaines, University of Colorado, D. W. Hardy, New Mexico State University, and K. F. Klopfenstein, Wabash College, have been appointed Assistant Professors; Assistant Professor P. W. Mielke, Jr. has been promoted

to Associate Professor; Professor K. G. Medearis has been appointed Professor of Civil Engineering at Colorado State University; Mr. V. A. Nelson has been appointed Assistant Professor at South Dakota School of Mines and Technology.

Duke University: Associate Professor T. M. Gallie has been promoted to Professor; Assistant Professor L. A. Hinrichs has been appointed Associate Professor at the University of Victoria.

East Carolina University: Drs. K. G. Johnson, Virginia Polytechnic Institute, and W. R. Spickerman, Spindle Top Research Foundation, have been appointed Associate Professors; Assistant Professors D. F. Bailey and Katye O. Sowell have been promoted to Associate Professors.

Florida Presbyterian College: Dr. G. W. Lofquist, Louisiana State University, has been appointed Assistant Professor; Assistant Professor L. D. Graber has been promoted to Associate Professor.

Georgia State College: Assistant Professor F. A. Massey, Auburn University, has been appointed Assistant Professor; Assistant Professors K. E. Whipple and W. W. Leonard have been promoted to Associate Professors.

University of Hawaii: Professor N. H. McCoy, Smith College, has been appointed Visiting Professor; Drs. D. L. Outcalt, University of California, Santa Barbara, and R. I. Sandler, University of Illinois, have been appointed Visiting Associate Professors; Mr. W. T. Stout, University of Illinois, has been appointed Assistant Professor; Mr. J. K. M. Siu has been promoted to Assistant Professor; Assistant Professor Z. Z. Yeh has been promoted to Associate Professor.

Indiana University: Assistant Professors Morton Lowengrub, Wesleyan University, and D. M. Topping, University of Washington, have been appointed Associate Professors; Dr. A. W. Schurle, University of Kansas, has been appointed Assistant Professor; Assistant Professor P. A. Fillmore has been promoted to Associate Professor.

University of Iowa: Assistant Professor W. A. Kirk, University of California, Riverside, has been appointed Associate Professor; Assistant Professor Marilyn J. Zweng has been promoted to Associate Professor; Professor R. H. Oehmke has been appointed Chairman of the Mathematics Department.

Jamestown College: Assistant Professor Elaine Cook has been promoted to Associate Professor; Assistant Professor L. R. Tanner has been promoted to Associate Professor and Chairman of the Mathematics Department.

Kansas State College: Assistant Professor H. L. Thomas has been promoted to Associate Professor; Associate Professor D. W. Hight has been promoted to Professor; Associate Professor Helen K. Kriegsman has been promoted to Professor and Chairman of the Mathematics Department.

Lock Haven State College: Assistant Professor R. E. Whitney has been promoted to Associate Professor; Associate Professor R. L. Duncan has been appointed Associate Professor at Pennsylvania State University.

Marquette University: Associate Professor C. C. Braunschweiger, University of Delaware, has been appointed Professor; Associate Professor C. B. Hanneken has been promoted to Professor.

Massachusetts Institute of Technology: Professor G. C. Rota, Rockefeller University, has been appointed Professor; Dr. T. J. Lardner has been promoted to Assistant Professor; Associate Professor F. B. Hildebrand has been promoted to Professor.

University of Minnesota-Institute of Technology: Assistant Professors J. I. Richards and G. R. Sell have been promoted to Associate Professors; Associate Professor C. A. McCarthy has been promoted to Professor.

University of Missouri: Dr. Walter Leighton, Western Reserve University, has been appointed Professor; Drs. R. B. Grafton, Brown University, J. A. Huckaba, University of Iowa, A. M. Kirch, University of Minnesota, and D. C. Taylor, University of Ken-

tucky, have been appointed Assistant Professors; Assistant Professor D. J. Rodabaugh has been promoted to Associate Professor.

University of Montana: Assistant Professors D. O. Loftsgaarden, Western Michigan University, and I. K. Yale, Morehouse University, have been appointed Assistant Professors.

University of New Hampshire: Professor Emil Grosswald, University of Pennsylvania, has been appointed Professor; Associate Professor S. L. Ross has been promoted to Professor.

North Carolina State University: Dr. L. K. McDowell, University of Illinois, has been appointed Assistant Professor; Assistant Professor R. E. Chandler has been promoted to Associate Professor; Associate Professor P. A. Nickel has been promoted to Professor.

Northeast Louisiana State College: Mr. George Berzsenyi, Texas Christian University, has been appointed Assistant Professor; Associate Professor R. A. Hickman has been promoted to Professor; Associate Professor D. R. Bedgood has been appointed Chairman of the Mathematics Department at East Texas State University.

Northwestern State College: Mr. Stanley Van Steenvoort and Mr. T. E. Ikard have been promoted to Assistant Professors.

Ripon College: Mr. J. T. Teska, University of Oklahoma, has been appointed Assistant Professor; Mr. N. E. Aiken has been promoted to Assistant Professor.

St. Lawrence University: Mr. J. F. Burke, University of Vermont, has been appointed Assistant Professor; Associate Professor R. G. Van Meter, Florida Presbyterian College, has been appointed Associate Professor; Professor F. D. Parker has been appointed Chairman of the Mathematics Department.

Seton Hall University: Mr. J. J. Saccoman has been promoted to Assistant Professor; Assistant Professor C. H. Franke has been promoted to Associate Professor.

State University College of New York at Geneseo: Professor P. T. Schaefer, SUNY at Albany, has been appointed Professor; Mr. D. D. Eckert has been promoted to Assistant Professor.

State University College of New York at Oswego: Assistant Professor R. W. Deming, Idaho State University, has been appointed Associate Professor; Assistant Professor P. L. Dussere, Idaho State University, has been appointed Assistant Professor; Associate Professor R. D. Mayer, Idaho State University, has been appointed Professor and Chairman of the Mathematics Department.

Stephen F. Austin State College: Assistant Professor R. G. Dean, University of Texas at Arlington State, has been appointed Associate Professor; Assistant Professor F. D. Alexander has been promoted to Professor.

University of Tulsa: Associate Professor T. W. Cairns has been appointed Head of the Department of Mathematics; Professor W. A. Rutledge has been appointed Head of the Department of Mathematics at Old Dominion College.

University of Utah: Assistant Professors R. E. Barnhill and J. O. Sather have been promoted to Associate Professors; Associate Professor D. H. Tucker has been promoted to Professor.

Wellesley College: Assistant Professor Torsten Norvig has been promoted to Associate Professor; Dr. Vivian Y. Kraines has been appointed Assistant Professor at Drexel Institute of Technology.

West Chester State College: Mr. T. J. Ahlborn, University of Rochester, and Mr. J. F. Faulkner, University of Wisconsin, have been appointed Assistant Professors; Professor A. E. Filano, Chairman of the Mathematics Department, is also Director of the Division of Science and Mathematics.

University of Western Ontario: Dr. T. M. Viswanathan, Queen's University, has been appointed Assistant Professor; Associate Professor W. E. Bonnice, University of New Hampshire, has been appointed Associate Professor; Senior Professor David Borwein has been promoted to Professor and Head of the Mathematics Department.

Western State College: Mr. R. L. Harris, University of New Mexico, has been appointed Assistant Professor; Assistant Professor C. O. Mallory, St. Cloud State College, has been appointed Assistant Professor.

Western Washington State College: Drs. Sara J. Kelley, Washington State University, and R. G. Levin, University of California, Davis, have been appointed Assistant Professors; Mr. N. F. Lindquist, Oregon State University, has been appointed Assistant Professor; Associate Professor W. R. Abel has been promoted to Professor.

Wichita State University: Assistant Professor B. A. Johns has been promoted to Associate Professor; Professor W. M. Perel has been appointed Head of the Mathematics Department.

University of Windsor: Associate Professor Nathan Shklov, University of Saskatchewan, has been appointed Professor; Assistant Professor K. A. Zischka has been promoted to Associate Professor; Assistant Professor S. P. Singh has been appointed Associate Professor at the Memorial University of Newfoundland.

Winthrop College: Associate Professor B. G. Hodges has been promoted to Professor; Assistant Professor G. B. Lampton, Jr., has been promoted to Associate Professor.

University of Wisconsin-Milwaukee: Drs. R. L. Hall, Columbia University, K. M. Kapp, University of Wisconsin, Madison, and E. A. Schwandt, University of Minnesota, have been appointed Assistant Professors; Mr. E. G. Schuld has been promoted to Assistant Professor; Assistant Professor G. G. Walter has been promoted to Associate Professor.

Mr. Jacques Allard, University of Sherbrooke, has been promoted from professeur adjoint to professeur agrégé.

Assistant Professor D. W. Bailey, Amherst College, has been promoted to Associate Professor.

Mr. T. D. Boyll, Treasure Valley Community College, has been appointed Head of the Mathematics Department.

Miss Nancy J. Capozzolo, Millersville State College, has been promoted to Assistant Professor.

Assistant Professor R. L. Cooley, Wabash College, has been promoted to Associate Professor.

Mr. G. S. Cunningham, Greater Cleveland Research Council, has been appointed Associate Professor at the University of Maine.

Mr. J. C. Davis, Mesa College, has been appointed Chairman of the Division of Mathematics and Engineering.

Sister Diane Drufenbock, Alverno College, has been promoted from Assistant Professor to Associate Professor.

Dr. F. H. Eng, Douglas Aircraft Co., has been appointed Professor at Nicholls State College.

Mr. G. J. Etgen, NASA, has been appointed Assistant Professor at the University of Houston.

Mr. Stanley Fasbinder, Mercy College of Detroit, has been promoted to Assistant Professor.

Dr. Norma M. Gilbert, Drew University, has been promoted to Assistant Professor.

Assistant Professor M. D. Gutzmer, Northwest Missouri State College, has been appointed Acting Chairman of the Mathematics Department.

Associate Professor J. C. Hickman, University of Iowa, has been promoted to Professor.

Professor W. E. Hoff, Northeast Missouri State Teachers College, has been appointed Professor at Concord College.

Assistant Professor Brindell Horelick, Lafayette College, has been appointed Associate Professor at SUNY College at Cortland.

Mr. L. C. House, University of Connecticut, has been appointed Assistant Professor at Canisius College.

Assistant Professor W. H. Jamison, Rocky Mountain College, has been promoted to Associate Professor and appointed Acting Chairman of the Division of Natural Sciences and Mathematics.

Assistant Professor Rosalie S. Jensen, Shorter College, has been appointed Chairman of the Mathematics Department.

Professor H. B. Keller, Courant Institute of Mathematical Sciences, has been appointed Professor at the California Institute of Technology.

Associate Professor M. R. Kenner, Southern Illinois University, has been appointed Chairman of the Mathematics Department at Stephens College.

Assistant Professor B. L. Lercher, SUNY at Binghamton, has been promoted to Associate Professor.

Associate Professor F. M. Lister, Western Washington State College, has been appointed Professor at Southern Oregon College.

Sister Marie Genevieve Love, Mount Saint Mary College, has been promoted from Assistant Professor to Associate Professor.

Dr. J. P. Mayberry, of Mathematica, Princeton, has been promoted from Senior Mathematician to Director of Mathematical Research Services.

Associate Professor R. A. McCartney, Suffolk County Community College, has been promoted to Professor.

Assistant Professor R. G. McDermot, Duquesne University, has been promoted to Associate Professor.

Assistant Professor B. B. Peterson, Middlebury College, has been promoted to Associate Professor and appointed Dean of Men.

Assistant Professor G. W. Polites, Madison College, has been appointed Associate Professor at Illinois Wesleyan University.

Professor O. W. Rechard, University of Wyoming, has been appointed Professor at Fort Lewis College.

Assistant Professor G. M. Rosenstein, Jr., Case-Western Reserve University, has been appointed Assistant Professor at Franklin and Marshall College.

Assistant Professor R. W. Roth, Malone College, has been appointed Director of the Computer Center at Taylor University.

Assistant Professor R. P. Savage, Tennessee Technological University, has been promoted to Associate Professor.

Assistant Professor C. J. Searcy, Central State College, has been appointed Assistant Professor at Highlands University.

Mr. E. V. Sherman, formerly lieutenant, USAF, has been appointed Assistant Professor at Morningside College.

Dr. M. A. Snyder, Courant Institute, has been appointed Assistant Professor at Bryn Mawr College.

Professor A. H. Sprague, Amherst College, has been appointed Visiting Professor at Hollins College.

Assistant Professor J. H. Stoddard, University of the South, has been appointed Associate Professor at Kenyon College.

Assistant Professor J. B. Stroud, Davidson College, has been promoted to Associate Professor.

Professor R. F. Tidd, North Dakota State University, has been appointed Chairman of the Mathematics Department.

Assistant Professor B. E. Trumbo, California State College at Hayward, has been promoted to Associate Professor.

Dr. R. A. Vandervelde, University of Iowa, has been appointed Assistant Professor at Hope College.

Dr. T. L. Vickrey, Oklahoma State University, has been appointed Assistant Professor at Central Missouri State College.

Professor G. L. Weiss, Washington University, has been appointed Chairman of the Mathematics Department.

Assistant Professor E. L. Wilson, Wittenberg University, has been appointed Visiting Assistant Professor at the University of the South.

Mrs. Emilie H. Wohlenberg, Seattle Pacific College, has been promoted to Assistant Professor.

Professor Emeritus J. W. Bradshaw, University of Michigan, died on June 10, 1967. He was a Charter Member of the Association.

Professor D. F. Coulter, Hartnell College, died on July 2, 1967. He was a member of the Association for sixteen years.

Professor Emeritus T. R. Hollcroft, Wells College, died on September 1, 1967. He was a member of the Association for forty-six years.

Miss Sydney Myers, Central Bucks High School, died on February 20, 1967. She was a member of the Association for eight years.

Professor D. L. Robb, Baldwin Wallace College, died on September 11, 1967. He was a member of the Association for seventeen years.

Professor C. P. Sousley, Rose Polytechnic Institute, died on August 21, 1967. He was a Charter Member of the Association.

Professor Emeritus Mary C. Suffa, Elmira College, died in May, 1967. She was a Charter Member of the Association.

Professor J. S. Taylor, Technical University of Santa Maria, Valparaiso, died on August 15, 1967. He was a member of the Association for forty-seven years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE LESTER R. FORD FUND

To honor the late Professor Lester R. Ford, the Association has established a fund to be known as the *Lester R. Ford Fund* to replace the fund previously carried in the Treasurer's records as the *Fund Established by an Anonymous Donor.*

The capital of this fund consists of contributions made by Professor Ford during his lifetime which have been listed anonymously at his request, a bequest in his will, and by gifts made in his memory by his family and friends. It is expected that the principal of the fund will be about $25,000.

Former colleagues and students of Professor Ford may wish to contribute to the fund. Payments should be sent to the MAA Buffalo office marked for the *Lester R. Ford Fund.* Since the Association is a non-profit organization, all such gifts are deductible for income tax purposes.

Income from the Ford Fund will be used for payment of the Ford Awards to authors of expository articles published in the MONTHLY and MATHEMATICS MAGAZINE (see this MONTHLY, Aug.–Sept. 1967, p. 908), and for such publication and other purposes as may be voted by the Board of Governors.

CALENDAR OF FUTURE MEETINGS

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

Fifty-second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Indiana University of Pennsylvania, Indiana, April 27, 1968.

FLORIDA, Miami-Dade Junior College, South Campus, March 22–23, 1968.

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA, Ball State University, Muncie, May 4, 1968.

IOWA, Wartburg College, Waverly, April 19, 1968.

KANSAS, Marymount College, Salina, March 23, 1968.

KENTUCKY, University of Kentucky, Lexington, April 27, 1968.

LOUISIANA-MISSISSIPPI, Broadwater Beach Hotel, Biloxi, Mississippi, February 16–17, 1968.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Staten Island Community College, Staten Island, March 16, 1968.

MICHIGAN, Grand Valley State College, Allendale, March 23, 1968.

MINNESOTA, College of St. Teresa, Winona, May 4, 1968.

MISSOURI, Lindenwood College, St. Charles, April 27, 1968.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY, Rider College, Trenton, May 4, 1968.

NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.

NORTHERN CALIFORNIA

OHIO, Miami University, Oxford, April 26–27, 1968.

OKLAHOMA-ARKANSAS, Federal Aviation Agency, Oklahoma City, March 29–30, 1968.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.

ROCKY MOUNTAIN, University of Denver, Colorado, May 10–11, 1968.

SOUTHEASTERN, East Carolina University, Greenville, North Carolina, March 29–30, 1968.

SOUTHERN CALIFORNIA, Loyola University of Los Angeles, Los Angeles, March 9, 1968.

SOUTHWESTERN, New Mexico State University, University Park, April 12–13, 1968.

TEXAS, Texas A and M University, College Station, April 19–20, 1968.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 11, 1968.

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, University of Wisconsin, Madison, Aug. 27–30, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, Chicago, August 20–22, 1968.

ASSOCIATION FOR SYMBOLIC LOGIC, University of California, Los Angeles, March 22, 1968.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS, University of Wisconsin, Madison, August 1968.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Hall, Philadelphia, April 17–20, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, St. Francis Hotel, San Francisco, May 1–3, 1968.

PI MU EPSILON, University of Wisconsin, Madison, August 27–28, 1968.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, King Edward Sheraton Hotel, Toronto, Canada, June 11–14, 1968. (Symposium on Optimization.)

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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CONTENTS

FEBRUARY 1968

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. A. ROSENBAUM, Wesleyan University, Middletown, Conn. 06457.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with det $[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $8.00. For nonmembers the subscription price is $15.00.

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THE MATHEMATICS OF PHYSICAL QUANTITIES

PART I: MATHEMATICAL MODELS FOR MEASUREMENT

HASSLER WHITNEY, Institute for Advanced Study

INTRODUCTION

1. Purpose of this paper. To set up a physical theory, one constructs a mathematical model, and considers its relation to certain aspects of the physical world. There is a variety of models associated with the concept of measurement. Certain systems of numbers are important; for instance:

N, the natural numbers;	R^+, the positive reals;
Q^+, the positive rationals;	R, the reals.

Commonly one takes R^+ or R as a model for measurement. There are disadvantages in this, however. These models contain a specific number 1, and there is no natural way of putting this number in correspondence with a particular measurement; moreover, the models contain an operation of multiplication, with no natural physical counterpart.

Let us consider the problem of choosing a model M for masses. An object A has a certain property which we call its "mass"; why not let this property itself be an element of the model? As far as the structure of the model is concerned, we need not theorize on what "mass" really is; we need merely give it certain properties in the model. For instance, if we have distinct objects A and B, with masses m_A and m_B, we may think of the objects as forming a single object C; its mass m_C should then be m_A+m_B. Therefore M should contain an operation of addition, and any further properties we choose.

The two types of models that best fit in most situations we shall call "rays" and "birays." A ray (like a half line) is used for positive measurements, and a biray (like an oriented line with starting point), for measurements of quantities both positive and negative. It turns out that numbers appear in a natural way as operators on the model (see the next section).

In this paper we set up the theory of rays and birays; the real number system is constructed along with the models in a natural manner. In fact, this approach gives a simple and elegant way of introducing the reals and finding their basic properties.

We shall not give applications of the models in the body of the paper; some remarks on the subject will be made in this introduction. In this connection, see Part II, which will appear in the next issue of this MONTHLY.

2. Numbers as operators on the models. If we choose a stick of length l, and wish to use it to measure another stick, we lay out the first stick along the second several times; we thus form $l+l$, $l+l+l$, and so on. We also call these lengths $2l$, $3l$, $\cdots$. Thus N appears as a natural set of operators on our model. For masses, there is a different physical process of addition; but again we may use $2m=m+m$ and so on, with the same set N of numbers.

If we have a stick, whose length we call l, and we find a shorter stick, of length l', such that $3l'=l$, then we wish to give an expression of l' in terms of l. It is natural to set $l'=(1/3)l$. We may now set $2l'=(2/3)l$, and thus introduce Q^+ as operators. Finally, if our model has a certain completeness property, we may enlarge Q^+ to R^+ as operator system, and if we have negative quantities, we may enlarge R^+ to R.

3. Working in the model. Various properties of a model and its operations have obvious meaning in the applications. For instance we have distributive and associative laws:

$$5 \text{ cakes} + 2 \text{ cakes} = (5+2)\text{cakes} = 7 \text{ cakes}$$
$$2 \text{ yd} = 2(3 \text{ ft}) = (2 \times 3)\text{ft} = 6 \text{ ft}.$$

The fact that "2 yd" and "6 ft" name the same element of the model enables us to say they are equal; there is no need for such mysterious phrases as "2 yd measures the same as 6 ft."

4. The use of units. If we wish to use R^+ as a model for measuring (positive) lengths, we must decide which length 1 corresponds to; this length will then be called our "unit length." The more natural model is a ray L (on which R^+ operates); since the elements of L themselves are "lengths," the above question does not arise.

If we choose a length $l_0 \in L$, and compare other lengths with it, we may call l_0 our "unit"; this serves merely to remind us that l_0 is being kept fixed for a period. Suppose we now find certain other lengths, for instance $5l_0$, $2l_0$, $7l_0$. If we wish to shorten our notations, and call these lengths 5, 2, 7, we are then replacing L by R^+. We can then say "the length 5 really means the length $5l_0$." More awkwardly, one could say "the length is 5 when measured in terms of l_0."

Suppose we wish to "change units," say from ft to in. Then since, for any $a \in R^+$, a ft $=a(12$ in$)=12a$ in, we would replace "the length a" by "the length $12a$." If any problems about units arise, they are at once resolved by going back to the explicit phrase "a ft."

5. The postulational treatment. Though all rays (and all birays) have the same structure, one may wish to use several rays in a single investigation. For instance, in mechanics, one uses separate rays M, L, T for measurement of mass, length and time. (We study structures containing several rays in Part II.) Hence we introduce our models postulationally; the definitions show whether or not a given structure is a ray or a biray. However, only a single structure R (or one of its subsets) is needed for operators; hence we introduce R constructively. (We give the characterization of R as a complete ordered field at the end.)

A basic theorem in the subject is an *isomorphism theorem*; a homomorphism of one ray into another is necessarily an isomorphism onto, and has certain additional properties (and similarly for birays). This theorem is a great aid in setting up the theory; in particular, with its use, multiplication in R^+ and in R is introduced and its properties derived with a minimal effort.

The postulates used for rays and birays are few in number and simple in character, and correspond to simple experimental phenomena.

6. Other models. We introduce only the most important models. With the real numbers at our disposal, and the facts about rays and birays, other similar models are easily studied. For instance, in measuring masses, one wishes to allow the mass zero (not present in a ray). This extra element may be introduced and related to the remaining elements in the obvious manner.

A model of a somewhat different nature is an oriented affine one-dimensional space T^*; this is the natural model for instance for moments in time (or positions on a line). There is a corresponding biray T of translations of T^*; this is the natural model for intervals of time (or directed lengths). We do not consider models including for instance 3-dimensional space; the term "measurement" is not the best term here.

If the measures of some type of quantity form a progression, as in counting, it is natural to use N for a model. However, if several such types of quantities are considered together, it is better to use several isomorphic models. For a plebeian illustration, suppose there will be six children at a party. We wish each to have two balloons and three cookies. What is the total supply needed? The answer is:

$$6(2 \text{ bl} + 3 \text{ ck}) = 6(2 \text{ bl}) + 6(3 \text{ ck}) = 12 \text{ bl} + 18 \text{ ck}.$$

7. A finite model. We have no infinite sets available in our environment. What happens if we have a set G with a large number n of elements, called $1'$, $2'$, $\cdots$, n', and wish it to approximate to N? We could define $a'+b'$ to be $(a+b)'$, or n' if $n<a+b$. Note that we may set $a'b'=ab'$; now G operates on itself, thus defining multiplication in G. We find that these operations are commutative and associative, and the distributive laws hold. However, the cancellation laws fail.

We give an instance from everyday life. Helen is setting the table for lunch for four; she places two spoons at each place. Mother answers the doorbell; it is Mr. and Mrs. Jones. Perhaps they will stay; Helen needs 4 more spoons. There are only two left in the drawer, so Helen puts them out. (She thus makes $8_s+4_s=10_s$.) Hearing the visitors say goodbye, Helen thinks, take away four spoons. She then realizes that, actually, she must take away only two. In her model, $8_s+4_s=8_s+2_s$.

CHAPTER I. DIVISIBLE SEMI-GROUPS

This chapter is preliminary in nature; certain basic properties of rather general structures are derived. It is shown how N appears as a system of operators on any commutative semi-group, and Q^+, if the semi-group is "uniquely divisible." To save space and help in the grasping of concepts, the proofs pertaining to N are given in rather intuitive fashion. However, the Peano postulates are seen to hold for N; hence one may replace the proofs by the usual more formal proofs where desired.

8. Commutative semi-groups. We begin with the definition.

DEFINITION 8A. *A* semi-group $(G, +)$ *is a nonvoid set* G *and an associative binary operation* $+$ *in* G. *The semi-group is* commutative *if addition is commutative. Only the commutative case will be considered here.*

We may define $x+y+z$ to mean either $(x+y)+z$ or $x+(y+z)$, since these are the same. More generally, as is well known and easy to see, in any sum, the terms may be written in any order, and parentheses may be inserted or removed at will.

We wish to introduce a shorthand notation for such expressions as $x+x$, $x+x+x$, etc. Since the letter x plays no role here, let us think of it as replaced by a dot. This suggests the expressions

$$\begin{array}{llll} x & x+x & x+x+x & x+x+x+x\cdots \\ (\cdot) & (\cdot\quad\cdot) & (\cdot\quad\cdot\quad\cdot) & (\cdot\quad\cdot\quad\cdot\quad\cdot) \end{array} \tag{8·1}$$

We now consider the new expressions as being names for new objects, forming a set N. It does not matter what these objects are; only their relation to G (or any other semi-group) counts. We also set $1=(\cdot)$, $2=(\cdot\cdot)$, $3=(\cdot\cdot\cdot)$.

We next let N operate on G as follows:

$$(\cdot)x = x, \quad (\cdot\cdot)x = x+x, \quad (\cdot\cdot\cdot)x = x+x+x, \text{ etc.} \tag{8.2}$$

We give an elementary property of the operation:

$$\begin{aligned} (\cdot)(x+y) &= x+y = (\cdot)x + (\cdot)y, \\ (\cdot\cdot)(x+y) &= (x+y)+(x+y) = (x+x)+(y+y) = (\cdot\cdot)x+(\cdot\cdot)y, \\ (\cdot\cdot\cdot)(x+y) &= (x+y)+(x+y)+(x+y) \\ &= (x+x+x)+(y+y+y) = (\cdot\cdot\cdot)x+(\cdot\cdot\cdot)y, \end{aligned}$$

and clearly, in general,

$$a(x+y) = ax+ay \qquad (a\in N; \quad x, y \in G). \tag{8.3}$$

With the obvious definition of "successor function" σ in N, the Peano postulates are clear:

(N_1) *The element* $(\cdot)=1$ *is not a successor*: $\sigma x \neq 1$ *for all* $x\in N$.
(N_2) *For all* $x\neq y$, $\sigma x\neq\sigma y$; *that is, the function* σ *is one-one.*
(N_3) *For any* $N'\subset N$, *if* $1\in N'$, *and* $x\in N'$ *implies* $\sigma x\in N'$, *then* $N'=N$.

We could now give proofs involving N in the classical manner, using mathematical induction, defining addition in N, and showing how to give definitions by induction. We shall, however, continue to give intuitive derivations.

If we look at the pattern

$$\begin{array}{lcl} (x+x+x) + (x+x) &=& x+x+x+x+x \\ (\cdot\quad\cdot\quad\cdot)\quad(\cdot\quad\cdot) && (\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot) \end{array}$$

this suggests defining

$$(\cdots) + (\cdot\cdot) = (\cdot\cdot\cdot\cdot\cdot).$$

The general rule is clear: To add two elements of N, place the corresponding symbols beside each other, and remove the inner parentheses. The commutative and associative laws are evident; hence $(N, +)$ is a commutative semi-group. Moreover, because of our definition,

$$(\cdots)x + (\cdot\cdot)x = (\cdot\cdot\cdot\cdot\cdot)x = [(\cdots) + (\cdot\cdot)]x,$$

and more generally,

$$(8.4) \qquad (a + b)x = ax + bx \qquad (a, b \in N, x \in G).$$

Since $(N, +)$ is a commutative semi-group, we may operate on it by N itself, thus defining *multiplication* in N. This gives, for instance,

$$(\cdot\cdot)(\cdots) = (\cdots) + (\cdots), \qquad (\cdots)(\cdots) = (\cdots) + (\cdots) + (\cdots).$$

As a consequence,

$$[(\cdot\cdot)(\cdots)]x = [(\cdots) + (\cdots)]x = (\cdots)x + (\cdots)x = (\cdot\cdot)[(\cdots)x],$$

and more generally,

$$(8.5) \qquad (ab)x = a(bx) \qquad (a, b \in N, x \in G).$$

Letting N operate on itself, (8.5) and (8.4) give the associative law for multiplication and the distributive law. The commutative law for multiplication must be proved separately. The cancellation law: $x+u=y+u$ implies $x=y$, is obvious from the representation of the elements with dots; a proof with the Peano postulates is easily given.

9. Order in N. If we think of the elements of N as laid out in their natural order, then $x<y$ means that x comes before y. The usual properties of order are clear. For later purposes, we show how to derive the properties from the two following properties of addition in N:

(a) For all $a, b \in N$, $a+b \neq a$.
(b) If $a \neq b$, then either $a+c=b$ or $b+c=a$, for some c.

Now write $a<b$ if $a+c=b$ for some c. The following properties follow at once from the definition:

$$(9.1) \qquad a < a + b.$$

$$(9.2) \qquad \text{If } a < b \text{ and } b < c \text{ then } a < c.$$

$$(9.3) \qquad \text{If } a_i < b_i (i = 1, \cdots, n) \text{ then } \Sigma a_i < \Sigma b_i.$$

The *trichotomy* property is: For all $a, b \in N$, exactly one of the following is true:

$$a < b, \qquad a = b, \qquad b < a.$$

For if $a \neq b$, then (b) shows that $a<b$ or $b<a$. That at most one of these is true follows from (a). (If $a+c=b$, $b+d=a$, then $a+(c+d)=a$, contradicting (a).)

Order is related to addition by the property (writing "iffif" for "if and only if")

$$(9.4) \qquad a < b \quad \text{iffif} \quad a + c < b + c.$$

This is evident, using the cancellation law. Finally,

$$(9.5) \qquad a < b \quad \text{iffif} \quad na < nb;$$

for by (9.3), $a<b$, $a=b$, $b<a$ imply respectively $na<nb$, $na=nb$, $nb<na$.

Rather than discuss *subtraction* here, we give the equivalent discussion when studying rays, in section 14.

10. Some examples of semi-groups. We give first a general kind of example:

Example 10A. Let G contain the elements $1, 2, 3, \cdots, m, \cdots, n$ from N. Let the successor function in G be as in N, except that we set $\sigma n=m$. Now addition is defined in terms of this function; we require

$$(10.1) \qquad x + 1 = \sigma x, \qquad x + \sigma y = \sigma(x + y).$$

(These relations are used in defining addition in N through the Peano postulates.) For instance, with $m=5$, $n=7$, the elements are 1, 2, 3, 4, 5, 6, 7; adding 3 and adding 4 gives respectively the sequences

$$4, 5, 6, 7, 5, 6, 7; \qquad 5, 6, 7, 5, 6, 7, 5.$$

Example 10B. Take $m=1$ in the last example. Then we have the ring of integers mod n; n is the zero element.

Example 10C. Take $m=n$ in Example 10A. This gives the example of Section 7.

If G and G' are semi-groups, their *direct sum* consists of all pairs (x, x') with $x \in G$, $x' \in G'$. Define addition componentwise:

$$(x, x') + (y, y') = (x + y, x' + y').$$

This is clearly a semi-group, commutative if G and G' are.

Example 10D. There are ten teaspoons and six dessert spoons in a drawer. This gives us semi-groups G_t and G_d as in section 7. In the direct sum G^*, we have for instance

$$(5_t, 5_d) + (3_t, 3_d) = (8_t, 6_d).$$

If now we do not need to differentiate between teaspoons and dessert spoons, we have sixteen spoons, forming a semi-group G'. There is a natural mapping of G^* into G', in which for instance

$$(5_t,\ 5_d) \to 10_s, \qquad (3_t,\ 3_d) \to 6_s, \qquad (8_t,\ 6d) \to 14s.$$

Note that this mapping is not a homomorphism: $10_s+6_s \neq 14_s$.

11. Divisible semi-groups. In the following definition, we make use of the fact that N acts on any semi-group.

DEFINITION 11A. *We say that the commutative semi-group* $(G, +)$ *is* divisible *if for each* $x \in G$ *and* $n \in N$ *there is a* $y \in G$ *such that* $ny = x$. *We say that* $(G, +)$ *is* uniquely divisible *if the above* y *is unique.*

Example 11B. The group N_m of integers mod $m(m>1)$ is not divisible; nor is N itself. The semi-groups Q^+, R^+, R are uniquely divisible. The group of dyadic rationals, containing all $k/2^n$ (k an integer, $n \in N$) is not divisible. The group of rationals mod 1 is divisible, but not uniquely; the same is true of the reals mod 1, or equivalently, of the multiplicative group of complex numbers of absolute value 1.

The direct sum of a finite set of uniquely divisible commutative semi-groups is uniquely divisible; in particular, any vector space over the reals is so.

DEFINITION 11C. *An element* x *of* G *is* idempotent *if* $x+x=x$. *We say* G *is* idempotent *if all its elements are idempotent.*

In a group, only the identity element is idempotent.

THEOREM 11D. *If* G *is idempotent, it is uniquely divisible.*

First, induction shows at once that $nx=x$ for all n and x. Now given x and n, set $y=x$; then $ny=x$. If also $ny'=x$, then $y'=ny'=x=ny=y$.

Note that, therefore, the operation of N on an idempotent semi-group is trivial.

Example 11E. Let U be a set, and let S be a set of subsets of U such that if A and B belong to S, so does their union $A \cup B$. Then $(S, \cup)$ is an idempotent commutative semi-group.

REMARK 11F. It is a theorem that every idempotent commutative semi-group is of the form of Example 11E.

DEFINITION 11G. *Let us say that* $(G, +)$ separates N *if for any two distinct natural numbers* m, n *there is an* $x \in G$ *such that* $mx \neq nx$.

We cannot then reduce N to a smaller set of operators on G, as was done for instance in Section 7.

THEOREM 11H. *Any divisible commutative semi-group* $(G, +)$ *which is not idempotent separates* N.

Suppose not. Then for some m and n, $n=m+h$ and $mx=nx$ for all x. For some a and k, $m+a=kh$. Now take any $y \in G$. Choose x so that $khx=y$. Since $mx=(m+h)x$, adding ax gives $khx=(k+1)hx$. Adding hx gives $(k+1)hx=(k+2)hx$. Continuing gives $khx=2khx$, i.e. $y=2y$. Thus G is idempotent, a contradiction.

12. Introduction of Q^+. We assume here that $(G, +)$ is a uniquely divisible commutative semi-group which is not idempotent. Then, using Theorem 11H,

we have the properties

(12.1) $$ax = ay \text{ implies } x = y \qquad (a \in N;\ x, y \in G),$$

(12.2) $$ax = bx \text{ for all } x \text{ implies } a = b \qquad (a, b \in N;\ x \in G).$$

Given x and a, there is a unique y such that $ay=x$; let us denote this y by the expression $\bar{a}x$. Now, by definition,

(12.3) $$a(\bar{a}x) = x \qquad (a \in N;\ x \in G).$$

We may now form $a(\bar{b}x)$. We would like to write this in the form $(a\bar{b})x$. To this end, we must define new elements $\bar{b}$, and give the whole expression meaning.

First, we show that (writing $\bar{b}'$ to denote $\overline{b'}$)

(12.4) $$a(\bar{b}x) = a'(\bar{b}'x) \quad (\text{all } x \in G) \quad \text{iffif} \quad ab' = a'b.$$

For, using (8.5) and (12.3) gives

$$(bb')(a(\bar{b}x)) = (ab')(b(\bar{b}x)) = (ab')x,$$
$$(bb')(a'(\bar{b}'x)) = (a'b)(b'(\bar{b}'x)) = (a'b)x.$$

Now if the left hand side of (12.4) holds, then the above equations give $(ab')x = (a'b)x$ (all x), and $ab'=a'b$, by (12.2). The converse follows also, using (12.1).

Now consider the expressions $a\bar{b}$ as denoting new objects. Because of (12.4), we wish an equivalence relation between these objects:

(12.5) $$a\bar{b} \sim a'\bar{b}' \quad \text{iffif} \quad ab' = a'b.$$

(To prove that the relation is transitive, multiply $ab'=a'b$ by $a'b''=a''b'$, giving $ab'a'b''=a'ba''b'$, and apply the cancellation law in N, giving $ab''=a''b$.) Denote the equivalence class of $a\bar{b}$ by a/b. Now let Q^+ be the set of equivalence classes thus obtained. (We could shorten $1\bar{b}$ to $\bar{b}$.) Note that $ac/bc=a/b$.

We now let Q^+ operate on G, by setting

(12.6) $$\frac{a}{b}x = a(\bar{b}x) \qquad (a, b \in N;\ x \in G).$$

Because of (12.4), the result is independent of the name a/b chosen for the given element of Q^+.

Note that $\bar{1}x=x$; hence $(a/1)x=a(\bar{1}x)=ax$. We therefore *identify* the element $a/1$ of Q^+ with the element a of N, thus imbedding N in Q^+, and the operation of N on G is preserved. (This is permissible; for (12.5) shows that $a/1 \neq b/1$ if $a \neq b$.)

If $\bar{b}x=\bar{b}y$, then $x=b(\bar{b}x)=b(\bar{b}y)=y$. Hence also, using (12.1),

(12.7) $$rx = ry \text{ implies } x = y \qquad (r \in Q^+;\ x, y \in G.)$$

The relation (12.4) gives:

(12.8) $$rx = sx \text{ for all } x \in G \text{ implies } r = s \ (r, s \in Q^+.)$$

13. Properties of Q^+ and G. First we note that for any $r=a/b \in Q^+$, using (8.5), $b(rx)=b(a(\bar{b}x))=a(b(\bar{b}x))=ax$. Hence

$$b[r(x+y)] = a(x+y) = ax + ay = b(rx) + b(ry) = b(rx + ry),$$

and applying (12.1) gives

$$r(x+y) = rx + ry \qquad (r \in Q^+;\, x, y \in G). \tag{13.1}$$

Next, since

$$\frac{a}{b}x + \frac{c}{b}x = a(\bar{b}x) + c(\bar{b}x) = (a+c)(\bar{b}x) = \frac{a+c}{b}x,$$

we have

$$\frac{a}{b}x + \frac{c}{d}x = \frac{ad}{bd}x + \frac{bc}{bd}x = \frac{ad+bc}{bd}x. \tag{13.2}$$

Hence it is natural to define

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}. \tag{13.3}$$

Taking $b=d=1$ shows that this extends the definition in N. Now (13.2) gives the distributive law

$$(r+s)x = rx + sx \qquad (r, s \in Q^+;\, x \in G). \tag{13.4}$$

That the definition of $r+s$ is independent of the manner of writing r and s may be verified directly; it also follows from (13.2) and (12.8) (using some G).

Since addition and multiplication in N are commutative, (13.3) shows that addition in Q^+ is commutative. Similarly, addition is associative. (Using G, these properties follow easily, on applying (12.8).) Hence Q^+ is a commutative semi-group.

From the definition of addition in Q^+ we have

$$\left(\frac{a}{b} + \frac{a}{b}\right) + \frac{a}{b} = \frac{2a}{b} + \frac{a}{b} = \frac{3a}{b},$$

and similarly, in general,

$$n\frac{a}{b} = \frac{na}{b}. \tag{13.5}$$

We can solve $ns=r$ for s: If $r=a/b$, set $s=a/nb$. Thus Q^+ is divisible. If $n(a/b)=n(a'/b')$, then (13.5) and (12.5) give $na/b=na'/b'$, $nab'=na'b$, $ab'=a'b$, $a/b=a'/b'$; thus division (by elements of N) is unique. Clearly Q^+ is not idempotent.

Now Q^+ operates on itself; we call this operation multiplication. This extends the operation of multiplication in N. Because of (13.5),

$$bd\left(\frac{a}{b}\left(\frac{c}{d}\right)\right) = ad\left(\frac{b}{b}\left(\frac{c}{d}\right)\right) = ad\left(\frac{c}{d}\right) = ac = bd\frac{ac}{bd};$$

hence (12.1) gives

$$\frac{a}{b}\,\frac{c}{d} = \frac{ac}{bd}\,. \tag{13.6}$$

We now prove the general associative law:

$$r(sx) = (rs)x \qquad (r, s \in Q^+; x \in G). \tag{13.7}$$

Say $r=a/b$, $s=c/d$. Then, with the help of (12.6) and (12.3),

$$bd\left(\frac{a}{b}\left(\frac{c}{d}\,x\right)\right) = ad\left(\frac{c}{d}\,x\right) = acx = bd\left(\frac{ac}{bd}\,x\right);$$

this gives the result.

In particular, multiplication in Q^+ is associative; it is clearly commutative. The distributive laws follow from (13.1) and (13.4). The cancellation law for multiplication follows from (12.7). The existence and uniqueness of division is clear: The solution of $(a/b)u=c/d$ is

$$u = \frac{c}{d} \div \frac{a}{b} = \frac{cb}{da}\,. \tag{13.8}$$

In particular, $1/(a/b)=b/a$.

The subtraction property in Q^+ is:

(13.9) If $r \neq s$, then either $r+t=s$ or $s+t=r$, for some t.

For we may write $r=a/c$, $s=b/c$; then $a\neq b$, hence $b=a+d$ or $a=b+d$, and we may use $t=d/c$.

The cancellation law for addition is also easy to prove: Suppose $r+s=r+t$. Write $r=a/d$, $s=b/d$, $t=c/d$. Then $a+b=a+c$, hence $b=c$, and $s=t$.

The relation $r+s=r$ in Q^+ is impossible. Hence we may define order in Q^+ as in N (section 9), and Q^+ is now simply ordered.

CHAPTER II. RAYS

A "semi-ray" has certain properties needed for the measurement of positive quantities: addition and subtraction, order, and the existence of arbitrarily small elements. If the semi-ray is Archimedean, it may be completed to form a ray. The operation of Q^+ on a ray is extended to the operation by R^+; we find the basic properties of R^+ here.

14. Semi-rays. We shall not need divisibility in the definition; it will appear as a consequence of completeness in the next section.

DEFINITION 14A. *A* semi-ray *L is a commutative semi-group such that:*

(R_1) *For all x and y in L, $x+y\neq x$.*

(R_2) *For all x and y in L with $x\neq y$, we can find u and v in L such that $x+u+v=y$ or $y+u+v=x$.*

DEFINITION 14B. *$x<y$ means that $x+u=y$ for some u.*

Since we now have properties (a) and (b) of section 9, we may deduce the properties of *order* given there. We also have the *cancellation law*:

$$\text{(14.1)} \qquad \textit{If } x + u = y + u \textit{ then } x = y.$$

For if $x\neq y$, then $x<y$ or $y<x$, by trichotomy, and hence $x+u<y+u$ or $y+u<x+u$, contrary to $x+u=y+u$.

We now introduce *subtraction.* If $x<y$, then there is a unique element u such that $x+u=y$; we call this element $y-x$. Now

$$\text{(14.2)} \qquad (y - x) + x = y \qquad \text{if } x < y.$$

We give some simple properties:

$$\text{(14.3)} \qquad (x - y) - z = x - (y + z) \qquad \text{if } y + z < x,$$

$$\text{(14.4)} \qquad (x + y) - z = \begin{cases} x + (y - z) & \text{if } z < y, \\ x - (z - y) & \text{if } y < z < x + y. \end{cases}$$

To prove each one, add a quantity to each side which will remove the minus signs. Thus, for the last equality, add z:

$$\begin{aligned}[x - (z - y)] + z &= [x - (z - y)] + [(z - y) + y] \\ &= [(x - (z - y)) + (z - y)] + y = x + y = [(x + y) - z] + z;\end{aligned}$$

now apply (14.1).

We now show that, owing to the particular form of (R_2), L contains arbitrarily small elements:

THEOREM 14C. *For each $x\in L$ and $n\in N$ there is some $y\in L$ with $ny<x$.*

First, $x+x\neq x$; also $x+x+u+v\neq x$ for all u and v, by (R_1); hence, by (R_2), we may find u and v so that

$$x + u + v = x + x.$$

By (14.1), $u+v=x$. Either $u\leqq v$ or $v\leqq u$; say $u\leqq v$. Then $u+u\leqq u+v=x$, and $u<2u\leqq x$; the statement holds with $n=1$. For the general case, we use induction: Say $nz<x$. Find y such that $2y\leqq z$. Then

$$(n + 1)y = ny + y \leqq n(y + y) \leqq nz < x.$$

DEFINITION 14D. *The semi-ray L is* Archimedean *if for each x and y in L there is an $n\in N$ such that $y<nx$.*

Example 14E. Let L consist of all ordered pairs $(x, 0)$ with $x \in R^+$, and (x, y) with $x \in R$ and $y \in R^+$ (or $y \in N$); define addition component-wise. Then $(L, +)$ is clearly a commutative semi-group, and (R_1) holds. To prove (R_2), take $(x, y) \neq (x', y')$. If $y = y'$, then say $x < x'$; now $(x, y) + (x' - x, 0) = (x', y')$. If $y \neq y'$, say $y < y'$; then $(x, y) + (x' - x, y' - y) = (x', y')$. Now take $u = v = \frac{1}{2}(x' - x, 0)$ or $\frac{1}{2}(x' - x, y' - y)$. Since $n(1, 0) = (n, 0) < (0, 1)$ for all n, $(L, +)$ is not Archimedean.

15. Rays. We introduce completeness through Dedekind cuts.

DEFINITION 15A. *An* upper set *U in a semi-ray L is a set such that*

(U_1) *$U \neq \varnothing$ and $U \neq L$.*
(U_2) *If $x \in U$ and $x < y$ then $y \in U$.*

We say the upper set U is strict *if also*
(U_3) *If $x \in U$ then there is some $y \in U$ with $y < x$.*

DEFINITION 15B. *Given $S \subset L$, z is a* lower bound *for S if $z \leqq x$ for all $x \in S$. Also, z is a* greatest lower bound (g.l.b. for short) *for S is z is a lower bound, but no $z' > z$ is.*

THEOREM 15C. *Any subset of L has at most one* g.l.b.

This is trivial.

DEFINITION 15D. *The semi-ray L is* complete *if any strict upper set has a* g.l.b.

THEOREM 15E. *If L is complete, then any upper set has a* g.l.b.

Suppose the upper set U is not strict. Then for some $z \in U$, $x < z$ implies $x \notin U$. Clearly z is the g.l.b. for U.

DEFINITION 15F. *A* ray *is a complete semi-ray.*

THEOREM 15G. *A ray is Archimedean.*

For suppose not. Say $nx \leqq y$ for all n. Set

$$U = \{u \colon nx \leqq u \text{ for all } n \in N\}.$$

Since $x \notin U$ and $y \in U$, (U_1) holds. Clearly (U_2) holds also; hence U is an upper set. By Theorem 15E, U has a g.l.b., say z. We show now that $z < nx$ for some n. If $z \leqq x$, then $z < 2x$. Otherwise, $x < z$, and we may write $x + y = z$; now $y < z$ also. Therefore $y \notin U$, and we have $y < mx$ for some m; hence $z = x + y < (m+1)x = nx$. But nx is a lower bound for U, contradicting the definition of z.

THEOREM 15H. *Any ray L is uniquely divisible.*

Take any x and any $n > 1$; suppose that $ny \neq x$ for all y. Set

$$U = \{u \in L \colon x < nu\}.$$

By Theorem 14C, $U \neq L$. Since $x \in U$, (U_1) holds. Using (9.5) in L shows that (U_2) holds; thus U is an upper set, and has a g.l.b., say z.

Suppose $nz < x$; say $nz + y = x$. Find h by Theorem 14C so that $nh < y$; then $n(z+h) < nz + y = x$ and (9.5) shows that $z+h$ is a lower bound for U, a contradiction. Hence $nz > x$, and we may write $x + y = nz$. Choose h so that $nh < y$. Now $nh < nz$, hence $h < z$, and we may write $v + h = z$; now

$$x + y = nz = n(v + h) < nv + y,$$

hence $x < nv$, and $v \in U$, contradicting $v < z$. We have proved that L is divisible. Since $z' < z$ implies $nz' < nz$, division is unique.

Since $x < x + x$, L is not idempotent. Hence (section 12) *Q^+ operates on any ray L.*

We give two facts about inequalities. For $r, s \in Q^+$ and $x, y \in L$,

$$r < s \quad \text{iffif} \quad rx < sx, \tag{15.1}$$

$$x < y \quad \text{iffif} \quad rx < ry. \tag{15.2}$$

If $r < s$, say $r + t = s$; then $rx + tx = sx$, and $rx < sx$. Similarly, if $r \geqq s$, then $rx \geqq sx$; hence if $rx < sx$ then $r < s$. (15.2) follows similarly.

Finally, we prove a *density* theorem:

(15.3) *If $x, y, z \in L$, $x < y$, then for some $r \in Q^+$, $x < rz < y$.*

Say $x + u = y$. For some $n \in N$, $nu > z$. Set $h = (1/n)z$; then $h < u$. Also $h < y$, and $mh > y$ for some $m \in N$. Hence, for some $k \in N$, $kh < y \leqq (k+1)h$. Set $r = k/n$; then $rz = kh < y$. Also $x + u = y \leqq rz + h < rz + u$, and hence $x < rz$.

16. Completion of an Archimedean semi-ray. Such a completion is a ray; in particular, we use this later to construct R^+ from Q^+.

LEMMA 16A. *If L is a semi-ray and $x \in L$, then $U(x) = \{u \in L: x < u\}$ is a strict upper set.*

For $x \notin U(x)$, $2x \in U(x)$, and (U_1) holds. (U_2) is clear. Say $y \in U(x)$; then $x < y$. By (R_2), we may write $x + u + v = y$. Now $x + u \in U(x)$ and $x + u < y$, so that (U_3) holds.

LEMMA 16B. *If U is an upper set in the Archimedean semi-ray L and $h \in L$, we may find $z \in L$ such that $z \notin U$ and $z + h \in U$.*

Choose $x \notin U$, $y \in U$. Say $nh > y$; then $x + nh \in U$. Hence, for z, we may take one of the elements $x, x+h, x+2h, \cdots, x+(n-1)h$.

LEMMA 16C. *If U and V are strict upper sets in the semi-ray L, then*

$$U + V = \{u + v: u \in U, v \in V\} \tag{16.1}$$

is a strict upper set; we have:

(16.2) *If $u \notin U$, $v \notin V$, then $u + v \notin U + V$.*

First, given $u \notin U$, $v \notin V$, take any $z \in U+V$, and write $z = u' + v'$, with $u' \in U$, $v' \in V$; then $u < u'$, $v < v'$, hence $u+v < z$, and $u+v \notin U+V$.

Because of this, $U+V \neq L$. Clearly $U+V \neq \varnothing$. Properties (U_2) and (U_3) for $U+V$ are clear.

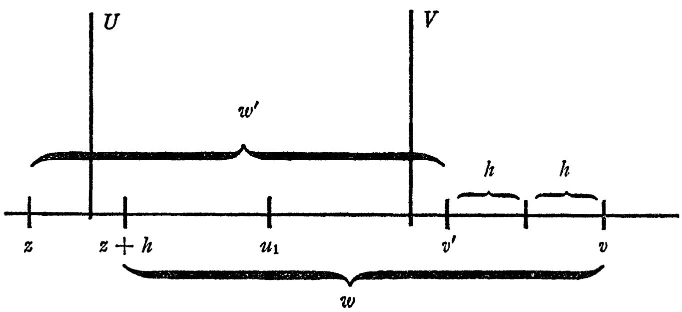
THEOREM 16D. *Let* $(L, +)$ *be an Archimedean semi-ray, let* L^* *be the set of all strict upper sets in* L, *and define addition in* L^* *by* (16.1); *then* $(L^*, +)$ *is a ray. The mapping* $x \rightarrow U(x)$ *of* L *into* L^* (see Lemma 16A) *is one-one and preserves addition; it thus imbeds* L *in* L^*.

With addition in L^* defined by (16.1), we clearly have a commutative semigroup. To prove (R_1), take any strict upper sets U and V. Take $h \notin V$, and find z for U by Lemma 16B. By (16.2), $z+h \notin U+V$; since $z+h \in U$, $U \neq U+V$.

Next we introduce subtraction in L^*. Suppose $u_1 \in U$, $u_1 \notin V$. Then set

$$W = \{w: \text{for some } w' < w,\ w' + u \in V \text{ for all } u \in U\}.$$

We have $V \subset W$; hence $W \neq \varnothing$. We may write $u_1 = u + h$, with $u \in U$; then $h \notin W$, and $W \neq L$. Clearly (U_2) and (U_3) hold for W; hence W is a strict upper set.



Clearly $U+W \subset V$. Conversely, take any $v \in V$. By (U_3) for V and Theorem 14C, we may write $v = v' + 2h$, with $v' \in V$. By Lemma 16B, find $z \notin U$ such that $z+h \in U$. By (U_2) for U and for V, $z < u_1 < v'$; hence we may write $z + w' = v'$. Set $w = w' + h$. Now if $u \in U$, then $u > z$, and hence $w' + u > w' + z = v' \in V$; thus $w' + u \in V$. This shows that $w \in W$. Also $z+h \in U$, $w \in W$, and $z+h+w = v$; thus $v \in U+W$, proving that $U+W = V$.

Next, with W as above, choose x so that $2x \notin W$. Since $2x \in U(x)$, we can find W' by the proof above so that $U(x) + W' = W$. Now we prove (R_2). Given U and V with $U \neq V$, either $U \not\subset V$ or $V \not\subset U$, say the former. The proof above gives $U + U(x) + W' = V$, as required. Thus $(L^*, +)$ is a semi-ray.

To prove that L^* is complete, take any strict upper set $U^* \subset L^*$. The elements of U^* are strict upper sets in L; let S be their union. Since $U^* \neq \varnothing$, $S \neq \varnothing$. Since $U^* \neq L^*$, there is a strict upper set $U \notin U^*$. Take $x \notin U$. For any $U' \in U^*$, $U < U'$ (in L^*), by (U_2) for U^*; hence (Definition 14B) $U + W = U'$

for some W, and thus $x \notin U'$. This shows that $x \notin S$, and $S \neq L$. Properties (U_2) and (U_3) for S are clear; thus S is a strict upper set in L, i.e. $S \in L^*$. Since $U \in U^*$ implies $U \subset S$ and hence $S \leqq U$ in L^*, S is a lower bound for U^* in L^*. If $S < S'$ in L^*, then $S + W = S'$, and there is some $x \in S$ with $x \notin S'$. Now $x \in U$ for some $U \in U^*$, and hence $U < S'$ in L^*; thus S' is not a lower bound for U^*. Therefore S is the g.l.b. of U^* in L^*, proving completeness.

Suppose $x < y$. Then say $x + u + v = y$; it follows that $x + u \in U(x)$, $x + u \notin U(y)$, and $U(x) \neq U(y)$. Thus the mapping $x \to U(x)$ is one-one.

We must show still that $U(x+y) = U(x) + U(y)$. Suppose $z \in U(x+y)$; then $x + y < z$, and we may write $x + y + u + v = z$. Now $x + u \in U(x)$ and $y + v \in U(y)$, and thus $z \in U(x) + U(y)$. Thus $U(x+y) \subset U(x) + U(y)$. The converse is clear, and the proof is complete.

Suppose we form the completion L^* of the ray L. Each element U of L^* is a strict upper set in L; it has a g.l.b., say x. The properties of strict upper sets show at once that $U = U(x)$ (compare Lemma 16A). Thus the mapping $x \to U(x)$ of L into L^* is onto, and we may identify L^* with L itself.

· **17. The isomorphism theorem.** We shall show that there is essentially only one kind of ray; any two rays are isomorphic. All homomorphisms are isomorphisms; we find them all. For further information, see Section 19.

DEFINITION 17A. *Let $(G, +)$ and $(G', +)$ be commutative semi-groups. A mapping ϕ: $G \to G'$ is a* homomorphism *if $\phi(x+y) = \phi(x) + \phi(y)$ for all x and y in G. It is an* isomorphism *if it is also one-one, and is* onto *G' if the image of G is all of G'.*

THEOREM 17B. *Let ϕ be a homomorphism of the ray L into the ray L'. Then*

$$\text{if } x < y \quad \text{then } \phi(x) < \phi(y), \tag{17.1}$$

$$\text{if } r \in Q^+ \quad \text{then } \phi(rx) = r\phi(x). \tag{17.2}$$

If $x < y$, write $x + u = y$; then $\phi(x) + \phi(u) = \phi(y)$, and hence $\phi(x) < \phi(y)$. Next for any n, $\phi(x + \cdots + x) = \phi(x) + \cdots + \phi(x)$ (n terms in the sums); hence $\phi(nx) = n\phi(x)$. Also, if $y = (1/m)x$, i.e. $x = my$, then $\phi(x) = m\phi(y)$ and hence $\phi(y) = (1/m)\phi(x)$, from which (17.2) follows.

THEOREM 17C. *Let L and L' be rays, and suppose $w \in L$, $w' \in L'$. Then there is a unique homomorphism ϕ of L into L' such that $\phi(w) = w'$. Any homomorphism of L into L' is an isomorphism onto L'.*

For each $x \in L$, set

$$U'_x = \{y' \in L' : \text{for some } r \in Q^+, x < rw \text{ and } rw' < y'\}.$$

We see at once that U'_x is a strict upper set in L'; hence (see the end of Section 16) for a certain $x' \in L'$, $U'_x = U'(x')$, with $U'(x')$ defined as in Lemma 16A. Set $\phi(x) = x'$.

We show that

(17.4) $$U'_{x+y} = U'_x + U'_y.$$

If $z' \in U'_x + U'_y$, then $z' = x' + y'$, with $x' \in U'_x$, $y' \in U'_y$. For some $r, s \in Q^+$,

$$x < rw, \qquad rw' < x'; \qquad y < sw, \qquad sw' < y'.$$

Now $x+y<(r+s)w$, $(r+s)w'<z'$, and hence $z' \in U'_{x+y}$. Conversely, given $z' \in U'_{x+y}$, we may find r, u, v so that $x+y+u+v=rw$, $rw'<z'$. By (15.3) we may find $s, t \in Q^+$ so that

$$x < sw < x + u, \qquad y < tw < y + v.$$

Now $(s+t)w<rw$, and by (15.1), $s+t<r$, $(s+t)w'<rw'<z'$. We may write $(sw'+u')+(tw'+v')=z'$. Since $sw'+u' \in U'_x$ and $tw'+v' \in U'_y$, we have $z' \in U'_x + U'_y$, proving (17.4).

Recalling that $U'_z = U'(\phi(z))$ and $U'(x')+U'(y')=U'(x'+y')$ (see the end of Section 16), (17.4) gives

$$U'(\phi(x + y)) = U'_{x+y} = U'(\phi(x)) + U'(\phi(y)) = U'(\phi(x) + \phi(y));$$

hence $\phi(x+y)=\phi(x)+\phi(y)$, and ϕ is a homomorphism. Now we may apply (17.1), showing that ϕ is one-one.

Next we show that if ϕ' is any homomorphism of L' into L such that $\phi'(w')=w$, then $\phi(\phi'(x'))=x'$ for all $x' \in L'$. Set $x=\phi'(x')$, $y'=\phi(x)$. If $y'<x'$ we may find $r \in Q^+$ such that $y'<rw'<x'$. Then by Theorem 17B for ϕ', $rw=\phi'(rw')<\phi'(x')=x$, and hence, applying ϕ, $rw'<y'$, a contradiction. Similarly $x'<y'$ is false; hence $y'=x'$.

Because of this, ϕ maps L onto L'. Also, for another ϕ_1 like ϕ, we have $\phi_1(\phi'(x'))=x'=\phi(\phi'(x'))$, and since ϕ' is onto L, $\phi_1(x)=\phi(x)$ for all x; thus ϕ is uniquely determined, and the proof is complete.

18. Introduction of R^+. We saw in Section 13 that Q^+ is a completely divisible commutative semi-group; also, that $r+s=r$ is impossible in Q^+, so that (R_1) holds. Given $r, s \in Q^+$, choose $t \in Q^+$ by (13.9); set $u=v=t/2$; then either $r+u+v=s$ or $s+u+v=r$, proving (R_2). Therefore Q^+ is a semi-ray. Given $r, s \in Q^+$, we may write $r=a/c$, $s=b/c$; then since $ba \geqq b$, (13.5) shows that $br \geqq s$. Therefore Q^+ is Archimedean. We may now apply Theorem 16D, giving the completion R^+ of Q^+; R^+ is a ray, and we may consider Q^+ as imbedded in R^+, by the definition $r \to \{s \in Q^+ \colon r<s\}$.

19. The operation of R^+ on L. For each x in the ray L, let ϕ_x be the homomorphism of R^+ into L given by Theorem 17C, such that $\phi_x(1)=x$. By (17.2),

$$\phi_x(r) = \phi_x(r \cdot 1) = r\phi_x(1) = rx, \qquad (r \in Q^+);$$

hence, if we set

(19.1) $$ax = \phi_x(a) \qquad (a \in R^+, x \in L),$$

this extends the operation of Q^+ to an operation of R^+ on L. Note that

$$1x = x. \tag{19.2}$$

Since ϕ_x is a homomorphism, we have

$$(a + b)x = \phi_x(a + b) = \phi_x(a) + \phi_x(b) = ax + bx. \tag{19.3}$$

Given x and y in L, set $\psi(a) = \phi_x(a) + \phi_y(a)$ (all $a \in R^+$); then ψ is a homomorphism, and $\psi(1) = x + y$. By uniqueness in Theorem 17C, $\psi = \phi_{x+y}$. Hence

$$a(x + y) = \phi_{x+y}(a) = \psi(a) = \phi_x(a) + \phi_y(a) = ax + ay. \tag{19.4}$$

Since R^+ is a ray, R^+ operates on itself as above; let Φ_c be the corresponding functions. We call this operation *multiplication* in R^+. Thus

$$ab = \Phi_b(a); \qquad \Phi_b(1) = b. \tag{19.5}$$

Since the operation of R^+ on R^+ extends the operation of Q^+ on R^+ and hence of Q^+ on Q^+, and the latter operation is multiplication in Q^+, multiplication in R^+ extends that in Q^+.

The above distributive laws give, in R^+,

$$(a + b)c = ac + bc, \qquad a(b + c) = ab + ac. \tag{19.6}$$

Set $\Psi(a) = a$; Ψ is a homomorphism, and $\Psi(1) = 1$. By uniqueness, $\Psi = \Phi_1$. Hence $a1 = \Phi_1(a) = \Psi(a) = a$. Also, since $1 \in Q^+$ (or, by (19.5)), $1a = a$. Thus

$$1a = a1 = a. \tag{19.7}$$

We can *divide* in R^+: Given $a, b \in R^+$, since Φ_a is onto R^+, we can find $c \in R^+$ such that $ca = \Phi_a(c) = b$.

Given $b \in R^+$ and $x \in L$, let ψ be the composite mapping $\phi_x \circ \Phi_b$; this is a homomorphism of R^+ into L. Since

$$\psi(1) = \phi_x(\Phi_b(1)) = \phi_x(b) = bx,$$

uniqueness shows that $\psi = \phi_{bx}$. Hence we find the general associative law:

$$a(bx) = \phi_{bx}(a) = \psi(a) = \phi_x(\Phi_b(a)) = \phi_x(ab) = (ab)x. \tag{19.8}$$

In particular, taking $L = R^+$,

$$a(bc) = (ab)c. \tag{19.9}$$

If we set $\Psi_b(a) = \Phi_a(b)$, then the second part of (19.6) shows that Ψ_b is a homomorphism. Since $\Psi_b(1) = b1 = b = \Phi_b(1)$, $\Psi_b = \Phi_b$ and

$$ba = \Phi_a(b) = \Psi_b(a) = \Phi_b(a) = ab. \tag{19.10}$$

THEOREM 19A. *The operation of R^+ on L extends the operation of Q^+; we have properties* (19.1) *through* (19.10), *and also*:

(V_1) *For each x and y in L there is an $a \in R^+$ such that $ax = y$.*

(V_2) *If $ax = ay$ then $x = y$.*

(V_3) *If $ax = bx$ then $a = b$.*

To prove (V_1), recall that ϕ_x is onto L; hence for some a, $ax=\phi_x(a)=y$. To prove (V_2), set $c=1/a$; then $ax=ay$ gives $c(ax)=c(ay)$, and by (19.8) and (19.2), $x=y$. Since ϕ_x is one-one, (V_3) holds.

We now extend (17.2):

THEOREM 19B. *If ϕ is a homomorphism of the ray L into the ray L', then*

$$\phi(ax) = a\phi(x) \qquad (a \in R^+,\ x \in L). \tag{19.11}$$

To show this, take any fixed $x\in L$, and set $\psi(a)=\phi(ax)$, $\theta(a)=a\phi(x)$. Using (19.3) in L and in L' shows that ψ and θ are homomorphisms of R^+ into L'. Since $\psi(1)=\theta(1)$, uniqueness in Theorem 17C shows that $\psi=\theta$, as required.

CHAPTER III. BIRAYS

A biray is constructed from a ray by adjoining a zero element and negative elements. The biray constructed from the ray R^+ is the group of real numbers; multiplication in R is defined through the operation of R on itself.

20. Definition of birays. Essentially, a biray B is a commutative group containing a ray B^+, such that if $x\neq 0$ then x or $-x$ is in B^+.

DEFINITION 20A. *A* biray $(B, B^+, +)$ *is a set B, a subset B^+, and an operation of addition in B, such that:*

(B_1) *$(B, +)$ is a commutative semi-group.*
(B_2) *$(B^+, +)$ is a ray.*
(B_3) *For each x, $y\in B$ there is a $z\in B$ such that $x+z=y$.*
(B_4) *If $x\neq y$, $x+z=y$, and $y+z'=x$, then $z\in B^+$ or $z'\in B^+$.*

We prove first the *cancellation law*:

$$\text{If } x + z = x + z' \quad \text{then} \quad z = z'. \tag{20.1}$$

For suppose $z\neq z'$. By (B_3), we may write $z+u=z'$, $z'+v=z$. By (B_4), one of u, v is in B^+; say $u\in B^+$. Now use (B_3) to write $x+z+w=u$. These relations give

$$u + u = x + z + w + u = x + z' + w = x + z + w = u.$$

But $u\in B^+$, contrary to (B_2) and (R_1) (see Section 14).

We now find the *zero* element of B. Choose $x_0\in B^+$. Choose $0\in B$ so that $x_0+0=x_0$, by (B_3). We show that for all $x\in B$,

$$x + 0 = x. \tag{20.2}$$

By (B_3), we may write $x=x_0+y$. Now $x+0=y+x_0+0=y+x_0=x$. Moreover,

$$\text{if } x + u = x \quad \text{then} \quad u = 0, \tag{20.3}$$

by the cancellation law; hence the zero element 0 is uniquely defined by (20.2).

We now know that $(B, +)$ is a commutative group.

DEFINITION 20B. *For $x\in B$, $-x$ is the element such that*

$$x + (-x) = 0; \tag{20.4}$$

set also

$$x - y = x + (-y). \tag{20.5}$$

We have

$$(x - y) + y = x + [(-y) + y] = x + 0 = x. \tag{20.6}$$

Using this gives

$$y = z - x \quad \text{iffif} \quad x + y = z \tag{20.7}$$

(add x in the first equation; add $-x$ in the second). Also

$$-0 = 0, \qquad -(-x) = x; \tag{20.8}$$

for $0+0=0$; also $(-x)+x=0$, $(-x)+(-(-x))=0$, and the cancellation law gives the second part of (20.8). Some further elementary properties are:

$$-(x + y) = (-x) + (-y) = -x - y, \tag{20.9}$$

$$-(x - y) = y - x, \tag{20.10}$$

$$(x - z) + (y - w) = (x + y) - (z + w). \tag{20.11}$$

We may prove each directly from (20.4) and (20.5). Or, we may add something to both sides and apply (20.1). For instance,

$$-(x - y) + x = -(x - y) + [(x - y) + y] = 0 + y = y,$$

and also $(y-x)+x=y$; hence (20.10) follows.

21. Order in birays. We first consider negative elements.

DEFINITION 21A. *B^- is the set of all x such that $-x \in B^+$. The elements of B^+ are* positive; *those of B^- are* negative.

We prove trichotomy: Each element of B is zero, positive or negative, and is only one of these.

Given x, at least one holds; for if $x \neq 0$, then since $x+(-x)=0$ and $0+x=x$, applying (B_4) shows that x is in B^+ or in B^-. At most one is true. For, since $(B^+, +)$ satisfies (R_1) (Section 14), and $0+0=0$, $0 \notin B^+$. Also, if $x \in B^- \cap B^+$, then $-x$ and x are in B^+, and by (B_2), $0=x+(-x) \in B^+$, a contradiction.

DEFINITION 21B. *$x<y$ means that $y-x \in B^+$.*

Because of (B_2), order is transitive. The trichotomy above is equivalent to trichotomy in terms of order (Section 9). Thus B is simply ordered. The properties (9.2), (9.3) and (9.4) clearly hold. Note that

$$-x < -y \quad \text{iffif} \quad y < x. \tag{21.1}$$

One may introduce *absolute values* in terms of order. Note that if $x \leqq y$ and $y \leqq x$, then $x=y$. Hence the following definition makes sense:

$$\text{If } x \leqq y, \quad \text{then set } \inf\{x, y\} = x, \quad \sup\{x, y\} = y; \tag{21.2}$$

set

$$|x| = \sup\{x, -x\}. \tag{21.3}$$

The derivation of properties of absolute value is standard; we need not go into it here. The definitions of $\inf\{x_1, \cdots, x_n\}$, $\sup\{x_1, \cdots, x_n\}$ are clear.

22. The isomorphism theorem. We prove:

THEOREM 22A. *Let $(B, B^+, +)$ and $(B', B'^+, +)$ be birays. Take any $w \neq 0$ in B and any w' in B'. Then there is a unique homomorphism ϕ of $(B, +)$ into $(B', +)$ such that $\phi(w) = w'$. If $w' \neq 0$, then ϕ is an isomorphism onto B'. If w and w' are both positive or both negative, then ϕ carries B^+ onto B'^+ and B^- onto B'^-, and*

$$\phi(x) < \phi(y) \quad \textit{iffif} \quad x < y; \tag{22.1}$$

if one of w, w' is positive and the other negative, ϕ has the opposite effect.

See also Theorem 24A.

Suppose first that w and w' are positive. By Theorem 17C, there is a unique homomorphism ϕ^+ of B^+ into B'^+ such that $\phi^+(w) = w'$; hence the restriction $\phi|B^+$ of ϕ to B^+ must be ϕ^+. We must have $\phi(x) = \phi(x+0) = \phi(x) + \phi(0)$, and hence $\phi(0) = 0$ (letting 0 denote the zero element in both birays). Also we must have $\phi(x) + \phi(-x) = \phi(x + (-x)) = \phi(0) = 0$, and hence $\phi(-x) = -\phi(x)$. Thus ϕ, if it exists, is unique. We may use these equations to define ϕ outside B^+.

To prove that ϕ is a homomorphism, we examine $\phi(\alpha+\beta)$ for any $\alpha, \beta \in B$. The case that $\alpha = 0$ or $\beta = 0$ is clear, and the case $\alpha, \beta \in B^+$ is known. The remaining cases are as follows: For $x, y \in B^+$,

$$\begin{aligned} \phi(-x + (-y)) = \phi(-(x+y)) &= -\phi(x+y) = -[\phi(x) + \phi(y)] \\ &= -\phi(x) - \phi(y) = \phi(-x) + \phi(-y); \end{aligned}$$

$$\phi(x + (-x)) = \phi(0) = 0 = \phi(x) - \phi(x) = \phi(x) + \phi(-x);$$

$$\text{if } y < x \quad \text{then} \quad \phi(x + (-y)) = \phi(x - y) = \phi(x) - \phi(y) = \phi(x) + \phi(-y);$$

$$\begin{aligned} \text{if } x < y \quad \text{then} \quad \phi(x + (-y)) &= \phi(-(y - x)) = -\phi(y - x) \\ &= -[\phi(y) - \phi(x)] = \phi(x) + \phi(-y); \end{aligned}$$

also $\phi((-x)+y) = \phi(y+(-x))$, and the above applies.

Since ϕ^+ is an isomorphism onto B'^+, ϕ carries B^+ onto B'^+ and B^- onto B'^-. Thus ϕ is onto B'. That ϕ is also one-one and is therefore an isomorphism is clear. If $x<y$, say $x+z=y$, $z \in B^+$. Expanding $\phi(x+z)$ shows that (22.1) holds.

Next suppose that w is positive and w' is negative. Set $w'' = -w'$, and let ϕ' be the homomorphism of B into B' with $\phi'(w) = w''$; then we see easily that $\phi(x) = -\phi'(x)$ is the unique required isomorphism.

If w is positive and $w' = 0$, we show that $\phi(w) = 0$ for all $x \in B$. If $x \in B^+$ and $\phi(x) \in B^+$, then by Theorem 17C, $\phi(y) \in B^+$ for all $y \in B^+$, contrary to $\phi(w) = 0$. If $x \in B^+$ and $\phi(x) \in B^-$, then $\phi' = -\phi$ is a homomorphism with $\phi'(w) = 0$ and

$\phi'(x) \in B^+$, and we again have a contradiction. Thus $\phi(x)=0$, all $x \in B^+$. The case that $x \in B^-$ is similar.

Finally, if w is negative, using the pair $(-w, -w')$ in place of (w, w') gives again the required properties.

23. Construction of a biray from a ray. Given a ray $(B^+, +)$, we construct a corresponding biray $(B, B^+, +)$. The elements of B are those of B^+, called positive, a new element 0, and for each element x of B^+, a new element x^*; the latter form B^-, the negative elements of B. We shall use $x, y, \cdots$ to denote elements of B^+, and $\alpha, \beta, \cdots$ to denote elements of B (including B^+). We define addition in B as follows:

$$\alpha + 0 = \alpha, \tag{23.1}$$

$$x + x^* = 0, \tag{23.2}$$

$$x + y^* = \begin{cases} x - y & \text{if } y < x, \\ (y - x)^* & \text{if } x < y, \end{cases} \tag{23.3}$$

$$x^* + y^* = (x + y)^*; \tag{23.4}$$

requiring addition to be commutative gives the remaining cases.

To prove properties of addition it is convenient to introduce new names for elements of B (corresponding to the "ordered pair" definition of negative numbers): For any $x, y \in B^+$, set

$$[x, y] = x + y^*. \tag{23.5}$$

As a consequence, for $x, y, z \in B^+$,

$$[y + z, y] = z, \qquad [x, x] = 0, \qquad [x, x + z] = z^*; \tag{23.6}$$

this shows that each element of B has new names.

We show now that

$$[x, y] = [x, y'] \quad \text{iffif} \quad y = y'. \tag{23.7}$$

Suppose $[x, y] = [x, y']$. From (23.5) and trichotomy we see first that whichever case in (23.2), (23.3) applies to x and y also must apply to x and y'; now if $y<x$, $x-y=x-y'$ implies $y=y'$, and the situation is similar in the other cases.

Applying (20.11) to (23.2) or (23.3) shows that

$$[x + z, y + z] = [x, y]. \tag{23.8}$$

Next, since $[x, y] = [x+u, y+u]$ and $[u, v] = [x+u, x+v]$, (23.7) gives

$$[x, y] = [u, v] \quad \text{iffif} \quad x + v = y + u. \tag{23.9}$$

We are now ready to prove that

$$[x, y] + [u, v] = [x + u, y + v]. \tag{23.10}$$

For $[x, y]$ there are three cases to consider: $[y+z, y]$, $[x, x]$ and $[x, x+z]$; similarly for $[u, v]$. Some of these cases are:

$$[x, x] + [u, v] = 0 + [u, v] = [u, v] = [x + u, x + v],$$
$$[y + z, y] + [v + w, v] = z + w = [y + z + v + w, y + v],$$
$$[x, x + z] + [u, u + w] = z^* + w^* = (z + w)^* = [x + u, x + z + u + w],$$

if $w < z$ then

$$[y + z, y] + [u, u + w] = z + w^* = z - w = [y + (z - w), y]$$
$$= [y + z + u, y + u + w],$$

if $w < z$ then

$$[y + z, y] + [u, u + w] = (w - z)^* = [y, y + (w - z)]$$
$$= [y + z + u, y + u + w];$$
$$[y + z, y] + [u, u + z] = z + z^* = 0 = [y + z + u, y + u + z];$$

the remaining cases are taken care of by commutativity.

Applying (23.10) twice gives

$$([x, y] + [u, v]) + [p, q] = [(x + u) + p, (y + v) + q];$$

now associativity in B^+ gives associativity in B. We now have (B_1) and (B_2) for $(B, B^+, +)$.

To prove (B_3), we need merely note that

$$(23.11) \quad [x, y] + [u + y, v + x] = [x + u + y, y + v + x] = [u, v].$$

Next we prove the cancellation law, through the implications

$$[x, y] + [u, v] = [x, y] + [p, q] \Rightarrow [x + u, y + v] = [x + p, y + q]$$
$$\Rightarrow x + u + y + q = y + v + x + p \Rightarrow u + q = v + p \Rightarrow [u, v] = [p, q].$$

To prove that (B_4) holds, suppose that

$$[x, y] \neq [u, v], \qquad [x, y] + \alpha = [u, v], \qquad [u, v] + \beta = [x, y].$$

By (23.11) and the cancellation law, we then have

$$\alpha = [u + y, v + x], \qquad \beta = [x + v, y + u].$$

Now by (23.9), $x+v \neq y+u$; hence either $x+v < y+u$ and $\alpha \in B^+$ or $x+v > y+u$ and $\beta \in B^+$.

We have now proved that $(B, B^+, +)$ is a biray.

REMARK: An alternative proof may be given as follows: First show that

$$\text{if } z + \alpha = z + \beta, \qquad z > |\alpha| + |\beta|, \quad \text{then } \alpha = \beta;$$

this is easy. Next, considering several cases, we find:

$$\text{If } z > |\alpha| + |\beta| \quad \text{then} \quad z + (\alpha + \beta) = (z + \alpha) + \beta.$$

Using these gives at once:

If $z > |\alpha| + |\beta| + |\gamma|$ then $z + [(\alpha + \beta) + \gamma] = z + [\alpha + (\beta + \gamma)]$;

this gives the associative law. To prove (B_3), use $x+(x^*+\beta)=\beta$, $x^*+(x+\beta)=\beta$. Finally, (B_4) is proved by considering several cases.

There is essentially only one biray containing a given ray:

THEOREM 23A. *If* $(B, B^+, +)$ *and* $(B', B^+, +)$ *are birays, then there is an isomorphism* ϕ *of the first onto the second, defined by* $\phi(0)=0'$, *and* $\phi(x)=x$, $\phi(-x)=-x$, $(x\in B^+)$.

Here, the first "$-x$" is interpreted in B, and the second, in B'; see Definition 20B. The theorem follows at once from Theorem 22A, choosing some $w=w'$ in B^+.

24. The operation of R on a biray. First, define $(R, R^+, +)$ to be the biray constructed from $(R^+, +)$ as in Section 23; by Theorem 23A, this is (up to isomorphisms) the only biray containing $(R^+, +)$ as its positive part.

We now define the operation of R on any biray $(B, B^+, +)$. For each $\alpha\in B$, let ϕ_α be the homomorphism of $(R, R^+, +)$ into $(B, B^+, +)$ given by Theorem 22A, with $\phi_\alpha(1)=\alpha$; set

$$(24.1)\qquad a\alpha = \phi_\alpha(a) \qquad (a \in R, \alpha \in B).$$

This extends the operation of R^+ on the part B^+ of B. Exactly as in Section 19, we find (always using $a, b, \cdots \in R; \alpha, \beta, \cdots \in B$)

$$(24.2)\qquad (a + b)\alpha = a\alpha + b\alpha, \qquad a(\alpha + \beta) = a\alpha + a\beta.$$

Since $(R, R^+, +)$ is a biray, it operates on itself; this operation we call *multiplication* in R. Now

$$(24.3)\qquad (a + b)c = ac + ac, \qquad a(b + c) = ab + ac.$$

There is no harm in identifying the zero element in different birays; in particular, in R and in B. Since ϕ_α is a homomorphism, $\phi_\alpha(0)=0$. We now have (compare Section 19)

$$(24.4)\qquad 1\alpha = \alpha, \qquad 0\alpha = \alpha 0 = 0,$$

$$(24.5)\qquad 1a = a1 = a, \qquad 0a = a0 = 0.$$

Again as in Section 19,

$$(24.6)\qquad a(b\alpha) = (ab)\alpha, \qquad a(bc) = (ab)c, \qquad ab = ba.$$

We now show that we can *divide* in R. Given $a, b\in R$ with $a\neq 0$, the homomorphism Φ_a of R into itself with $\Phi_a(1)=a$ is an isomorphism onto R (Theorem 22A); hence for some $c\in R$, $ca=\Phi_a(c)=b$.

THEOREM 24A. *If ϕ is a homomorphism of the biray B into the biray B', then*

$$\phi(a\alpha) = a\phi(\alpha) \qquad (a \in R, \alpha \in B). \tag{24.7}$$

The proof of Theorem 19B applies, using Theorem 22A (which holds also for the case that $\phi(\alpha) \equiv 0$).

THEOREM 24B. *$(R, +, \times)$ is a complete ordered field, unique up to isomorphisms.*

We have proved that we have a field; it is ordered. Since R^+ is complete, the proof that R is complete (using Dedekind cuts) is simple. Suppose $(R', +, \times)$ is another such field. Then $1 \in R^+$, $1' \in R'^+$. Let ϕ be the corresponding homomorphism of birays, with $\phi(1) = 1'$; this is an isomorphism onto R', by Theorem 22A.

Now take any fixed b, and set

$$\psi(a) = \phi(ab), \qquad \theta(a) = \phi(a)\phi(b).$$

Then

$$\begin{aligned}\psi(a_1 + a_2) &= \phi((a_1 + a_2)b) = \phi(a_1b + a_2b)\\ &= \phi(a_1b) + \phi(a_2b) = \psi(a_1) + \psi(a_2),\end{aligned}$$

and similarly θ is a homomorphism. Since $\theta(1) = 1'\phi(b) = \phi(b) = \psi(1)$, Theorem 22A shows that $\psi = \theta$; that is, ϕ is a multiplicative homomorphism. Since $\phi(1) = 1'$, ϕ maps R^+ onto R'^+. Thus ϕ is an isomorphism of ordered fields.

Note that ϕ is the only such isomorphism; for we must have $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$, and since $\phi(1) \neq 0'$, $\phi(1) = 1'$.

THEOREM 24C. *For any biray B, we have:*

(V_1^*) *For each α and β in B, $\alpha \neq 0$, there is an $a \in R$ such that $a\alpha = \beta$.*

(V_2^*) *If $a\alpha = a\beta$, $a \neq 0$, then $\alpha = \beta$.*

(V_3^*) *If $a\alpha = b\alpha$, $\alpha \neq 0$, then $a = b$.*

The proofs are like the corresponding proofs in Theorem 19A; we use the fact that if $\alpha \neq 0$ then ϕ_α is onto B and is one-one.

THEOREM 24D. *Any biray $(B, B^+, +)$, with the operation of R, is an oriented one-dimensional vector space over R, and conversely.*

By (24.2), (24.6) and (24.4), the biray is a vector space; it is oriented by the choice of B^+ as "positive" part. Because of (V_1^*), it is one-dimensional. Conversely, any such space satisfies the postulates for a biray, and the operation by R is the same as that defined here.

WHAT MAKES A LOOP A GROUP?

H. ZASSENHAUS, Ohio State University

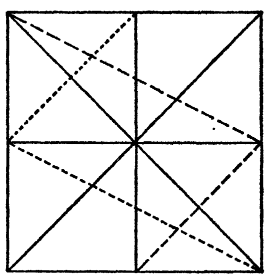
In a paper with the title: When is a Loop a Group? F. D. Parker [1] quotes the associativity conditions: $a_{ij}\, a_{jk} = a_{ik}$ for a loop G of n elements given by a normal multiplication table with element a_{ik} at the intersection of the ith row and the kth column ($i, k = 1, 2, \cdots, n$) (see [3]). He uses the $(n-2)(n-1)$ conditions

$$a_{1j}\, a_{jk} = a_{1k} = a_{jk}\, a_{k1} \qquad (1 < j < k) \tag{1}$$

in order to establish an algorithmic test of associativity.

But these are only necessary conditions.

Indeed, any triple system T (they are defined as nonempty sets in which for any two distinct elements P, Q there is a third element PQ of the system such that: $PQ \neq P$, $PQ = QP$, $P(PQ) = Q$) gives rise to a loop G satisfying (1) which is formed by an additional element $a_{11} = 1$ and the elements of T subject to the additional multiplication rules: $1 \cdot 1 = 1$, $1 \cdot P = P \cdot 1 = P$, $PP = 1$ $(P \in T)$. Since the square of each element is 1 it follows that the loop corresponding to a finite triple system is a group only if the number of its elements is a power of 2. (Indeed, the n-vector space over the field of 2 elements forms an additive group of order 2 in which every nonzero vector is of order 2. Moreover the nonzero vectors form a triple system of the triples $(u, v, u+v)$ where u, v may be any two distinct nonzero n-vectors. The best known among these triple systems is the Steiner triple systems of 7 elements distributed among 7 triples.) But there is a triple system of 9 elements viz.



The corresponding loop is not a group.

On the other hand let 1 be the element that is repeated n times in the diagonal of a normal multiplication table of a loop G of n elements. It is the neutral element of multiplication. Let a^{-1} be the element of G for which $aa^{-1} = 1$ $(a \in G)$. Then (1) is equivalent to

$$a(a^{-1}b) = b = (ba)a^{-1}. \tag{2}$$

Since this rule is satisfied by the loops that correspond to the triple systems the conditions (2) do not suffice to make a loop a group.

In this paper a set of conditions sufficient to make a finite loop a group is given which consists of less than $n(\log_2 n)^2$ associativity equations together with a transitivity condition and less than n^2 equations between loop elements. (Light's test ([2], p. 7) requires the verification of about $n^2 \log_2 n$ associativity equations.) The work involved in testing these conditions is substantially less than the test of the n^3 associativity equations

$$a(bc) = (ab)c \qquad (a, b, c \in G) \tag{3}$$

which by definition provide the test of the group property of the loop G of n elements.

The conditions are based on the remark that for any element a of a loop G the left multiplication $L(a) = ({}_a{}^x{}_x)$ and the right multiplication $R(a) = ({}_x{}^x{}_a)$ are permutations of the set G, (for the notation see [3] p. 5). The loop G is a group if and only if it satisfies one of the three equivalent conditions:

$$L(ab) = L(a)L(b) \tag{4}$$

$$R(ab) = R(b)R(a) \tag{5}$$

$$L(a)R(b) = R(b)L(a) \tag{6}$$

for all elements a, b, of G.

The conditions (6) will be applied to a set of generators of G. In order to achieve and to evaluate economy the following remarks are interpolated.

Any expression of the form,

$$W = a_{\alpha_1} \cdots a_{\alpha_d} (1 \leqq \alpha_i \leqq d;\ 1 \leqq i \leqq 1)$$

is called a *word of length* l formed from letters of the letter set $a_1, a_2, \cdots, a_d$. If the letters are elements of a loop G then we define the *value of the word* W recursively as follows: If $l = 0$, i.e. if $W = \phi$ is the empty word, then the value of W is 1, if $l = 1$ then the value of W is equal to the element a_{α_1} of G and if $l > 1$ then the value of W is equal to the product of a_{α_1} and the value of the word $a_{\alpha_2} \cdots a_{\alpha_d}$ of length $l-1$.

The *compound* of the *root words* W of length l and $W' = a_{l+1} \cdots a_{l+m}$ of length m is defined as the word $WW' = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_{l+m}}$ of length $l+m$. This composition of words is associative with the empty word ϕ acting as neutral element. If G is a group then the value of a compound is equal to the product of the values of the roots. In general this does not remain true for loops.

The words in a_1 are ordered in the order of the sequence

$$\phi < a_1 < \overset{2}{a_1} < \cdots,$$

where we define recursively

$$\overset{m}{a_1} = a_1 \overset{m-1}{a_1} \qquad (1 < m).$$

Assuming that $d > 1$ and that the words in $a_1, a_2, \cdots, a_{d-1}$ already are

ordered we extend the ordering by giving precedence to any word W_1 in a_1, $a_2, \cdots, a_d$ over a word W_2 in the same letters that employs the letter a_d more often than W_1. Two words that employ the letter a_d an equal positive number m of times may be written as: $W_1 = W_1' a_d W_1''$, $W_2 = W_2' a_d W_2''$ where W_1'', W_2'' are words in $a_1, \cdots, a_{d-1}$ and W_1', W_2' are words in $a_1, \cdots, a_d$ that employ a_d just $(m-1)$-times. We give W_1 precedence to W_2 if either W_1' precedes W_2' or if W_1' and W_2' are identical words, but W_1'' precedes W_2''.

The ordering relation of the words in $a_1, \cdots, a_d$ is multiplicative in the sense that $W_1 < W_1'$ implies $W_1 W_2 < W_1' W_2$ as well as $W_2 W_1' < W_2 W_1'$ for all words W_2.

For each element x of G that belongs to the subset $[a_1, a_2, \cdots, a_d]$ that is formed by the values of the words in $a_1, \cdots, a_d$ there is an earliest word in $a_1, \cdots, a_d$ with value x, say the word $W(x; a_1, \cdots, a_d)$. If G is finite then it is not difficult to construct an ordered list of the words $W(x; a_1, \cdots, a_d)$ and of their values.

Observe that the subset $[a_1, \cdots, a_d]$ of G contains 1 and is closed under left multiplication by the elements $a_1, \cdots, a_d$ of G. It follows that the presence of a word $a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_l}$ of length $l > 0$ in the list implies the presence of the tail word $a_{\alpha_2} \cdots a_{\alpha_l}$ of length $l-1$ in the list. The words making up the list for $[a_1, \cdots, a_{d-1}]$ also appear in the list for $[a_1, \cdots, a_d]$. If G happens to be a finite group then $[a_1, \cdots, a_d]$ will be a finite group generated by $a_1, a_2, \cdots, a_d$. Moreover, if a word of the form $W = W_1' a_d W_1''$ appears in the list in which W_1'' is a word in the letters $a_1, \cdots, a_{d-1}$ then for all members W^* of the list for $[a_1, \cdots, a_{d-1}]$ the word $W_1' a_d W^*$ appears in the list for $[a_1, \cdots, a_d]$.

A loop with only one element is a group. From now on let G be a loop with finite cardinality $n > 1$. It is easy to find elements $a_1, a_2, \cdots, a_e$ of G such that

$$1 = G_0 < G_1 = [a_1] < G_2 = [a_1, a_2] < \cdots < G_e = [a_1, \cdots, a_e] = G.$$

The first set of conditions necessary for a group are the divisibility conditions

$$|G_{i-1}| \,/\, |G_i| \qquad (i = 1, 2, \cdots, e) \tag{7}$$

on the cardinalities of the subsets $G_0, \cdots, G_e$. Since these subsets form a properly ascending chain of subsets of G beginning at $\{1\}$ and terminating at G it follows that

$$2^e \leqq n. \tag{8}$$

It is clear from the construction that the permutation group L generated by the permutations $L(a_1), \cdots, L(a_e)$ of G is transitive. The second requirement for G in order to be a group consists in the transitivity of the permutation group R generated by the permutations $R(a_1), R(a_2), \cdots, R(a_e)$ of G. (An alternative for conditions 2 and 3 are the $nd(1+d)$ equations: $xa_i^{-1} = (a_i x^{-1})^{-1}$, $(a_i x)a_k^{-1} = a_i(xa_k^{-1})(1 \leqq i \leqq d,\ 1 \leqq k \leqq d,\ x \in G)$.) Indeed if G is a group, then the mapping of a onto $R(a^{-1})$ is an isomorphism of G onto R, which is a regular permutation group on the elements of G.

The third set of conditions necessary for the group property of the finite loop G consists of the conditions

$$L(a_i)R(a_j) = R(a_j)L(a_i) \qquad (1 \leqq i \leqq e, 1 \leqq j \leqq e) \tag{9}$$

which amount to the verification of d^2n associativity relations for the elements of G.

If all three sets of conditions are satisfied then it follows from a theorem of Birkhoff-Jordan on transitive permutation groups (see [3], ch. II Theorem 5) that both L and R are regular permutation groups of G such that the mapping of $L(a_i)$ onto $R(a_i)^{-1}$ for $i=1, 2, \cdots, e$ can be extended to an isomorphism of L onto R.

If G is a group then it must be isomorphic to L and to R.

But G may not be a group, e.g. the loop $G=[a_1]$ given by the normal multiplication table:

1	a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1
a_1	1	a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2
a_2	a_1	1	a_9	a_4	a_8	a_7	a_6	a_5	a_3
a_3	a_2	a_1	1	a_9	a_7	a_8	a_5	a_6	a_4
a_4	a_3	a_6	a_1	1	a_9	a_2	a_8	a_7	a_5
a_5	a_4	a_2	a_3	a_1	1	a_9	a_7	a_8	a_6
a_6	a_5	a_3	a_2	a_8	a_1	1	a_9	a_4	a_7
a_7	a_6	a_4	a_5	a_2	a_3	a_1	1	a_9	a_8
a_8	a_7	a_5	a_4	a_3	a_2	a_6	a_1	1	a_9
a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	1

with 10 rows and 10 columns for which $L(a_1)=R(a_1)$ is a 10-cycle. *N.B.*: This loop is commutative and satisfies Parker's condition. Therefore a fourth set of conditions is required in order that G be the group with L as the left multiplication group. It consists of the $(n-d-1)^2$ equations:

$$W(x; L(a_1), \cdots, L(a_e)) \quad (y) = xy \qquad (x, y \in G;\ x, y \neq 1, a_1, a_2, \cdots, a_e) \tag{10}$$

where $W(x;\ L(a_1), \cdots,\ L(a_e))$ denotes the permutation $L(a_{\alpha_1}) \cdots L(a_{\alpha_l})$ which is obtained by substitution of $L(a_i)$ for a_i in the word $W(x; a_1, \cdots, a_e) = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_l}$.

Indeed the fourth set of conditions states explicitly that the mapping of x onto $W(x; L(a_1), \cdots L(a_e))$ provides an isomorphism of G onto L.

References

1. F. D. Parker, When is a loop a group? this MONTHLY, 72 (1965) 765–766.
2. Clifford and Preston, Algebraic theory of semigroups, Amer. Math. Soc., 1961.
3. H. Zassenhaus, The Theory of Groups, Chelsea, New York, 1958.
4. D. Watson, Condition for a loop to be a group, this MONTHLY, 74(1967)843–844.

PASCAL'S THEOREM ON AN OVAL

R. ARTZY, SUNY at Buffalo (Now at Temple University)

Ovals in projective planes have been studied from various points of view (cf. [1], [2], [3], [4]). Recently F. Buekenhout [2] proved that the validity of Pascal's theorem on an oval makes the projective plane pappian. An essential part of his proof is based on a theorem of J. Tits concerning transitivity properties of the projective group. Here a direct and self-contained proof will be given using the coordinatization method introduced by the author in [1].

A *projective plane* π consists of a finite or infinite set of points and lines such that any two distinct points P and Q lie on a unique line called PQ and, dually, any two distinct lines k and j go through a unique point called $k\times j$. In order to avoid trivial cases we will assume that π contains at least 4 points no 3 of which colline.

An *oval* in a projective plane is a nonempty set $\mathcal{C}$ of points of the plane satisfying:

I. Each point P of $\mathcal{C}$ is on just one line which contains no other point of $\mathcal{C}$. This line is called PP, the *tangent* at P. We denote $\delta\mathcal{C}=\{PP|P\in\mathcal{C}\}$.

II. No 3 points of $\mathcal{C}$ colline.

δI. Each line of $\delta\mathcal{C}$ contains just one point which is not also on another tangent.

δII. No 3 lines of $\delta\mathcal{C}$ concur.

Obviously the set of all points mentioned in δI is $\mathcal{C}$. It follows from I and δI that a point is on $\mathcal{C}$ if and only if it lies on just one tangent. If it lies on no tangent it is called *interior*, and if it lies on two distinct tangents it is *exterior*. It is easy to show [1] that $\mathcal{C}$ contains at least 3 distinct points. A line intersecting $\mathcal{C}$ in 2 points is called a *secant*. If P and Q are distinct points of $\mathcal{C}$, and R another point on the secant PQ, we denote $PR\times\mathcal{C}=Q$.

Let the couple $(\pi, \mathcal{C})$ designate a projective plane π containing an oval $\mathcal{C}$. We say that in $(\pi, \mathcal{C})$ the *Pascal property* holds if for every choice of 6 points $A_1, A_2, A_3, B_1, B_2, B_3$ of $\mathcal{C}$, the points $A_iB_k\times A_kB_i$ $(i, k=1, 2, 3;\ i\neq k)$ colline. The property as described here will be referred to as "Pascal's property for $(A_1, A_2, A_3; B_1, B_2, B_3)$." If some of the 6 points coincide, the statement may either become trivial (as in the case $A_1=B_1$), or it may involve tangents (as in the case $A_1=B_2$) without being trivial.

The rest of this note will be devoted to the proof of

THEOREM. *If in a projective plane with oval, $(\pi, \mathcal{C})$, the Pascal property holds, then π is pappian (and hence desarguesian), and $\mathcal{C}$ is a conic.*

First we have to discuss a limited concept of polarity with respect to $\mathcal{C}$. If P is an exterior point, it lies on exactly two tangents, say QQ and RR. Then QR is called the *polar* of P, and P is called the *pole* of the secant QR.

Note that polars have been defined for exterior points only. By no means should the existence of polars for interior points be taken for granted (cf. [3]).

LEMMA 1. *If P and Q are distinct exterior points, if p is the polar of P and q the polar of Q, and if Q lies on p, then P lies on q.*

Proof. Let p have the points A and C on $\mathfrak{C}$, and q the points B and D on $\mathfrak{C}$. Then Pascal's property for $(ACB;\ CAD)$ yields the collinearity of the points $CC\times AA=P$, $AB\times CD=F$, and $AD\times BC=G$. Pascal's property for $(ABD;\ CDB)$ results in the collinearity of F, G, and $BB\times DD=Q$. Therefore P, F, G, and Q are collinear. Now, by Pascal's property for $(BCD;\ CDA)$, the points F, Q, and $BD\times CC=P'$ colline. But $CC\times FQ=P$ and consequently $P=P'$, and P lies on BD.

The next step will be the introduction of coordinates in $(\pi,\ \mathfrak{C})$. We label all points of $\mathfrak{C}$. Three of them will be called 0, 1, $\infty\,(0\neq 1\neq \infty\neq 0)$. We call $\mathfrak{C}-\{\infty\}=\Sigma$ and define addition and multiplication of elements of Σ as follows, using the notation $(r)(s)$ for the line joining the points labeled r and s.

$$((\infty)(\infty)\times(a)(b))(0)\times\mathfrak{C}=a+b,$$

$$((0)(\infty)\times(a)(b))(1)\times\mathfrak{C}=ab.$$

The only operation that cannot be thus introduced is multiplication by 0, and we define $a\cdot 0=0\cdot a=0$.

LEMMA 2. *All elements of Σ form an abelian group under addition, and all nonzero elements of Σ form an abelian group under multiplication.*

Proof. Commutativity of addition and multiplication, closure, and unique subtractibility and divisibility follow directly from the definitions. Obviously 0 and 1 are neutral elements of addition and multiplication, respectively. The associativity of addition follows from Pascal's property for $(0,\ a,\ c;\ b,\ a+b,\ c+b)$, yielding the collinearity of $(a)(b)\times(0)(a+b)$, $(0)(c+b)\times(c)(b)$, and $(a)(c+b)\times(c)(a+b)$. Since the first two intersections lie on $(\infty)(\infty)$, so also does the third, and this implies $(a+b)+c=a+(c+b)$. The associativity of multiplication is obtained in an analogous way from the Pascal property for $(1,\ a,\ c;\ b,\ ab,\ cb)$, with the 3 collinear points of intersection on the line $(0)(\infty)$.

LEMMA 3. *For all c in Σ, $(-1)c=-c$.*

Proof. For $b\in\Sigma$ let $(b)(b)\times(-b)(-b)=B$. Then B is the pole of $(b)(-b)$. The pole A of $(0)(\infty)$ is the point $(0)(0)\times(\infty)(\infty)$. Since $b+(-b)=0$, the line $(b)(-b)$ passes through A. Both A and B are exterior points, and hence, by Lemma 1, B lies on $(0)(\infty)$. But then $b^2=(-b)^2$, and since b was arbitrarily chosen, this holds for all b in Σ. Now, for all c in Σ, $(-1)c(-1)c=(-1)(-1)cc$, that is, $((-1)c)^2=(-1)^2c^2=1c^2=c^2$. This means that the tangent at c and the tangent at $(-1)c$ meet on $(0)(\infty)$. But the same point is also the intersection of the tangent at $-c$ and $(0)(\infty)$ in view of $(-c)^2=c^2$, and by δII no 3 distinct tangents meet in one point. Hence either $(-1)c=-c$ or $(-1)c=c$. The second alternative can hold only if $c=0$, because otherwise $1=-1$ and consequently

$1+1=0$. This would cause the tangents at ∞, 0, and 1 to meet in one point, contradicting δII. (For later use note that $1+1=2\neq 0$.)

We now introduce coordinates for the lines of the plane, excluding at first the lines through ∞. Every other line j intersects the lines $(0)(\infty)$ and $(\infty)(\infty)$ in 2 distinct points, say U and V, respectively. If $U\neq 0$, let $U(1)\times\mathcal{C}=u$ and $V(0)\times\mathcal{C}=v$. Then we say that $j=[u, v]$; j has the *line coordinates* u and v. If $U=0$, we define $j=[0, v]$. In the remaining case, where j passes through ∞, we define $j=[0]$ if $j=(\infty)(\infty)$, $j=[\infty]$ if $j=(0)(\infty)$, and $j=[a^{-1}]$ if $j=(a)(\infty)$ for $0\neq a\neq\infty$.

Evidently the secant through p and q is $[pq, p+q]$, and the tangent through p is $[p^2, p+p]$ if p and q are both in Σ.

LEMMA 4. *For all lines $[u, v]$ through a fixed point $P\neq\infty$ on $(0)(\infty)$ the first coordinate u equals a fixed element of Σ. The coordinates of all lines $[u, v]$ through a point P not on $(0)(\infty)$ satisfy an equation $v=xu+y$ with fixed x and y. In particular, all lines through a point on $\mathcal{C}$ other than ∞ and 0 satisfy equations of the form $v=xu+x^{-1}$.*

Proof. The first statement follows trivially from the definition. For the second statement suppose at first P not on $\mathcal{C}$ and not on $(\infty)(\infty)$. Then let $P(0)\times\mathcal{C}=y$ and $P(\infty)\times\mathcal{C}=x^{-1}$. Let $j=[u, v]$ be any line through P (but not through ∞), and let $j\times(0)(\infty)=U$. By definition of the coordinates, $(1)(u)\times(0)(\infty)=U$, and as a consequence of the associativity of multiplication, $(x^{-1})(xu)\times(0)(\infty)=U$. Pascal's property for $(0, \infty, xu; x^{-1}, y, \infty)$ yields collinearity of P, U, and $(y)(xu)\times(\infty)(\infty)=Q$, say. But this means that $[u, xu+y]=QU$ passes through P, and $v=xu+y$. In the case when $P=y$ is on $\mathcal{C}$, the same proof applies with $y=x^{-1}$. Finally, when P lies on $(\infty)(\infty)$ and again $P(0)\times\mathcal{C}=y$, then by definition every line $[u, v]$ through P satisfies $v=y=0u+y$.

Lemma 4 permits us to introduce *point coordinates* for all points P not on $(0)(\infty)$. If for all lines $[u, v]$ through P the relation $v=xu+y$ holds, define $P=(x, y)$. Thus $y=-xu+v$ is the *equation of the line* $[u, v]$. Note that all the points of $\mathcal{C}$, other than 0 and ∞, and only those, have coordinates (x, x^{-1}).

LEMMA 5. *$(\Sigma, +, \cdot)$ is distributive.*

Proof. Let $p\neq 1$ and $q\neq 0$ be elements of Σ. Let $B=(x, y)=(1)(q)\times(0)(p+q)$. Then $B=[q, 1+q]\times[0, p+q]$, which after some computation yields

$$(*)\qquad x = q^{-1}(1-p),\quad y = p+q.$$

If B is on $\mathcal{C}$, then $B=y=q$, $p=0$, and $x=y^{-1}$, which conforms with (*). Pascal's property for $(1, \infty, p; x^{-1}, q, \infty)$ results in the collinearity of B, $S=(1)(\infty)\times(p)(x^{-1})$, and $T=(p)(q)\times(\infty)(\infty)=(0, p+q)$. The line $(p)(x^{-1})$ has coordinates $[px^{-1}, p+x^{-1}]$, and hence $S=(1, p+x^{-1}-px^{-1})$. Both B and T lie on the line $[0, p+q]$ and certainly do not coincide. Thus S also lies on this line and therefore $p+q=p+x^{-1}-px^{-1}$. After substitution of (*) this becomes $p+q$

$=p+q(1-p)^{-1}-pq(1-p)^{-1}$. Put $p=1+s$ and $q=r(-s)$. Then $r(-s)=r-(1+s)r$, and by Lemma 3, $r-(1+s)r=(-1)sr=-rs$, that is, $r(1+s)=r+rs$ for all nonzero r and s in Σ. Then for all a, b, $c\neq 0$ in Σ,

$$a(b+c)=a(b(1+b^{-1}c))=ab(1+b^{-1}c)=ab+ac.$$

If a, b, or c is zero, distributivity is trivial.

It follows from Lemmas 2 and 5 that $(\Sigma, +, \cdot)$ is a field. Now homogeneous coordinates can be introduced such that the points (x, y), $(1)(m)\times(0)(\infty)$, and ∞ become $(1, x, y)$, $(0, 1, m)$, and $(0, 0, 1)$, respectively. The line equations are linear and homogeneous, and each such linear homogeneous equation represents a line. The result is a projective plane which is coordinatized by a field Σ of characteristic $\neq 2$ and therefore pappian (and *a fortiori* desarguesian). $\mathcal{C}$ consists of the points whose homogeneous coordinates are $(x_0, x_1, x_2)=(1, x, x^{-1})$ for all nonzero x in Σ and the points $(0, 1, 0)$ and $(0, 0, 1)$, that is, all the points whose coordinates satisfy $x_0^2=x_1x_2$. Hence $\mathcal{C}$ is a conic.

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References

1. R. Artzy, Non-euclidean incidence planes, Israel J. Math., 4 (1966) 43–53.
2. F. Buekenhout, Plans projectifs à ovoides pascaliens, Arch. Math., 17 (1966) 89–93.
3. T. G. Ostrom, Ovals, dualities, and Desargues's theorem, Canad. J. Math., 7 (1955) 417–431.
4. B. Segre, Lectures on Modern Geometry, Edizione Cremonese, Rome, 1961.

APPLICATIONS OF THE COMPANION MATRIX

LOUIS BRAND, University of Houston

1. Introduction. In an article in this MONTHLY [1] it was shown that the polynomial

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \tag{1}$$

corresponds to its companion matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \tag{2}$$

whose characteristic and minimum equation is

$$\det(\lambda I - A) = f(\lambda) = 0. \tag{3}$$

If we define the column vector

$$(4)\qquad \mathbf{e}(\lambda) = (1, \lambda, \lambda^2, \cdots, \lambda^{n-1})^T,$$

equation (3) may be put in matrix form

$$(5)\qquad (\lambda I - A)\mathbf{e}(\lambda) = (0, 0, \cdots, 0, f(\lambda))^T = 0.$$

The main results of the article cited may be summarized in two theorems.

THEOREM 1. *The eigenvalues of A are the n zeros λ_i of $f^{(\lambda)}$. Each simple zero λ_i corresponds to the proper eigenvector $\mathbf{e}_i = \mathbf{e}(\lambda_i)$. A k-tuple zero λ_1 corresponds to the set of one proper and $k-1$ generalized eigenvectors*

$$\mathbf{e}_1 = \mathbf{e}(\lambda_1); \qquad \mathbf{e}_{j+1} = \frac{1}{j!}\mathbf{e}^{(j)}(\lambda_1), \qquad j = 1, 2, \cdots, k-1,$$

which satisfy the k equations $A\mathbf{e}_1 = \lambda_1\mathbf{e}_1$; $A\mathbf{e}_{j+1} = \lambda_1\mathbf{e}_{j+1} + \mathbf{e}_j$, $j = 1, 2, \cdots, k-1$.
The n eigenvectors of A so determined form a linearly independent set.

THEOREM 2. *If B is the matrix formed from the n eigenvectors of A as columns, each group for a multiple zero having increasing indices, then*

$$(6)\qquad B^{-1}AB = J$$

is the corresponding Jordan form of A which has $k-1$ ones for each k-tuple zero.

When $n=3$, the Jordan forms of A when λ_i are distinct, $\lambda_2=\lambda_3$, $\lambda_1=\lambda_2=\lambda_3$, are respectively

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix};$$

and the corresponding B matrices are

$$B = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ \lambda_1^2 & 2\lambda_1 & 1 \end{bmatrix}.$$

2. Solution of $f(D)x(t)=0$. If we define the vector

$$\mathbf{x}(t) = (x, x', \cdots, x^{(n-1)})^T = (1, D, \cdots, D^{n-1})^T x(t),$$

the differential equation becomes

$$(7)\qquad (DI - A)\mathbf{x} = 0$$

by virtue of (5). We shall now solve (7) in matrix form when the initial vector $\mathbf{x}(0) = (x, x', \cdots, x^{(n-1)})(0)$ is given.

From the operational calculus of Mikusiński [2], $D\mathbf{x} = s\mathbf{x} - \mathbf{x}(0)$. Hence (7) becomes

$$(sI - A)\mathbf{x} = \mathbf{x}(0), \tag{8}$$

and

$$\mathbf{x} = (sI - A)^{-1}\mathbf{x}(0). \tag{9}$$

This solution is better adapted for calculation if we use (6) to put $A = BJB^{-1}$; then since

$$(sI - BJB^{-1})^{-1} = \{B(sI - J)B^{-1}\}^{-1} = B(sI - J)^{-1}B^{-1},$$

the solution of (7) is given by

$$\mathbf{x} = B(sI - J)^{-1}B^{-1}\mathbf{x}(0). \tag{10}$$

This result is also given by Chen and Yates in a recent paper [3] on the inversion of a rational Laplace transform.

Once the matrix $B(sI-J)B^{-1}$ has been computed, it can be applied to diverse initial vectors $\mathbf{x}(0)$. Moreover it is of immediate use in solving the inhomogeneous problem with the driving function $g(t)$. We then add the vector $\boldsymbol{g} = (0, 0, \cdots, 0, G(s))^T$ to the right side of (8); here $G(s)$ is the Laplace transform of $g(t)$. The solution is now

$$\mathbf{x} = B(sI - J)B^{-1}[\mathbf{x}(0) + \boldsymbol{g}]. \tag{11}$$

Note that *the initial vector* $\mathbf{x}(0)$ *enters only in the solution* (10) *of the homogeneous problem, whereas the particular solution*

$$\xi = B(sI - J)B^{-1}\boldsymbol{g} \tag{12}$$

is characterized by the property $\xi(0) = \mathbf{0}$. In both (10) and (12) only the *first* row of the matrix $B(sI-J)B^{-1}$ need be computed, for this yields $x(t)$ or $\xi(t)$. The successive rows give derivatives up to order $n-1$. If, however, we have an integro-differential equation

$$[D^{n-1} + a_1D^{n-2} + \cdots + a_{n-1} + a_nD^{-1}]x(t) = g(t),$$

the *second* row of the matrix must be used to get $x(t)$.

To compute the particular solution ξ in (12), we may express the right member entirely in terms of the operator s, and then transform back to obtain $\xi(t)$. Or, alternatively, we may only express the matrix $B(sI-J)^{-1}B^{-1}$ in terms of t, and then compute the convolution

$$\xi(t) = B(sI - J)^{-1}B^{-1} * \boldsymbol{g}(t). \tag{13}$$

In fact the latter procedure *must* be used when $g(t)$ has no Laplace transform $G(s)$, as when $g(t) = \exp(t^2)$. This method is adopted in the following example.

2. Example 1. $f(D) = (D-2)(D-1)^2$.

To solve the differential equation $f(D)x = 0$ with the initial vector $(0, 1, -1)$, we take $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$; then

$$B=\begin{bmatrix}1&1&0\\2&1&1\\4&1&2\end{bmatrix},\qquad B^{-1}=\begin{bmatrix}1&-2&1\\0&2&-1\\-2&3&-1\end{bmatrix},\qquad J=\begin{bmatrix}2&0&0\\0&1&1\\0&0&1\end{bmatrix};$$

$$(sI-J)^{-1}=\begin{bmatrix}(s-2)^{-1}&0&0\\0&(s-1)^{-1}&(s-1)^{-2}\\0&0&(s-1)^{-1}\end{bmatrix}.$$

In Mikusiński's calculus [2, p. 33]

$$(s-2)^{-1}=e^{2t},\qquad (s-1)^{-1}=e^{t},\qquad (s-1)^{-2}=te^{t};$$

hence from (10)

$$(14)\qquad x(t)=(1,1,0)\begin{bmatrix}e^{2t}&0&0\\0&e^{t}&te^{t}\\0&0&e^{t}\end{bmatrix}\begin{bmatrix}1&-2&1\\0&2&-1\\-2&3&-1\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$
$$=(e^{2t},e^{t},te^{t})(-3,3,4)^{\mathrm{T}}$$

or

$$(15)\qquad x(t)=-3e^{2t}+3e^{t}+4te^{t}.$$

To solve the integro-differential equation $f(D)(D^{-1}x)=0$ with $(D^{-1}x, x, Dx)\cdot(0)=(0,1,-1)$, we replace $(1, 1, 0)$ in (14) by the second row $(2, 1, 1)$ of B; then

$$(16)\qquad x(t)=(2e^{2t},e^{t},(t+1)e^{t})(-3,3,4)^{\mathrm{T}}=-6e^{2t}+7e^{t}+4te^{t}.$$

To solve the inhomogeneous equation $f(D)x=t$ with the same initial vector $(0, 1, -1)$ we have, from (13), the particular solution

$$\xi(t)=(e^{2t},e^{t},te^{t})\begin{bmatrix}1&-2&1\\0&2&-1\\-2&3&-1\end{bmatrix}*\begin{bmatrix}0\\0\\t\end{bmatrix}$$
$$=(e^{2t},e^{t},te^{t})*(t,-t,-t)^{\mathrm{T}}$$
$$=e^{2t}*t-e^{t}*t-te^{t}*t$$
$$=(\tfrac{1}{4}e^{2t}-\tfrac{1}{2}t-\tfrac{1}{4})-(e^{t}-t-1)-(te^{t}+t-2e^{t}+2).$$

Hence

$$(17)\qquad \xi(t)=\tfrac{1}{4}e^{2t}-te^{t}+e^{t}-\tfrac{1}{2}t-\tfrac{5}{4},$$

and we readily verify the characteristic property $\xi(0)=\xi'(0)=\xi''(0)=0$. The complete solution is given by the sum of (15) and (17).

3. Solution of $f(E)x(t)=0$. If we define the vector

$$\mathbf{x}(t) = (1, E, \cdots, E^{n-1})^{\mathrm{T}} x(t)^{\mathrm{T}}, \; E = 1+\Delta$$

where t is a nonnegative integer, the difference equation becomes

$$(EI - A)\mathbf{x} = \mathbf{0} \tag{18}$$

by virtue of (5). We shall now solve (18) in matrix form when the initial vector $\mathbf{x}(0)=(1, E, \cdots, E^{n-1})^{\mathrm{T}}x(0)$ is given.

Instead of the calculus of Mikusiński we now employ the analogous division algebra for sequences developed by the author [4]. But instead of the shift operator $s=\{0, 1, 0, \cdots\}$ used in the article cited, it is simpler to use its inverse $r=1/s$. The basic formula [4, (13)] is now

$$Ex(t) = rx(t) - rx(0),$$

and (18) becomes

$$(rI - A)\mathbf{x} = r\mathbf{x}(0) \tag{19}$$

and

$$\mathbf{x} = (rI - A)^{-1} r\mathbf{x}(0),$$

or since $A=BJB^{-1}$,

$$\mathbf{x} = B(rI - J)B^{-1} r\mathbf{x}(0). \tag{20}$$

When the matrix $B(rI-J)B^{-1}$ has been computed, it can be used with diverse initial vectors. It is also of immediate use in solving the inhomogeneous problem with the input function $g(t)$. We then add the vector $\mathbf{g}=(0, 0, \cdots, G(r))^{\mathrm{T}}$ to the right hand side of (19); here $G(r)$ is the operational form of $g(t)$. The solution is now

$$\mathbf{x} = B(rI - J)B^{-1}(r\mathbf{x}(0) + \mathbf{g}). \tag{21}$$

Note again that the initial vector $\mathbf{x}(0)$ enters only in the solution (19) of the homogeneous problem, whereas the particular solution

$$\boldsymbol{\xi} = B(rI - J)B^{-1}\mathbf{g} \tag{22}$$

is characterized by the property $\boldsymbol{\xi}(0)=\mathbf{0}$. In both (19) and (22) only the first row of the matrix need be computed, for this yields $x(t)$ or $\xi(t)$.

To compute $\boldsymbol{\xi}$ in (22) we may express the right member entirely in terms of r and then transform back if the inverse functions in t are known. As an alternative, we may only express the matrix $B(rI-J)^{-1}B^{-1}$ in terms of t, and then compute $\boldsymbol{\xi}$ by sequence convolution. The first method is adopted in the following example.

4. Example 2. $f(E)=(E-2)(E-1)^2$.

To solve the difference equation $f(E)\mathbf{x}=\mathbf{0}$ with the initial vector $(0, 1, -1)$,

note that B, B^{-1}, J are the same as in Example 1, and from $(sI-J)^{-1}$ we have

$$(rI - J)^{-1}r = \begin{bmatrix} r(r-2)^{-1} & 0 & 0 \\ 0 & r(r-1)^{-1} & r(r-1)^{-2} \\ 0 & 0 & r(r-1)^{-1} \end{bmatrix}.$$

From [4, 28] we have

$$\frac{r}{r-2} = \{2^t\}, \qquad \frac{r}{r-1} = \{1\}, \qquad \frac{r}{(r-1)^2} = \{t\};$$

hence from (20),

$$x(t) = (1, 1,0)\begin{bmatrix} \{2^t\} & 0 & 0 \\ 0 & \{1\} & \{t\} \\ 0 & 0 & \{1\} \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{23}$$
$$= (\{2^t\}, \{1\}, \{t\})(-3, 3, 4)^T$$

or

$$x(t) = -3\cdot 2^t + 3 + 4t. \tag{24}$$

Finally, consider the inhomogeneous equation $f(E)x=t$. The particular solution ξ is now given by (22). Since

$$(1, 1, 0)(rI - J)^{-1} = ((r-2)^{-1}, (r-1)^{-1}, (r-1)^{-2}),$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ r(r-1)^{-2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} r(r-1)^{-2}$$

we have

$$\xi(t) = r\left(\frac{1}{(r-2)(r-1)^2} - \frac{1}{(r-1)^3} - \frac{1}{(r-1)^4}\right)$$
$$= \frac{r}{r-2} - \frac{r}{r-1} - \frac{r}{(r-1)^2} - \frac{r}{(r-1)^3} - \frac{r}{(r-1)^4}$$
$$= \{2^t\} - \{1\} - \{t\} - \tfrac{1}{2}\{t^{(2)}\} - \tfrac{1}{6}\{t^{(3)}\};$$

from [4, (19)]; hence

$$\xi(t) = 2^t - 1 - t - \tfrac{1}{2}t(t-1) - \tfrac{1}{6}t(t-1)(t-2).$$

The property $\xi(0)=\xi(1)=\xi(2)=0$ is readily verified.

For both D and E equations, $\xi(t)=0$ follows from its expression as a convolution.

References

1. Louis Brand, The companion matrix and its properties, this MONTHLY, 71 (1964) 629–634.
2. Jan Mikusiński, Operational Calculus, Pergamon Press, London and New York, 1959.
3. C. F. Chen and R. E. Yates, A new matrix formula for the inverse Laplace transformation, Paper No. 66, ASME convention, New York, December, 1966.
4. Louis Brand, A division algebra for sequences and its associated operational calculus, this MONTHLY, 71 (1964) 719–728.

ON A QUESTION CONCERNING FIXED POINTS

A. L. YANDL, Seattle University

1. Introduction. Suppose that X and Y are continua which have the fixed-point property (f.p.p.) and that the intersection $X\cap Y$ is also a continuum. Several authors have sought additional conditions to guarantee that the union $X\cup Y$ also has the f.p.p. If X, Y and $X\cap Y$ are all absolute retracts, then the well-known theorem of Borsuk asserts that $X\cup Y$ is also an absolute retract and hence has the f.p.p. If X and Y are one dimensional and $X\cap Y$ is a tree, a recent theorem of Štanko [3] asserts that $X\cup Y$ has the f.p.p. L. Ward [4] has asked whether $X\cup Y$ must have the f.p.p. if $X\cap Y$ is a retract of $X\cup Y$. The present note contains three examples which are of interest in connection with the general problem mentioned above. The first example, based on one of Kinoshita [1], shows the answer to Ward's question is negative. Specifically, there exist two 2-dimensional continua X and Y in E^3 such that X, Y and $X\cap Y$ all have the f.p.p. and such that $X\cap Y$ is a retract of $X\cup Y$, although $X\cup Y$ lacks the f.p.p. We note that the problem is still unsolved when the intersection is an arc. However, the second example exhibits two (noncompact) plane sets X and Y such that X, Y and $X\cap Y$ all have the f.p.p., in fact $X\cap Y$ is an arc, and yet $X\cup Y$ fails to have the f.p.p. The third example, based on one of Klee [2], shows that "tree" cannot be replaced by "chainable continuum" in Štanko's theorem. Specifically, there exist two arcwise connected one-dimensional plane continua X and Y such that X, Y and $X\cap Y$ all have the f.p.p. and $X\cap Y$ is in fact a chainable continuum, although $X\cup Y$ lacks the f.p.p. The author wishes to express his gratitude to Professor V. L. Klee for the guidance he provided during the preparation of this note. Propositions 1, 2 and 3 were suggested by him. These propositions are related to examples 2 and 3.

2. The Propositions. The following three propositions give some conditions under which a space which is either a continuous biunique image of the half line $[0, \infty[$ or the union of such an image with some other set, has the f.p.p.

PROPOSITION 1. *Suppose that ϕ is a continuous biunique mapping of $[0, \infty[$ onto a space A such that every arc in A is contained in $\phi([0, t])$ for some $t<\infty$.*

Suppose that f is a continuous mapping of A into A. Let $\eta=\phi^{-1}f\phi$: $[0, \infty[\to[0, \infty[$ and let $R=\{r\in[0, \infty[: \eta(r)\leqq r\}$. Then f has a fixed point if and only if the set R is nonempty.

Proof. The condition that R is nonempty is obviously necessary since $\phi(r)$ is a fixed point for f if and only if r is a fixed point for η. To show that it is sufficient, let us suppose that $R\neq\varnothing$ and let $r_0=\inf R$; we claim that $\eta(r_0)=r_0$. For each positive integer k, choose some $r_k\in R\cap[r_0, r_0+1/k]$. Then we have $\eta(r_k)\leqq r_k\leqq r_0+1$ and therefore $f\phi(r_k)\in\phi([0, r_0+1])$ for each k. But ϕ and f are continuous and ϕ^{-1} is continuous on $\phi([0, r_0+1])$. Thus $\eta(r_k)\to\eta(r_0)$ and since $r_k\to r_0$ we have $\eta(r_0)\leqq r_0$.

Now suppose that $\eta(r_0)<r_0$, and note that by the definition of r_0, $\eta(r)>r$ for all $r<r_0$. Choose u such that $\eta(r_0)<u<r_0$. Then $f\phi(]u, r_0])$ is an arcwise connected subset of A which includes $\phi\eta(r_0)$ and includes $\phi(t)$ for some $t\geqq r_0$, but does not include $\phi(u)$. This contradicts the assumption about arcs in A. Hence $\eta(r_0)\not< r_0$, whence $\eta(r_0)=r_0$ and the proof of proposition 1 is complete.

PROPOSITION 2. *Suppose that ϕ and A are as in proposition* 1, *and that at least one of the following conditions is satisfied*: (a) *A is compact*; (b) *ϕ is not a homeomorphism, but there exists a $t<\infty$ such that $\phi|[t, \infty[$ is a homeomorphism. Then A has the f.p.p.*

Proof. Suppose that f is a continuous mapping of A into A and let η and R be as in proposition 1. Let S denote the set of all points $s\in[0, \infty[$ such that there exists a sequence $s_k\to\infty$ with $\phi(s_k)\to\phi(s)$. Then S is nonempty and under both (a) and (b), S has a first member; further, under (b) S has a last member.

We want to show that $R\neq\varnothing$. Assume, on the contrary, that $\eta(r)>r$ for all $r\in[0, \infty[$, and consider the case in which A is compact. Let s_0 be the first member of S and let $A'=A-\phi(]0, s_0[)$. Then A' is a compact subset of A. For each k, $f\phi([s_k, \infty[)$ is an arcwise connected subset of $\phi(]s_k, \infty[)$, and from the assumption about arcs in A it follows that $f\phi([s_k, \infty[)$ contains $\phi([s_m, \infty[)$ for almost all m. Thus $\phi(s_0)\in\overline{f(A')}$, and since A' is compact it follows that $\phi(s_0)\in f(A')$; that is, $\phi(s_0)=f\phi(s)$ for some $s\geqq s_0$. But then $\eta(s)=s_0\leqq s$, whence $s\in R$ and the proof of case (a) is complete.

To show that $R\neq\varnothing$ in case (b), let t_0 be the last member of S and suppose that $\eta(r)>r$ for all $r\in[0, \infty[$. There is a sequence $s_k\to\infty$ such that $\phi(s_k)\to\phi(t_0)$, and therefore $f\phi(s_k)\to f\phi(t_0)$. But then $\phi\eta(s_k)\to\phi\eta(t_0)$, and with $\eta(s_k)>s_k$ and $\eta(t_0)>t_0$ this contradicts the choice of t_0. The proof of proposition 2 is now complete.

PROPOSITION 3. *The space X has the f.p.p. if it is the union of disjoint subsets A and B which satisfy the following conditions*:

(i) *A is dense in X*;

(ii) *B has the f.p.p. and is closed in X*;

(iii) *There is a homeomorphism ϕ of $[0, \infty[$ onto A such that every arc in X is contained in B or in $\phi([0, t])$ for some $t<\infty$.*

Proof. Let f be a continuous mapping of X into itself; we want to show that there exists a point x_0 such that $f(x_0)=x_0$. If $f(B)\subset B$, the existence of such a point follows from the fact that B has the f.p.p. Thus we may assume that $f(B)\cap A\neq\emptyset$. From (i) and (ii) it follows that $f(A)$ intersects A, and then since $f(A)$ is arcwise connected it follows from (iii) that $f(A)\subset A$. With η and R as in proposition 1, the desired conclusion will follow if we can show that R is nonempty.

Let $b\in B$ with $f(b)\in A$; say $f(b)=\phi(r)$. Since A is dense in X and since $\phi([0, t])$ is closed in X for each $t<\infty$, there exists a sequence $s_k\to\infty$ such that $\phi(s_k)\to b$. Then of course $f\phi(s_k)\to\phi(r)$. Since B is closed and $\phi(r)\notin B$, we have $f\phi(s_k)\in A$ for almost all k, and then since ϕ is a homeomorphism it follows that $\eta(s_k)\to r$. But then $\eta(s_k)<s_k$ for almost all k, and the proof of proposition 3 is complete.

3. The Examples.

Example 1. Using cylindrical coordinates, let

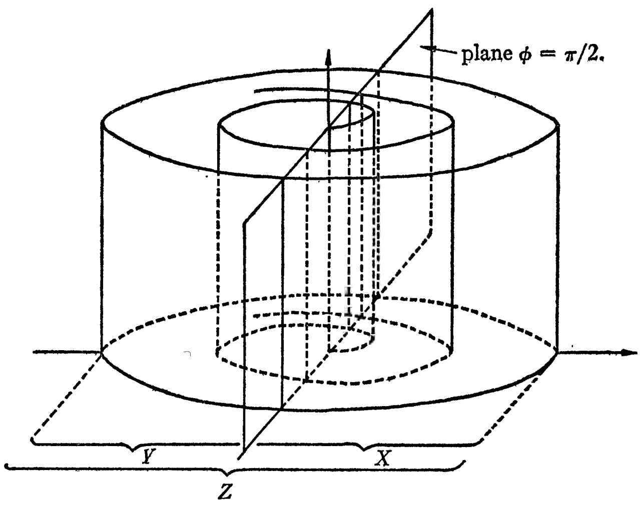
$X_1=\{(r, \phi, 0):\ 0\leqq r<1\}$,

$X_2=\{(r, \phi, z):\ r=(2/\pi)\ \mathrm{Tan}^{-1}\,\phi, 0\leqq\phi<\infty, 0\leqq z\leqq 1\}$,

$X_3=\{(r, \phi, z):\ r=1, 0\leqq z\leqq 1\}$,

$A=\{(r, \phi, z):\ 0\leqq r\leqq 1, -\pi/2\leqq\phi\leqq\pi/2, 0\leqq z\leqq 1\}$,

$B=\{(r, \phi, z):\ 0\leqq r\leqq 1, \pi/2\leqq\phi\leqq 3\pi/2, 0\leqq z\leqq 1\}$, and $Z=X_1\cup X_2\cup X_3$.



Kinoshita [1] proved that Z lacks the f.p.p. Let $X=Z\cap A$ and $Y=Z\cap B$. We claim that X and Y are 2-dimensional continua in E^3 which have the f.p.p., that $X\cap Y$ has the f.p.p. and is in fact a retract of $X\cup Y=Z$.

We first show that X has the f.p.p. Let d denote the Euclidean metric and suppose that X does not have the f.p.p. Then there exists a continuous mapping f of X into itself such that $f(x) \neq x$ for each $x \in X$. By compactness, there is some $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$ for each $x \in X$. Clearly there is a continuous mapping g of A into X such that $d(x, g(x)) < \epsilon$ for each $x \in X$. Let $h = fg$ and note that h is a continuous mapping of A into itself. Thus, by Brouwer fixed-point theorem, there exists a point $x_0 \in A$ such that $h(x_0) = x_0$. Since $h(A) \subset X$, $x_0 \in X$ and therefore $d(x_0, g(x_0)) < \epsilon$. Hence

$$d(f(g(x_0)), g(x_0)) = d(h(x_0), g(x_0)) = d(x_0, g(x_0)) < \epsilon.$$

This is a contradiction which completes the proof that X has the f.p.p. Similarly, Y has the f.p.p. For convenience in showing that $X \cap Y$ is a retract of $X \cup Y$, let us make the following definitions. For each positive integer n, let

$$F_n = \{(r, \phi, z) : r = (2/\pi) \operatorname{Tan}^{-1} \phi, (4n-1)\pi/2 \leqq \phi \leqq (4n+1)\pi/2, 0 \leqq z \leqq 1\},$$
$$G_n = \{(r, \phi, 0) : 0 \leqq r \leqq (2/\pi) \operatorname{Tan}^{-1} \phi, (4n-1)\pi/2 \leqq \phi \leqq (4n+1)\pi/2\} \quad \text{and}$$
$$I_n = \left\{ \left(\frac{2}{\pi} \operatorname{Tan}^{-1} \frac{(2n-1)\pi}{2}, \frac{(2n-1)\pi}{2}, z \right) : 0 \leqq z \leqq 1 \right\}.$$

Further let

$$F_0 = \{(r, \phi, z) : r = (2/\pi) \operatorname{Tan}^{-1} \phi, 0 \leqq \phi \leqq \pi/2, 0 \leqq z \leqq 1\},$$
$$G_0 = \{(r, \phi, 0) : 0 \leqq r \leqq (2/\pi) \operatorname{Tan}^{-1} \phi, 0 \leqq \phi \leqq \pi/2\},$$
$$F_\infty = \{(r, \phi, z) : r = 1, -\pi/2 \leqq \phi \leqq \pi/2, 0 \leqq z \leqq 1\},$$
$$G_\infty = \{(r, \phi, 0) : 0 \leqq r \leqq 1, -\pi/2 \leqq \phi \leqq \pi/2\},$$
$$I_0 = \{(0, 0, z) : 0 \leqq z \leqq 1\}, I_{-1} = \{(1, \pi/2, z) : 0 \leqq z \leqq 1\},$$
$$I_{-2} = \{(1, -\pi/2, z) : 0 \leqq z \leqq 1\} \quad \text{and} \quad I_{-3} = \{(r, \pi/2, 0) : -1 \leqq r \leqq 1\}.$$

We shall refer to F_n as the nth sheet and G_n as the nth base. Clearly, $X \cap Y = \bigcup_{k=-3}^{\infty} I_k$ and for each $s \in \{0, 1, 2, \cdots\}$, I_{2s}, I_{2s+1} are vertical edges of the sth sheet F_s. Clearly, for each s, there is a retraction r_s of the sth sheet F_s onto the union of the two vertical edges I_{2s} and I_{2s+1} of the sheet and the bottom edge $F_s \cap G_s$ such that the middle third of the top edge is mapped onto the bottom edge and each of the other two thirds of the top edge is sent onto a vertical edge. Similarly, there is a retraction r_∞ of the "limit" sheet F_∞ onto $I_{-2} \cup (G_\infty \cap F_\infty) \cup I_{-1}$. Now if we define $R: X \to (X \cap Y) \cup G_\infty$ by $R(x) = r_s(x)$ if $x \in F_s$ and $R(x) = x$ if $x \in G_\infty$, then R is a retraction of X onto $(X \cap Y) \cup G_\infty$. Further, define $H: (X \cap Y) \cup G_\infty \to X \cap Y$ by $H(x) = x$ if $x \in X \cap Y$ and $H(x) = y$ if $x \in G_\infty$ and y is the foot of the perpendicular from x to I_{-3}. Then H is a retraction onto. Now let $r_X = HR$. Then r_X is a retraction of X onto $X \cap Y$. Similarly, we can describe a retraction $r_Y: Y \to (X \cap Y) - I_0$. Finally, let $r = r_X \cup r_Y$. Then r is a retraction of $X \cup Y$ onto $X \cap Y$. The proof that $X \cap Y$ is a retract of $X \cup Y$ is now complete.

Example 2. For each point $t \in T = (0, 3]$, let the point $\phi(t)$ in the Cartesian plane be described as follows:

$$\text{for } t \in \begin{cases}]0, 2], \\ [2, 3], \end{cases} \qquad \phi(t) = \begin{cases} (t, \sin 4\pi/t) \\ (2\cos(-\pi t + 2\pi), 2\sin(-\pi t + 2\pi). \end{cases}$$

Let $X_1 = \{\phi(t): t \in T\}$, $X = X_1 \cup \{(0, 1)\}$. Further, let Y_1 be symmetric to X_1 with respect to the y-axis and $Y = Y_1 \cup \{(0, -1)\}$. Clearly $X \cap Y$ is an arc. It is in fact the lower half of the circle $\{(x, y): x^2 + y^2 = 4\}$. From proposition 3, it follows that X and Y have the f.p.p. We now show that $X \cup Y$ lacks the f.p.p. For each positive integer i, let P_i be the point $(8/(3+2i), \sin((3+2i)\pi/2))$. For each negative integer j, let P_j be the symmetric of P_{-j} with respect to the y-axis. Finally let P_0 be the point $(0, -2)$ and let R denote the real line $]-\infty, \infty[$. Define η to be a continuous biunique mapping of R onto $X_1 \cup Y_1$ with the property that for each integer i, $\eta(i) = P_i$. Let $g: R \to R$ be defined by $g(s) = s - 1$ for each $s \in R$ and let $h = \eta g \eta^{-1}$. Finally define $f: X \cup Y \to X \cup Y$ as follows:

$$f(z) = \begin{cases} h(z), & \text{if } z \in X_1 \cup Y_1, \\ (0, 1), & \text{if } z = (0, -1) \\ (0, -1), & \text{if } z = (0, 1). \end{cases}$$

Clearly, f is continuous and leaves no point fixed. Thus $X \cup Y$ lacks the f.p.p.

Example 3. Let $X_1 = \{(x, y): y = \sin(\pi/x), 0 < x \leqq 1\}$, $X_2 = \{(0, y): -1 \leqq y \leqq 1\}$, $X_3 = \{(1, y): -1 \leqq y \leqq 0\}$, $X_4 = \{(x, y): (x - 1/2)^2 + (y+1)^2 = 1/4, y \leqq -1\}$ and $X_5 = \{(x, y): 4(x - 1/2)^2 + (y+1)^2 = 1, y \leqq -1\}$. Further, let $Z = X_1 \cup X_2 \cup X_3$, $X = Z \cup X_4$ and $Y = Z \cup X_5$. By proposition 2, X and Y have the f.p.p. Clearly, $X \cap Y = Z$ is a chainable continuum and thus has the f.p.p. However, $X \cup Y$ lacks the f.p.p. as can easily be seen by projecting Z onto X_4 and following this by a "rotation" of $X_4 \cup X_5$. In this way a fixed point free continuous mapping of $X \cup Y$ into itself is obtained. Thus, we have shown that in Štanko's theorem, one cannot replace "$X \cap Y$ is a tree" by "$X \cap Y$ is a chainable continuum."

References

1. S. Kinoshita, On some contractible continua without fixed-point property, Fund. Math., 40 (1953) 96–98.

2. V. L. Klee, Jr., Stability of the fixed-point property, Coll. Math., 8 (1961) 43–46.

3. M. Štanko, Continua with the fixed-point property, Dokl. Akad. Nauk SSSR, 154 (1964) 1291–1293, translated in the Soviet Math., 5 (1964) 303–305.

4. L. E. Ward, Jr. Personal communication.

MATHEMATICAL NOTES

ON SOME ANALOGIES TO A THEOREM OF BLICHFELDT IN THE GEOMETRY OF NUMBERS

JOSEPH HAMMER, University of Sydney

In what follows we shall restrict ourselves to bounded closed sets in two-dimensional Euclidean space, and we shall call them simply "regions." A lattice of points is chosen so that the basis parallelograms of the lattice-plane are unit squares.

Blichfeldt's theorem [1] states that any region whose area is greater than one can be placed into such a position by a parallel displacement that it covers at least two lattice points. If, instead of the notion of *parallel displacement* we use the notion of *rotation around a point*, we can have the following theorems:

THEOREM I. *Any central-symmetric convex region of area greater than π, which is symmetric about a lattice point, can be brought into such a position by rotating it around the centre that it contains, besides the centre, at least two other lattice points.*

It can be noticed that this theorem corresponds to Minkowski's fundamental theorem [4], which states that any central-symmetric convex region of area greater than 4 which is symmetric about a lattice point, covers, besides the centre, at least two other lattice points. This theorem can be interpreted as saying that *if we rotate a region satisfying the above condition around its centre, it covers, in any position of rotation,* at least two lattice points in addition to the centre.

Proof of Theorem I. (We shall call the greatest distance between two points of the region the "diameter" of the region.) It is known [5] that of all regions with equal area, the circle has the smallest diameter. The diameter of a circle whose area is greater than π is greater than 2. In a central symmetric convex region, every diameter passes through the centre of symmetry. Therefore by rotating the region around the centre, the diameter can be brought in one line with a lattice-line where the distance of two lattice points is unity. In this position the region contains, besides the centre, at least two further lattice points.

Of course it is *necessary* that the area be greater than π: to see this, take a circle with area less than π with centre on a lattice point.

In the same way, from Ehrhart's theorem [2] can be derived a corresponding theorem. Ehrhart's theorem states that any convex region of area greater than 4.5, whose centre of gravity is a lattice point, covers besides the centre, at least two further lattice points, i.e. by rotating the region around its centre of gravity, *in any position* of rotation besides the centre, two further lattice points are covered. The corresponding theorem is the following:

THEOREM II. *Given any convex region of area greater than $(9/8)\pi$ whose centre of gravity is a lattice point. By rotating it around this point, the region can be brought into such a position that it covers, besides the centre, two further lattice points.*

We omit the proof of this theorem since it can be derived from Theorem I in exactly the same way as Ehrhart's theorem was derived from Minkowski's in [2].

The following theorems are also related:

THEOREM III. *A convex region whose area is greater than $\pi/2$, by rotation around an arbitrary inner point, can be brought into such a position that it covers at least one lattice point.*

Proof. It is clear that in a lattice square, the farthest point from a lattice point is the intersection of the diagonals of the square, the distance is $\frac{1}{2}\sqrt{2}$. Now, if the area of the region is greater than $\pi/2$, then its diameter is greater than $\sqrt{2}$. This means that the distance of any point of the region from a lattice point is less than $\frac{1}{2}\sqrt{2}$. Thus at least in one position of rotation around any point, the region will pass through a lattice point.

Again, it is necessary for the area to be greater than $\pi/2$. Take a circle of area less than $\pi/2$ rotating around its centre which is at the intersection of the diagonals of the square.

THEOREM IV. *In any convex region whose area is greater than $\pi/18$, a point can be found inside the region so that by rotating it around that point, the region can be brought into a position that makes it cover at least one lattice point.*

Proof. Let us divide the diagonal (i.e. the diameter) of the lattice square into three equal parts. Each is of length $\frac{1}{3}\sqrt{2}$. The middle section is the longest segment in the square, which lies farthest from the vertices in the sense that if we rotate it around any of its points, the segment will not pass through any vertex of the square.

Now, since the area of the region is greater than $\pi/18$ its diameter is greater than $\frac{1}{3}\sqrt{2}$. By rotating the region around one of the end points of the diameter, one point of the diameter will pass through a lattice point in some position of rotation.

Again, it is necessary for the area to be greater than $\pi/18$. Take a circle of area less than $\pi/18$ with centre at the intersection of the diagonals of the square.

Application. It is known how effective Minkowski's and also Blichfeldt's theorems are in applications of many problems in Number Theory. It will be interesting to see how our theorems can be applied. In this paper we will demonstrate the applicability of Theorem I only by one problem [3b].

The problem is: approximate the given irrational number α by rational numbers; i.e. find integers u and v such that $|\alpha-(u/v)|$ is as small as possible.

To be able to understand the solution of this problem the following comments should be made:

(1) The cartesian coordinates of the points of a parallelogram-lattice are

$$x = a_1u + b_1v$$

$$y = a_2u + b_2v$$

where a_1, b_1, a_2, b_2 are given real numbers and u, v run through all the integers. On the other hand, for each lattice point (x, y) belongs one and only one pair of (u, v) of the integers in the lattice. A lattice is called unit lattice if

$$|a_1b_2 - a_2b_1| = 1.$$

(For some more particulars the reader is advised to consult for instance [3] (a) or (b).)

(2) In the proof of Theorem I it was not necessary for the basis parallelograms to be squares. It was done to conform Theorems III and IV.

Now we solve the problem after these comments. For this purpose we construct the lattice consisting of the lattice points with coordinates

$$x = \frac{\alpha v - u}{\delta}, \qquad y = \delta v. \tag{1}$$

This is a unit lattice for any positive δ. Let us choose δ in such a way that the minimum distance between the lattice points be not greater than one.

Let us first apply Minkowski's theorem. For this purpose we construct a square of side 2 with centre at the origin and with sides parallel to the coordinate axes. This region can be expressed by the inequalities

$$|x| \leqq 1, \qquad |y| \leqq 1. \tag{2}$$

By (1), (2) can be written

$$\left|\frac{\alpha v - u}{\delta}\right| \leqq 1, \qquad |\delta v| \leqq 1. \tag{3}$$

By Minkowski's theorem there is at least one pair of integers of u, v, not both zero which satisfies these inequalities. By eliminating δ from (3), we obtain the inequality

$$\left|\alpha - \frac{u}{v}\right| \leqq \frac{1}{v^2}\cdot \tag{4}$$

Let us now apply Theorem I. For this purpose take, in the same lattice as above, a square with side $\sqrt{\pi}$ in the same position as the previous one. In that position the diameter, i.e. the diagonal of the square, lies on a lattice line. (This is how the lattice was constructed.) So, by Theorem I the inequalities

$$\left|\frac{\alpha v - u}{\delta}\right| \leqq \frac{\sqrt{\pi}}{2}, \qquad |\delta| \leqq \frac{\sqrt{\pi}}{2}$$

have at least one pair of integers of u, v, as solutions. By eliminating δ we obtain

$$\left|\alpha - \frac{u}{v}\right| \leqq \frac{\pi}{4}\,\frac{1}{v^2}, \tag{5}$$

which is a better approximation than (4).

Moreover in (5) the coefficient $\pi/4$ can be replaced by a smaller value for certain classes. To understand this we would like to point out that in Theorem I the distance between two lattice points is a characteristic element. For the sake of clarity, we have had the distance as unity. Of course it can easily be seen that if the minimum distance is less than unity, then the area of the region can be smaller accordingly. Now, there are unit lattices for which the minimum distance is arbitrarily small; take for instance the lattices generated by a rectangle with sides d and $1/d$, thus $\sqrt{\pi}$ for the side of the square can be replaced by an arbitrary small value. This flexibility of area is an important feature of Theorem I.

The n-dimensional case. We give the n-dimensional version of Theorem I.

THEOREM IA. *Any central symmetric convex region of volume greater than*

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^n \Big/ \Gamma\left(1+\frac{n}{2}\right),$$

where Γ denotes Gamma function, which is symmetric about a lattice point, can be brought into such a position by rotating it around the centre, so that it contains, besides the centre, at least two further lattice points.

The proof of this theorem is parallel to that of Theorem I.

Similarly Theorems III and IV have n-dimensional versions. Ehrhart's theorem has not as yet been extended to the n-dimensional case, and so the same applies to our Theorem II.

References

1. H. F. Blichfeldt, A new principle in the geometry of numbers with some applications, Trans. Amer. Math. Soc., 15 (1914) 227–235.

2. M. Eugen Ehrhart, Une généralisation du théorème de Minkowski, Comptes Rendus Acad. Sci., Paris, 240 (1955) 483–485.

3. (a) G. H. Hardy, and E. N. Wright, An introduction to the theory of numbers, Oxford, 1959, 23–37. (b) D. Hilbert, und S. Cohn-Vossen, Anschauliche Geometrie, Springer Verlag, Berlin, 1932, pp. 38–39. English translation: Geometry and Imagination, Chelsea, New York.

4. H. Minkowski, Gesammelte Abhandlungen, Leipzig, 1911, p. 255.

5. I. M. Yaglom, and V. G. Boltyanskii, Convex Figures, Holt, Rinehart and Winston, New York, 1961, pp. 239–240.

A PROOF OF A CLASSICAL THEOREM IN MULTIPLE INTEGRATION

P. STEIN, London

1. The theorem on the change of variables in a multiple integral runs as follows:

Let f be a real-valued continuous function on E_n. Suppose g is a 1-1 *real-valued function defined on an open subset S of E_n such that, if*

$$Y = g(X),\ (Y = (y_1, \cdots, y_n),\ X = (x_1, \cdots, x_n)),$$

the partial derivatives $(\partial x_i/\partial y_j)(i, j=1, \cdots, n)$ are continuous on S and the Jacobian

$$J = \det \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(y_1, y_2, \cdots, y_n)}$$

does not vanish at any point of S. Then, for any bounded, closed, connected subset T of S,

$$\int_T f\, dx_1 dx_2 \cdots dx_n = \int_{g(T)} f \,|J|\, dy_1 dy_2 \cdots dy_n. \tag{1.1}$$

We shall refer to a bounded, closed, connected subset of E_n as a "solid" in E_n.

In this paper, we show that, if V_X is the volume of a solid T in S and V_Y is the volume of the corresponding solid $g(T)$, then

$$V_Y \leqq |J|(1+\epsilon)V_X \tag{1.2}$$

provided the diameter of T does not exceed $\delta = \delta(\epsilon)$, and J is computed at any point in T. From 1.2, and the corresponding result for g^{-1}, we deduce (1.1).

2. We prove (1.2) first for a simplex and when g is affine.

We may without confusion use J both for the matrix and its determinant.

If $X_1, X_2, \cdots, X_{n+1}$ are $n+1$ points on an n dimensional space, and the rank of

$$\begin{pmatrix} X_1, & X_2, & \cdots, & X_{n+1} \\ 1, & 1, & \cdots, & 1 \end{pmatrix}$$

is $n+1$, we may define the simplex in this space with vertices $X_1, X_2, \cdots, X_{n+1}$ as the set of points

$$X = \sum_{r=1}^{n+1} \alpha_r X_r, \qquad \alpha_r \geqq 0,\ r = 1, 2, \cdots, n+1 \quad \text{and} \quad \sum_{r=1}^{n+1} \alpha_r = 1.$$

It is known that the volume of this simplex is:

$$\pm \frac{1}{n!} \begin{vmatrix} X_1, & X_2, & \cdots, & X_{n+1} \\ 1, & 1, & \cdots, & 1 \end{vmatrix}$$

If we make the affine transformation g with $g^{-1}\ X \to Y$ given by $X = AY + B$, A nonsingular, we have the matrix

$$\begin{aligned}(X_2 - X_1, X_3 - X_1, \cdots, X_{n+1} - X_1) \\ = A(Y_2 - Y_1, Y_3 - Y_1, \cdots, Y_{n+1} - Y_1).\end{aligned} \tag{2.1}$$

Again

$$\begin{vmatrix} X_1, & X_2, & \cdots, & X_{n+1} \\ 1, & 1, & \cdots, & 1 \end{vmatrix} = (-1)^{n+1} |X_2 - X_1, X_3 - X_1, \cdots, X_{n+1} - X_1| \tag{2.2}$$

and

$$\begin{vmatrix} Y_1, & Y_2, & \cdots, & Y_{n+1} \\ 1, & 1, & \cdots, & 1 \end{vmatrix} = (-1)^{n+1} \left| Y_2 - Y_1, Y_3 - Y_1, \cdots, Y_{n+1} - Y_1 \right|. \tag{2.3}$$

Further, $A = J$. So from (2.1), (2.2), (2.3), we obtain (1.2) in the form $V_X = |J| V_Y$.

3. We now take a simplex in S when the transformation is not necessarily affine but satisfies the conditions stated in Section 1. We require two lemmas. If $S(X)$ is a simplex with vertices $X_1, X_2, \cdots, X_{n+1}$, and $\sigma > 0$, we denote by $S(\sigma X)$ the simplex whose vertices are $\sigma X_1, \sigma X_2, \cdots, \sigma X_{n+1}$.

LEMMA 1. *If* $O = (0, 0, \cdots, 0) \in S(X)$, *then* $S(\rho X) \supset S(\sigma X)$, *provided* $\rho \geqq \sigma > 0$.

Proof. Under the hypotheses, there are some nonnegative real numbers $\beta_1, \beta_2, \cdots, \beta_{n+1}$ with $\sum \beta_r = 1$ and $0 = \sum \beta_r X_r$. Any point in $S(\sigma X)$ can be written in the form $\sum \sigma \alpha_r X_r$, where $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ are nonnegative real numbers and $\sum \alpha_r = 1$. If we write $\gamma_r = (\rho - \sigma)\beta_r + \sigma \alpha_r$, we have $\gamma_r \geqq 0$, $\sum \gamma_r = \rho$, so that

$$\sum \gamma_r X_r = \sum \frac{\gamma_r}{\rho} \cdot \rho X_r \in S(\rho X).$$

However, $\sum \gamma_r X_r = \sum \alpha_r (\sigma X_r)$.

LEMMA 2. *Let* $O = (0, 0, \cdots, 0) \in S(X)$ *and let* $M > 0$, *be the distance of* O *from the boundary of* $S(X)$, *then if* $Y \in S(\rho X)$ *for* $\rho > 0$, *and* Z *is a vector of length not exceeding* $p > 0$, *then* $Y + Z \in S(\rho + (p/M))X$.

Proof. Since $\|Z\|$ is $\leqq p$, $\|(M/p)Z\| \leqq M$ so that the point $O + (M/p)Z \in S(X)$. It follows that $O + Z \in S((p/M)X)$. By the argument used in Lemma 1, $Y + Z \in S((\rho + (p/M))X)$.

4. We suppose the partial derivatives of the transformation g continuous and J nonsingular. We take

$$X_0 = \frac{X_1 + X_2 + \cdots + X_{n+1}}{n + 1}.$$

Then $X_0 \in S(X)$ and the distance of X_0 from the boundary of $S(X) = M > 0$. Let $Y_0 = g(X_0)$ and J_0 the Jacobian at X_0. To simplify the notation we write ϵ for any positive number which can be chosen arbitrarily small; thus if k is any finite positive number, we write ϵ for $k\epsilon$.

We have

$$Y - Y_0 + Z = J_0^{-1}(X - X_0),$$

where $\|Z\| < \epsilon \|X - X_0\|$ provided $\|X - X_0\| \leqq \delta = \delta(\epsilon)$. Hence, if the diameter d of $S(X) \leqq \delta$, by Lemma 2,

$$V_{Y-Y_0} \leqq \left| J_0^{-1} \right| (1 + \epsilon)^n V_{X-X_0}$$

by the special case of an affine transformation.

But $V_Y = V_{Y-Y_0}$ and $V_X = V_{X-X_0}$ and so

$$V_Y \leqq |J_0^{-1}|(1+\epsilon)^n V_X = |J_0^{-1}|(1+\epsilon)V_X. \tag{4.1}$$

5. Suppose we have a solid in S which can be covered by a finite number of simplexes which do not overlap except possibly at their boundaries. An n dimensional cube can be so covered. Corresponding to the volume of each of these simplexes there is a corresponding volume in $g(T)$ given by (4.1). Since J_0^{-1} is continuous, then provided the solid in T is of diameter $d \leqq \delta = \delta(\epsilon)$, J_0 in each simplex can be replaced by J where J may be calculated at any point in this solid, the same point for each simplex. Thus (4.1) holds for any permissible solid of small diameter. We write (4.1) as

$$|J| V_Y \leqq (1+\epsilon)V_X. \tag{5.1}$$

Let $f^+ = f$ when $f \geqq 0$ and $f^+ = 0$ when $f < 0$.
Since f^+ and J are continuous and ϵ is arbitrary

$$\int_{g(T)} |J| f^+ dy_1, dy_2, \cdots, dy_n \leqq \int_T f^+ dx_1, dx_2, \cdots, dx_n. \tag{5.2}$$

By starting with a simplex in $g(T)$ by similar arguments we would deduce

$$\int_T f^+ dx_1, dx_2, \cdots, dx_n \leqq \int_{g(T)} |J| f^+ dy_1, dy_2, \cdots, dy_n \tag{5.3}$$

from (5.2) and (5.3) we get

$$\int_T f^+ dx_1, dx_2, \cdots, dx_n = \int_{g(T)} |J| f^+ dy_1, dy_2, \cdots, dy_n. \tag{5.4}$$

If $f^- = f$ when $f \leqq 0$ and $f^- = 0$ when $f > 0$ a similar argument will show

$$\int_T f^- dx_1, dx_2, \cdots, dx_n = \int_{g(T)} |J| f^- dy_1, dy_2, \cdots, dy_n. \tag{5.5}$$

Since $f = f^+ + f^-$, (1.1) follows from (5.4) and (5.5).

I have to thank Dr. M. Fiedler for reading a part of this manuscript and suggesting some simplifications of the proofs.

A FORMULA FOR THE SPECTRAL RADIUS OF AN OPERATOR

R. B. Holmes, Wright-Patterson AF Base, Ohio

Let $(X, \|\cdot\|)$ be a complex Banach space and $B(X)$ the space of all bounded linear operators which map X into itself. If $T \in B(X)$, the operator norm of T is

defined as

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

This choice of norm makes $B(X)$ into a Banach space (actually a Banach algebra, since

$$\|T_1T_2\| \leqq \|T_1\| \|T_2\|).$$

If $T \in B(X)$ the spectrum of T, $\sigma(T)$, consists of those complex numbers λ for which $(\lambda I - T)^{-1}$ is *not* an element of $B(X)$; here I denotes the identity operator on X: $Ix = x$. It is well known that $\sigma(T)$ is a nonempty compact subset of the complex plane. Hence the number $\sup \{|\lambda| : \lambda \in \sigma(T)\}$ exists. It is called the spectral radius of T and denoted $r_\sigma(T)$. A familiar formula for the spectral radius asserts that $r_\sigma(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$, where T^n means $T \circ T \circ \cdots \circ T$, n times. Since $\|T^n\| \leqq \|T\|^n$ we have in particular that $r_\sigma(T) \leqq \|T\|$. The purpose of this note is to establish an alternative formula for $r_\sigma(T)$ and to consider some of its consequences. A complete discussion of the notions of spectral theory used here may be found in [1] or [4].

Before stating our result we need one additional concept—that of equivalent norm. If $|\cdot|$ is another norm on X, we say that it is equivalent to $\|\cdot\|$ if it induces the same (metric) topology on X as does $\|\cdot\|$. For this to occur it is necessary and sufficient that there exist $\beta > 0$ such that $|x| \leqq \beta\|x\|$ for all $x \in X$. Indeed, if this be true, then as a consequence of the open mapping principle, there also exists $\alpha > 0$ such that $\alpha\|x\| \leqq |x|$ for all x. We finally note that if $|\cdot|$ and $\|\cdot\|$ are equivalent norms on X, then the corresponding operator norms are equivalent on $B(X)$.

THEOREM. *Let $T \in B(X)$. Then $r_\sigma(T) = \inf |T|$, where the infimum is taken over all norms $|\cdot|$ on X equivalent to $\|\cdot\|$.*

Proof. We have already observed that $r_\sigma(T) \leqq \|T\|$. Now by definition, the spectrum of an operator in $B(X)$ is completely determined by the linear and topological structure of X. Since this structure is not altered if we replace $\|\cdot\|$ by an equivalent norm $|\cdot|$, we see that $r_\sigma(T) \leqq |T|$. Thus $r_\sigma(T)$ is not greater than the infimum in question. We will complete the proof by showing that if $s > r_\sigma(T)$ there is a norm $|\cdot|$ on X, equivalent to $\|\cdot\|$ for which $|T| < s$. By setting $U = s^{-1}T$ we see that $r_\sigma(U) < 1$; so it will suffice to produce an equivalent norm $|\cdot|$ for which $|U| < 1$. To this end, we note that the series $\sum_{n=0}^\infty \|U^n\|$ converges, since

$$\lim_{n\to\infty} \|U^n\|^{1/n} = r_\sigma(U) < 1.$$

Then for any $x \in X$, we define $|x| = \sum_{n=0}^\infty \|U^n x\|$. Clearly $|\cdot|$ is a norm on X, and since

$$\|x\| \leqq |x| \leqq \left(\sum_{n=0}^{\infty}\|U^n\|\right)\|x\|,$$

we see that it is equivalent to $\|\cdot\|$. Further, because

$$|Ux| = \sum_{n=0}^{\infty}\|U^{n+1}x\| = |x| - \|x\|,$$

we obtain, if $x \neq 0$,

$$\frac{|Ux|}{|x|} = 1 - \frac{\|x\|}{|x|} \leqq 1 - \inf_{x\neq 0}\frac{\|x\|}{|x|} \leqq 1 - \left(\sum_{n=0}^{\infty}\|U^n\|\right)^{-1}.$$

Thus we indeed have $|U| < 1$, and the proof is complete.

As an immediate consequence of this result we have the

COROLLARY. *X can be renormed (with an equivalent norm) so that an operator T is a strict contraction if and only if $r_\sigma(T) < 1$.*

For, since T is linear, T is a strict contraction with respect to the norm $|\cdot|$ iff $|T| < 1$.

It is obvious that $r_\sigma(cT) = |c| r_\sigma(T)$ for all scalars c. It also follows quite directly from our result that *$r_\sigma(\cdot)$ is an upper semi-continuous function on* $(B(X), \|\cdot\|)$. In general we cannot say more about the continuity for r_σ. For example, there is a construction due to Kakutani [2, p. 282] which shows that r_σ is never lower semi-continuous on $B(X)$ when X is a separable Hilbert space. If r_σ is not continuous on $B(X)$, then, of course, it is not a semi-norm, although it is known that $r_\sigma(S+T) \leqq r_\sigma(S) + r_\sigma(T)$ whenever S and T commute [3, p. 426].

As an application of the theorem we cite the following result, which is of some interest in the modern theory of differential equations.

Suppose, for some $T \in B(X)$, that $\sigma(T) \cap \{\lambda: |\lambda| = 1\}$ is void. Then there exist closed subspaces X_1 and X_2, positive numbers ϵ and δ, and an equivalent norm $|\cdot|$ on X such that

(a) $X = X_1 \oplus X_2$;

(b) $T(X_i) \subset X_i$ *(i.e., T is reduced by the subspaces X_1 and X_2)*;

(c) *if $x \in X_1$, $|Tx| \leqq (1-\epsilon)|x|$, while if $x \in X_2$, $|Tx| \geqq (1+\delta)|x|$.*

That is, in the $|\cdot|$ norm, T strictly contracts on X_1 and strictly expands on X_2.

Proof. Set $\sigma_1 = \sigma(T) \cap \{\lambda: |\lambda| < 1\}$ and $\sigma_2 = \sigma(T) \cap \{\lambda: |\lambda| > 1\}$. By the assumption about $\sigma(T)$ there exists a function f, analytic on a neighborhood of $\sigma(T)$ and satisfying $f = 1$ on a neighborhood of σ_1, $f = 0$ on a neighborhood of σ_2. Via the operational calculus for T the operators $f(T)$ and $I - f(T)$ are complementary projections in $B(X)$ (because $f = f^2$ on a neighborhood of $\sigma(T)$). Hence, if we set $X_1 = \text{range}\,(f(T))$ and $X_2 = \text{range}\,(I - f(T))$, we obtain (a) and (b). Let T_i be the restriction of T to X_i; then $\sigma(T_i) = \sigma_i$, $i = 1, 2$. Thus $r_\sigma(T_1) < 1$ and by the spectral mapping theorem (which asserts that $\sigma(f(T)) = f(\sigma(T))$ if f is ana-

lytic on a neighborhood of $\sigma(T)$), we also have $r_\sigma(T_2^{-1})<1$. Applying our theorem to the operators T_1 and T_2^{-1} we obtain equivalent norms $|\cdot|_i$ on X_i for which $|T_1|_1<1$ and $|T_2^{-1}|_2<1$. We conclude by defining $|x|=|x_1|_1+|x_2|_2$ if $x\in X$ and $x=x_1+x_2$ with $x_i\in X_i$. This norm is easily seen to be equivalent to the given one on X.

References

1. N. Dunford and J. Schwartz, Linear Operators, Part I, Interscience, New York, 1958 (Chapter 7).
2. C. Rickart, General Theory of Banach Algebras, Van Nostrand, Princeton, 1960.
3. F. Riesz and B. Sz.-Nagy, Functional Analysis, translated by L. Boron, Ungar, New York, 1955.
4. A. Taylor, Introduction to Functional Analysis, Wiley, New York, 1958 (Chapter 5).

A CALCULUS COUNTEREXAMPLE

N. W. RICKERT, Yale University

Most calculus books prove at least one form of l'Hospital's rule, but few seem to mention whether the converse is true. One exception is [1], where the author claims to prove the converse. More precisely, he claims to prove that if f, g are continuously differentiable functions with $f(a)=g(a)=0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

in the sense that if either limit exists, so does the other, and the two limits are equal.

The purpose of this note is to exhibit functions f and g, both infinitely differentiable, for which $f(0)=g(0)=0$, $\lim_{x\to 0}f(x)/g(x)=0$, but $\lim_{x\to 0}f'(x)/g'(x)$ does not exist. In fact, define $f(x)=x\sin 1/x^4\exp\{-1/x^2\}$, $(x\neq 0)$, $f(0)=0$, and

$$g(x)=\exp\left\{-\frac{1}{x^2}\right\}\qquad(x\neq 0)$$
$$g(0)=0.$$

It is not difficult to show that f and g are C^∞ functions and that $f(x)/g(x)\to 0$ as $x\to 0$. Furthermore a simple calculation shows that for small x,

$$\frac{f'(x)}{g'(x)}=-\frac{2}{x}\cos\frac{1}{x^4}+\epsilon(x),$$

where $\epsilon(x)\to 0$ as $x\to 0$. Thus $f'(x)/g'(x)$ has every real number as a limit point as $x\to 0$.

Reference

1. Herbert S. Wilf, Calculus and linear algebra, Harcourt, Brace and World, New York, 1966.

THE COMPACT GRAPH THEOREM

I. I. Kolodner, Carnegie-Mellon University

The closed graph theorem is a well-known and much appreciated theorem of linear analysis. Apparently little known (and perhaps not at all noticed) but still useful is the simple *compact graph theorem* which is proved below. Before a statement of the theorem is given, let us consider a typical question which is answered by it. Let f: $[0, 1] \rightarrow R$ be a function whose graph, $G(f) = \{(x, f(x)) \mid x \in [0, 1]\}$ is a closed and bounded set. Is it true that $G(f)$ is a connected set? The answer is "yes" since, as will be evident, f is continuous.

Theorem. *Let X and Y be topological spaces and let f: $X \rightarrow Y$ be a function with domain X, range $R(f) \subset Y$ and graph $G(f) \subset X \times Y$. Then:*

(i) *X and $R(f)$ are compact if $G(f)$ is compact;*

(ii) *$G(f)$ is compact if X is compact, f is continuous and Y is a Hausdorff space;*

(iii) *f is continuous if $G(f)$ is compact and X is a Hausdorff space.*

Proof. Let p_1 and p_2 be the coordinate projections of $X \times Y$ on X and Y. They are both continuous.

(i) $X = p_1(G(f))$ and $R(f) = p_2(G(f))$ are both compact if $G(f)$ is compact since they are images of a compact set under continuous functions.

(ii) Since f is continuous and Y is a Hausdorff space, $G(f)$ is closed. Since X is compact and f is continuous, $R(f)$ is compact and so $X \times R(f)$ is compact. Thus $G(f) \subset X \times R(f)$ is a closed subset of a compact set, and thus it is compact.

(iii) Consider the restriction $q = p_1 \mid G(f)$: $G(f) \rightarrow X$. Since $(x, f(x)) \neq (y, f(y))$ implies $x \neq y$, q is one-to-one and thus q^{-1} exists. Assume now that X is a Hausdorff space and that $G(f)$ is compact. Then q is a homeomorphism, and thus q^{-1} is continuous. Since $f = p_2 \circ q^{-1}$, f is continuous.

A NOTE ON GROUP THEORY

Kung-Wei Yang, Western Michigan University

Quite often the inclusion of a carefully chosen nonroutine theorem in an elementary textbook could provide students with a valuable tool for solving less routine problems. We would like to illustrate this fact by giving a simple proof of a proposition in group theory by using a lemma in [1]. First of all we state the

Lemma ([1] p. 62). *If G is a finite group, and $H \neq G$ is a subgroup of G such that $o(G) \nmid i(H)!$ (where $o(G)$ is the order of G and $i(H)$ is the index of H in G), then H must contain a nontrivial normal subgroup of G.*

Proposition. *If p is the smallest prime number dividing $o(G)$ and H is a subgroup with $i(H) = p$, then H is normal in G.*

Proof. We use induction on $o(G)$.

$o(G)=1$. The proposition is vacuously true.

$o(G)>1$. Assume that the proposition is true for all finite groups G' such that $o(G')<o(G)$. Let p and H be as stated in the proposition. If $o(H)=1$, the proposition is trivially true. We may assume $o(H)>1$. Then, clearly, $o(G)\nmid i(H)!$ By the above lemma, H contains a nontrivial normal subgroup N of G. Let $\overline{G}=G/N$ and $\overline{H}=H/N$. Then, $o(\overline{G})<o(G)$, and $i(\overline{H})=p=$the smallest prime number dividing $o(\overline{G})$. By the induction assumption, $\overline{H}$ is normal in $\overline{G}$. By Lemma 2.17 of [1] (p. 54), H is normal in G.

I would like to thank the referee for the improvements on the style of the presentation of this note.

Reference

1. I. N. Herstein, Topics in Algebra, Blaisdell, New York, 1964.

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the Monthly. Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.

A PROPERTY OF RIEMANN-INTEGRABLE FUNCTIONS

K. E. Hirst, University of Southampton, England

In this note we indicate how the method of nested intervals can be used to show that a Riemann integrable function on an interval $[a, b]$ must be continuous somewhere in that interval. It is of course easy to prove a stronger result using ideas of sets of measure zero (see M. E. Munroe, Introductory Real Analysis pp. 169–170. Addison-Wesley, 1965), but traditional real analysis courses do not seem to include any implications of integrability as far as continuity is concerned.

Outline of the method. With the usual notations for upper and lower sums we know that if $\{D_n\}$ is any sequence of subdivisions of $[a, b]$ for which $\mu(D_n)\to 0$, then if f is integrable over $[a, b]$, $S(D_n, f)-s(D_n, f)\to 0$ as $n\to\infty$. This relation will be satisfied if we let D_n be the subdivision of $[a, b]$ into n equal parts. Thus, given $\epsilon>0$, there exists m such that, for all $n\geqq m$ we have

$$\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})<\epsilon. \tag{1}$$

We choose $\epsilon=\frac{1}{2}(b-a)$ and so we can find $m\geqq 4$ such that (1) holds. We use the special choice of D_n to deduce that

$$\sum_{i=1}^{m} (M_i - m_i) < \tfrac{1}{2}m \leqq m - 2.$$

It follows that there are at least three integers i with $1 \leqq i \leqq m$ for which $M_i - m_i < 1$. We may thus choose $[x_{i-1}, x_i]$ so that $x_{i-1} \neq a$, $x_i \neq b$. Denote this interval by $[a_1, b_1]$. We now generate inductively a nested sequence of intervals with the following properties:

(i) $$[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$$

(ii) $$a_n \neq a_{n-1}, \quad b_n \neq b_{n-1}$$

(iii) $$b_n - a_n \leqq (b - a)/4^n$$

(iv) $$\sup_{x \in [a_n, b_n]} f(x) - \inf_{x \in [a_n, b_n]} f(x) < 1/n.$$

This nested sequence of intervals then converges to a single point x_0, and we can prove that f is continuous at x_0. We need the above use of *three* intervals to obtain property (ii), which is needed to ensure full continuity rather than one-sided continuity. As a corollary we have the fact that the set of points of continuity of f is dense in $[a, b]$.

Mail Address: Department of Mathematics, University of Southampton, Southampton SO9 5NH, England.

A NOTE ON GENERALIZED APPELL POLYNOMIALS

J. W. Brown, Oberlin College and J. L. Goldberg, University of Michigan

Associated with any simple set of polynomials

$$p_n(x) = \sum_{k=0}^{n} \alpha_{n,k} x^k \qquad (n \geqq 0)$$

are the uniquely determined inverse relations $x^n = \sum_{k=0}^{n} \beta_{n,k} p_k(x)$. The related polynomials $q_n(x) = \sum_{k=0}^{n} \beta_{n,k} x^k$ $(n \geqq 0)$ form the *inverse set* which we examine when the original one has a generalized Appell representation [1, 2]. That is, there exist power series $\Phi(t) = \sum_{k=0}^{\infty} \phi_k t^k$ $(\phi_k \neq 0)$ and $\Psi(t) = \sum_{k=0}^{\infty} \psi_k t^k$ $(\psi_k \neq 0)$ such that for some $A(t)$ and $B(t)$

$$A(t)\Psi(xB(t)) = \sum_{n=0}^{\infty} \phi_n p_n(x) t^n.$$

It is understood that $A(t)$ and $B(t)/t$ have power series expansions with nonzero initial coefficients. Our main result is the following

Theorem. *If a simple polynomial set has a generalized Appell representation, then so does its inverse.*

To prove this, we exhibit the generalized Appell representation

$$[1/A(\tilde{B}(t))]\Phi(x\tilde{B}(t)) = \sum_{n=0}^{\infty} \psi_n q_n(x) t^n$$

where $\tilde{B}(t)$ is defined by $\tilde{B}(B(t)) = B(\tilde{B}(t)) = t$.

The authors are indebted to the late Professor Earl D. Rainville. Research supported in part by the National Science Foundation.

Mail Address: Department of Mathematics, Oberlin College, Oberlin, Ohio 44074 *or* Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104.

References

1. R. P. Boas, Jr. and R. C. Buck, Polynomials defined by generating relations, this MONTHLY, 63 (1956) 626–632.
2. ———, Polynomial expansions of analytic functions, 2nd ed., Academic Press, New York, 1964.

A NOTE ON FERMAT'S LAST THEOREM

M. PERISASTRI, M. R. College, India

In this paper we give an extension of the results proven by Stone [1], Gandhi [2], and Christilles [3].

THEOREM. *If $p>51$ and $8p+1$ are primes, and a, b, and c are pairwise coprime integers such that $(abc, 8p+1)=1$, then $a^p+b^p+c^p \neq 0$.*

Mail Address: Dept of Math., M. R. College, Vizianagaram, India.

References

1. D. E. Stone, On Fermat's last theorem, this MONTHLY, 70 (1963) 976.
2. J. M. Gandhi, A Note on Fermat's last theorem, this MONTHLY, 73 (1966) 1106.
3. W. E. Christilles, A note concerning Fermat's conjecture, this MONTHLY, 74 (1967) 292.

IF TWO COWS WALK ON HOMOTOPIC PATHS, THEY HAVE TO PASS A COMMON GATE

EMIL STAMM, University of Toronto

Consider a meadow G(=open connected set in the plane $\mathbb{R}^2$) separated by a fence l(=straight line in $\mathbb{R}^2$, determining two half planes H_1, H_2 such that $H_1 \cap G$ and $H_2 \cap G$ are nonempty). Then $l \cap G = \bigcup_{m=1}^{r} G_m$, $r \leqq \infty$, G_m = open interval, called a gate.

THEOREM. *Let $p_1 \in H_1 \cap G$, $p_2 \in H_2 \cap G$ be two points; w_1, w_2 paths homotopic in G relative to their common end points p_1, p_2, the homotopy being $H: I \times I \to G$. Then there is a gate G_m such that $w_1 \cap G_m$ and $w_2 \cap G_m$ are nonempty.*

Proof. Since G is an infinite polyhedron (Runge) one can assume w_1, w_2 to be "nice" edge paths in G. Assume w_j crosses G_m k_{jm}-times. $\sum_m k_{jm}$ $(j=1, 2)$ are odd numbers. Two cases are possible: I. All the numbers $k_{1m}+k_{2m}$ are even. II. There is a gate G_m such that $k_{1m}+k_{2m}$ is odd.

One shows by parity arguments that in case I the theorem is true. Case II is impossible: Let m_0 be the smallest integer such that $k_{1m_0}+k_{2m_0}$ is odd. Let g be the right end point of G_{m_0}. Looking at the points of intersection of the ray on l, starting at g and pointing to the "left," one sees that g has odd order relative to the closed curve $w_1+(-w_2)$. Hence by Kronecker's existence theorem (Alexandroff-Hopf, *Topologie*, p. 467) $g\in$(Image of H)$\subset G$. But $g\notin G$.

Mail address: Department of Mathematics, University of Toronto, Toronto 5, Ont., Canada.

CERTAIN TRANSFORMATIONS AND SUMMATION FORMULAE FOR BASIC BILATERAL HYPERGEOMETRIC SERIES $_2\Psi_2$

MANJARI UPADHYAY, Lucknow University, India

1. The object of this note is to find the sum of a general $_2\Psi_2$ analogous to the sum of a $_2H_2$ (see [3]: p. 180).

2. The sum of a $_2\Psi_2$. Replacing d and e by aq/d, aq/e and then taking the limit as $a\to 0$ in the known sum of a well-poised $_6\Psi_6$ given in ([3], p. 248), we get

$$(1)\qquad {}_2\Psi_2\begin{bmatrix} b, c;\ de/qbc \\ d, e \end{bmatrix} = \prod\begin{bmatrix} d/b, d/c, e/b, e/c, q; \\ d, e, q/b, q/c, de/qbc \end{bmatrix}.$$

This result reduces when $q\to 1$ to the sum of a $_2H_2$ ([3], p. 180).

Also, using (1) in a transformation given by Bailey ([1], (2.3)), we get

$$(2)\qquad {}_2\Psi_2\begin{bmatrix} \alpha, \gamma/q;\ z \\ \beta, \gamma \end{bmatrix} = \prod\begin{bmatrix} \gamma/\alpha, q\beta/\gamma, q, q; \\ \gamma, q/\alpha, \beta, q^2/\gamma \end{bmatrix}.$$

Next, consider the relation

$$(3)\qquad \begin{aligned}(1-c)\,{}_3\Psi_3\begin{bmatrix} a, b, cq;\ t \\ d, e, c \end{bmatrix} &= (1-cq/d)\,{}_2\Psi_2\begin{bmatrix} a, b;\ t \\ d, e \end{bmatrix} \\ &\quad - c(1-q/d)\,{}_2\Psi_2\begin{bmatrix} a, b;\ t \\ d/q, e \end{bmatrix},\end{aligned}$$

the truth of which can easily be verified.

Using the sum (1) in (3), we get

$$(4)\qquad \begin{aligned}{}_3\Psi_3\begin{bmatrix} a, b, c;\ (q) \\ d, e, c/q \end{bmatrix} &= \left\{1 - \frac{(1-aq/d)(1-(bq/d))(c)}{(1-c/d)(1-q^2ab/de)(e)}\right\} \\ &\quad \cdot\frac{(1-c/d)}{(1-c/q)}\,a\prod\begin{bmatrix} q, d/a, d/b, e/a, e/b; \\ d, e, q/a, q/b, de/qab \end{bmatrix}\end{aligned}$$

$|de/q^2ab|<1$, which extends a result due to Bailey [2] for a $_3H_3$.

3. A general transformation theorem: When $\theta_s = 0(x^s)$, $0<|b/a|<|q|<|x|<|q(\alpha/\beta)|<1$,

$$\prod\begin{bmatrix}\beta/\alpha,\ \beta b/q,\ q/a\alpha,\ b/a,\ q; \\ \beta,\ q/\alpha,\ b,\ q/a,\ b\beta/qa\alpha\end{bmatrix} {}_1\Psi_1\begin{bmatrix}a\alpha;\ 1/\alpha E \\ b\beta/q\end{bmatrix}\theta_0 \tag{5}$$
$$= \sum_{s=-\infty}^{\infty}\frac{(\alpha)_s}{(\beta)_s}(\beta/qa)^s{}_1\Psi_1\begin{bmatrix}a;\ E \\ b\end{bmatrix}\theta_s,$$

where E is the displacement operator $E^n\theta_s=\theta_{s+n}$ and θ_s is any arbitrary sequence.

A particular case of the above theorem is

$$ {}_2\Psi_2\begin{bmatrix}A,\ B;\ z \\ c,\ D\end{bmatrix} = \prod\begin{bmatrix}qc/zB,\ q/aA,\ c/A,\ z,\ aD,\ D/B; \\ q/A,\ c,\ qc/zaAB,\ zaD/q,\ q/B,\ D\end{bmatrix} \tag{6}$$
$$\times\ {}_2\Psi_2\begin{bmatrix}q/z,\ aA;\ zB \\ aD,\ qc/zB\end{bmatrix},$$

which generalises a transformation due to Bailey ([1]; (2.3)).

Mail Address: Lucknow University, P. O. Lucknow. (U.P.) India.

References

1. W. N. Bailey, On the basic bilateral hypergeometric series ${}_2\Psi_2$, Quart. J. Math. Oxford Ser., (2) 1(1950).

2. ———, On the sum of a particular bilateral hypergeometric series ${}_3H_3$, Quart. J. Math. Oxford Ser., (2) 10 (1959) 92–94.

3. L. J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, 1966.

AN EXTENSION OF LEIBNITZ'S RULE FOR INTEGRALS

J. N. HUNT, University of Reading, England

Continuity equations for tidal flow in estuaries require the evaluation of derivatives of integrals, with respect to a parameter, of the general form

$$I_x = \frac{\partial}{\partial x}\iint_{A(x,t)} f(x,\ t;\ y,\ z)\,dy\,dz. \tag{1}$$

The cross-sectional area of flow $A(x, t)$ in the (y, z) plane varies along the length of the estuary $(0<x<X)$, and with time t. The integrand f is usually the transport of water or salt. We wish to extend the well-known Leibnitz rule for the derivative of an integral with respect to a parameter,

$$\frac{\partial}{\partial x}\int_{\psi_1(x)}^{\psi_2(x)} g(x;\ y)dy = \int_{\psi_1(x)}^{\psi_2(x)}\frac{\partial}{\partial x}g(x;y)dy + \frac{\partial\psi_2}{\partial x}g(x;\psi_2) - \frac{\partial\psi_1}{\partial x}g(x;\psi_1), \tag{2}$$

to the case of a surface integral, under suitable conditions. Erroneous forms of the result have recently been published. The boundary of $A(x, t)$ is denoted by

two curves $z_1(x, t; y)$, $z_2(x, t; y)$, such that $z_2 > z_1$ in $y_1 < y < y_2$ and $z_1 = z_2$ at $y = y_1$, $y = y_2$. The integral (1) then becomes

$$I_x = \frac{\partial}{\partial x}\int_{y_1(x,t)}^{y_2(x,t)}\left\{\int_{z_1(x,t;y)}^{z_2(x,t;y)} f(x, t; y, z)dz\right\} dy. \tag{3}$$

Denoting path length by s, then $z = z_1$, $z = z_2$, are the s-eliminant of say, $y = \phi_1(x, t; s)$ and $z = \phi_2(x, t; s)$. Thus on z_1 and z_2, for the inner integral at constant y,

$$z_x = (\phi_2)_x + (\phi_2)_s s_x, \qquad 0 = (\phi_1)_x + (\phi_1)_s s_x.$$

Repeated application of (2) to the two integrals in (3) then leads to the result

$$\begin{aligned} \frac{\partial}{\partial x}\iint_{A(x,t)} f(x, t; y, z)dy\, dz &= \iint_{A(x,t)} \frac{\partial}{\partial x} f(x, t; y, z)dy\, dz \\ &\quad + \oint \left\{\frac{\partial \phi_1}{\partial s}\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_2}{\partial s}\frac{\partial \phi_1}{\partial x}\right\} f(x, t; \phi_1, \phi_2)ds \end{aligned} \tag{4}$$

where the contour enclosing A is described in a clockwise sense. Suitable continuity conditions on f, f_x, $(\phi_1)_x$, $(\phi_2)_x$ must be satisfied.

Mail Address: Dept. of Math., University of Reading, England

A NOTE ON FOURIER SERIES OF FUNCTIONS OF Z CLASS

A. H. SIDDIQI, Aligarh Muslim University, India

1. A function $f(x)$ is said to belong to Z or $\wedge^*$ class if it is continuous and satisfies the condition $f(x+h)+f(x-h)-2f(x)=O(h)$ uniformly in x, as $h \to 0$.

We write

$$T(x) = \frac{a_0}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

and denote nth-Cesàro mean of order 1 of this series by $\sigma_n(x)$.

2. In this note we prove the following theorems:

THEOREM 1. *Sufficient condition that a trigonometric series $T(x)$ be the Fourier series of $f(x) \in Z$ is that $\sigma_n(x) - \sigma_m(x) = O(n^{-1})$ uniformly in $[0, 2\pi]$ for all $m > n$.*

THEOREM 2. *If a trigonometric series $T(x)$ be the Fourier series of $f(x) \in Z$, then $\sigma_n(x) - \sigma_m(x) = O(n^{-1} \log n)$ uniformly in $[0, 2\pi]$ for all $m > n$.*

It may be remarked that the corresponding problem for Lip α class has been examined by Duplessis [3].

3. We require the following lemmas for the proof.

LEMMA 1. *If $\sigma_n(x) - f(x) = O(n^{-1})$ uniformly, then $f(x) \in Z$.*

Proof. Obviously the trigonometric polynomial of best approximation deviates from f by $O(n^{-1})$. It follows from a result of Zygmund [1, p. 103] that $f \in Z$.

LEMMA 2 [2, p. 91]. *If* $f(x) \in Z$, *then* $\sigma_n(x) - f(x) = O(n^{-1} \log n)$.

4. *Proof of Theorem* 1. From the hypothesis we have $\sigma_n(x) - \sigma_m(x) \to 0$ uniformly in x as $m, n \to \infty$, which implies that there exists a function $f(x)$ such that $\sigma_n(x) \to f(x)$ uniformly. From this it follows that $\sigma_n(x) - f(x) = O(n^{-1})$ and hence by Lemma 1, $f(x) \in Z$. It can be also shown that

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \begin{Bmatrix} \cos kt \\ \sin kt \end{Bmatrix} dt,$$

and this completes the proof of Theorem 1.

Proof of Theorem 2. Since $f(x) \in Z$ we have by Lemma 2, $\sigma_n(x) - f(x) = O(n^{-1} \log n)$, $n \to \infty$. Therefore it follows that $\sigma_n(x) - \sigma_m(x) = O(n^{-1} \log n)$, uniformly in x and for all $m > n$.

Mail Address: Dept. of Math., Aligarh Muslim University, Aligarh (UP) India.

References

1. I. P. Natanson, Constructive theory of functions, Leningrad, 1949. Translation series U.S.A. Atomic Energy Commission, 1961.
2. A. Zygmund, Trigonometric series, Vol. 1, Cambridge University Press, 1959.
3. N. Duplessis, A note about functions in Lip α, Proc. Edinburgh Math. Soc., 10 (1954) 100.

CLASSROOM NOTES

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT06268.

THE EXISTENCE-UNIQUENESS THEOREM FOR AN nTH ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION

D. WILLETT, University of Utah

In this note we will present a new proof of the existence and uniqueness of a solution to the initial value problem

$$\begin{aligned} &y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0, \\ &y(x_0) = y_0,\ y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1}, \end{aligned} \tag{1}$$

where the functions $a_k(x)$ are defined and continuous on a bounded interval I containing the point x_0.

The classical way (see, e.g., [1]) of obtaining the existence and uniqueness of a solution to (1) is as a special case of the corresponding result for the more

general nth order system of equations. Since the proof for a system is almost identical with the proof for the corresponding nonlinear problem in which a uniform Lipschitz condition is assumed to hold, one can as well let the differential equations be nonlinear. However, in doing this, there is added the problem of showing the solution for the linear problem can be uniquely extended throughout the interval I.

The classical proof applied directly to (1) is computationally complicated. An elegant proof in this manner requires the use of vector or matrix notation. Quite often students in beginning courses in differential equations do not have the required linear algebra background to appreciate such a proof. A common practice in many elementary textbooks (see, e.g., [2], [4], [6]) is to set up the proof of the theorem for the general nth order problem as n different scalar successive approximations. This eliminates the introduction of vectors, but because of the resulting computational complexities, details of the proof are left out of the text. Some authors (see, e.g., [3]) are satisfied with just a statement of the theorem, and leave the proof to more advanced treatises.

In this note we will use the scalar nature of (1) to help supply a proof which does not require the use of vectors or matrix-type computations. We will need successive approximations for the simplest scalar situation and the method of variation of parameters.

LEMMA 1. *If $g(x, t)\in C(I\times I)$ and $u(x)\in C(I)$, then there exists a solution $y(x)\in C(I)$ to the equation*

$$y(x) = u(x) + \int_{x_0}^{x} g(x, t)y(t)dt \qquad (x_0 \in I).$$

LEMMA 2. *If $a(x)\in C(I)$, then there exists a solution $y(x)\in C'(I)$ to the initial value problem*

$$y' + a(x)y = 0, \qquad y(x_0) = y_0 \qquad (x_0\in I),$$

and y and y' are continuous functions of (x, x_0) on $I\times I$.

LEMMA 3. *If $y(x)\in C^n(I)$ and $y(x_0)=y'(x_0)= \cdots = y^{(n-1)}(x_0)=0$ $(n\geq 1$, $x_0\in I)$, then*

$$\int_{x_0}^{x} \left| y(t)y^{(n)}(t) \right| dt \leq \frac{(x - x_0)^n}{2}\int_{x_0}^{x} \left| y^{(n)}(t) \right|^2 dt \qquad (x \geq x_0).$$

Lemma 1 is the fundamental existence theorem for linear Volterra integral equations with numerical functions and can be easily established by successive approximations; the most exotic mathematical steps in its proof probably are the realization that the continuous function $g(x, t)$ is bounded on the bounded set $I\times I$ and that integral and limit can be interchanged when the limit is uniform. Lemma 2 is the first order linear initial value problem for which the solution can be explicitly displayed. Lemma 3 is a generalization of Opial's inequal-

ity ($n=1$) [5] and can be easily proven using only the Schwarz inequality for integrals. The proof is as follows: Let

$$z(x) = \int_{x_0}^{x}\int_{x_0}^{t_{n-1}} \cdots \int_{x_0}^{t_1} | y^{(n)}(t) | \, dt\, dt_1 \cdots dt_{n-1},$$

so that $z^{(n)}(x) = |y^{(n)}(x)|$ and

$$\int_{x_0}^{x} | y(t)y^{(n)}(t) | \, dt \leqq \int_{x_0}^{x} z(t)z^{(n)}(t)dt \leqq \int_{x_0}^{x} (t - x_0)z'(t)z^{(n)}(t)dt$$

$$\leqq \int_{x_0}^{x} (t - x_0)^{n-1}z^{(n-1)}(t)z^{(n)}(t)dt \leqq (x - x_0)^{n-1} \frac{[z^{(n-1)}(x)]^2}{2}$$

$$= \frac{(x - x_0)^{n-1}}{2}\left[\int_{x_0}^{x} z^{(n)}(t)dt\right]^2 \leqq \frac{(x - x_0)^n}{2}\int_{x_0}^{x} [z^{(n)}(t)]^2 dt, \qquad x \geqq x_0.$$

With the above three lemmas established, we can now prove without a great deal of difficulty the main theorem:

THEOREM. *If $a_k(x) \in C(I)$, $k=1, 2, \cdots, n$, then there exists a unique solution $y(x) \in C^n(I)$ to the initial value problem* (1); *and y and the first n derivatives of y are continuous functions of (x, x_0) on $I \times I$.*

Proof. We will prove the uniqueness first. If (1) has two solutions $y_1(x)$ and $y_2(x)$, then $z(x) = y_1(x) - y_2(x)$ is a solution of the differential equation such that $z(x_0) = z'(x_0) = \cdots = z^{(n-1)}(x_0) = 0$. We can assume without loss of generality that x_0 is the left endpoint of I. Multiplying the differential equation by $z^{(n-1)}$ and integrating between x_0 and x, we obtain

$$[z^{(n-1)}(x)]^2 + 2\int_{x_0}^{x} [a_1(t)z^{(n-1)}(t) + \cdots + a_n(t)z(t)]z^{(n-1)}(t)dt = 0.$$

Since the functions $a_k(x)$, $k=1, 2, \cdots, n$, are continuous functions on I and I is bounded, there exists a constant M_0 such that

$$[z^{(n-1)}(x)]^2 \leqq M_0 \int_{x_0}^{x} \big[| z^{(n-1)}(t) |^2 + | z^{(n-2)}(t)z^{(n-1)}(t) | + \cdots + | z(t)z^{(n-1)}(t) | \big] dt.$$

But by Lemma 3,

$$\int_{x_0}^{x} | z^{(n-k)}(t)z^{(n-1)}(t) | \, dt \leqq \frac{(x - x_0)^{k-1}}{2}\int_{x_0}^{x} | z^{(n-1)}(t) |^2 dt \qquad (k = 2, \cdots, n),$$

and so there exists a constant M_1, such that

$$[z^{(n-1)}(x)]^2 \leqq M_1 \int_{x_0}^{x} [z^{(n-1)}(t)]^2 dt \qquad (x \in I).$$

This can be possible only if $z^{(n-1)}(x)=0$ for all $x\in I$, and from the homogeneous initial conditions satisfied by z this in turn implies that $z(x)=0$ for all $x\in I$; hence, $y_1=y_2$.

The proof of existence is by induction on the order n. For $n=1$, the theorem is simply Lemma 2. Suppose the theorem is true for $n=k$ and consider (1) when $n=k+1$.

For each $t\in I$, let $w(x)$ be the unique solution of the initial value problem with differential equation

$$w^{(k)}+a_1(x)w^{(k-1)}+\cdots+a_k(x)w=0 \tag{2}$$

and initial conditions

$$w(t)=w'(t)=\cdots=w^{(k-2)}(t)=0,\ w^{(k-1)}(t)=1. \tag{3}$$

Such a function exists by the induction hypothesis, and furthermore, can be considered as a continuous function of x and t. Let

$$g(x,t)=\int_t^x w(s,t)ds. \tag{4}$$

Clearly, $g(x,t)$ is k times differentiable with respect to x, and

$$g(x,x)=\frac{\partial g}{\partial x}(x,x)=\cdots=\frac{\partial^{k-1}g}{\partial x^{k-1}}(x,x)=0,\quad \frac{\partial^k g}{\partial x^k}(x,x)=1\qquad (x\in I). \tag{5}$$

From the induction hypothesis g and these k derivatives are continuous functions of (x,t) on $I\times I$. Actually, $g(x,t)$ is the Green's function for the initial value problem with differential operator $y^{(k+1)}+a_1(x)y^{(k)}+\cdots+a_k(x)y'$.

Next, let $v(x)$ be the unique solution of the kth order initial value problem with differential equation (2) and initial conditions

$$v(x_0)=y_1,\ v'(x_0)=y_2,\cdots,v^{(k-1)}(x_0)=y_k,$$

and let

$$u(x)=y_0+\int_{x_0}^x v(t)dt.$$

Using (2)–(5) and the Leibnitz formula for differentiating integrals, we can now easily establish that any solution of the Volterra integral equation

$$y(x)=u(x)-\int_{x_0}^x g(x,t)a_{k+1}(t)y(t)dt \tag{6}$$

is a solution of (1) when $n=k+1$. But by Lemma 1, (6) has a solution for $y(x)$. Hence, (1) when $n=k+1$ has a solution, and the existence part of the theorem follows for all n by the principle of finite induction.

Note that Lemma 3 was used only for establishing uniqueness. We included this particular proof for uniqueness only because of the novelty of using the generalized Opial's inequality to obtain the conclusion. If one is willing to give a more thorough preliminary discussion of Green's functions, as for example in [3], then the uniqueness can be incorporated into the induction argument used to obtain existence. There are also other ways to conclude uniqueness independent of existence.

References

1. E. A. Coddington, An Introduction to Ordinary Differential Equations, Prentice-Hall, Englewood Cliffs, N. J., 1961.

2. L. R. Ford, Differential Equations, 2nd ed., McGraw-Hill, New York, 1955.

3. D. L. Kreider, R. G. Kuller, D. R. Ostberg, and F. W. Perkins, An Introduction to Linear Analysis, Addison-Wesley, Reading, Mass., 1966.

4. W. Leighton, Ordinary Differential Equations, 2nd ed., Wadsworth, Belmont, Cal., 1963.

5. Z. Opial, Sur une inégalité, Ann. Polon. Math., 8 (1960) 29–32.

6. S. L. Ross, Differential Equations, Blaisdell, New York, 1964.

ON THE EQUIVALENCE OF COMPACTNESS AND FINITENESS IN TOPOLOGY

NORMAN LEVINE, The Ohio State University

1. Introduction. We shall term a topological space X a cf space (compact finite) iff compactness and finiteness are equivalent in X. To show that a space is cf, it clearly suffices to prove that every compact set is finite (or that every infinite set is noncompact). Of course finite spaces are always cf spaces. We list a few more cf spaces.

Example 1. Let $(X, \mathfrak{J})$ be any discrete topological space.

Example 2. Let X be the set of positive integers and $\mathfrak{J}$: $\emptyset$, (1), (1, 2), $\cdots$, $(1, 2, \cdots, n), \cdots, X$.

Example 3. Let X be any set and let $\mathfrak{J}$ consist of all subsets of X which are empty or have countable complements.

Example 4. Let X be any set and let x^* be a fixed point in X. Let $\mathfrak{J}$ consist of all subsets of X which are empty or contain x^*.

cf is not invariant under continuous, 1-1, onto maps as shown by

Example 5. Let X be the set of positive integers with the discrete topology and let Y consist of $0, 1, 1/2, \cdots, 1/n, \cdots$ with the relative topology. Let $f(1)=0, f(2)=1, f(3)=1/2, \cdots, f(n)=1/(n-1)$ for $n>1$.

cf is not invariant under continuous, open, onto maps as shown by

Example 6. Let X be the unit interval and let $\mathfrak{J}$ consist of all subsets of X which are empty or have countable complements. Let Y be the set of positive integers with the *indiscrete* topology. Let $f(x)=1$ if $x=0$ or $x\in(1/2, 1]$, $f(x)=2$ if $x\in(1/3, 1/2]$, $\cdots$, $f(x)=n$ if $x\in(1/(n+1), 1/n]$, $\cdots$.

cf however is invariant under open, 1-1, onto transformations.

THEOREM 1. *Let X and Y be topological spaces and let X be* cf. *If $f: X\rightarrow Y$ is open*, 1-1, *and onto, then Y is* cf.

Proof. Let A be a compact subset of Y. Now f^{-1}: $Y \to X$ is continuous and thus $f^{-1}[A]$ is compact in X. Since X is cf, $f^{-1}[A]$ is finite and thus A is finite.

2. Product spaces.

THEOREM 2. *Let X and Y each be* cf *spaces. Then $X \times Y$ is* cf.

Proof. Let A be a compact subset of $X \times Y$. Then $P_x[A]$ and $P_y[A]$ are compact in X and Y respectively, P_x and P_y denoting the projection maps. Now $A \subset P_x[A] \times P_y[A]$ and since $P_x[A]$ and $P_y[A]$ are each finite, it follows that A is finite.

Theorem 2 cannot be extended to the infinite case as is shown by

Example 7. For each positive integer n, let X_n be a two point discrete topological space. Then $\times X_n$ is compact and infinite and thus is not a cf space.

The converse of Theorem 2, however, is true as implied by

THEOREM 3. *Let $\{X_\alpha : \alpha \in \triangle\}$ be a nonempty family of nonempty spaces. If $\times X_\alpha$ is* cf, *then X_α is* cf *for each $\alpha \in \triangle$.*

Proof. Let $\alpha^* \in \triangle$ and let A be a compact subset of X_{α^*}. For each $\alpha \neq \alpha^*$, select x_α arbitrarily in X_α. By the Tychonoff theorem. $P_{\alpha^*}^{-1}[A] \cap \bigcap_{\alpha \neq \alpha^*} P_\alpha^{-1}[(x_\alpha)]$ is compact and hence finite in $\times X_\alpha$. It follows then that A is finite.

3. Additivity theorems.

THEOREM 4. *Let X be a topological space and let F_1 and F_2 be closed subsets which are each* cf. *If $X = F_1 \cup F_2$, then X is* cf.

Proof. Let A be a compact subset of X. Then $A \cap F_1$ is compact in F_1 and therefore finite. Likewise $A \cap F_2$ is finite and thus $A (= A \cap F_1 \cup A \cap F_2)$ is finite.

COROLLARY. *Let X be a topological space and suppose $F_1, \cdots, F_n$ are each closed in X and* cf. *If $X = F_1 \cup \cdots \cup F_n$, then X is* cf.

THEOREM 5. *Let X be a topological space and let U and V be open* cf *subspaces. If $X = U \cup V$, then X is* cf.

We first present

LEMMA 1. *Let X be a* cf *space and suppose A is an infinite subset of X. If $a_1, \cdots, a_n$ are in A, there exist open sets $O_1, \cdots, O_n$ for which $a_i \in O_i$ and $A - (O_1 \cup \cdots \cup O_n)$ is infinite.*

Proof of lemma. If no such open sets exist, then A is compact contrary to X being a cf space.

Proof of theorem. Suppose that X is not cf. Then there exists an infinite compact set A in X. Then $A - U$ is a closed subset of A and is thus compact in V (and hence finite). Likewise $A - V$ is finite. Denote $(A - U) \cup (A - V)$ by $\{a_1, \cdots, a_n\}$. By Lemma 1 there exist open sets $O_1, \cdots, O_n$ such that $a_i \in O_i$ and $A - (O_1 \cup \cdots \cup O_n)$ is an infinite subset of $U \cap V$. Since $U \cap V$ is cf, it follows that $A - (O_1 \cup \cdots \cup O_n)$ is not compact in $U \cap V$ and thus there exists

an infinite cover Φ open in $U \cap V$ (and hence also open in X) with no finite subcover. Then $\{O_1, \cdots, O_n\}$ together with Φ is an open cover of A with no finite subcover. This contradicts A being compact.

It is clear that neither Theorem 4 nor Theorem 5 can be extended to the infinite case.

In regard to Theorems 4 and 5, it seems appropriate that we present

Example 8. Let X consist of the positive integers and let $\mathfrak{J}$ consist of all subsets of $\{2, 3, \cdots, n, \cdots\}$ together with all subsets of X containing 1 which have finite complements. Let $F=\{1\}$ and let $G=\{2, 3, \cdots, n, \cdots\}$. Then G is open, F is closed, F and C are each cf, $X=F\cup G$, but X is not cf.

4. First axiom spaces.

THEOREM 6. *A topological space is discrete iff it is T_1, first axiom and* cf.

Proof. The necessity is obvious. To show the sufficiency, let A be an arbitrary subset of X. We shall show that the derived set of A (denoted by A') is empty. If $x_0 \in A'$, there exists an infinite sequence of distinct points x_n $(n=1, 2, \cdots)$ in A such that $x_0=\lim x_n$. Then x_n: $n=0, 1, \cdots$ is compact and infinite contrary to X being cf.

COROLLARY. *A metrizable space is* cf *iff it is discrete.*

THEOREM 7. *Let X be a first axiom space. Then X is* cf *iff every infinite sequence of distinct points in X is free of cluster points.*

Proof. Necessity. Suppose $\{x_n\}$ is an infinite sequence of distinct points and suppose x^* is a cluster point. Let O be a monotone decreasing open base at x^*. Choose $x_{n_1} \in O_1$. Take $n_2 > n_1$ and $x_{n_2} \in O_2$, etc. Then $x^*=\lim x_{n_j}$. It follows then that $\{x^*\} \cup \{x_{n_j}: j=1, 2, \cdots\}$ is infinite and compact contrary to X being a cf space.

Sufficiency. Suppose there exists in X an infinite compact subset A. Select an infinite sequence of distinct points a_n in A. Since the sequence is free of cluster points, for each a in A, there exists an open set O_a such that $a \in O_a$ and $a_n \in O_a$ for at most a finite number of n. Then $\{O_a\}$ is clearly an open cover of A with no finite subcover contrary to A being compact.

References

1. D. W. Hall and G. L. Spencer II, Elementary Topology, Wiley, New York, 1955.
2. John L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.

A CHARACTERIZATION OF FINITE SUPERSOLVABLE GROUPS

W. E. DESKINS, Michigan State University

In a recent note [3] in this journal C. V. Holmes presented a new characterization of a finite nilpotent group, viz., *a finite group G is nilpotent if and only if G contains a normal subgroup of order d for each divisor d of the order $|G|$ of G.*

Here we show that a characterization of a supersolvable group is obtained when (roughly speaking) the normality condition is dropped. All groups considered are finite.

DEFINITION. *A group G is supersolvable if it contains a chain $\langle 1\rangle = H_0 < H_1 < \cdots < H_n = G$ of subgroups normal in G such that H_i/H_{i-1} is cyclic for $i=1, 2, \cdots, n$.*

Two well-known characterizations are:

1. (IWASAWA) A group G is supersolvable if and only if the maximal subgroup chains of G all have the same length, ([2], page 342).

2. (HUPPERT) A group G is supersolvable if and only if each of its maximal subgroups has index a prime, ([2], page 162).

In this note we give another characterization.

THEOREM 1. *A group G is supersolvable if and only if each subgroup $H \leqq G$ contains a subgroup of order d for each divisor d of $|H|$.*

The following alternate formulation, clearly equivalent, is more easily treated.

THEOREM 1′. *A group G is supersolvable if and only if each subgroup $H \leqq G$ contains a subgroup of index p for each prime divisor p of $|H|$.*

Proof. First suppose that G is supersolvable. Then the following facts ([2], pages 158, 159) are known about G:

(i) Its subgroups and factor groups are supersolvable.

(ii) If p is the maximal prime factor of $|G|$ and p^r is the order of the p-Sylow subgroups of G, then G contains a normal subgroup P of order p^{r-1}. Moreover the p-Sylow subgroup G_p of G is normal.

We prove the necessity now by induction on $|G|$. If H is a proper subgroup of G then it is supersolvable by (i) and so by the induction hypothesis contains a subgroup of prime index for each prime factor of $|H|$. Thus we need only show that G contains a subgroup of index q whenever q is a prime factor of $|G|$. If $q<p$, consider the group $\overline{G}=G/G_p$, where G_p is the subgroup described in (ii). By (i) $\overline{G}$ is supersolvable and since $|\overline{G}| < |G|$, $\overline{G}$ contains a subgroup $\overline{K}$ of index q. Clearly K, the pre-image of $\overline{K}$ in G, has index q. If on the other hand $q=p$, then we use a well-known property of solvable groups (P. Hall; [2], page 141):

(iii) If S is a solvable group of order mn, $(m, n)=1$, then S contains a subgroup of order m.

Since G is certainly solvable it contains a subgroup T of order $|G|/p^r$. Then the subgroup PT has index p in G.

The sufficiency is proved by induction on the order of G also. Let q be the least prime factor of $|G|$. By hypothesis G contains a subgroup K of index q in G, and by a standard exercise ([1], page 45) K is normal in G. Since a subgroup of K is also a subgroup of G we conclude from the induction hypothesis that K is supersolvable. Then from (ii) above we know that the p-Sylow subgroup K_p

of K is normal in K and hence also in G, where p is the maximal prime factor of $|G|$. If $p=q$ then G is a p-group and hence supersolvable, so we need only consider the case $q\neq p$. Then K_p is the p-Sylow subgroup of G.

Let M be a minimal normal subgroup of G which lies in the center of K_p. We shall show that $|M|=p$. By hypothesis G contains a subgroup N of index p. Then either $N\cap M=\langle 1\rangle$ or $N\cap M=M$ since M is abelian and minimal normal in G. If $N\cap M=\langle 1\rangle$ then $|M|=|G|/|N|=p$, so suppose $N\cap M=M$. Now by the induction hypothesis N is supersolvable so that M contains a subgroup M_1 which is normal in N and of order p. Furthermore, since N has index p in G there exists an element x of K_p which lies outside of N. But $M_1\leqq M\leqq$ center of K_p, so that x normalizes M_1. Thus M_1 is normal in G and $M_1=M$ because of the minimality of M. So G contains a normal subgroup M of order p in all cases.

Now consider $G^*=G/M$. If G^* is supersolvable then it is clear from the definition above that G is supersolvable. To prove G^* supersolvable we shall show that each subgroup H^* of G^* contains a subgroup of prime index for every prime divisor of $|H^*|$. If H^* is a proper subgroup of G^* then its pre-image H in G is also proper and, as noted above, supersolvable. Then H^* is supersolvable by (i) and so contains the required subgroups. Consequently we need only show that G^* contains the required maximal subgroups.

If r is a prime factor of $|G^*|$ then r divides $|G|$. Therefore G contains a subgroup R of index r. If $R\geqq M$ then $R^*=R/M$ has the same index in G^* as R does in G. If $R\ngeqq M$ then $R\cap M=\langle 1\rangle$ as above, so that $G=RM$ and $G^*=RM/M\cong R$. Since R is a proper subgroup of G it is supersolvable by induction, and G^* isomorphic with R clearly implies that G^* has all the required subgroups. Thus in all cases G^* is supersolvable, proving that G is supersolvable.

It should be noted that a group G is *not* necessarily supersolvable if the condition, "each subgroup $H\leqq G$ contains a subgroup of every possible order," is replaced by "G contains a subgroup of every possible order." For example, the symmetric group on four letters, S_4, has the latter property but is not supersolvable.

This work was supported in part by the National Science Foundation GP-5416.

References

1. W. Burnside, Theory of Groups of Finite Order, 2nd ed., Dover, New York, 1955.
2. M. Hall, The Theory of Groups, Macmillan, New York, 1959.
3. C. V. Holmes, A characterization of finite nilpotent groups, this MONTHLY, 73 (1966) 1113–1114.

TOWARDS MODERNIZATION OF THE DEFINITION OF A FUNCTION

HARRY GONSHOR, Rutgers—The State University

The inadequacy of the classical definition of a function as a set of ordered pairs has become clear from the spirit of category theory. Actually this is evident from the point of view of classical algebraic topology.

Let us summarize this point of view. The basic technique of algebraic topology is to study topological systems by means of operators which map them into algebraic systems. For our purpose we need consider only the one dimensional homology functor with the integers as coefficient group. This operator maps topological spaces into abelian groups and, in addition, continuous mappings between spaces into homomorphisms between the corresponding groups. Furthermore this operator preserves composition and takes identity maps into identity maps.

As examples, consider the unit circle, $C=[(X, Y): X^2+Y^2=1]$ and the unit cell $S=[(X, Y): X^2+Y^2\leqq 1]$ and the two following functions:

(1) The identity map $C\rightarrow C$,

(2) The inclusion map $C\rightarrow S$.

The above operator maps C into the group of integers Z and S into the zero group. The identity map $C\rightarrow C$ is mapped into the identity map $Z\rightarrow Z$ and the map $C\rightarrow S$ into the zero map $Z\rightarrow 0$.

We are now ready to examine the problem created by the classical definition of a function. It is clear that as a set of ordered pairs both functions are the same, yet in algebraic topology they are regarded as being distinct. Furthermore this is *not* a matter of careless terminology. The above functor operating on these two functions lead to functions which are unquestionably distinct, namely an identity function and a zero function. Thus it is logically essential to regard them as distinct.

One idea I have in mind is to distinguish between function and mapping. (Since the two words already exist in the language as well as the two concepts, it is convenient to do so.) A function is still defined as a set of ordered pairs. A map is defined as an ordered triple (X, Y, F) where F is a function with dom $F=X$ and range $F\subset Y$. This would be more consistent with present mathematical usage. Also with this definition the above functor is unquestionably well-defined on maps.

An obvious possible objection to my definition is the apparent superfluousness of X. In fact F determines X uniquely for all functions F. However, it is natural to generalize this to relations. (Although category theory has traditionally emphasized abstractions of functions, general relations have also occurred. A discussion of the advantages of using relations instead of functions only is given in (1).) Then the X is needed. Thus to retain the fact that a function is a special case of a relation the X is kept. (Incidentally, it would be nice to have two words for "relation" for this purpose.)

Reference

1. D. Puppe, Korrespondenzen in abelschen Kategorien, Math. Ann., 148 (1962) 1–30.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS

COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D.C. 20005.

VARIABLES AND FUNCTIONS

C. H. DOWKER, Birkbeck College, University of London

Variables, in the sense of time dependent measurable properties, do not occur in pure mathematics. But whereas the abstract concept corresponding to a physical straight line is called a straight line, the abstract concept corresponding to a variable is called a function. This name is well established in pure mathematics and cannot now be changed. The abstract analogs of constants and independent variable are constant functions and identity function; $f(x)$ and f and x are all functions. Function value is ambiguous and should not be used unless the context makes the meaning clear. Sometimes it means function and sometimes it means a number. A number is neither a function nor a variable. Dependent variable, variable and quantity are more or less synonymous. Likewise transformation, mapping and function are more or less synonymous. Since function is the abstract analog of variable, it is quite proper to use the same adjective "increasing" for variables and for functions.

The problem of mathematical definition can be considered in a simpler case. A rational number is usually defined to be a certain kind of set of ordered pairs of integers. It follows that it is a function. Though nobody believes that a rational number is really either a function or a set of pairs of integers, it is fascinating to see the formal properties of the rational numbers derived from the definition. Similarly some students find it enlightening to see the elements of the calculus derived by the consistent use of the definition of function as a set of ordered pairs, even when they observe the tongue in the professor's cheek. Definitions which are never used have no educational value. It is doubtful if it makes any sense to ask what a function, or any other abstract concept, really is.

The word "variable" is used in mathematical logic in a sense totally different from that in applied mathematics. Whenever the two different meanings of variable occur in the same context, frightful confusion reigns. It would be helpful if authors of books likely to fall into the hands of applied mathematicians would avoid ever using the word "variable" in its mathematical logic sense. It would also be decent of them to respect the applied mathematician's copyright and never use variable for any pure mathematics concept.

Comments.

1. A quantity is a property regarded as capable of measurement, e.g. humidity, density, viscosity. When certain variable quantities are so related that giving the value of one of them determines the values of the others, the one is

called the independent variable and the others are, or were, called functions or dependent variables. These functions may have derivatives and also physical dimensions. The independent variable has a derivative too; thus all the variables in this context are functions in this sense. The abstract functions of pure mathematics may have derivatives but do not have physical dimensions.

In some calculus books a variable is defined to be a symbol which stands for any element of a class; that is, "variable" means "common noun." This grammatical variable should not be confused with the variable quantity of applied mathematics.

2. A rational number is defined, e.g. by E. Landau, Foundations of Analysis, New York, 1960, to be an equivalence class of fractions, such as $\{\frac{2}{3}, \frac{4}{6}, \frac{10}{15}, \cdots\}$. A fraction is an ordered pair of integers. Distinct but equivalent fractions have different denominators. Thus the equivalence class of ordered pairs is a function.

Reference

1. C. P. Nicholas, A dilemma in definition, this MONTHLY, 73 (1966) 762–768.

SOME MODERN MATHEMATICAL METHODS IN THE THEORY OF LION HUNTING[1]

OTTO MORPHY, D.Hp. (Dr. of Hypocrisy)

It is now 30 years since the appearance of H. Pétard's classic treatise [2] on the mathematical theory of big game hunting. These years have seen a remarkable development of practical mathematical techniques. It is, of course, generally known that it was Pétard's famous letter to the president in 1941 that led to the establishment of the Martini Project, the legendary crash program to develop new and more efficient methods for search and destroy operations against the axis lions. The Infernal Bureaucratic Federation (IBF) has recently declassified certain portions of the formerly top secret Martini Project work. Thus we are now able to reveal to the world, for the first time, these important new applications of modern mathematics to the theory and practice of lion hunting. As has become standard practice in the discipline [2] we shall restrict our attention to the case of lions residing in the Sahara Desert [3]. As noted by Pétard, most methods apply, more generally, to other big game. However, method (3) below appears to be restricted to the genus Felis. Clearly, more research on this important matter is called for.

1. (Surgical method) A lion may be regarded as an orientable three-manifold with a nonempty boundary. It is known [4] that by means of a sequence of surgical operations (known as "spherical modifications" in medical parlance) the lion can be rendered contractible. He may then be signed to a contract with Barnum and Bailey.

2. (Logical method) A lion is a continuum. According to Cohen's theorem [5] he is undecidable (especially when he must make choices). Let two men

approach him simultaneously. The lion, unable to decide upon which man to attack, is then easily captured.

3. (Functorial Method) A lion is not dangerous unless he is somewhat gory. Thus the lion is a category. If he is a small category then he is a kittygory [6] and certainly not to be feared. Thus we may assume, without loss of generality, that he is a proper class. But then he is not a member of the universe and is certainly not of any concern to us.

4. (Method of differential topology) The lion is a three-manifold embedded in euclidean 3-space. This implies that he is a handlebody [7]. However, a lion which can be handled is tame and will enter the cage upon request.

5. (Sheaf theoretic method) The lion is a cross-section [8] of the sheaf of germs of lions [9] on the Sahara Desert. Merely alter the topology of the Sahara, making it discrete. The stalks of the sheaf will then fall apart releasing the germs which attack the lion and kill it.

6. (Method of transformation groups) Regard the lion as a surface. Represent each point of the lion as a coset of the group of homeomorphisms of the lion modulo the isotropy group of the nose (considered as a point) [10]. This represents the lion as a homogeneous space. That is, this representation homogenizes the lion. A homogenized lion is in no shape to put up a fight [11].

7. (Postnikov method) A male lion is quite hairy [12] and may be regarded as being made up of fibers. Thus we may regard the lion as a fiber space. We may then construct a Postnikov decomposition [13] of the lion. This being done, the lion, being decomposed, is dead and in bad need of burial.

8. (Steenrod algebra method) Consider the mod p cohomology ring of the lion. We may regard this as a module over the mod p Steenrod algebra. Doing this requires the use of the table of Steenrod cohomology operations [14]. Every element must be killed by some of these operations. Thus the lion will die on the operating table.

9. (Homotopy method) The lion has the homotopy type of a one-dimensional complex and hence he is a $K(\pi, 1)$ space. If π is noncommutative then the lion is not a member of the international commutist conspiracy [15] and hence he must be friendly. If π is commutative then the lion has the homotopy type of the space of loops on a $K(\pi, 2)$ space [13]. We hire a stunt pilot to loop the loops, thereby hopelessly entangling the lion and rendering him helpless.

10. (Covering space method) Cover the lion by his simply connected covering space. In effect this decks the lion [16]. Grab him while he is down.

11. (Game theoretic method) A lion is big game. Thus, *a fortiori*, he is a game. Therefore there exists an optimal strategy [17]. Follow it.

12. (Group theoretic method) If there are an even number of lions in the

Sahara Desert we add a tame lion. Thus we may assume that the group of Sahara lions is of odd order. This renders the situation capable of solution according to the work of Thompson and Feit [18].

We conclude with one significant nonmathematical method:

13. (Biological method) Obtain a number of planarians and subject them to repeated recorded statements saying: "You are a planarian." The worms should shortly learn this fact since they must have some suspicions to this effect to start with. Now feed the worms to the lion in question. The knowledge of the planarians is then transferred to the lion [19]. The lion, now thinking that he is a planarian, will proceed to subdivide. This process, while natural for the planarian, is disastrous to the lion [20].

Ed. note: Prof. Morphy is the namesake of his renowned aunt, the author of the famous series of epigrams now popularly known as Auntie Otto Morphisms or euphemistically as epimorphisms.

Footprints

1. This report was supported by grant #007 from Project Leo of the War on Puberty.
2. H. Pétard, A contribution to the mathematical theory of big game hunting, this MONTHLY, (1938).
3. This restriction of the habitat does not affect the generality of the results because of Brouwer's theorem on the invariance of domain.
4. Kervaire and Milnor, Groups of homotopy spheres, I, Ann. of Math., (1963).
5. P. J. Cohen, The independence of the continuum hypothesis, Proc. N.A.S. (63-64).
6. P. Freyd, Abelian Categories, Harper and Row, New York, 1964.
7. S. Smale, A survey of some recent developments in differential topology, Bull. A.M.S., (1963).
8. It has been experimentally verified that lions are cross.
9. G. Bredon, Sheaf Theory, McGraw-Hill, New York, 1967.
10. Montgomery and Zippin, Topological Transformation Groups, Interscience, 1955.
11. E. Borden, Characteristic classes of bovine spaces, Peripherblatt für Math., (1966BC).
12. Eddy Courant, Sinking of the Mane, Pantz Press, 1898.
13. E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.
14. Steenrod and Epstein, Cohomology Operations, Princeton, 1962.
15. Logistics of the Attorney General's list, Band Corp. (1776).
16. Admiral, T. V., (USN Ret.), How to deck a swab, ONR tech. rep. (classified).
17. von Neumann and Morgenstern, Theory of Games . . . , Princeton, 1947.
18. Feit and Thompson, Solvability of groups of odd order, Pac. J. M. (1963).
19. J. V. McConnell, ed., The Worm Re-turns, Prentice-Hall, Englewood Cliffs, N. J., 1965.
20. This method must be carried out with extreme caution for, if the lion is large enough to approach critical mass, this fissioning of the lion may produce a violent reaction.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before June 30, 1968. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2055. *Proposed by M. F. Capobianco, St. John's University*

Consider a real polynomial equation of degree n. Attention is paid to whether the roots are real and unequal, real and equal (in various combinations), or simple or multiple complex conjugates. If $n=2$ there are but three possibilities, namely, all roots real and equal, all roots real and unequal, and all roots complex. If $n=3$, there are four possibilities: three equal, two equal, three unequal, two complex. For $n=4$, there are nine possibilities. How many possibilities are there for general n?

E 2056. *Proposed by A. W. Walker, Toronto, Canada*

Two given circles have four real common tangents passing in pairs through two points K and K' on the line of centers. A third circle C touches the given circles at points Q and R antihomologous with respect to the homothetic center K. Then the four points where the two common tangents through K' meet circle C form a quadrangle with one pair of opposite sides that are parallel to the common tangents through K and meet at a point on the line KQR. (Part of this result was given by V. Thébault, Mathesis, 62 (1953) 112–114.)

E 2057. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

(1) If $f''(x)>0$ throughout the closed interval $[a, b]$, then for each point $\xi\in(a, b)$, there is a point $x_0\in(a, b)$ such that either

$$f'(\xi)=\frac{f(b)-f(x_0)}{b-x_0} \quad\text{or}\quad f'(\xi)=\frac{f(a)-f(x_0)}{a-x_0}\,.$$

(2) If $f''(x)>0$ in $[a, b]$ and $f'(x_0)=0$ for $x_0 \in (a, b)$, then there is a point $x_1 \in (a, b)$ such that either $f(a)=f(x_1)$ or $f(b)=f(x_1)$.

E 2058. *Proposed by J. E. Desmond, Florida State University*

Prove that b^{a-j+1} divides $\binom{b^a}{j}$ for positive integers a, $b>1$ and $j \leqq a+1$.

E 2059. *Proposed by A. Oppenheim, University of Reading, England*

For any two triangles prove the inequalities

$$\text{(1)} \qquad \frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c} \geqq 2\left(\frac{\cos A'}{a}+\frac{\cos B'}{b}+\frac{\cos C'}{c}\right),$$

$$\text{(2)} \qquad r_1+r_2+r_3 \geqq 2(h_1 \cos A' + h_2 \cos B' + h_3 \cos C'),$$

(where r_1 is the exradius corresponding to A, h_1 is the altitude from A, etc.) with equality if and only if both triangles are equilateral.

Further, deduce from (1) Barrow's inequality

$$\text{(3)} \qquad x+y+z \geqq 2(u+v+w),$$

where x, y, z are the distances of an internal point O from the vertices of a triangle XYZ and u, v, w are the distances of O from YZ, ZX, XY measured along the internal bisectors of the angles YOZ, ZOX, XOY. (See also problem 3740 by Paul Erdös [1937, 252], Solution by D. F. Barrow.)

E 2060. *Proposed by F. Leuenberger, Feldmeilen, Switzerland*

Let t_a, t_b, t_c be the internal anglebisectors and m_a, m_b, m_c the medians of a triangle T, r and R its in- and circum-radius, p its semiperimeter. Prove that

$$\sum t_a^6 \leqq p^4(p^2-12rR) \leqq \sum m_a^6,$$

with equality if and only if T is equilateral.

E 2061. *Proposed by David Singmaster, American University of Beirut, Lebanon*

For a Mercator map, it is well known that a straight line does not yield the path of shortest distance. What kind of curve does correspond to a path of shortest distance?

E 2062. *Proposed by Dale Peterson, Student, Mira Loma High School, Sacramento, Calif.*

Prove that there is a set of n composite integers in arithmetic progression, relatively prime in pairs, for any integer n.

E 2063. *Proposed by W. G. Dotson, Jr., North Carolina State University*

Prove: If S is a set of real numbers such that every continuous function from

S into S has a fixed point, then S either consists of a single point or is a closed bounded interval.

E 2064. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let A_n be an $n\times n$ determinant in which the entries, 1 to n^2, are put in order along the diagonals. For example

$$A_4 = \begin{vmatrix} 1 & 2 & 4 & 7 \\ 3 & 5 & 8 & 11 \\ 6 & 9 & 12 & 14 \\ 10 & 13 & 15 & 16 \end{vmatrix}.$$

Show that if $n=2k$ then $A_n=\pm k(k+1)$, and if $n=2k+1$, $A_n=\pm(2k^2+2k+1)$.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Triangle Construction

E 1915 [1966, 890]. *Proposed by Kaidy Tan, Fukien Normal College, Foochow, China*

Construct a triangle ABC given side BC, median AM, and the angle bisector AT (that is, given a, m_a, t_a).

Solution by J. M. Quoniam, Saint-Etienne, France. The known relation $b^2+c^2=2m_a^2+a^2/2$ allows one to construct $k=(b^2+c^2)^{1/2}$. Also, from the known relation

$$t_a = \frac{2}{b+c}\sqrt{bcs(s-a)},$$

one obtains $t_a^2(k^2+2bc)=bc(k^2-a^2+2bc)$, from which one can construct $l=\sqrt{bc}$. Finally, one can construct b and c from k and l.

Also solved by Walter Bluger, Norman Miller, Simeon Reich (Israel), Dimitrios Vathis (Greece), and Miss M. Sugunamma (India).

An Interesting Number-theoretic Function

E 1916 [1966, 891]. *Proposed by J. E. Lewis, Acadia University, Wolfville, N.S., Canada*

Given any positive integer n, then if both $(a, n)=1$ and $(a+2, n)=1$, we will call a and $a+2$ a *special pair*; and we define the function $\psi(n)$ as the number of special pairs less than n. Prove the

THEOREM. $\psi(n)+1=n\prod_{i=1}^{k}(1-2/p_i)$, *where* $n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ *is the canonical decomposition of* n, $(2, n)=1$;

$$\psi(m)+1 = 2^{a-1}n\prod_{i=1}^{k}(1-2/p_i), \quad where \quad m=2^a n,\ (2,n)=1.$$

Solution by M. G. Beumer, Technological University, Delft, the Netherlands. The theorem is a direct consequence of V. Schemmel's generalization of the Euler function $\phi(n)$. A detailed treatment can be found in Paul Bachmann, *Niedere Zahlentheorie, I* (Leipzig, 1902), pp. 91–93. A further generalization of Schemmel's function is treated in Edouard Lucas, *Théorie des nombres, I* (Paris, 1891), p. 402. See also Dickson, *History of the Theory of Numbers, I*, p. 147 (references 190–193) for further literature.

Also solved by William Forman, Jerry Goodman, M. G. Greening (Australia), Donald Jeffords, D. C. B. Marsh, L. J. Marx, S. F. Robinson, Miss M. Sugunamma (India), D. R. Stark, Charles Wexler, and the proposer.

Editorial Note. For completeness we give Lucas' generalization of Schemmel's function: Let $e_1, e_2, \cdots, e_k$ be any integers. Let $\Psi(n)$ denote the number of those integers h, chosen from $0, 1, \cdots, n-1$, such that $h-e_1, h-e_2, \cdots, h-e_k$ are prime to n. Then $\Psi(p)\Psi(q)=\Psi(pq)$ if $(p, q)=1$ and $\Psi(n)=n\prod_i(1-(\lambda_i/p_i))$, where $n=\prod_i p_i^{a_i}$ and λ_i is the number of distinct residues of $e_1, \cdots, e_k$ modulo p_i.

The problem here corresponds to the case $k=2$, $e_1=0$, $e_2=-2$, where $\Psi(n)=\psi(n)+1$. (Note that $\Psi(n)$ includes the pair $[n-1, n+1]$.)

A Divisibility Problem

E 1918 [1966, 891]. *Proposed by L. J. Warren and Jerry Tice, San Diego State College*

Let p be a prime larger than 3. Show that there is no positive integer k such that $3p \mid \sigma_k(3p)$, where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n.

Solution by Stanley Rabinowitz, Far Rockaway, N. Y. The only divisors of $3p$ are 1, 3, p, and $3p$. Now $3p \mid (1+3^k+p^k+3^kp^k)$ implies $3 \mid p^k+1$, which implies $p\equiv 2 \pmod 3$ and k is odd. If $p \mid 3^k+1$, then we would have $3^{k+1}\equiv -3 \pmod p$ with $k+1$ being even, but this contradicts the known fact that if $p\equiv 2 \pmod 3$ then -3 is a quadratic nonresidue $(\text{mod } p)$. See problem 3721 [1936, 583].

Also solved by D. C. Beverage, Brother Alfred Brousseau, L. Carlitz, H. M. Edgar, William Forman, M. G. Greening (Australia), Donald Jeffords, Andrzej Makowski (Poland), Daniel Marcus, D. C. B. Marsh, Alfred Pietrowski, Bob Prielipp, S. F. Robinson, A. P. Shah & A. M. Vaidya (India), D. R. Stark, Miss M. Sugunamma (India), and the proposers.

A Sextic Solvable in Radicals

E 1919 [1966, 891]. *Proposed by D. R. Rao, Secunderabad, India*

Find the roots of the equation $x^6-px^4+qx^3-rx^2+s=0$, having given $p(q^2-4s)=r^2$.

Solution by William Forman, Brooklyn College. If $q^2-4s=0$, then $r=0$ and the equation has the form $(x^3+q/2)^2=px^4$. The roots can now be obtained from the routine solution of the two cubic equations $x^3\pm p^{1/2}x^2+q/2=0$.

If $q^2-4s\neq 0$, let $D^2=q^2-4s$. Then the equation may be written in the form

$$\left(x^3+\frac{q}{2}\right)^2=\left(\frac{r}{D}x^2+\frac{D}{2}\right)^2$$

and again the roots may be obtained by solving two cubics.

Also solved by Neil Cameron (Australia), L. Carlitz, J. P. Celenza, G. C. Dodds, Donald Jeffords, Daniel Marcus, D. C. B. Marsh, Norman Miller, P. L. Montgomery, Walter Penney, I. R. Purdy, Simeon Reich (Israel), Marilyn Rodeen, P. H. Young, and the proposer.

Several solvers neglected the (trivial) case $p=0$. The proposer offered several other equations which yield to essentially the same method.

Eccentricity of a Certain Ellipse

E 1920 [1966, 891]. *Proposed by H. S. M. Coxeter, University of Toronto*

Find the eccentricity of an ellipse which passes through two of the cusps of its evolute.

Solution by Leon Bankoff, Los Angeles, California. The radius of curvature R of the ellipse $a^2y^2+b^2x^2=a^2b^2$ at $(0, \pm b)$ is a^2/b. If two cusps of the evolute lie at these points, then $R=2b$. Hence $a^2=2b^2$ and the eccentricity is $\sqrt{2}/2$.

Notes. (a) The other two cusps lie at the midpoints of the semi-major axes. (b) The minor eccentric circle passes through the foci. (c) The length of each arch of the evolute is $2b-a/2$.

Also solved by Charles Barber, Brother Thomas Flynn, Neil Cameron (New Zealand), J. P. Celenza, John Christopher, Marie D'Autrechy, Ragnar Dybvik (Norway), William Forman, R. B. Gasper, Michael Goldberg, M. G. Greening (Australia), Louise S. Grinstein, George G. Hay (Scotland), Lew Kowarski, R. V. Larson, D. C. B. Marsh, Norman Miller, D. S. Newman, J. R. Purdy, Stanley Rabinowitz, G. L. N. Rao (India), Simeon Reich (Israel), Sister M. Stephanie Sloyan, D. R. Stark, W. K. Viertel, Charles Wexler, and the proposer.

Sets of Points with Given Minimum Separation

E 1921 [1966, 891]. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove that there are at most three points on or inside the unit circle such that the distance between any two of them is greater than $\sqrt{2}$.

I. *Solution by M. G. Greening, University of New South Wales.* Choose n points $A_1, \cdots, A_n$ in the unit circle with center O. Then $OA_i \leqq 1$ and $A_iA_j > \sqrt{2}$. By the cosine rule

$$2 < OA_i^2 + OA_j^2 - 2\cdot OA_i\cdot OA_j\cdot \cos A_iOA_j$$

or $2\cdot OA_i\cdot OA_j\cdot \cos A_iOA_j < 0$; implying that the smaller angle subtended at O by any pair of points A_i, A_j be obtuse. This is impossible for $n>3$.

II. *Solution by Ron Graham, Bell Telephone Laboratories.* We prove somewhat more: *If $k \geqq 2$ points lie in a unit disk then some pair of these points are separated by a distance of no more than* $\max(1, 2 \sin \pi/k)$.

Proof: First we assert that if $a_1, \cdots, a_n$ are points which lie in a disk of radius 1, and x lies in the convex hull $(a_1, \cdots, a_n)$, then $d(x, a_j) \leqq 1$ for some j. This is immediate for $n=2$. For $n>2$, we can always choose three of the a_i, say a, b, c, such that $x \in$convex hull (a, b, c). But any point within the triangle T determined by a, b, c lies within a unit distance from some vertex of T. Now

assume that $a_1, \cdots, a_n$ lie in a unit disk and no a_i lies in the convex hull of the remaining a_j. If any a_i is not on the boundary of the disk then it is possible to move a_i to the boundary in such a way that the distance between a_i and each a_j does not decrease (this is easily seen, for example, by considering a line which separates a_i from the remaining points). Finally, if all the a_i lie on the boundary of the disk, dividing the circumference into n arcs, then some arc must have length $\leqq 2\pi/n$; hence some pair of points a_i, a_j have a separation of $\leqq 2 \sin (\pi/k)$. This proves the original statement.

The bound max $(1, 2 \sin \pi/k)$ is best possible for $2 \leqq k \leqq 7$.

III. *Solution by Jürg Rätz, University of Bern, Switzerland.* We prove the following generalization: *There are at most $n+1$ points in the closed unit ball U^n of Euclidean n-space R^n such that the distance between any two of them is greater than $\sqrt{2}$, n being a positive integer.*

Suppose that $A \subset R^n$ and card $A \geqq n+2$. Then by a theorem of H. Davenport and G. Hajós (communicated to me by H. Davenport; cf. also *Matematikai Lapok*, 2 (1951) 68) there are two elements $a_1, a_2 \in A\, (a_1 \neq a_2)$, the scalar product (a_1, a_2) of which is nonnegative. (In the case $n=2$ this can be shown in an elementary way by considering the angles aOb $(a, b \in A)$.) Now if also $A \subset U^n$, we have $|a_1-a_2|^2=(a_1-a_2, a_1-a_2)=|a_1|^2-2(a_1, a_2)+|a_2|^2 \leqq |a_1|^2+|a_2|^2 \leqq 1+1=2$, i.e. $|a_1-a_2| \leqq \sqrt{2}$. Now, our assertion follows.

On the other hand, there exist $n+1$ points in U^n with the required property, e.g., the vertices $e_0, e_1, \cdots, e_n$ of every regular n-simplex inscribed in U^n, for we have

$$(e_i, e_j) = \begin{cases} 1 & (i = j) \\ -1/n & (i \neq j) \end{cases}$$

Also solved by Marcia Ascher, Merrill Barnebey, Cloydine A. Beattie, R. J. Bonneau, J. L. Brown, Jr., M. G. Brown, J. P. Celenza, W. G. Dotson, Jr., William Forman, Michael Fredman, Michael Goldberg, Ned Harrell, G. A. Heuer, R. A. Jacobson, Donald Jeffords, M. S. Kaplan, P. S. Kornya, R. V. Larson, Dorembus Leonard (Israel), W. L. Lepowsky, Beatriz Margolis (Argentina), W. D. Markel, D. C. B. Marsh, L. J. Marx, B. J. Mattson, D. C. Mayne, Charles McCracken (two proofs), Brockway McMillan, Sam Newman, C. B. A. Peck, Stanley Rabinowitz (two proofs), Jürg Rätz (second proof), L. A. Ringenberg, G. F. Schumm, D. L. Silverman, Stephen Soldz, Z. Z. Uoiea, John Wessner, V. C. Williams, and the proposer.

A prime that can be put in many forms

E 1922 [1966, 891]. *Proposed by Gregory Wulczyn, Bucknell University*

Find a (least) prime which is simultaneously of each of the forms: x^2+y^2, $x^2+2y^2, \cdots, x^2+10y^2$.

Solution by A. M. Vaidya, Gujarat University, Ahmedabad, India. If $x^2+ry^2=p$, then $x^2 \equiv -ry^2 \pmod{p}$, so that $-ry^2$ and hence $-r$ is a quadratic residue of p. Thus the required prime satisfies $(-r/p)=1$ for $1 \leqq r \leqq 10$. These will be satisfied if -1, 2, 3, 5 and 7 are quadratic residues of p. These in turn lead to the congruences $p \equiv 1 \pmod{8}$, $p \equiv 1 \pmod{3}$, $p \equiv 1$ or $-1 \pmod{5}$, and $p \equiv 1$, 2,

or 4 (mod 7). These can be combined into $p\equiv 1$, 121, 169, 289, 361, or 529 (mod 840). The least prime satisfying this requirement is 1009, and we have

$$\begin{aligned}1009 &= 15^2+28^2 = 19^2+2\cdot 18^2 = 31^2+3\cdot 4^2 = 15^2+4\cdot 14^2 = 17^2+5\cdot 12^2\\ &= 25^2+6\cdot 8^2 = 1^2+7\cdot 12^2 = 19^2+8\cdot 9^2 = 28^2+9\cdot 5^2 = 3^2+10\cdot 10^2.\end{aligned}$$

A similar problem may be found in J. Hunter, *Number Theory*, Oliver and Boyd, 1964, ex. 6, p. 111.

Also solved by Robert Baillie, Merrill Barnebey, Brother Alfred Brousseau, John Christopher, G. C. Dodds, William Forman, M. G. Greening (Australia), P. L. Montgomery, Alfred Pietrowski, S. F. Robinson, D. R. Stark, and Charles Wexler.

W. F. Feeny, Donald Jeffords, D. C. B. Marsh, and the proposer cited larger primes which have the same forms.

A Law of Cosines in E_3

E 1923 [1966, 891]. *Proposed by I. J. Good, Trinity College, Oxford, England*

Prove that the square of the area of a face of any polyhedron is equal to the sum of the squares of the areas of the other faces, minus twice the sum, over every pair of the other faces, of the products of their areas times the cosine of the angle between them.

Solution by T. Teichmann, General Atomic Division, General Dynamics. Suppose that the polyhedron has $n+1$ faces, numbered from 0 to n, and let the kth face have area A_k. Choose a Cartesian coordinate system with the first axis normal to the "0" face, and the other two, therefore, perpendicular to it. Let a_k, b_k, c_k be the direction cosines of the normal to the kth face with respect to this system. Clearly $a_0=1$, $b_0=c_0=0$. Taking projections on the coordinate planes (orientation being taken into account), one has

$$-A_0 = \sum_{k=1}^{n} A_k a_k, \qquad \sum_{k=1}^{n} A_k b_k = 0, \qquad \sum_{k=1}^{n} A_k c_k = 0.$$

Squaring these equations, adding the results, and noting that

$$a_k^2+b_k^2+c_k^2 = 1, \qquad a_k a_j + b_k b_j + c_k c_j = -\alpha_{kj}$$

where α_{kj} is the cosine of the angle between the kth and jth faces, one finds

$$A_0^2 = \sum_{k=1}^{n} A_k^2 - 2\sum_{k>j\geq 1}\sum A_k A_j \alpha_{kj}$$

as required.

Comment by M. G. Beumer. This problem was solved by Lagrange in 1783, as a special case of a general theorem proposed by L. N. M. Carnot (*Geometrie der Stellung*, 2, nr. 257 and 261, p. 59 and 61). A detailed description can be found in Encyclopädie der mathematischen Wissenschaften, Band *Geometrie*, III, 1, 2 (Leipzig, 1914–1931) pp. 1013–1015, article written by M. Zacharias.

Also solved by Marcia Ascher, M. G. Greening (Australia), V. F. Ivanoff, C. W. Langley, Stanley Rabinowitz, Sidney Spital, E. W. Trost (Switzerland), J. M. Wessner, and the proposer.

Extensions to higher dimensions are noted by Ivanoff, Langley, and the proposer.

Covering Disks with Squares and Rectangles

E 1924 [1966, 891]. *Proposed by Mason Henderson, Montana State University*

Let A and B be unit disks, A cut by a chord into parts A_1 and A_2. (1) Is there a square of least area which covers A_1 and A_2, where A_1 and A_2 are placed so as not to overlap? (2) Is there a square of least area which covers B and both A_1 and A_2? (3) Replace square by rectangle in (1) and (2).

Solution by Michael Goldberg, Washington, D. C. (1) Since the circle A has unit radius, the smallest square enclosing the pieces A_1 and A_2 is the square of edge 2.

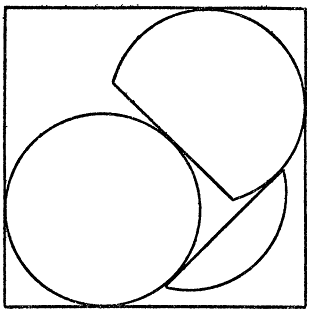


FIG. 1

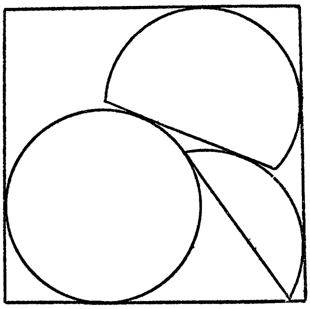


FIG. 2

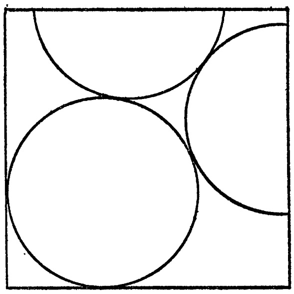


FIG. 3

(2) If h, the height of the smallest piece A_2, is small enough, the sought square is the smallest one enclosing B and A_1. There is ample room for the remaining piece A_2, as shown in Figure 1. The edge e of the square is given by $e=2+(2-h)/\sqrt{2}$. However, as h is increased, this arrangement cannot be used since there is not enough room for A_2. An arrangement similar to Figure 2 must now be used to minimize the size of the enclosing square. For still larger values of h, up to $h=1$, the arrangement of Figure 3 must be used to minimize the square. The transition values of h have not been computed.

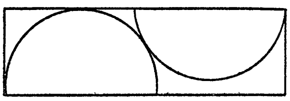


FIG. 4

(3a) For small h, the smallest rectangle enclosing A_1 and A_2 is the square circumscribing A_1. As h is increased, a value is reached in which Figure 4 gives the smallest rectangle. The transition value of h occurs when the areas of the rectangle and square are equal, namely, when

$$(2 - h)\left\{1 + \sqrt{4 - (2 - h)^2} + \sqrt{2h - h^2}\right\} = 4.$$

The solution of this equation is $h = 0.91$ approximately.

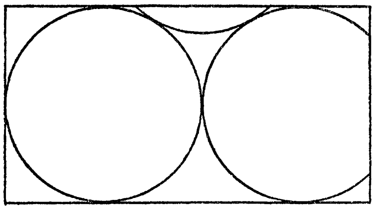


FIG. 5

(3b) The smallest rectangle enclosing B, A_1 and A_2 is shown in Figure 5 for $h \leqq 2 - \sqrt{3}$. The height of the rectangle is 2, and the length of the base is $4 - h$. For $h_1 \geqq h \geqq 2 - \sqrt{3}$, the arrangement shown in Figure 6 gives the smallest rectangle with dimensions $\sqrt{3} + h$ by $4 - h$. For still larger values $1 \geqq h \geqq h_1$, the arrangement shown in Figure 7 gives the smallest rectangle with dimensions 2 by $2 + \sqrt{4 - h^2}$. The transition value h_1 is obtained as the solution of the equation

$$(\sqrt{3} + h)(4 - h) = 4 + 2\sqrt{4 - h^2}.$$

This yields $h_1 = 0.54$ approximately.

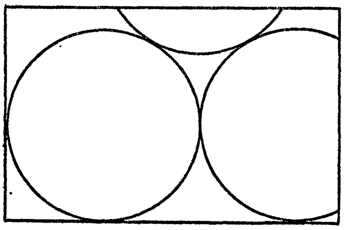


FIG. 6

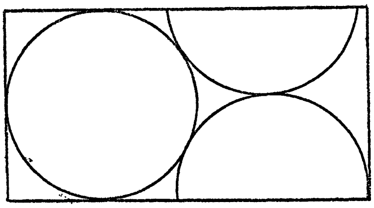


FIG. 7

Triangle Inequalities

E 1925 [1966, 1016]. *Proposed by W. J. Blundon, Memorial University of Newfoundland*

Let r and R denote the inradius and circumradius, and define $k = r/R$, so that $0 < k \leqq \frac{1}{2}$. Prove that, for every triangle ABC,

(1) $-1 + 4k - k^2 \leqq \cos A \cos B + \cos B \cos C + \cos C \cos A \leqq k + k^2,$

(2) $-1 + 2k - \frac{3}{2}k^2 \leqq \cos A \cos B \cos C \leqq \frac{1}{2}k^2,$

with equality only for equilateral triangles.

Solution by M. G. Greening, University of New South Wales. Let I, G, P represent the incenter, centroid, and orthocenter, respectively. Then

$4R^2 \cos A \cos B \cos C = 2r^2 - IP^2 = 9IG^2 + 12Rr - 4R^2 - 6r^2$, giving the inequality (2). As both $IP = 0$, and $IG = 0$ if, and only if, the triangle is equilateral, the full result follows.

For (1), we have

$$\begin{aligned} 2\cos A \cos B \cos C &= 1 - \cos^2 A - \cos^2 B - \cos^2 C \\ &= 2\sum \cos A \cos B - \left(\sum \cos A\right)^2 + 1 \\ &= 2\sum \cos A \cos B - (k+1)^2 + 1, \end{aligned}$$

giving $2\sum \cos A \cos B = 2\cos A \cos B \cos C + k^2 + 2k$. Substitution in (2) then gives (1) with the condition for equality preserved.

Also solved by Leon Bankoff, and Michael Goldberg.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate signed sheets and should be mailed before August 31, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5560. *Proposed by G. B. Seligman, Yale University*

Let F be a field of prime characteristic p. Consider the p-polynomials with coefficients in F:

$$\begin{aligned} q(X) &= X^{p^m} - a_{m-1}X^{p^{m-1}} - \cdots - a_1X^p - a_0X, \\ r(X) &= a_0^{p^{m-1}} X^{p^m} + a_1^{p^{m-2}} X^{p^{m-1}} + \cdots + a_{m-1}X^p - X, \end{aligned}$$

where $a_0 \neq 0$ (cf. Jacobson, Lie Algebras, p. 193).

(A) Show that $q(X)$ splits completely in F if and only if $r(X)$ splits completely in F. (B) Is it true that $q(X)$ has a nonzero root in F if and only if $r(X)$ has a nonzero root in F?

5561. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let $f(x)$ be an irreducible (normal) polynomial over a field R of characteristic zero and $\psi_j(x) \in R[x]$. Prove (or disprove): If $f(\psi_j(x)) = f(x)h(x)$, then $h(x)$ is irreducible (normal).

5562. *Proposed by Lawrence Kuipers, Delft, Netherlands*

Two sequences $P(m, n)$ and $Q(m, n)$ are defined as follows (m, n are integers). $P(m, 0) = 1$ for $m \geqq 0$, $P(0, n) = 0$ for $n \geqq 1$, $P(m, n) = 0$ for $m, n < 0$. $P(m, n) = \sum_{j=0}^{n} P(m-1, j)$ for $m \geqq 1$.

$$Q(m, n) = P(m-1, n) + P(m-1, n-1) + P(m-1, n-2) \quad \text{for } m \geqq 1.$$

Express $Q(m, n)$ in terms of m and n for $m \geqq 1$.

5563. *Proposed by D. E. Daykin and C. J. Eliezer, University of Malaya, Kuala Lumpur*

For which positive functions $f(x, y)$, $g(x, y)$ do we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqq \sum_{i=1}^{n} f(a_i, b_i) \sum_{i=1}^{n} g(a_i, b_i) \leqq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,$$

for all real a_i, b_i?

5564. *Proposed by R. B. Killgrove, California State College, Los Angeles*

Consider the set S of ordered pairs of reals. Define $\oplus$, $\otimes$ on elements of S as follows ($+$ is the usual addition for reals):

$$(a, b) \oplus (c, d) = (a + c, b + d),$$
$$(a, b) \otimes (c, d) = (f(a, b, c, d), g(a, b, c, d)).$$

Let f, g be continuous and the system $\mathcal{S}(S, \oplus, \otimes)$ be a, possibly nonassociative, ring with identity $(1, 0)$; i.e. under $\oplus$, S forms an Abelian group and both distributive laws hold and a specified element is identity for $\otimes$. Certainly the complex numbers form such an $\mathcal{S}$. Characterize all other systems $\mathcal{S}$.

5565. *Proposed by D. A. Hejhal, University of Chicago*

Let $f(x)$ be real and continuous over $[0, 1]$ with $f(0) = 0$. Then there exists a sequence of real polynomials $\{A_n(x)\}$, $A_n(x) = \sum_{k=0}^{\infty} a_{k,n} x^k$ [of course, for each n, $a_{k,n} = 0$ eventually], such that $A_n(x) \to f(x)$ uniformly over $[0, 1]$ as $n \to \infty$, but $\lim_{n\to\infty} a_{k,n} = 0$, $k = 0, 1, \cdots$.

5566. *Proposed by Forrest Dristy, Clarkson College of Technology*

Let X be the closed interval $[0, 1]$ with the topology consisting of the empty set $\varnothing$, X, and all subsets of X of the form $[0, a)$ where $0 < a \leqq 1$. Let Y be the space $\{0, 1\}$ with the topology $\{\varnothing, \{0\}, Y\}$, and let Z be the cartesian product of denumerably many copies of Y. Prove or disprove that X and Z are homeomorphic.

5567. *Proposed by Donna J. Seaman, Olympic College, Bremerton, Wash.*

Given a loop L_n with n elements, such that for every pair a, b in L_n (a, b need not be distinct):

(i) $a \circ b = b \circ a$; (ii) $a \circ (a \circ b) = b$.

(1) For which values of n do such loops exist?

(2) Is there a nonassociative loop L_n with properties (i) and (ii) for some $n = 2^k$?

5568. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York City*

Let the domain of the function $f_{ijk}(x) = ix^j + k$ consist of all integers $\geqq 2$. Denote the range of $f_{ijk}(x)$ by R_{ijk}. If p and m are arbitrary positive integers, prove that there exists an infinite number of primes which are not contained in the set R, where $R = \bigcup_{i=1}^{p} \bigcup_{k=1}^{m} \bigcup_{j=2}^{\infty} R_{ijk}$.

5569. *Proposed by Stephen Baron, McGill University*

It is known that at most 14 distinct sets may be obtained from a given subset of a topological space by the operations of closure and complementation applied as often as one likes, in any order. There are many examples of subsets of the real line where this maximum is obtained (e.g. $(0, 1) \cup (1, 2) \cup \{r \mid \text{rational}, 2 < r < 3)\} \cup \{4\})$.

Prove or disprove: Any such subset of the reals must contain a set of points which is dense in some interval and whose complement is dense in the same interval. Are there any other necessary conditions for such a subset?

SOLUTIONS OF ADVANCED PROBLEMS

Continued Fraction for e^k

5445 [1966, 1124]. *Proposed by A. J. Macintyre and C. I. Lubin, University of Cincinnati*

Show that

$$e = 1 + \frac{2}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+}\cdots$$

and the sequence of convergents to this fraction form a subsequence of the sequence of convergents of the regular fraction for e. Show also that

$$e^2 = 7 + \frac{2}{5+}\,\frac{1}{7+}\,\frac{1}{9+}\,\frac{1}{11+}\cdots$$

and the sequence of convergents of this fraction form a subsequence of the sequence of convergents of the regular fraction for e^2.

Solution by Joseph Gillis, Weizmann Institute, Rehoboth, Israel. For any positive integer k let

$$\begin{aligned} p_n &= \int_0^\infty x^n(x+k)^n e^{-x}dx = e^k \int_k^\infty x^n(x-k)^n e^{-x}dx \\ q_n &= \int_0^\infty x^n(x-k)^n e^{-x}dx. \end{aligned} \tag{1}$$

We note that $p_n > 0$, and that $p_n > q_n > 0$ if $|p_n - e^k q_n| < p_n$. Using integration by parts twice we obtain for $n \geqq 2$,

$$q_n = 2n(2n-1)q_{n-1} + k^2n(n-1)q_{n-2}$$

and a similar relation for p_n. Using the easily calculated values of p_0, p_1, q_0, q_1 it follows that p_n, q_n are divisible by n! Now, from the integrals defining p_n, q_n we obtain

$$\left|\frac{p_n}{q_n} - e^k\right| < \frac{ke^k}{|q_n|}\left(\frac{k^2}{4}\right)^n \tag{2}$$

so that, as soon as $k\cdot e^k\cdot(k^2/4)^n < n!$ it must follow that $q_n > 0$.

Now setting $Q_n = q_n/n!$, $P_n = p_n/n!$ we see that

$$Q_n = 2(2n-1)Q_{n-1} + k^2Q_{n-2}, \qquad P_n = 2(2n-1)P_{n-1} + k^2P_{n-2}. \tag{3}$$

We can verify directly that $(P_0, Q_0) = (1, 1)$, $(P_1, Q_1) = (2+k, 2-k)$, $(P_2, Q_2) = (12+6k+k^2, 12-6k+k^2)$.

Now consider some special cases:

(i) $k=1$. Then it follows from the initial values and (3) that P_n/Q_n are the convergents of the continued fraction given in the problem. To show also that they are a subsequence of the convergents of the regular continued fraction for e it will be enough to show that

$$\left|e - \frac{P_n}{Q_n}\right| < \frac{1}{2Q_n^2}.$$

But, by (2),

$$\left|e - \frac{P_n}{Q_n}\right| = \left|e - \frac{p_n}{q_n}\right| < \frac{1}{4^nq_n},$$

and so it remains to show that $Q_n^2 < 2^{2n-1}q_n$. Now $Q_n/Q_{n-1} = 2(2n-1) + Q_{n-2}/Q_{n-1}$. If $|Q_{n-1}/Q_{n-2}| \geqq 1$, then

$$2(2n-1) - 1 \leqq Q_n/Q_{n-1} \leqq 2(2n-1) + 1.$$

From $Q_1/Q_0 = 1$ it now follows that $Q_n/Q_{n-1} \geqq 1$ for all n and $Q_n \leqq \prod_{k\leq n}(4k-1) < 2^{2n}\cdot n!$. It is clear that q_n and $Q_n(=q_n/n!)$ are increasing as soon as they are assuredly positive, and thus we ultimately have the inequality for Q_n^2.

(ii) $k=2$. We write $\alpha_n = 2^{-n}P_n$, $\beta_n = 2^{-n}Q_n$. Then

$$\alpha_n = (2n-1)\alpha_{n-1} + \alpha_{n-2}, \qquad \beta_n = (2n-1)\beta_{n-1} + \beta_{n-2}$$

and the argument is closely similar to that in (i).

(iii) Repeating essentially the same argument we can show, quite generally that

$$e^k = 1 + \frac{2k}{2-k+}\,\frac{k^2}{6+}\,\frac{k^2}{10+}\,\frac{k^2}{14+}\cdots,$$

and that the convergents of this are a subsequence of the convergents of the regular continued fraction for e^k.

See also a paper by I. Shemer, *Gilyonot Mathematika* (Hebrew) III, 3(1966).
Also solved by H. E. Fettis, Robert Heller, and the proposer.

Finite Groups without Center

5458 [1967, 90]. *Proposed by C. C. Lindner, Coker College, Hartsville, S.C.*

Let G be a finite nonabelian group in which every Sylow subgroup of G is maximal in G. Prove that G has no center.

I. *Solution by Homer Bechtell, University of New Hampshire.* Since G cannot be nilpotent, there exists at least one self-normalizing Sylow subgroup. If there are two distinct primes for which the corresponding Sylow subgroups are self-normalizing, their intersection is the identity and contains the center. If Q is a normal Sylow q-subgroup, then $G=[Q]\{x\}$ for $x^p=1$, p a prime, $p\neq q$, and x not in the center of G. But $\{x\}$ is maximal, self-normalizing, and the intersection of its conjugates is the identity and contains the center. So G is centerless.

II. *Solution by Kenneth Yanosko, Ohio State University.* Assume that every Sylow subgroup of G is maximal, and let G have nontrivial center. Let P be any central subgroup of prime order p, and let Q be a Sylow q-subgroup of G, $q\neq p$. Since Q is maximal, $G=PQ$, and hence $|G|=pq^n$. Then P is a Sylow subgroup, and therefore maximal. Hence $|Q|=q$, so that Q is abelian, and hence $G=PQ$ is abelian.

III. *Extension by W. R. Scott, University of Utah.* It is possible to assert more than has been proved above. Suppose that G is not simple, and let N be a nontrivial normal subgroup. Let p be a prime dividing $o(G/N)$. If G/N is not a p-group, then $S<SN<G$ for some Sylow subgroup S (a Sylow p-subgroup works unless it contains N, otherwise use a Sylow q-subgroup with $q\neq p$.) Hence G/N is a p-group. It follows in turn that N contains a Sylow q-subgroup S, and that $N=S$. If $o(G/N)\neq p$, then S is not maximal and $o(G/N)=p$. Thus $G=ST$, S a normal Sylow q-subgroup, T a Sylow p-subgroup, and $o(T)=p$. If T is normal, then $G=S\times T$. Since T is maximal, this implies $o(S)=q$. But then $G=S\times T$ is Abelian (even cyclic), contrary to hypothesis. Hence T is not normal, and $N(T)=T$. It follows that G is a Frobenius group with kernel S and complement T. A reinterpretation is that S has an automorphism of order p without fixed point. Thus we are left with G simple, or a Frobenius group of the above type.

IV. *Further result by D. B. Parker, Ohio State University.* Using a theorem of Thompson (see Schenkman, *Group Theory*, p. 277) it is possible to prove the following: If G is a finite non-abelian group then all Sylow subgroups are proper maximal subgroups of G if and only if $G=PQ$ where $P\lhd G$, P is an elementary Abelian p-group, Q is a group of order q(p and q distinct primes), and Q acts irreducibly on P, and nontrivially.

Also solved by Elizabeth Appelbaum, D. M. Bloom, D. Ž. Djoković (Yugoslavia), Gene Gale, E. R. Gentile (Argentina), M. G. Greening (Australia), V. H. Keiser, Jr., T. P. Kezlan, Erwin

Just, J. H. Oppenheim, Tommy Rinfrow, P. K. Subramanian, Philip Trauber, J. R. Weaver, J. J. Zeltmacher, Jr., and the proposer.

Euler's Constant

5460 [1967, 206]. *Proposed by Adriano Behrmann, Sao Paulo, Brazil*

Prove the following series expansion for the Euler constant:

$$C = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{a_n}{(2n-2)(2n-1)(2n)},$$

where a_n satisfies $2^{a_n} < 2n-1 < 2^{a_n+1}$. Compare 4353 [1951, 116].

Solution by J. H. van Lint, Technological University, Eindhoven Netherlands. We use the following identities:

$$(1) \qquad \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{(2n-2)(2n-1)(2n)} = \frac{1}{2} \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{2n-1}\left(\frac{1}{2n-2} - \frac{1}{2n}\right)$$

$$= \frac{1}{2} \sum_{n=2^m+1}^{2^{m+1}} \left(\frac{1}{2n-2} - \frac{2}{2n-1} + \frac{1}{2n}\right) = \sum_{n=2^{m+1}}^{2^{m+2}-1} \frac{(-1)^n}{n} - \frac{1}{2^{m+3}},$$

$$(2) \qquad \sum_{n=1}^{\infty} \frac{m}{2^m} = 2.$$

We then have

$$\frac{1}{2} + \sum_{n=2}^{\infty} \frac{a_n}{(2n-2)(2n-1)(2n)} = \frac{1}{2} + \sum_{m=1}^{\infty} m \sum_{n=2^{m-1}+1}^{2^m} \frac{1}{(2n-2)(2n-1)(2n)}$$

$$= \frac{1}{2} + \sum_{m=1}^{\infty} m \sum_{n=2^m}^{2^{m+1}-1} \frac{(-1)^n}{n} - \sum_{m=1}^{\infty} \frac{m}{2^{m+2}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right] = C,$$

the last equality being the result of 4353.

Also solved by M. G. Beumer (Netherlands), L. Carlitz, M. G. Greening (Australia), and the proposer.

The proposer uses a partial sum of the series to obtain the bounds $.57718 < C < .57724$. Beumer notes that a solution of the problem may be found in J. C. Kluyver, *Over de algemene gedaante van de reeks van Vacca*, Proceedings Royal Academy of Sc., Amsterdam, 34 (1925), p. 685.

Orthogonal Hyperspheres

5461 [1967, 206]. *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

Show that it is possible in E_n to have $n+1$ mutually orthogonal spheres. What is the maximum number of such spheres?

Solution by Seymour Schuster, University of Minnesota. Consider a hypersphere of unit radius centered at the origin of a rectangular coordinate system in E_n. An inversion with respect to any hypersphere whose center is two units

from the origin transforms the n coordinate hyperplanes and the unit hypersphere into $n+1$ mutually orthogonal hyperspheres.

Conversely, any n mutually orthogonal hyperspheres can be inverted into n mutually orthogonal hyperplanes, and we recall that n is the maximum number of mutually orthogonal hyperplanes in n-space. Any two distinct hyperspheres orthogonal to n mutually orthogonal hyperplanes must be concentric. Thus $n+1$ is the maximum number of mutually orthogonal hyperspheres in E_n.

Also solved by K. A. Post (Netherlands), and the proposer.

J. D. E. Konhauser states that the number is $n+2$ if imaginary hyperspheres are admitted.

Finite Rings and Fields

5462 [1967, 207]. *Proposed by W. A. McWorter, University of British Columbia*

(Restated) A finite ring $R(\neq 0)$ is a field if and only if it has no nonzero nilpotent element and at most one nonzero idempotent.

I. *Solution by E. J. Taft, Rutgers—The State University.* Fields have this property. Conversely, let R be a finite ring with the property. It is enough to prove R is an integral domain, for then R is a division ring (Jacobson, vol. I, p. 55, Ex. 3) and so a field by Wedderburn's theorem. We use induction on the number n of elements in R. It is clear for $n=2$, as the only 2-element ring satisfying the property is the field of integers, modulo 2. Suppose R is not an integral domain, $ab=0$, $a\neq 0$, $b\neq 0$ in R. Then aR is a nonzero ring with fewer elements than R and satisfies the hypothesis placed on R. Thus aR is a field; write its identity element as ax. Similarly Rb is a field, with identity element yb. Since ax and yb are nonzero idempotents in R, $ax=yb$. Let wb be the inverse of b^2 in Rb, so $b^2wb=yb$. Then $ayb=ab^2wb=0$. Hence $a(ax)=ayb=0$, so $a^2(ax)=0=a^2$ since ax is the identity of aR. This is a contradiction.

II. *Solution by J. E. Delany, University of California, Irwin.* It is easily verified that the only nilpotent element of a field is 0 and the only idempotent elements are 0, 1.

If x is any element, then, applying the pigeon hole principle to the collection of powers of x, there are positive integers m, n such that $x^{m+n}=x^m$. Then $x^{am}x^{bn}=x^{am}$ for any positive integers a, b. Letting $a=n$, $b=m$ we have $(x^{nm})^2=x^{nm}$ and x^{nm} is idempotent. If $x\neq 0$ then $x^{nm}\neq 0$, since 0 is the only nilpotent. Thus if $x\neq 0$ there is a positive integer i such that $x^i=e$, the unique nonzero idempotent.

Suppose $xy=0$ with $x, y\neq 0$. Then $(yx)^2=yxyx=y0x=0$ and yx is nilpotent. Hence $yx=0=xy$. Let i, j be positive integers such that $x^i=y^j=e$. Since x and y commute we have $0=(xy)^{ij}=x^{ij}y^{ij}=e^je^i=e$, a contradiction. Thus the ring contains no zero divisors. It is well known that a finite ring with no zero divisors is a field.

We may also prove the following extension: A finite ring is a direct product of fields if and only if it has no nonzero nilpotent elements.

Also solved by C. B. Baytop & J. E. Joseph, Charles Chouteau, L. D. Crowson, Kenneth Dahlberg, Jim Diederich, M. A. Ettrick, M. F. Friedell, E. R. Gentile (Argentina), Charles Green, M. G. Greening (Australia), G. A. Heuer, R. A. Howland, T. L. Jenkins, T. P. Kezlan, Kwangil Koh, Daryl Kreiling, G. L. Loudner, Jr., D. L. Lutzer, Jiang Luh, M. D. Mavinkurve (India), L. F. Meyers, Barbara L. Osofsky, D. M. Parra, Stephen Price, L. J. Pratte, Simeon Reich (Israel), Charles Riley, Azriel Rosenfeld, David Ryeburn, P. P. Sanchez, W. R. Scott, S. O. Sickler, K. N. Sigmon, Al Somayajulu, L. E. Spence, D. P. Sumner, M. B. Suryanarayana, Philip Trauber, D. H. Underwood, Charles Wells, Kenneth Yanosko, and the proposer.

$$D^{2n}\log(1+x^{2n}) = 0,\ x=-1$$

5463 [1967, 207]. *Proposed by W. G. Spohn, Jr., Applied Physics Laboratory, Johns Hopkins University*

For $n=2, 4, 6, \cdots$, define $f_n(x)=\log(1+x^n)$. Prove $f_n^{(n)}(-1)=0$.

I. *Solution by A. S. Adikesavan, Regional Engineering College, Tiruchirapalli, India.* We write

$$f_n(x) = \log(1+x^n) = \sum_{k=1}^{n}\log(x-\omega_k),$$

where $\omega_k=\cos\{(2k-1)\pi/n\}+i\sin\{(2k-1)\pi/n\}$, $k=1, 2, \cdots, n$ and $i^2=-1$. Now, $f_n^{(n)}(x)=(-1)(n-1)!\sum_{k=1}^{n}(x-\omega_k)^{-n}$, whence

$$f_n^{(n)}(-1) = (-1)(n-1)!\sum_{k=1}^{n}(1+\omega_k)^{-n}, \tag{1}$$

and this, upon simplification, becomes

$$f_n^{(n)}(-1) = i\,\frac{(n-1)!}{2^n}\sum_{k=1}^{n}\frac{(-1)^k}{\cos^n\{(2k-1)\pi/2n\}}. \tag{2}$$

Now ω_k, $k=1, 2, \cdots, n$, are complex numbers occurring in conjugate pairs, because they are roots of $x^n+1=0$ and n is even. Thus $(1+\omega_k)^{-n}$, $k=1, 2, \cdots, n$, are also complex numbers occurring in conjugate pairs. It follows from (1) that $f_n^{(n)}(-1)$ is real, while, from (2), it is a real multiple of i. Hence $f_n^{(n)}(-1)=0$.

II. *Solution by H. W. Gould, West Virginia University.* It is known that the general chain rule of differentiation can be written in the form

$$D_x^n f(z) = \sum_{k=1}^{n} D_z^k f(z)\,\frac{(-1)^k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}z^{k-j}, \qquad n \geqq 1.$$

(See I. J. Schwatt, *Introduction to the Operations with Series*, Chelsea, New York, 1962, reprint of original 1924 edition, p. 12, (83).) This is more convenient to use than the older formula of Faà de Bruno (Quart. J. Math., 1(1857) 359–360). Now $z=1+x^n$ and $f(z)=\log z$. We have, when $x=-1$ and $n=2, 4, \cdots,$

$$D_x^n \log(1+x^n)\Big|_{x=-1} = \sum_{k=1}^{n} D_z^k \log z\Big|_{z=2} \frac{(-1)^k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}2^{k-j}.$$

The inner summation, by the binomial theorem, is clearly zero for $k\geqq 1$. One easily shows that $D_z^k \log z|_{z=2}$ exists as a finite number, so the entire expression must be zero.

Also solved by Einar Andresen (Norway), H. L. Baldwin, Jr., Robert Breusch, Louis Comtet (France), W. O. Egerland, M. A. Ettrick, N. J. Pine, M. G. Greening (Australia), F. W. Hartmann, D. A. Hejhal, Margaret M. LaSalle, R. K. Meany, W. J. Sarill, Philip Trauber, and the proposer.

Symmetry of the Set of Eigenvalues of a Square Matrix

5464 [1967, 207]. *Proposed by C. M. Petty and W. E. Johnson, Lockheed Aerospace Sciences Laboratory*

If A is a normal matrix (i.e., A commutes with its conjugate transpose), then the characteristic roots of A form a symmetric set with respect to the origin in the complex plane (i.e. if Z is a characteristic root of multiplicity r then $-Z$ is a characteristic root of multiplicity r) if and only if the trace $(A^{2k+1})=0$ for $k=0, 1, 2, \cdots$.

I. *Solution by P. V. Subba Rao and B. Ramàchandra Rao, Andhra University, Waltair, India.* Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the characteristic roots of A. It is known [Gantmacher, *Matrix Theory*, Chelsea, Vol. 1, p. 84, Th. 3] that $\lambda_1^{2k+1}, \lambda_2^{2k+1}, \cdots, \lambda_n^{2k+1}$ are the characteristic roots of $A^{2k+1}(k=1, 2, \cdots)$.

First observe that if $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is a symmetric set with respect to the origin, then the nonzero roots among them can be represented in the form λ_i, $-\lambda_i$, each of multiplicity t_i, $i=1, 2, \cdots, m$. We then have $\mathrm{Tr}(A^{2k+1}) =\lambda_1^{2k+1}+\cdots+\lambda_n^{2k+1}=0$, $k=0, 1, \cdots$.

Next consider the characteristic equation of A:

$$\chi(\lambda) \equiv \lambda^n + P_1\lambda^{n-1} + \cdots + P_{n-1}\lambda + P_n = 0.$$

If $S_r=\lambda_1^r+\lambda_2^r+\cdots+\lambda_n^r$, then by Newton's relations [loc. cit., p. 87, eq. 44] $rP_r=S_r-P_1S_{r-1}-P_2S_{r-2}-\cdots-P_{r-1}S_1$, $r=1, 2, \cdots, n$. If now $\mathrm{Tr}(A^{2k+1})=0$ for $k=0, 1, 2, \cdots$, it follows by induction that P's with odd subscripts vanish. Thus $\chi(\lambda)$ is an odd or an even polynomial so that, in either case, its roots form a symmetric set with respect to the origin.

It is to be noted that the proof does not require the normality condition on A.

II. *Solution by Olga Taussky, California Institute of Technology.* The assumption that A is normal is unnecessary. For any matrix A with characteristic roots $\lambda_1, \cdots, \lambda_n$ the following relation holds:

$$\mathrm{trace}(A^k) = \sum_{i=1}^{n} \lambda_i^k.$$

By assumption all $\sum_{i=1}^{n}\lambda_i^{2k+1}=0$, $k=0, 1, \cdots$. Let $\lambda_1, \cdots, \lambda_r$ be the subset of the λ_i obtained after removing all $\lambda_i=0$ and all pairs λ_i, λ_j $(i\neq j)$ with $\lambda_i+\lambda_j=0$. Then also

$$\sum_{i=1}^{r}\lambda_i^{2k+1} = 0, \qquad k = 0, 1, \cdots.$$

Let $\lambda_1, \cdots, \lambda_s$ be the distinct elements among the $\lambda_1, \cdots, \lambda_r$ each λ_i with multiplicity m_i. Then

$$\sum_{i=1}^{s} m_i\lambda_i^{2k+1} = 0, \qquad k = 0, 1, \cdots.$$

The λ_i are now distinct. Consider the first s of these equations. They can be looked upon as a set of linear homogeneous equations with matrix (λ_i^{2k+1}), yielding the nontrivial solution $m_1, \cdots, m_s$. Hence the determinant of the above matrix must vanish. This (Vandermonde) determinant can be expressed as a product

$$\pm \prod_{i=1}^{s}\lambda_i \prod_{i\neq j}(\lambda_i^2-\lambda_j^2)$$

which can vanish only if $\lambda_i=0$, or $\lambda_i+\lambda_j=0$, or $\lambda_i-\lambda_j=0$. Since no one of these is possible, the proof is complete.

Also solved by C. G. Cullen, M. G. Greening (Australia), A. S. Householder, T. P. Kezlan, M. D. Mavinkurve (India), R. K. Meany, M. F. Smiley, John H. Smith, R. C. Thompson, H. R. van der Vaart, and the proposer.

An Unbounded Sum of Sines

5466 [1967, 207]. *Proposed by Benjamin Volk, Yeshiva University*

Discuss the convergence, as $x\to\infty$, of

$$\sum_{n\le x}\frac{1}{n}\left(\sin\frac{x}{n}+\sin\frac{n}{x}\right).$$

Cf. 5203 [1965, 559].

Solution by M. E. Muldoon, York University, Toronto. The given expression becomes unbounded as $x\to\infty$. To see this we write it in the form

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{x}{n}-\sum_{n>x}\frac{1}{n}\sin\frac{x}{n}+\sum_{n\le x}\frac{1}{n}\sin\frac{n}{x}\cdot$$

The first sum here is unbounded by the result of 5203. The other two sums are bounded:

$$\sum_{n>x}\frac{1}{n}\sin\frac{x}{n}<\sum_{n>x}x/n^2<x\int_{x-1}^{\infty}t^{-2}dt=\frac{x}{x-1}, \qquad x>1,$$

$$\sum_{n\leqq x}\frac{1}{n}\sin\frac{n}{x}<\sum_{n\leqq x}\frac{1}{n}\left(\frac{n}{x}\right)\leqq 1,\qquad x\geqq 1.$$

Also solved by J. H. van Lint who observes that the order of the sum is determined by the infinite series used in the solution and refers to the paper of Hardy and Littlewood cited with the material on 5203, and to T. M. Fleet, Journal London Math. Society 25 (1950), p. 5 ff.

Equality of a Pair of Double Sums

5467 [1967, 207]. *Proposed by M. V. Subbarao, University of Alberta*

If $|x|<1$, prove that

$$x\left(\frac{1+x}{1-x}\right)+2x^4\left(\frac{1+x^2}{1-x^2}\right)+3x^9\left(\frac{1+x^3}{1-x^3}\right)+\cdots+nx^{n^2}\left(\frac{1+x^n}{1-x^n}\right)+\cdots$$
$$=x\frac{1-x}{(1-x)^2}+x^2\frac{(1-x^4)}{(1-x^2)^2}+x^3\frac{(1-x^9)}{(1-x^3)^2}+\cdots+x^n\frac{(1-x^{n^2})}{(1-x^n)^2}+\cdots.$$

Solution by M. G. Greening, University of New South Wales. A typical term on the left hand side can be expanded to give

$$rx^{r^2}(1+x^r)(1+x^r+x^{2r}+\cdots)=rx^{r^2}+\sum_{m=1}^{\infty}2rx^{r^2+mr}.$$

Then the coefficient of x^n arising from the expansion of the absolutely convergent expressions on the left hand side is

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}}2d+\epsilon_n\sqrt{n},\qquad \epsilon_n=\begin{cases}1, & n=h^2\\ 0, & n\neq h^2.\end{cases}$$

A typical term on the right hand side becomes

$$x^r(1-x^{r^2})(1+2x^r+3x^{2r}+\cdots)=\sum_{k\leqq r}kx^{kr}+\sum_{k>r}rx^{kr}.$$

The term in x^n arises again solely from those r for which $kr=n$; the coefficient of x^n equals

$$\sum_{\substack{k\mid n\\ k<\sqrt{n}}}k+\epsilon_n\sqrt{n}+\sum_{\substack{n=kr\\ k>\sqrt{n}}}r=\sum_{\substack{d\mid n\\ d<\sqrt{n}}}2d+\epsilon_n\sqrt{n},$$

and the proof is complete.

Also solved by Einar Andresen (Norway), Joseph Arkin, L. Carlitz, N. J. Fine, Jerry Fischer, Emil Grosswald, Edgar Karst, and the proposer.

Nonregular Separable Spaces

5468 [1967, 207]. *Proposed by S. W. Williams, Lehigh University*

A subset S of a linear space is called *radial at a point* x if and only if S contains a line segment through x in every direction. A set is *radially open* if it is radial at each of its points. Let T be the collection of radially open sets in the plane R^2. Show that the topological space (R^2, T) formed by T is separable, but neither Lindelöf, first countable, nor normal. Is it regular?

Solution by S. P. Franklin, Carnegie Institute of Technology. If α belongs to the T-open set U, there is a point β with rational abscissa on a horizontal segment through α contained in U. Similarly there is a point γ with rational ordinate on a vertical segment through β contained in U. Thus the rational points in the plane are dense and (R^2, T) is separable.

The circle $\{(x, y)|x^2+y^2=1\}$ is a closed uncountable discrete subset of (R^2, T) which cannot therefore be Lindelöf.

The existence of such a subset in a separable space also precludes the possibility of its being normal since separability insures that there are at most c distinct continuous real-valued functions while, if (R^2, T) were normal, the Tietze extension theorem would require that each of the 2^c distinct continuous real-valued functions on the subset have a continuous extension. We shall show that (R^2, T) is not even regular and hence, being Hausdorff, is not normal.

Construct a planar set S by attaching a segment of length one to the origin in each irrational direction, and a segment of length $1/q$ in the rational direction p/q (in lowest terms of course). (0, 0) lies in the closure of the complement of S but no sequence outside of S converges to it. Hence (R^2, T) is not first countable; it is not even a Fréchet space. It is sequential, however (and therefore a k-space) since T is the weak topology generated by the lines. Specifically, (R^2, T) is the quotient of a disjoint sum of lines which is a metric space.

Using the fact that the plane in its usual topology P is second countable, one may choose, one by one, a countable, P-dense subset, D, no three points of which are collinear. Clearly D is T-closed. If U is any T-neighborhood of α and $\alpha \in D$, let L be a closed horizontal segment through α contained in U. For each natural number n, let F_n be the set of points of L which are the center of some vertical segment of length $\geq 1/n$ which is contained in U. The Baire category theorem applied to L assures that for some n, the closure of F_n will contain an interval. From this it follows that the T-closure of U has a nonempty P-interior and hence contains a point of D. Thus (R^2, T) is not regular.

From this lack of regularity, we deduce that (R^2, T), being Hausdorff and sequential, cannot be locally countably compact. Thus (R^2, T) is an example of a k-space which is neither first countable nor locally compact.

Also solved by D. A. Hejhal, Ralph Jones, M. D. Mavinkurve (India), and the proposer. With his solution, Jones raises the question of the dimensionality of (R^2, T).

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

Applications of Undergraduate Mathematics in Engineering. Written and edited by Ben Noble. Mathematical Association of America, Macmillan, New York, 1967. xvii+364 pp. $9.00. (Telegraphic Review, Oct. 1967.)

This book is based on 45 contributions selected from a total of 200 submitted by engineers in universities and industries in the United States to the Commission on Engineering Education and the Committee on the Undergraduate Program in Mathematics.

The author's motivation for writing was a dissatisfaction with the manner in which mathematics is taught to engineers. It is his feeling that engineering mathematics can be both interesting and useful, even on an elementary level. Typical section headings are: Design of optical lenses, a yo-yo for de-spinning a satellite, the ignition system of an internal combustion engine, surge tanks in a hydroelectric system, the closure of a cavity in a salt dome, thermal explosions, film condensation and film boiling, concavity of resistance functions, determination of star positions from photographs, a waiting line problem in an aircraft factory, safe gaps at school crossings.

This is not a cut-and-dried problem and answer book. Professor Noble first explains the engineering context, then idealizes the problem so that it may be formulated mathematically. When the idealized mathematical model has been solved, the results are used to obtain insight into the original engineering problem. The book is divided into five parts: Elementary Mathematics; Ordinary Differential Equations; Field Problems; Linear Algebra; Probability Theory. Numerous illustrations emphasize and clarify each problem.

In the opinion of the reviewer, this book will be invaluable to engineers who seek a better understanding of how mathematics can be applied to engineering problems; to mathematicians who are interested in the relationship between mathematics and its applications; and to students who are not yet sophisticated in either field. The text is an outstanding contribution to both engineering and applied mathematics. It is one of the very few existing books that illustrate clearly and simply, by means of a wide range of examples, various ways in which elementary mathematics can be used to solve significant engineering problems.

L. A. PIPES, University of California at Los Angeles

Theory of Functions of a Complex Variable. Volumes I, II, and III. By A. I.Markushevich. Revised English edition translated and edited by Richard A. Silverman. Prentice-Hall, Englewood Cliffs, N. J. Vol. I (1965): xiv+459 pp.; $12.00. Vol. II (1965): xii+333 pp.; $12.00. Vol. III (1967): xi+360 pp.; $12.95. (Telegraphic Review, Dec. 1967)

This is an "enlightened translation," a rearranged and expanded version of the original Russian book. The translator has inserted new explanatory material and a great many excellent problems. The result is a lucid account of a wide range of topics in complex function theory.

Volume I: basic analysis and topology, elementary functions, contour integrals and Cauchy's theorem (with a clear and thoroughly convincing proof), and power series.

Volume II: Laurent series, argument principle, residue theorem (with good exercises), implicit functions, harmonic and subharmonic functions, applications to fluid dynamics,

Phragmén-Lindelöf theorems, Blaschke products, and an extensive treatment of entire functions. (The translator uses the term *indicator diagram* incorrectly on p. 279.)

Volume III: conformal mapping (in depth), approximation theory, elliptic functions, Riemann surfaces, and analytic continuation.

Each volume contains some relatively exotic topics: Lobachevskian geometry, the Plemelj formulas, boundary behavior of power series, functions of two complex variables, prime ends, . . . This wide scope obviously is an attractive feature, but it makes the book less useful as an introductory text. The most essential topics are distributed throughout the three volumes (for example, the Riemann mapping theorem is proved only in Volume III), and lengthy side-issues interrupt the flow of the basic theory. More appropriate as a text is the translator's abridged version *Introductory Complex Analysis* (Prentice-Hall, 1967; see MONTHLY, 74 (1967), p. 763). However, the three-volume work will remain a valuable reference both for students and for more experienced mathematicians. Volume III, in particular, contains a wealth of material not readily available elsewhere.

P. L. DUREN, University of Michigan

Elements of Numerical Analysis. By James Singer. Academic Press, New York, 1964. 395 pp. $8.75.

Here is another highly individual introductory text on numerical analysis. While the list of topics treated is fairly standard, the approach to each is nonstandard. The outstanding characteristic of the book is the presentation of general and special results on an absolute par, and finely intermixed. The spirit is one of love for tricky calculations. For example, the author begins the treatment of numerical quadrature by integrating the Newton forward-difference interpolation series over various intervals to obtain a large family of formulas using equally-spaced abscissas. (Most of them involve points lying outside the interval of integration, but the author takes no note of this.) Again, his development of Gaussian quadrature starts from a system of linear equations (pulled out of thin air) for the coefficients of the nth Legendre polynomial. Treatments of Weddle's formula, Simpson's rule, and other special topics are interspersed in these general developments.

It seems to me that in an ordinary college class situation this will not work. The side issues—which generally involve a line of reasoning and a batch of equations unrelated to the main development—only distract and confuse most students. It may be argued that this kind of presentation brings the student closer to the methods of the working mathematician, but the student does not have the researcher's preliminary understanding of the general situation or his intimate involvement in the details of the problem. Confronting him with a number of special results and formulas which he does not have time to appreciate, and which he feels he must memorize for the next test, tends, I believe, to prevent him from seeing the outlines of each subject and gaining an understanding of its main problems and methods.

The first chapter of the book is devoted to the usual opening discussion of significant figures and round-off errors (enlivened by some new notation and terminology). In practice this always comes down to the rule-of-thumb "use one or two extra significant figures in the course of the calculation," and I believe such chapters might well be reduced to single paragraphs. Any general rules stated for the propagation of round-off errors are of very little use in understanding the situations where these errors are really important. Only recently, through the work of Wilkinson, Dahlquist, and others, have we begun to gain this understanding, and I think any lengthy discussion of these errors should first be given in a concrete context, such as the problem of solving large linear systems.

SEYMOUR HABER, National Bureau of Standards

Linear Geometry. By K. W. Gruenberg and A. J. Weir. Van Nostrand, Princeton, N. J., 1967. viii+186 pp. $6.00. (Telegraphic Review, January 1968)

This book and several others recently published illustrate well the mutually beneficial interplay of algebra and geometry. They provide significant applications and point to further development of the linear algebra that has become an accepted component of the elementary college curriculum. The book is clear and stimulating, the treatment modern in spirit, the geometry largely coordinate-free. Several functors are considered in all but name. By staying on the rather narrow path they have chosen, the authors have achieved brevity and clarity, leaving the teacher to supplement if he wishes. There are many excellent exercises, the more challenging with outline solutions.

Following coverage of basic algebraic notions is a fine introduction to linear geometry via homogeneous vectors in both affine and projective forms. Coverage includes the following: transformations (including the structure of certain automorphism groups), dual spaces, duality in projective geometry. bilinear forms, structure for alternating and symmetric spaces, Sylvester's Law of Inertia, finite fields, quadrics and polarities (good exercises on the intersections of lines and conics, tangents, multiplicities, but no mention of the classical theorems of Pascal, Steiner, et al.), metrics arising from positive definite quadratic forms.

The final chapter, entitled "Modules" establishes the structure theorems, involving invariant factors and elementary divisors, for modules over $F[X]$, F any field. These are used to study similarity of linear maps over F, and the Jordan canonical form is found. Finally, after this long algebraic interlude, there is a return to geometry. It is shown that collineations have the same Segre symbol if and only if their invariant configurations correspond under some collineation.

A few revisions would improve this well written book. Determinants are never used, and are only mentioned in passing in the discussion of rotations in Euclidean space. Consequently, much of the discussion of eigenspaces is somewhat airy. The student may learn that minimal polynomials for linear maps exist, but may never be able to find one if he is unaware of characteristic polynomials. The authors continue the tradition of geometry books with no illustrations. While their claim that pictures are best when drawn by the reader is undoubtedly valid, a few pictures would help the novice. More examples completely worked out and a bibliography would strengthen the text.

ARNOLD ADELBERG, Grinnell College

Elements of Abstract Algebra. By Richard A. Dean. Wiley, New York, 1966. xiv+324 pp. $8.95. (Telegraphic Review, January 1967.)

(A review based on classroom use.)

One of the disadvantages of the recent profusion of new titles in abstract algebra is that a book as good as this one might be overlooked or given only casual attention. It is not likely to meet the needs or tastes of all of its potential audience but is deserving of serious consideration.

There is a wealth of mathematics here and much of it is treated in some depth. Beyond the by now standard introductions to groups, rings, and fields are included such topics as Euclidean rings, unique factorization, quaternions, the Wedderburn theorem, finitely generated abelian groups, and the Galois theory for finite fields and fields of characteristic zero. One notable omission is that nothing is said about the application of field theory to geometric construction.

This is not an easy book; particularly in the last two chapters which deal with groups and the Galois theory. Proofs are done in painstaking detail and there are several sections such as 1.9 or 10.1 where a number of important and varied results are presented in rapid succession. However, the author has made more than normal effort to motivate

proofs and this, together with an engaging informal style, may help to smooth the rough areas. There is even a flow chart given for a constructive proof for the decomposition of finitely generated abelian groups. Another feature is the exhibiting of a fifth degree polynomial with integer coefficients which is not solvable by radicals. From the student's point of view this is probably much to be preferred and appreciated when set against the later proof for the general polynomial of degree n.

The response to the book by the reviewer's students was quite enthusiastic; the only consistent (though mild) complaint was directed toward the examples. The feeling was expressed that these were often too limited and discussion so brief so as not to be of real value. A case could be made for an earlier treatment of elementary number theory since this could then serve as a source of examples, motivation, and a better appreciation of later abstractions. The exercises are generally good though occasionally one wishes for more variety and more problems of a computational nature. Misprints and errors are relatively minor; however, the statement of Theorem 11, page 161, should be corrected and Theorem 18, page 219, holds only if p has distinct roots.

JOHN SCHUE, Macalester College

Introduction to Analysis and Abstract Algebra. By John E. Hafstrom. Saunders, Philadelphia. 1967. xii+344 pp. $6.50. (Telegraphic Review, June 1967.)

This text should follow an intuition based calculus course. The author succeeds in providing a high level of sophistication and abstraction that should captivate the imagination of the serious potential mathematics major. Hopefully this material should be covered by the end of the sophomore year.

DAVID ROSEN, Swarthmore College

Modern Mathematical Analysis. By M. H. Protter and C. B. Morrey, Jr. Addison Wesley, Reading, Mass., 1964. x+790 pp. $10.75.

Modern Mathematical Analysis is a textbook covering "the more advanced portions of elementary calculus and the beginning portions of advanced calculus." It presupposes a knowledge of the material in the authors *Calculus and Analytic Geometry, A First Course.* Together, these texts take the student through the first sixteen semester hours of college level mathematics.

The topics from advanced calculus treated in the book under review are: uniform convergence, Fourier series, implicit function theorems, functions defined by integrals, vector field theory and the theorems of Green and Stokes. About 150 pages are devoted to linear algebra including a discussion of eigenvalue problems and complex vector spaces.

The advanced topics from elementary calculus include solid analytic geometry, partial differentiation, multiple integrals, infinite series and differential equations.

The two books together cover the algebra, analytic geometry and calculus that a well prepared student should learn in his first two years of college mathematics. Actually there is more than enough material for a substantial two year course. The increasing number of students entering college with training in analytic geometry and calculus should still find plenty to keep them busy. For less well prepared students, the books may form the basis for a three year course, though the authors do not suggest this. The inclusion of the topics from advanced calculus should do much to ease the frequently difficult transition to advanced calculus. A full fledged course in advanced calculus could then be offered in the senior year. For institutions that do not regularly offer advanced calculus these books provide an opportunity to extend the elementary calculus beyond its usual bounds.

The book under review can follow any competent first year text and should be thoughtfully examined by anyone concerned with the second year course, its relation to other disciplines, and the transition to more advanced work in mathematics.

H. M. MACNEILLE, Case Institute of Technology

The Language of Logic. By H. Freudenthal. American Elsevier, New York, 1966. 105 pp. $5.00.

The first three chapters give a very clear informal introduction to set theory, propositional logic, and predicate logic. The fourth chapter, "Formal Logic," is a remarkably compact and rigorous presentation of formal axiomatic systems for propositional and predicate logic, including a nice exposition of Henkin's proof of the completeness theorem. The last chapter, "Language and Metalanguage," touches very briefly on axiomatic set theory, Peano's postulates, the Löwenheim-Skolem "paradox," and the Liar and Richard-Berry paradoxes. Then, a method of arithmetization of metalinguistic concepts is given, and informal proofs of versions of Tarski's Theorem and Gödel's Incompleteness Theorem are sketched. These informal proofs seem to me, however, too informal to be understood by anyone not already familiar with the rigorous proofs. Exercises appear only in the first three chapters. The omission of any references to the literature is unfortunate, especially in the case of the Schröder-Bernstein Theorem (the proof of which is considered by the author too difficult to be included) and the inadequately handled theorems of Tarski and Gödel. Nevertheless, this book could serve as a very quick and readable introduction to logic, to be followed later by a more thorough treatment.

ELLIOTT MENDELSON, Queens College

Topology. By James Dugundji. Allyn and Bacon, Boston, 1966. xvi+447 pp. $13.25.

This is a good book. Among the spate of recent topology texts, it is one of the few which treats both general and algebraic topology (homotopy theory) in some depth. The author indicates in his preface that his text can easily be covered in a year's course by a student who has had one semester of rigorous analysis. The text, almost 450 pages, contains considerably more material than the reviewer would like to teach in a year. It reads easily. Although there are no figures of the "picture" type, and only an occasional mapping diagram, the many examples illustrating almost every definition keep the reader in touch with the geometric origins of topology and are a great help in fixing new concepts in the reader's mind. The proofs are marvels of concise clarity. The two years between the advertised, and actual publication must have been spent in polishing these proofs. An adequate supply of problems appears at the end of each chapter arranged by sections in the chapter. The contents may be abbreviated as: Chaps. I–II, 61 pp., Set Theory; Chaps. III–XIV, 253 pp., General Topology; Chaps. XV–XX, 95 pp., Homotopy Theory. There are two appendices on vector spaces and direct and inverse limits.

B. H. ARNOLD, Oregon State University

Evolution of Mathematical Thought. By H. Meschkowski. Translated from the German by Jane H. Gayl. Holden-Day, San Francisco, 1965. 157 pp. $5.95.

In this small book addressed to a wide audience the author traces the development of thought concerning the basic nature of mathematics and considers the philosophical consequences of the insights acquired. He begins with the Greek discovery of incommensurables, continues with the discovery of non-Euclidean geometry, problems of infinity, development of set theory and related antinomies, and then devotes almost one half of the book to intuitionism, mathematical logic, formalism, decision problems, operative mathematics. A chapter on the implications of all this for other areas of thought and an appendix with some reflections on education in this age bring the book to a close.

The book starts at an easy pace for the nonmathematician but, although not requiring a mathematical background, makes increasing demands on the reader's powers of attention and serious thought. Although the translator has tried hard to write unstilted English the resultant style is not so easy as one would like; at times a better mathematical term could have been chosen than the obvious translation of the German. The extensive bibliography appended has been appropriately revised for American readers.

This quite worthy little volume can be useful to the nonmathematician, the mathematics student not yet well-acquainted with this aspect of mathematical thought and, as the author suggests, to the mathematician "who completed his studies a long time ago."

ABBA V. NEWTON, Vassar College

Elements of Abstract Harmonic Analysis. By George Bachman. Academic Press, New York and London, 1964. xi+256 pp. $3.45.

This treatment of harmonic analysis begins with the Fourier transform of summable functions on the real line, and goes on to the Plancherel Theorem for square-summable functions. Then Gelfand's theory of Banach algebras is presented, followed by a review of general topology and a brief treatment of topological groups. This order of topics is odd but efficient. In later chapters come the Haar integral on locally compact groups, character theory and duality on locally compact abelian groups, and the beginning of the theory of Fourier transforms.

The book is written for students without advanced graduate training and it presents the basic methods and results of harmonic analysis clearly. Proofs are frequently omitted, however, so that this is a survey rather than an exposition. Only elementary material is included and no hint is given of the direction in which the field is currently moving.

HENRY HELSON, Univ. of California, Berkeley

Differential—und Integral—Ungleichungen und ihre Anwendung bei Abschatzungs—und Eindeutigkeitsproblemen, Springer Tracts in Natural Philosophy. By W. Walter. Springer-Verlag, Berlin-Göttingen—Heidelberg—New York, 1964. vol. 2, xiv+269 pp. DM 59.

The topic of this monograph is the study of Volterra integral equations and initial-value problems for differential equations by means of inequalities closely connected with the corresponding equations. The treatment of such problems involves: (a) existence questions, (b) questions of uniqueness and continuous dependence of solutions on the data, (c) quantitative and qualitative properties of solutions. The work at hand is limited primarily to questions of types (b) and (c) treated by the method of inequalities. These fruitful techniques were initiated by E. Hopf in 1927 and are typified by the work of M. Nagumo and T. Ważewski. The extensive bibliography appears to be rather comprehensive up to about 1960 and contains numerous later entries.

Professor Walter deserves praise on several counts. The material gives an impression of a scattered collection of loosely related techniques applied to particular problems. With great skill, however, the author has systematically organized the presentation of these techniques and has made very clear their common basis. The mode of development appealed to the reviewer: the author begins each topic with simple, concrete instances, well-chosen to motivate and outline the subsequent comprehensive development. There is a generous selection of applications. These merits seem particularly remarkable in view of the traditionally compact format of the several Ergebnisse series. A brief comparison with J. Szarski, Differential Inequalities, *Monografie Matematyczne,* vol 43, Warsaw, 1965, is in order. Professor Walter's monograph has a slightly applied flavor and furnishes a better initiation to the subject. For specialists the two works are almost complementary, due to their difference in emphasis.

This book is produced with the elegant typography, superior execution, and formidable price we have come to expect from Springer-Verlag.

WM. J. FIREY, Oregon State University

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year). A boldface star (*) marks a notable book that might be overlooked.

Analysis

Integrability Theorems for Trigonometric Transforms. By Ralph P. Boas, Jr. (Northwestern Univ.). Springer-Verlag, New York, 1967. 65 pp. $4.50. A survey of the substantial recent literature connecting integrability properties of a function and of the corresponding trigonometric series, supplemented by some questions posed by the author. P.

Conformal Mapping on Riemann Surfaces. By Harvey Cohn (Univ. of Arizona). McGraw-Hill, New York, 1967. xiv+325 pp. $12.95. Designed for a year or semester course pre-supposing complex analysis, this book appears to be a very interestingly written treatment of its subject including its various motivations and interrelation with diverse branches of mathematics and applications. Part 1 is a review of complex analysis; Part 2 is on Riemann manifolds. The next three parts are entitled Derivation of Existence Theorems, Real Existence Proofs, and Algebraic Applications. There are exercises and a selective bibliography. T (16–17), S, P.

Research Problems in Function Theory. By W. K. Hayman (University of London). Athlone Press, London (Oxford University Press in U.S.A. and Canada). vii+56 pp. $2.00. A collection of open problems, some famous, others of recent origin, in the classical theory of functions of one complex variable. S, P.

Differential-Gleichungen. By Sophus Lie. Chelsea, Bronx, New York, 1967. xiv+568 pp. $12.50. A reprint, textually unaltered except for corrections, of the classic "Vorlesungen über Differentialgleichungen, mit bekannten infinitesimalen Transformationen" edited and published by G. Scheffers in 1891. P.

The topology of the Calculus of Variations in the Large. By L. A. Ljusternik. Translations of Math. Monographs 16. American Mathematical Society, Providence, R.I., 1966. vii+96 pp. $9.30. A translation from the Russian by J. M. Danskin of Publication #19 of the Steklov Mathematical Institute (1947). P.

Approximate Methods for Solution of Differential and Integral Equations. By S. G. Mikhlin (Leningrad State Univ.) and K. L. Smolitskiy (Leningrad State Univ.) Translated by Scripta Technica, edited by Robert E. Kalaba (Rand Corporation). American Elsevier, New York, 1967. xi+308 pp. $14.00. The original was published in Moscow in 1965 and covers the most important and powerful methods for the approximate solution for boundary value problems (including the Cauchy problem) both ordinary and partial, as well as the most frequently encountered types of integral equations. Both numerical and analytical methods are described, and the entire domain of classical applications of mathematical analysis to mechanics, engineering, and physics is covered. S, P, L.

Analytic Theory of Continued Fractions. By H. S. Wall (Univ. of Texas). Chelsea, Bronx, New York, 1967. xiii+433 pp. $7.50. An unaltered reprint of this classic first published in 1948. The "Analytic" in the title indicates that the author deals with such matters as theory of equations, orthogonal polynomials, series, infinite matrices and quadratic forms, integrals, the moment problem, analytic functions, and divergent series, but not with the number theoretic aspects of continued fractions. S, P, L.

Applications

An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion. By Harold Crabtree. Third edition, a corrected reprint of the second edition of 1914. Chelsea, New York, 1967. xv+193 pp. $4.95. An exposition at the elementary college level. S, L.

Introduction to Operations Research. By Frederick S. Hillier (Stanford Univ.), and Gerald J. Lieberman (Stanford Univ.). Holden-Day, San Francisco, 1967. x+639 pp. $12.95. An introduction to methodology and techniques with emphasis on motivation rather than rigour or technical details. Aimed at advanced students with no previous knowledge of the field, it includes the necessary mathematics of probability, statistics, and linear algebra. Unfortunately there are few references and no bibliography. T, P, L.

Theoretical Physics: Applications of Vectors, Matrices, Tensors and Quaternions. By A. Kyrala (Arizona State Univ.). W. B. Saunders, Philadelphia, 1967. xiii+359 pp. $9.00. Of interest because of chapters on tensor analysis and curvi-linear coordinates, quantum mechanics, four-dimensional vector analysis, and quaternions. S, P.

Theoretical Analysis of Information Systems. By Borje Langefors (Saab Aircraft Company). Studentlitteratur, Lund, Sweden, 1966. 402 pp. $11.00 (paper). Based on courses on business data processing given in 1962, this book is intended both as a textbook and as a scientific penetration into a new field. Its three main parts are entitled Systems Theory, Information Systems Theory, and Some Data Processing Problems. T, P.

The Logic of Special Relativity. S. J. Prokhovnik (Univ. of New South Wales). Cambridge University Press, 1967. xiv+128 pp. $5.95. Of interest to mathematicians concerned with relativity. S, P.

History and Reprints of Classics

Differential Equations. By H. Bateman. Chelsea, New York, 1966. xi+306 pp. $4.95. A reprint with corrections of the original edition of 1918. L.

Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups. By John Edward Campbell. Chelsea, New York, 1966. xx+416 pp. $6.50. An unaltered (except for correction of errata) reprint of the original (1903). P.

The Doctrine of Chances or a Method of Calculating the Probabilities of Events in Play. By A. DeMoivre. Facsimile reprint of the second edition of 1738 plus a table of contents and index. Frank Cass, London, 1967. xvi+256 pp. £5/5/0. A handsome edition, but see next review.

The Doctrine of Chances or a Method of Calculating the Probabilities of Events in Play. By A. DeMoivre. Facsimile reprint of the third (final) edition of 1756 plus a biographical article on the author by Helen M. Walker. Chelsea, New York, 1967. xi+368 pp. $7.95. A well printed edition. L.

The almost simultaneous publication of facsimile reprints of two different editions of this classic in the history of probability is testimony to the chaotic state of the publishing industry. Libraries that point toward completeness should have both volumes. Where a choice must be made, the Chelsea reprint is the best buy.

Modern Science and Zeno's Paradoxes. By Adolf Grunbaum (Andrew Mellon Professor of Philosophy, University of Pittsburgh). Wesleyan University Press, Middletown, Conn., 1967. x+148 pp. $6.50. Written from a philosophical point of view, this book

may be of some interest to specialists in the foundations and to the much wider audience whom Zeno has intrigued for more than two thousand years. P.

The Mathematical Papers of Sir William Rowan Hamilton. Volume 3. Algebra. Edited for the Royal Irish Academy by H. Halberstam and R. E. Ingram. Cambridge University Press, 1967. xxiv+672 pp. $37.50. All of Hamilton's published papers and one previously unpublished paper on algebra are collected here and arranged according to subject. Among the many treasures in this volume are the first treatment of complex numbers as ordered pairs of reals, all his papers on quaternions, a paper supplementing Abel's proof of the insolubility of the quintic in terms of radicals, and his work on the icosian calculus, which is related to both group theory and modern graph theory. There is an eleven page introduction by the editors, numerous editorial footnotes, and a facsimile of the pocketbook pages in which Hamilton first wrote down the fundamental equations of quaternion algebra. The previous volumes on optics and mechanics and a forthcoming volume on geometry and analysis will complete the planned set. P, L.

* *A Mathematician's Apology.* By G. H. Hardy. Foreword by C. P. Snow. Cambridge University Press, 1967. 153 pp. $2.95. Hardy's beautifully written, moving, exciting, opinionated, snobbish essay is a well-known classic, here published in its fourth reprinting since the first edition of 1940, but this time with a fascinating and revealing fifty page reminiscence by C. P. Snow. Hardy was one of the most able mathematicians of his day and also one of the most paradoxical. He was a vigorous campaigner for rigor yet in many ways quite irrational in his private life. He expressed bitter disdain for "exposition, criticism, appreciation," yet he was an excellent expositor (his classic *Pure Mathematics* is an expository rather than a research achievement), an effective critic and analyst of mathematical ideas, and a student of the literature of the field in which he worked. Snow does not discuss these matters in general but provides anecdotal material that will fascinate every mathematician. Did you know that Ramanujan was rejected by two top British mathematicians before he reached Hardy? Do you know what subject engrossed Hardy's attention throughout his life even after he had lost interest in mathematics? Do you know that on the wall of his study was a portrait of . . . but I shall not spoil your pleasure in reading this gem on the human side of mathematics. S, P, L.

David Hilbert. Gesammelte Abhandlungen. Three volumes. Chelsea, Bronx, New York, 1965. I: xiv+539. II: viii+453. III: vii+435 pp. $25.00. An unaltered reprint of the original work (1932–1935) which included practically all his papers. L.

Celestial Mechanics. By Marquis de la Place. Translated with commentary by Nathaniel Bowditch. Four volumes. Chelsea, New York, 1966. I: 168 pp.+xxiv+746 pp. II: xviii+990 pp. III: xxix+910 pp. IV: xxxvi+1018 pp. $79.50. A photo off-set reprint of the original translation published in Boston in 1829–1839 under the title "Mécanique Céleste." The biography of Bowditch has been moved to the beginning of Volume I instead of at the end of Volume IV. Libraries can now place the original edition in the rare book room where it belongs and use this reprint for reference and circulation. L.

Cumulative Index. The Mathematics Teacher, 1908–1965 (Volumes I–LVIII). National Council of Teachers of Mathematics, Washington, D.C., 1967. 207 pp. $5.00. Of particular interest to college teachers of mathematics is the substantial number of articles on the history of mathematics, some of them very informative, that are here indexed under history of mathematics, famous mathematicians, miscellaneous, and topics. Unfortunately the articles on famous mathematicians are indexed under the

entire names starting with the first name, so that Monge is found under Gaspard! T, L.

* *The Mathematical Works of Isaac Newton.* Vol. 2. Assembled with an Introduction by Derek T. Whiteside. The Sources of Science, No. 3. Johnson Reprint Corporation, New York, 1967. xxix+173 pp. $17.50. Volume 1 was described in the telegraphic reviews for November 1967. This volume reprints in facsimile three influential treatises outside the field of calculus. They are "Universal Arithmetick: or, a Treatise of Arithmetical Composition and Resolution, written in Latin by Sir Isaac Newton, and translated by the late Mr. Ralphson, and revised and corrected by Mr. Cunn." (London 1728), "Curves, by Sir Isaac Newton," in *Lexicon Technicum. Or, an Universal Dictionary of Arts and Sciences.* By John Harris. Volume 2 (London 1710), and "Newton's Interpolation Formulas" by D. C. Fraser, Journal of the Institute of Actuaries, Volume 51 (London 1918). This last item is a rendering of Newton's *Methodus Differentialis*, probably written in 1676 and published with a tract by William Jones in 1711. These two volumes cover all of Newton's published mathematical works. P, L. (!)

Oeuvres Mathêmatiques. By Raphael Salem (University of Paris). Hermann, Paris, 1967. 645 pp. 90F. Raphael Salem (1898–1963), during a career which included business and teaching at MIT and at the Sorbonne, made important contributions to Fourier series and other branches of analysis. All these, except his two published books are included here along with a complete bibliography, portrait, facsimile of a manuscript and 24 pages of biography commentary by Antoni Zygmund and Jean-Pierre Kahane. P.

Probability and Statistics

The Theory of Gambling and Statistical Logic. By Richard A. Epstein (Hughes Aircraft). Academic Press, New York, 1967. xiii+492 pp. $10.00. An exposition in lively style of the theory of gambling (decision-making under risk conditions) including a historical introduction, expositions of basic principles, detailed discussion of numerous types of games and chapters on weighted statistical logic and statistical games, games of pure skill, competitive computers, fallacies and sophistries. S, P, L.

The Theory of Probability. By B. V. Gnedenko. Translated from the Russian by B. D. Seckler. Chelsea, New York. 1967. 529 pp. $9.50. This fourth English edition is a translation of the fourth Russian edition but differs from it in the inclusion of a chapter omitted from the Russian edition and some more general theorems. There is a new chapter on queueing. The book is suitable for an introductory course with rather broad coverage. It pre-supposes calculus but keeps the intuitive aspects as well as rigour in view. T (15–16), S, L.

An Introduction to Optimal Estimation. By Paul B. Liebelt (The Boeing Company). Addison-Wesley, Reading, Mass., 1967. ix+273 pp. $11.75. This book is not a presentation of the theory of estimation as a whole, but rather of minimum variance unbiased estimators. The theory is based on the Gauss-Markoff theorem and is related to the work of Wiener and Kalman. The references on estimation are all in the field of random processes and communication theory and no account appears to have been taken of the work of mathematical statisticians on the general problem of statistical estimation. The book is self-contained in the sense that it includes chapters on linear algebra, dynamic systems, and probability. S, P.

Exercises in Probability and Statistics for Mathematics Undergraduates. By N. A. Rahman (Univ. of Leicester). Hafner, New York, 1967. x+307 pp. $10.50. With textbooks becoming more abstract and less oriented toward problems and applications, there

is a growing need for workbooks and problem collections. This one is designed for a course at the post calculus level. It includes answers, hints, and references to the literature. T (15), S.

Dictionary Outline of Basic Statistics. By John E. Freund (Arizona State Univ.), and Frank J. Williams (San Francisco State Coll.). McGraw-Hill, New York, 1966. v+195 pp. $2.95 (paper), $5.95 (cloth). This contains 190 pages of fairly closely packed vocabulary plus 68 pages of formulae, a table of control chart constants and a short list of references. It is less complete and authoritative than the dictionary of statistical terms by Kendall and Buckland (Hafner, 1960), but it may be handy for students. S, P, L.

FILMS

After several years and several million dollars the basic lessons of mathematical filming seem to have been learned. Perhaps most important is that the mere filming of a lecture course is of little value. Putting on film a series of lectures designed to "cover" the material of a course merely highlights the weaknesses of this teaching device and results in even more acute boredom than a live lecture because people are nowadays accustomed to be entertained by film. However, filmed lectures of about an hour in length have proved quite successful when they recorded a presentation truly appropriate for the lecture medium. Examples are colloquium type lectures, demonstration classes, and presentations designed to motivate, provide insight, inspire, or teach by personal example. The most important element in these films appears to be a personal aspect that cannot be communicated non-visually. For the imparting of information, it appears that short films, no more than fifteen and perhaps as little as five minutes long, are most effective. Such films are becoming increasingly available, and the most successful ones make full use of the film medium. They seldom involve much lecturing but utilize colour and animation, sometimes without verbal explanation. When the problems of foolproof projection are solved (probably by inserting cartridges in automatic projectors that do not require darkening the room), short films of this type should become a standard part of classroom teaching.

The supply of films has so far been so small compared with the demand that producers have not been concerned with obtaining reviews. However, as the output of films increases, timely reviews are going to become essential. Since there is no tradition for reviewing, it is rather difficult to get started and we hope that volunteers will help. If you would like to review a film. please write to this department and the editor will try to arrange a showing if you have not already seen it. In order to get the ball rolling we review below one of the best films so far produced.

Let Us Teach Guessing. A demonstration with George Polya. Produced by the Committee on Educational Media, Mathematical Association of America. Available from Modern Learning Aids in the United States and Canada. 61 min., 16 mm., colour, sound. $400. Also for rent or lease to buy.

The film shows a class working on a problem under the guidance of Professor Polya. Because of the personality of the author, his teaching skill, and the way in which the film illustrates principles of discovery and heuristic reasoning, this film had very wide appeal and usefulness. It ought to be seen by every future mathematics teacher, whether destined for the first grade or graduate school. It would make an enjoyable program for a mathematics club or colloquium. It could be used in courses or offered by the department to the general campus public. It is one of the best of the many fine films produced by the Individual Lectures Project of the C.E.M. Like many of them it is also an historical document because it records a mathematician who has made important contributions.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor T. F. Kimes, Austin College, represented the Association at the inauguration of President J. O. Perpener of Jarvis Christian College on October 1, 1967.

Professor V. O. McBrien, College of the Holy Cross, represented the Association at the inauguration of President F. H. Jackson of Clark University on October 7, 1967.

Professor H. V. Monks, Northeastern State College, represented the Association at the inauguration of President G. D. Holstine of Bacone College on October 8, 1967.

Professor T. E. Mott, SUNY College at Buffalo, represented the Association at the Centennial Convocation at SUNY College at Fredonia on October 6, 1967.

Professor F. R. Olson, SUNY College at Fredonia, represented the Association at the inauguration of President R. A. Redlon, OFM, of St. Bonaventure University on October 4, 1967.

State College of Arkansas: Assistant Professor D. W. Adlong, McMurry College, has been appointed Assistant Professor; Professor F. M. Hudson, McMurry College, has been appointed Professor and Chairman of the Mathematics Department.

University of Arkansas: Associate Professor B. J. Attebery, Louisiana Polytechnic Institute, has been appointed Associate Professor; Assistant Professor Tetsundo Sekiguchi has been promoted to Associate Professor.

Austin Peay State University: Professor G. H. Lundberg, Vanderbilt University, has been appointed Professor; Assistant Professors G. L. Brotherton and W. A. Crabtree have been promoted to Associate Professors.

Ball State University: Dr. R. A. Kallman, University of Minnesota at Duluth, has been appointed Assistant Professor; Assistant Professor R. L. McCormick has been promoted to Associate Professor.

Berea College: Mr. L. C. Grunwald, Madison College, has been appointed Assistant Professor; Mr. L. R. Henson has been promoted to Assistant Professor and Director of the Computer Center.

Boston College: Associate Professor G. G. Bilodeau has been appointed Chairman of the Mathematics Department; Associate Professor S. S. Holland has been appointed Associate Professor at the University of Massachusetts at Amherst.

Bradley University: Assistant Professor F. P. Prokop, U. S. Naval Academy, has been appointed Assistant Professor; Associate Examiner G. P. Speck, Educational Testing Service, has been appointed Assistant Professor.

Brooklyn College: Assistant Professor M. E. Levenson has been promoted to Associate Professor; Associate Professor A. W. Landers has been promoted to Professor.

California State College (Pa.): Assistant Professor J. S. Gibson, Jr., Waynesburg College, has been appointed Assistant Professor; Mr. J. S. Skocik, Jr., West Virginia University, has been appointed Associate Professor.

California State Polytechnic College-Pomona: Associate Professor C. C. Bogue has been promoted to Professor; Associate Professor Sidney Spital has been appointed Associate Professor at California State College at Hayward.

California State Polytechnic College-San Luis Obispo: Associate Professors K. G. Fuller, W. B. Judd, and G. R. Mach have been promoted to Professors; Mr. G. M. Lewis, University of Southern California, has been appointed Assistant Professor.

University of California-Berkeley: Professor P. E. Thomas has been appointed Vice Chairman of the Mathematics Department; Professor Lipman Bers, Columbia University, has been appointed Research Professor in the Miller Institute for Basic Research; Professor Murray Protter has been appointed Research Professor in the Miller Institute for Basic Research.

University of California-Santa Barbara: Dr. R. W. Chaney, University of Washington, has been appointed Assistant Professor; Visiting Assistant Professor J. A. Ernest, Tulane University, has been appointed Associate Professor; Professor Leo Moser, University of Alberta, has been appointed Visiting Professor; Professor S. T. C. Moy, Syracuse University, has been appointed Professor.

Central Washington State College: Associate Professor D. R. Comstock has been appointed Acting Chairman of the Mathematics Department; Associate Professor B. L. Martin has been appointed Assistant Dean of Arts and Sciences.

Chico State College: Mr. D. I. Goslin has been promoted to Assistant Professor; Assistant Professor R. C. Frascatore has been appointed Assistant Professor at SUNY College at Buffalo.

The Citadel: Assistant Professor G. L. Crumley has been promoted to Associate Professor; Professor G. E. Reves has been appointed Head of the Mathematics Department.

University of Colorado: Assistant Professors J. W. Bebernes and E. R. Krueger have been promoted to Associate Professors; Associate Professor G. H. Meisters has been promoted to Professor; Associate Professor Roberta F. Johnson, Colorado State University, has been appointed Associate Professor.

Emory University: Assistant Professor W. S. Mahavier has been promoted to Associate Professor; Associate Professor Henry Sharp, Jr., has been promoted to Professor.

Franklin College: Miss Myra B. McFadden, Bucknell University, has been appointed Assistant Professor; Dr. P. T. Nugent, University of Kentucky, has been appointed Assistant Professor.

George Washington University: Professor Choy-tak Taam, Georgetown University, has been appointed Professor; Assistant Professor Dagmar R. Henney has been promoted to Associate Professor; Associate Professor Hewitt Kenyon has been promoted to Professor.

Illinois State University: Assistant Professor K. A. Retzer has been promoted to Associate Professor; Assistant Professor K. C. Ha, University of South Florida, has been appointed Associate Professor; Dr. A. G. White, Jr., University of St. Louis, has been appointed Assistant Professor.

Indiana University of Pennsylvania: Associate Professor Gus Di Antonio, Northern Illinois University, has been appointed Professor; Senior Professor J. P. Hoyt, U. S. Naval Academy, has been appointed Professor.

Iowa State University: Associate Professor G. W. Peglar has been promoted to Professor; Dr. D. E. Arganbright, University of Washington, has been appointed Assistant Professor; Associate Professor Dorothy A. Bollman, University of Puerto Rico, has been appointed Assistant Professor; Associate Professor A. M. Fink, University of Nebraska, has been appointed Associate Professor; Dr. R. L. Johnson, University of Kansas, has been appointed Assistant Professor; Assistant Professor and Computer Center Director R. F. Keller, University of Missouri, has been appointed Associate Professor of Mathematics and Computing.

Kent State University: Associate Professor E. L. Bethel, Clemson University, has been appointed Assistant Professor; Assistant Professor J. A. Fridy, Rutgers University, has been appointed Associate Professor; Associate Professors T. N. Bhargava and R. Y. Iwanchuk have been promoted to Professors.

University of Kentucky: Assistant Professor Lynn C. Kurtz has been appointed Assistant Professor at Arizona State University; Assistant Professors D. Z. Spicer, Uni-

versity of California at Los Angeles, and T. J. Suffridge, Miltonvale Wesleyan College, have been appointed Assistant Professors; Drs. L. E. Bragg, University of Minnesota and J. G. Caughran, Bowling Green State University, have been appointed Assistant Professors; Associate Professor H. C. Howard, University of Wisconsin, has been appointed Associate Professor; Professor C. E. Langenhop, Southern Illinois University, has been appointed Professor; Assistant Professor R. E. Powell, University of Kansas, has been appointed Visiting Research Assistant.

King's College: Mr. H. F. Gleim, Penn State University, has been appointed Assistant Professor; Assistant Professor M. J. Hudak has been promoted to Associate Professor.

Louisiana State University-Baton Rouge: Assistant Professor S. B. Nadler, Jr., Wayne State University, has been appointed Assistant Professor; Assistant Professor J. R. Dorroh has been promoted to Associate Professor.

Louisiana State University-New Orleans: Associate Professor D. M. Nead has been promoted to Professor; Assistant Professor M. P. Berri, Tulane University, has been appointed Associate Professor and Acting Chairman of the Mathematics Department.

Marshall University: Mr. S. H. Hatfield has been promoted to Assistant Professor; Assistant Professors Layton Thompson and Berfitt Jordan have been promoted to Associate Professors.

Northeast Missouri State Teachers College: Mr. W. J. Weber, Montana State University, has been appointed Assistant Professor; Mr. E. C. Pringle has been appointed Assistant Professor at SUNY College at Oneonta.

University of Missouri at Rolla: Associate Professor T. L. Hicks, Illinois State University, has been appointed Assistant Professor; Assistant Professor C. Y. Ho, St. Mary's College, has been appointed Associate Professor.

Newark College of Engineering: Dr. Gideon Peyser, Hofstra University, has been appointed Professor; Associate Professor Phyllis Fox has been promoted to Professor.

University of North Carolina-Chapel Hill: Assistant Professor R. J. Troyer has been promoted to Associate Professor; Dr. L. D. Geissinger, Purdue University, has been appointed Associate Professor; Dr. J. A. Pfaltzgraff, Indiana University, has been appointed Assistant Professor; Professor Johann Sonner, University of South Carolina, has been appointed Professor.

University of North Carolina-Charlotte: Assistant Professor Lucio Artiaga, Dalhousie University, has been appointed Assistant Professor; Dr. T. L. Markham, Auburn University, has been appointed Assistant Professor.

Ohio University: Assistant Professor C. B. Mehr has been promoted to Associate Professor; Associate Professor S. J. Jasper has been promoted to Professor and appointed Chairman of the Mathematics Department; Drs. Joaquin Bustoz, Arizona State University, and M. S. K. Sastry, University of Rochester, have been appointed Assistant Professors; Professor R. L. Blair, Purdue University, has been appointed Professor.

Olivet Nazarene College: Assistant Professor B. F. Hobbs, Purdue University, has been appointed Associate Professor and Chairman of the Mathematics Department; Mr. D. L. Strawn, Royal Oak, Michigan, has been appointed Assistant Professor.

Pace College: Assistant Professor Allan Gewirtz has been promoted to Associate Professor; Dr. L. V. Quintas, City University, has been appointed Professor; Dr. H. R. Cooley, New York University, has been appointed Visiting Professor Emeritus.

Pacific University: Dr. M. C. Clock, Oklahoma State University, has been appointed Assistant Professor; Mr. J. W. Marsh has been promoted to Assistant Professor.

University of Redlands: Mr. D. B. Bragg, Arizona State College, has been appointed Assistant Professor; Mr. R. R. Poole has been promoted to Assistant Professor.

Rice University: Assistant Professor M. Q. Jacobs, Brown University, has been appointed Assistant Professor; Professor Jim Douglas, Jr., has been appointed Professor at the University of Chicago; Assistant Professor R. C. O'Neil has been promoted to Associate Professor.

University of Rochester: Mr. Samuel Merrill, III, has been promoted to Assistant Professor; Professor R. A. Raimi has been appointed Associate Dean for Graduate Studies in the College of Arts and Science.

Sacramento State College: Assistant Professors R. L. Alves and G. S. Silberman have been promoted to Associate Professors; Associate Professors T. Y. Chow and John Christopher have been promoted to Professors.

St. Louis University: Assistant Professor J. C. Cantwell, University of Iowa, has been appointed Assistant Professor; Associate Professor R. W. Freese has been promoted to Professor.

San Diego State College: Drs. N. S. Morez, University of Cincinnati and D. E. Ryan, Bowling Green State University, have been appointed Assistant Professors; Assistant Professors H. G. Bray, S. I. Drobnies and Albert Romano have been promoted to Associate Professors; Associate Professors E. I. Deaton and R. L. Van de Wetering have been promoted to Professors.

Shippensburg State College: Assistant Professor J. S. Mowbray, Jr. has been promoted to Associate Professor; Associate Professor J. L. Sieber has been promoted to Professor.

South Dakota School of Mines and Technology: Mr. D. W. Ballew, University of Illinois, and Mr. V. A. Nelson, Colorado State University, have been appointed Assistant Professors; Associate Professor B. L. McAllister has been appointed Associate Professor at Montana State University.

University of Southern California: Drs. A. F. Abrahamse, University of Michigan and J. S. Alin, University of Nebraska, have been appointed Assistant Professors; Assistant Professor G. R. Sell, University of Minnesota, has been appointed Associate Professor; Associate Professor S. E. Dickson, University of Nebraska, has been appointed Associate Professor; Associate Professor T. S. Pitcher has been promoted to Professor.

Stevens Institute of Technology: Assistant Professor M. E. White has been promoted to Associate Professor; Associate Professor R. S. Pinkham has been promoted to Professor.

Sweet Briar College: Mr. P. M. Kannan, Union Carbide Corporation, has been appointed Assistant Professor; Assistant Professor Leonora A. Wikswo has been promoted to Associate Professor.

Temple University: Professor Rafael Artzy, SUNY at Buffalo, has been appointed Professor; Dr. Theodore Mitchell, SUNY at Buffalo, has been appointed Associate Professor; Mr. F. J. Sholomskas has been promoted to Assistant Professor.

East Texas State University: Professor W. F. Hill, Tarleton State College, has been appointed Professor; Mr. W. W. Taylor has been promoted to Assistant Professor.

Texas Technological College: Assistant Professors T. L. Boullion, University of Southwestern Louisiana, M. H. Hall, UCLA, T. G. Newman, Methodist University and F. E. Tidmore, Baylor University, have been appointed Assistant Professors; Dr. G. S. Innis, Los Alamos Scientific Lab., has been appointed Associate Professor; Assistant Professor W. T. Ford, University of Houston, has been appointed Associate Professor.

University of Texas at El Paso: Mr. Bernard Martin-Williams, West Georgia College, has been appointed Assistant Professor; Mr. S. E. Ball has been promoted to Assistant Professor.

University of Wyoming: Assistant Professors A. D. Porter and D. R. Anderson have been promoted to Associate Professors; Professor R. E. Carr, University of Alaska, has been appointed Professor; Associate Professor J. R. Hanna, University of Colorado, has been appointed Professor; Dr. J. H. George, Marshall Flight Space Center, has been appointed Associate Professor; Assistant Professor R. E. Smithson, University of Florida, has been appointed Associate Professor.

York University: Dr. Israel Kleiner has been promoted to Assistant Professor; Dr. Morton Abramson, McGill University, has been appointed Assistant Professor; Assistant Professor J. A. Ewell, Long Beach State College, has been appointed Assistant

Professor; Professor R. G. Stanton, University of Waterloo, has been appointed Professor of Mathematics and Computer Science.

Associate Professor H. J. Arnold, Bucknell University, has been appointed Associate Professor at Oakland University.

Associate Professor Lieutenant Colonel H. J. Arnold, U. S. Air Force Academy, has been appointed Associate Professor at Henderson State College.

Mr. A. M. Bazik, Elmhurst College, has been promoted to Assistant Professor.

Mr. A. F. Brownell, Jr., Wisconsin State University, has been promoted to Assistant Professor.

Mr. M. L. Brubaker, Susquehanna University, has been appointed Assistant Professor at Moravian College.

Assistant Professor R. I. Canavan, SJ, St. Peter's College, has been promoted to Associate Professor.

Dr. R. C. Davis, Jr., Tulane University, has been appointed Assistant Professor at Southern Methodist University.

Associate Professor S. F. Dice, Wittenberg University, has been appointed Chairman of the Mathematics Department.

Assistant Professor Sister Marie Augustine Dowling, College of Notre Dame of Maryland, has been promoted to Associate Professor.

Assistant Professor L. A. Dryden, Ft. Hays Kansas State College, has been promoted to Associate Professor.

Dr. L. K. Durst, Executive Director, CUPM, has been appointed Professor at Claremont Men's College.

Dr. A. N. Feldzamen, Research Foundation of the State University of New York, has been appointed Vice President and Treasurer of the Broadcasting Foundation of America.

Professor Morris Friedman, Northern Michigan University, has been appointed Professor at Hofstra University.

Dr. J. M. Gary, National Center for Atmospheric Research, has been appointed Visiting Professor at the University of Colorado.

Mr. J. A. Glasenapp, Rochester Institute of Technology, has been promoted to Assistant Professor.

Assistant Professor Orville Goering, Southern Illinois University, has been promoted to Associate Professor.

Assistant Professor Martin Goldsworth, Southern Colorado State College, has been promoted to Associate Professor.

Mr. D. T. Graves, University of Virginia, has been appointed Assistant Professor at Skidmore College.

Assistant Professor Siegfried Grosser, Cornell University, has been appointed Assistant Professor at the University of Minnesota.

Professor L. D. Kovach, Pepperdine College, has been appointed Visiting Professor at the U. S. Naval Postgraduate School.

Dr. G. C. Marley, University of Arizona, has been appointed Assistant Professor at California State College at Fullerton.

Assistant Professor D. E. McLeod, California State College at San Bernardino, has been appointed Assistant Professor at Colgate University.

Assistant Professor R. A. Melter, University of Massachusetts, has been appointed Associate Professor at the University of South Carolina.

Assistant Professor K. V. Menon, Texas Technological College, has been appointed Assistant Professor at Dalhousie University.

Assistant Professor Z. C. Motteler, Gonzaga University, has been promoted to Associate Professor.

Dr. Wanda J. Mourant, North American Aviation, has been appointed Assistant Professor at Denison University.

Dr. P. V. O'Neil, University of Minnesota, has been appointed Assistant Professor at the College of William and Mary.

Professor Herman Rubin, Michigan State University, has been appointed Professor at Purdue University.

Associate Professor Helga H. Schirmer, University of New Brunswick, has been appointed Associate Professor at Carleton University.

Professor Gaston Smith, University of Southern Mississippi, has been appointed Head of the Mathematics Department at William Carey College.

Mr. C. E. Snygg, University of Michigan, has been appointed Assistant Professor at Morehouse College.

Assistant Professor J. A. Synowiec, Illinois Institute of Technology, has been appointed Assistant Professor at the University of Indiana.

Assistant Professor J. D. Tarwater, Western Michigan University, has been appointed Assistant Professor at North Texas State University.

Assistant Professor Barbara E. Taucci, University of Omaha, has been appointed Associate Professor at the University of Akron.

Dr. M. C. Tews, University of Washington, has been appointed Assistant Professor at the College of the Holy Cross.

Professor Emeritus W. E. Cederberg, Augustana College, died on December 17, 1966. He was a Charter Member of the Association.

Professor Emeritus A. F. Kovarik, Yale University, died in 1965. He was a Charter Member of the Association.

Professor L. L. Lowenstein, Arizona State University, died on August 23, 1967. He was a member of the Association for forty years.

Professor Emeritus Pauline Sperry, University of California, Berkeley, died on September 24, 1967. She was a member of the Association for thirty-three years.

MATHEMATICAL SCIENCES ADMINISTRATIVE DIRECTORY—1968

The Directory is one of the annual publications of the American Mathematical Society. It lists the names and mailing addresses of chairmen of departments of mathematical sciences in the U. S. and Canada, heads of industrial mathematics groups, key personnel in government agencies involved in the mathematical sciences, officers and committees of sixteen mathematical societies, editorial boards of AMS journals, managing or corresponding editors of selected mathematical journals, and officers of organizations for mathematical education. Listings for several new journals, the Canadian Mathematical Congress, the Cambridge Conference on School Mathematics, and NASA have been added to this year's Directory. The new Directory will be available at the end of January 1968.

The Directory consists of about 140 pages, paper bound. The list price is $5.00. The price to AMS members is $1.00. Send orders accompanied by payment to American Mathematical Society, P.O. Box 6248, Providence, R.I. 02904.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

CALENDAR OF FUTURE MEETINGS

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

Fifty-second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Indiana University of Pennsylvania, Indiana, April 27, 1968.

FLORIDA, Miami-Dade Junior College, South Campus, Miami, March 22–23, 1968.

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA, Ball State University, Muncie, May 4, 1968.

IOWA, Wartburg College, Waverly, April 19, 1968.

KANSAS, Marymount College, Salina, March 23, 1968.

KENTUCKY, University of Kentucky, Lexington, April 27, 1968.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Old Dominion College, Norfolk, Virginia, April 27, 1968.

METROPOLITAN NEW YORK, Staten Island Community College, Staten Island, March 16, 1968.

MICHIGAN, Grand Valley State College, Allendale, March 23, 1968.

MINNESOTA, College of St. Teresa, Winona, May 4, 1968.

MISSOURI, Lindenwood College, St. Charles, April 27, 1968.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY, Rider College, Trenton, May 4, 1968.

NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.

NORTHERN CALIFORNIA

OHIO, Miami University, Oxford, April 26–27, 1968.

OKLAHOMA-ARKANSAS, Federal Aviation Agency, Oklahoma City, March 29–30, 1968.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.

ROCKY MOUNTAIN, University of Denver, Colorado, May 10–11, 1968.

SOUTHEASTERN, East Carolina University, Greenville, North Carolina, March 29–30, 1968.

SOUTHERN CALIFORNIA, Loyola University of Los Angeles, Los Angeles, March 9, 1968.

SOUTHWESTERN, New Mexico State University, University Park, April 12-13 1968.

TEXAS, Texas A and M University, College Station, April 19–20, 1968

UPPER NEW YORK STATE, Hamilton College, Clinton, May 11, 1968.

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, University of Wisconsin, Madison, August 27–30, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, Chicago, Illinois, August 20–22, 1968.

ASSOCIATION FOR SYMBOLIC LOGIC, University of California, Los Angeles, March 22, 1968.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS, University of Wisconsin, Madison, August 27–28, 1968.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Hall, Philadelphia, Pennsylvania, April 17–20, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, St. Francis Hotel, San Francisco, May 1–3, 1968.

PI MU EPSILON, University of Wisconsin, Madison, August 27–28, 1968.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, King Edward Sheraton Hotel, Toronto Canada, June 11–14, 1968. (Symposium on optimization.)

ANNOUNCING THE FORTHCOMING PUBLICATION OF—

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CONTENTS

MARCH 1968

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED in 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. A. ROSENBAUM, Wesleyan University, Middletown, Conn. 06457. Because of a large backlog, a moratorium has been called on the submission of new papers. The May issue of the MONTHLY will carry an announcement about the procedure with new submissions.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

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A CLASS OF ADDITIVE FUNCTIONS

M. V. SUBBARAO, University of Alberta (Edmonton)

1. Introduction. Throughout this note, let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the representation of $n > 1$ as a product of powers of distinct primes, and define $\Omega_k(n) = a_1^k + \cdots + a_r^k$; $w(n) = \Omega_0(n)$; $\Omega(n) = \Omega_1(n)$. In a series of interesting papers ([1]–[5]) R. L. Duncan considered these functions $\Omega_k(n)$ and in particular obtained identities ([4], Theorem 1) which generalize some results in Titchmarsh ([6] Ch. 1, eqs 1.6.2 and 1.6.3). In the paper [5] to appear, an advance copy of which he kindly sent me, he considers an even more general class of additive functions, given by

$$(1.1) \qquad a(n) = G(a_1) + \cdots + G(a_r) \qquad \text{for } n > 1; \quad a(1) = 0.$$

Here $G(n)$ is an arbitrary arithmetic function for which $G(0) = 0$ and $G(n) \geqq G(n-1)$ for $n \geqq 1$. Duncan establishes the following result: if

$$(1.2) \qquad b(n) = \sum_{d/n} \{G(d) - G(d-1)\} d\mu(n/d),$$

then

$$(1.3) \qquad \sum_{n=1}^{\infty} a(n) n^{-s} = \zeta(s) \sum_{n=1}^{\infty} \frac{b(n)}{n} \log \zeta(ns),$$

where $\zeta(s)$ is the Riemann zeta function, if both series converge absolutely. He then considers various estimates involving $a(n)$. We attempt here to generalize (1.3) and thus obtain a fairly general theorem applicable for a wide class of additive functions which include Duncan's $a(n)$.

2. The Theorem. We recall that an arithmetic function h(i.e. a complex-valued function on the positive integers) is said to be additive, provided $h(mn) = h(m) + h(n)$ whenever $(m, n) = 1$. For such a function, we have, obviously, $h(1) = 0$.

THEOREM. *Let h be an additive arithmetic function. For $m, n \geqq 1$, $r \geqq 0$, let p_m denote the m-th prime; set $H(m, r) = h(p_m^r)$, and let $E(m, n)$ denote the highest power of p_m dividing n.*

If the double series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H(m, E(m, n)) n^{-s}$$

is absolutely convergent for Re s *sufficiently large, say for* Re $s > \sigma_0$ *then the Dirichlet series* $\sum_{n=1}^{\infty} h(n) n^{-s}$ *converges to an analytic function $f(s)$ in the half-plane* Re $s > \sigma_0$, *and we have the representation*

$$f(s) = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} G(m, r) p_m^{-rs},$$

where $G(m, r) = H(m, r) - H(m, r-1)$, *and* $\zeta(s)$ *is the Riemann zeta-function.*

If, in addition, $H(m, r)$ *is independent of* m, *so that we can write* $G(m, r) = g(r) = h(2^r) - h(2^{r-1})$, *then in the half-plane* $\operatorname{Re} s > \sigma_0$ *we have*

$$f(s) = \zeta(s) \sum_{n=1}^{\infty} \frac{\log \zeta(ns)}{n} \sum_{d \mid n} d\mu\left(\frac{n}{d}\right) g(d),$$

where μ *is the Möbius function.*

Proof. By virtue of unique factorization of n into primes, we can write for all $n \geq 1$.

$$h(n) = \sum_{m=1}^{\infty} h(p_m^{E(m,n)}),$$

remembering that $h(1) = E(m, 1) = 0$ for all $m \geqq 1$. We have also $H(m, 0) = 0$ for all $m \geqq 1$, and

$$h(n) = \sum_{m=1}^{\infty} H(m, E(m, n)), \qquad n \geqq 1.$$

Hence for $s > s_0$ we have, formally:

$$(1) \qquad \sum_{n=1}^{\infty} h(n) n^{-s} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H(m, E(m, n)) n^{-s}.$$

We now assume in all that follows that $\operatorname{Re} s > \sigma_0$. Then the absolute convergence of the double series on the right validates all our subsequent steps. We pick up those terms for which m and $E(m, n) = r$ are fixed. These are the terms with

$$n = n_1 \cdot p_m^r, \quad (n_1, p_m) = 1.$$

The contribution of these terms to the right-hand side of (1) is:

$$H(m, r) \sum_{n_1=1}^{\infty} n_1^{-s} \cdot p_m^{-rs} = H(m, r) [\zeta(s) - p_m^{-s} \zeta(s)] p_m^{-rs}$$
$$= \zeta(s) H(m, r) [p_m^{-rs} - p_m^{-(r+1)s}].$$

Hence we have

$$(2) \qquad \sum_{n=1}^{\infty} h(n) n^{-s} = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} H(m, r) [p_m^{-rs} - p_m^{-(r+1)s}].$$

Applying Abel partial summation to the right side of (2) we obtain

$$(3) \qquad \sum_{n=1}^{\infty} h(n) n^{-s} = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} G(m, r) p_m^{-rs},$$

where $G(m, r)=H(m, r)-H(m, r-1)=h(p_m^r)-h(p_m^{r-1})$. This is as far as one can go for arbitrary additive functions, and establishes the first part of the theorem.

We now assume that H is independent of its first place. Then so also is G, and we can write $G(m, r)=g(r)$. Hence we have

$$\begin{aligned}\sum_{n=1}^{\infty} h(n)n^{-s} &= \zeta(s) \sum_{m=1}^{\infty}\sum_{r=1}^{\infty} g(r)p_m^{-rs} \\ &= \zeta(s) \sum_{r=1}^{\infty} g(r) \sum_{m=1}^{\infty} p_m^{-rs} \\ &= \zeta(s) \sum_{r=1}^{\infty} g(r) \sum_{m=1}^{\infty} \log \zeta(rms) \frac{\mu(m)}{m}\end{aligned}$$

(see, for example, Titchmarsh [6] Chapter 1).

Write the above expression in the right side as a double series in the form

$$\zeta(s) \sum_{r=1}^{\infty}\sum_{m=1}^{\infty} \log \zeta(rms) \frac{\mu(m)}{m} g(r),$$

and then set $rm=n$ and sum on n to obtain, finally:

$$\sum_{n=1}^{\infty} h(n)n^{-s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\log \zeta(ns)}{n} \sum_{d\mid n} d\mu\left(\frac{n}{d}\right) g(d). \tag{4}$$

This completes the proof of the theorem.

3. Special cases. I. Setting $h(n)=a(n)$ defined in (1.1), we obtain Duncan's formula (1.3). Duncan ([4], [5]) pointed out that some well-known results are immediate consequences of (1.3). The following are two other noteworthy deductions, which the author has not seen mentioned in the literature.

Let $t(n)$ and $t_1(n)$ denote respectively the number of divisors and the number of unitary divisors of n. Setting $G(n)=\log(1+n)(n\geq 0)$ and $G(n)=\log 2(n>0)$, $G(0)=0$ respectively, we obtain

$$\sum \frac{\log t(n)}{n^s} = \zeta(s) \sum \frac{1}{n} \log \zeta(ns) \log F(n), \tag{3.1}$$

where

$$F(n) = \prod_{d\mid n}\left(1+\frac{1}{d}\right)^{d\mu(n/d)},$$

$$\sum \frac{\log t_1(n)}{n^s} = (\log 2)\zeta(s) \sum \frac{1}{n} \log \zeta(ns)\mu(n). \tag{3.2}$$

II. If $G(n)$ is an arbitrary function and

$$V(n) = \sum_{d\mid n} d(G(d) - G(d-1))\phi(n/d),$$

$$h(n) = \sum_{i=1}^{n} \sum_{x=1}^{a_i} \sum_{D|x} (G(D) - G(D-1)),$$

then $\sum_{n=1}^{\infty} h(n)/n^s = \zeta(s) \sum_{n=1}^{\infty} \log \zeta(ns)\, V(n)/n$. This is easily deduced from the theorems. In particular, setting $G(n) = \pi(n)$ the number of primes not exceeding n, and $G(n) = \sum_{p \leqq n} p$, p being a prime, we obtain respectively,

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{r} w(1) + w(2) + \cdots + w(a_i) \right) \Big/ n^s$$
$$= \zeta(s) \sum_{n=1}^{\infty} \log \zeta(ns) \left(\sum_{p|n} \phi(n/p)(n/p) \right);$$

and

$$\sum_{n=1}^{\infty} \sum_{i=1}^{r} (\beta(1) + \beta(2) + \cdots + \beta(a_i)) \Big/ n^s = \zeta(s) \sum_{n=1}^{\infty} n \log \zeta(ns) \sum_{p|n} \phi(n/p)(p^2/n^2),$$

where $\beta(n)$ is the sum of the distinct prime divisors of n.

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References

1. R. L. Duncan, Note on the divisors of a number, this MONTHLY, 68 (1961) 356–359.
2. ———, A class of additive functions, this MONTHLY, 69 (1962) 34–36.
3. ———, Note on a class of arithmetic functions, this MONTHLY, 69 (1962) 991–992.
4. ———, Generating functions for a class of arithmetical functions, this MONTHLY, 72 (1965) 882–884.
5. ———, A general class of arithmetic functions, this Monthly, 72 (1965) 882–884.
6. E. C. Titchmarsh, The theory of the Riemann Zeta function, Oxford, 1951, Ch. 1.

VERY MAGIC SQUARES

D. E. KNUTH, California Institute of Technology

A square matrix A is "magic" in the weakest sense if it has a generalized "doubly stochastic" property, namely when all of its row sums and column sums have the same value s. The matrix is even more "magic" when the sums of its elements along certain diagonals are also equal to s.

If A is an $m \times m$ matrix whose elements are denoted by A_{xy} for $1 \leqq x, y \leqq m$, let us say a *generalized diagonal* of A is the set of all A_{xy} such that $ax + by \equiv c$ (modulo m), for some given integers a, b, c with a and b relatively prime to each other. For example, a 5×5 matrix has 30 distinct generalized diagonals, namely the sets of elements of the same value in the following squares:

11111	23451	35241	42531	54321	12345
22222	34512	41352	53142	15432	12345
33333	45123	52413	14253	21543	12345
44444	51234	13524	25314	32154	12345
55555	12345	24135	31425	43215	12345
$x \equiv c$	$x + y \equiv c$	$x + 2y \equiv c$	$x + 4y \equiv c$	$x + 4y \equiv c$	$y \equiv c.$

When m is not prime, further types of generalized diagonals appear; e.g., for $m=6$ we have:

111111	234561	351351	414141	531531	654321
222222	345612	462462	525252	642642	165432
333333	456123	513513	636363	153153	216543
444444	561234	624624	141414	264264	321654
555555	612345	135135	252525	315315	432165
666666	123456	246246	363636	426426	543216
$x \equiv c$	$x + y \equiv c$	$x + 2y \equiv c$	$x + 3y \equiv c$	$x + 4y \equiv c$	$x + 5y \equiv c$
345612	525252	165432	456123	513513	123456
561234	141414	321654	123456	246246	123456
123456	363636	543216	456123	513513	123456
345612	525252	165432	123456	246246	123456
561234	141414	321654	456123	513513	123456
123456	363636	543216	123456	246246	123456
$2x + y \equiv c$	$2x + 3y \equiv c$	$2x + 5y \equiv c$	$3x + y \equiv c$	$3x + 2y \equiv c$	$y \equiv c.$

Clearly the rows of a matrix are a special case of a generalized diagonal, with $a=1$, $b=0$; the columns have $a=0$, $b=1$; and the principal diagonals have $a=1$, $b=1$, $c=1$ and $a=1$, $b=-1$, $c=m$.

The object of this note is to give a short proof of the fact that *a square has equal sums along all of its generalized diagonals if and only if all of its elements are equal.* In other words, only trivial squares can be "magic" with respect to all generalized diagonals. A proof was given by Rosser and Walker [1] when m is prime, who also stated that "This theorem is true for squares of any order. The proof is very complicated, so it is not included here. A typewritten copy of the proof has been deposited in the Cornell University library $\cdots$ "

More generally we have the following result for n-dimensional arrays of numbers:

THEOREM 1. *The m^n complex numbers $A(x_1, \cdots, x_n)$, for $1 \leq x_1, \cdots, x_n \leq m$, satisfy the condition*

$$(1)\qquad \sum \{A(x_1, \cdots, x_n): \quad a_1x_1 + \cdots + a_nx_n \equiv c(\text{mod } m)\} = s,$$

for all sets of integers $a_1, \cdots, a_n, c$ *with* $gcd(a_1, \cdots, a_n) = 1$, *if and only if*

$$(2)\qquad A(x_1, \cdots, x_n) = sm^{1-n} \qquad \textit{for all } x_1, \cdots, x_n.$$

Proof. A proof that (1) implies (2) can be based on the idea of a "finite Fourier transform." Let $\omega = e^{2\pi i/m}$, and define

$$(3)\qquad a(t_1, \cdots, t_n) = \sum_{1 \leqq x_1, \cdots, x_n \leqq m} \omega^{-\beta} A(x_1, \cdots, x_n),$$

where $\beta = (t_1x_1 + \cdots + t_nx_n)$. It follows that, when $1 \leqq x_1, \cdots, x_n \leqq m$,

$$(4)\quad A(x_1, \cdots, x_n) = 1/m^n \sum_{1 \leqq t_1, \cdots, t_n \leqq m} \omega^{(\alpha t)} a(t_1, \cdots, t_n), \alpha(t) = (t_1x_1 + \cdots + t_n x_n)$$

since the latter sum is

$$\sum_{1 \leqq t_1, \cdots, t_n \leqq m} \sum_{1 \leqq y_1, \cdots, y_n \leqq m} \omega^{\beta(t, y)} A(y_1, \cdots, y_n),$$

where $\beta(t, y) = (t_1(x_1 - y_1) + \cdots + t_n(x_n - y_n))$; and $\sum_{1 \leqq t_j \leqq m} \omega^{t_j(x_j - y_j)}$ is a geometric series whose value is 0 unless $x_j \equiv y_j$ (modulo m). Now (1) implies, when $(t_1, \cdots, t_n) = (t_1'd, \cdots, t_n'd)$ and $\gcd(t_1', \cdots, t_n') = 1$, that

$$a(t_1, \cdots, t_n) = \sum_{1 \leqq c \leqq m} \sum \{\omega^{-dc} A(x_1, \cdots, x_n) : t_1'x_1 + \cdots + t_n'x_n \equiv c \ (\text{mod } m\}$$

$$= s \sum_{1 \leqq c \leqq m} \omega^{-dc} = \begin{cases} ms, & \text{if } d \equiv 0 \ (\text{mod } m) \\ 0, & \text{if } d \not\equiv 0 \ (\text{mod } m) \end{cases}$$

In other words, $a(t_1, \cdots, t_n)$ is zero except when $t_1 \equiv \cdots \equiv t_n \equiv 0$ (modulo m). Therefore the sum in (4) reduces to the single term with $t_1 = \cdots = t_n = m$, and (2) is immediate.

To prove that (2) implies (1), we must show the congruence

$$(5)\qquad a_1x_1 + \cdots + a_nx_n \equiv c \ (\text{mod } m), 1 \leqq x_1, \cdots, x_n \leqq m$$

has precisely m^{n-1} solutions when $\gcd(a_1, \cdots, a_n) = 1$, regardless of the choice of c. Since there are integers $a_1', \cdots, a_n'$ such that $a_1a_1' + \cdots + a_na_n' = 1$, we may take $x_1 \equiv ca_1', \cdots, x_n \equiv ca_n'$ to obtain one solution of (5), and then all solutions are obtained by adding solutions of (5) with $c = 0$ to this particular solution. Therefore there are the same number of solutions for $c = 0, 1, \cdots, m-1$, and this number must be m^{n-1} since m^n combinations of $x_1, \cdots, x_n$ are possible.

This completes the proof of the theorem. Theorem 1 is the discrete analogue of results on continuous functions obtained by Rényi (see [2], [3]).

Let us now consider whether Theorem 1 has hypotheses that are too strong, i.e. if it is possible to show the array has constant values if we insist only on constant sums along certain of the generalized diagonals. The next result gives

further information which follows from a slightly deeper analysis of the structure of generalized diagonals:

THEOREM 2. *Let* $\omega = e^{2\pi i/m}$, *and for* $1 \leqq x_1, \cdots, x_n \leqq m$ *let*

$$A(x_1, \cdots, x_n) = \omega^{\beta(x)} \qquad \beta(x) = t_1x_1 + \cdots + t_nx_n \tag{6}$$

where $t_1, \cdots, t_n$ *are integers with* $\gcd(t_1, \cdots, t_n) = 1$. *Then the sum*

$$\sum \{A(x_1, \cdots, x_n): \quad a_1x_1 + \cdots + a_nx_n \equiv c \pmod{m}\} = 0, \tag{7}$$

for all sets of integers $a_1, \cdots, a_n, c$ *with* $\gcd(a_1, \cdots, a_n) = 1$, *except when*

$$(a_1, \cdots, a_n) \equiv (\xi t_1, \cdots, \xi t_n) \mod m \tag{8}$$

for some integer ξ *relatively prime to* m.

Proof. Consider first the case when $m = p^e$ is a prime power. Then some a_k, say a_1, is not a multiple of p, and we can find b so that $a_1b \equiv 1$ (modulo m). The sum (7) is therefore

$$\sum_{1 \leqq x_2, \cdots, x_n \leqq m} \omega^{\eta(x)}, \qquad \eta(x) = t_1bc + (t_2 - bt_1a_2)x_2 + \cdots + (t_n - bt_1a_n)x_n.$$

Since ω is a primitive mth root of unity, this sum vanishes unless

$$t_2 \equiv bt_1a_2, \cdots, t_n \equiv bt_1a_n \pmod{m}.$$

And in this case t_1 cannot be a multiple of p, since $t_1 \equiv bt_1a_1$ and $\gcd(t_1, \cdots, t_n) = 1$; so (8) holds if $\xi bt_1 \equiv 1$ (modulo m).

Now if m is not a prime power, let the canonical factorization of m into primes be

$$m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}, \tag{9}$$

and for $1 \leqq j \leqq r$ determine constants b_j by the Chinese remainder theorem such that

$$b_j \equiv 1 \pmod{p_j^{e_j}}; \qquad b_j \equiv 0 \pmod{p_k^{e_k}}, \qquad k \neq j. \tag{10}$$

Then for any integer y,

$$y \equiv y_j \pmod{p_j^{e_j}}, 1 \leqq j \leqq r, \text{ implies } y \equiv b_1y_1 + b_2y_2 + \cdots + b_ry_r \pmod{m}. \tag{11}$$

It follows that we can write the sum in (7) as

$$\sum \omega_1^{t_1x_{11}+\cdots+t_nx_{n1}} \omega_2^{t_1x_{12}+\cdots+t_nx_{n2}} \cdots \omega_r^{t_1x_{1r}+\cdots+t_nx_{nr}}, \tag{12}$$

where $\omega_j = \omega^{b_j}$ and where the sum is over all sets of integers x_{ij} such that $1 \leqq x_{ij} \leqq p_j^{e_j}$, and $a_1x_{1j} + a_2x_{2j} + \cdots + a_nx_{nj} \equiv c$ (modulo $p_j^{e_j}$). Therefore the sum (12) is $S_1S_2 \cdots S_r$, where each S_j is a sum like (7) for the case $m = p_j^{e_j}$ with ω_j substituted for ω. By (10), ω_j is a primitive $p_j^{e_j}$-th root of unity; so if none of the S_j

are zero, we know by the previous argument that there exist integers $\xi_1, \xi_2, \cdots, \xi_r$ such that

$$(a_1, \cdots, a_n) \equiv (\xi_j t_1, \cdots, \xi_j t_n) \bmod p_j^{e_j}, \qquad 1 \leqq j \leqq r.$$

Finally, let $\xi = b_1\xi_1 + \cdots + b_r\xi_r$ to obtain (8), as desired.

Theorem 2 can be interpreted in the following way: Let us say any set of integers $a_1, \cdots, a_n$ with $\gcd(a_1, \cdots, a_n) = 1$ determines a *family* of generalized diagonals, namely the diagonals $a_1x_1 + \cdots + a_nx_n \equiv c$ for various values of c. The families determined by $(a_1, \cdots, a_n)$ and $(t_1, \cdots, t_n)$ are *equivalent* if (8) holds for some ξ prime to m; equivalent families are in fact essentially equal. Our matrix example above shows the 6 possible families when $m=5$ and the 12 possible families when $m=6$. (The number of families is $m(1+p_1^{-1}) \cdots (1+p_r^{-1})$ when m has the form (9) and $n=2$.)

The content of Theorem 2 is that there are arrays $A(x_1, \cdots, x_n)$ which do not have all elements equal, but which have constant sums on all generalized diagonals except for those in one family. Therefore Theorem 1 is "best possible" in the sense that we cannot prove (2) if we leave any one family of diagonals out of condition (1).

Since (7) is a system of linear equations in the $A(x_1, \cdots, x_n)$ with integer coefficients, and since (6) is a nonzero solution in terms of complex values, there must exist integer-valued solutions of (7). For example, the 6×6 square

123321
211233
332112
123321
211233
332112

is "magic" with respect to each of the 72 possible generalized diagonals, except for 4 of the diagonals belonging to the family $2x+5y\equiv c \pmod 6$.

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References

1. Barkley Rosser and R. J. Walker, The algebraic theory of diabolic magic squares, Duke Math. J., 5 (1939) 705–728.

2. A. Rényi, On projections of probability distributions, Acta Math. Sci. Hungar., 3 (1952) 131–142.

3. John W. Green, On the determination of a function in the plane by its integrals over straight lines, Proc. AMS, 9 (1959) 758–762.

MATHEMATICAL NOTES

PERIODIC MAPS VIA EQUICONTINUITY

GERALD JUNGCK, Bradley University

Introduction. A map $g: X \to X$ of a metric space (X, d) into itself is said to be nonexpansive (contractive) iff

$$d(g(x), g(y)) \leqq d(x, y)(d(x, y) > d(g(x), g(y)))$$

for all x, y in X, and g is ϵ-nonexpansive (ϵ-contractive) iff the inequalities hold whenever $d(x, y) < \epsilon$. In each of these instances the family $G = \{g^n : n \in N\}$, where N denotes the set of positive integers, is equicontinuous on X.

Let us say that a map g of a metric space X into itself is *composition equicontinuous* (c.e.) iff the family G of compositions of g is equicontinuous. By the assertion that g is periodic on X we shall mean that there exists a positive integer k such that $g^k(x) = x$ for all x in X.

In this note we present two theorems pertaining to c.e. maps on compact metric spaces. Theorem 1 provides a sufficiency condition for the existence of periodic points, and has as a corollary a result by Edelstein [1] on ϵ-contractive maps. Theorem 2 yields a characterization of periodic maps in terms of c.e. maps and pseudo-expansive maps—a concept we define later.

We require the following lemma which we state without proof, since it is a consequence of a theorem of Edrei [2].

LEMMA. *If g is a c.e. map of a compact metric space (X, d) onto itself, then g is a homeomorphism. Moreover, for any $\epsilon > 0$, there is a positive integer k such that $d(x, g^k(x)) < \epsilon$ for all x in X.*

Initial result and corollaries. Our first result, which to our knowledge is new, involves the concept of commuting maps. Maps $f, g: X \to X$ are said to commute iff $f(g(x)) = g(f(x))$ for all x in X.

THEOREM 1. *Let g be a c.e. map of a compact metric space (X, d) into itself, and let F be the family of all maps $f: X \to X$ which commute with g. If $\exists \epsilon > 0$ such that $0 < d(x, y) < \epsilon$ implies that*

$$d(g(x), g(y)) < d(f(x), f(y)) \text{ for at least one } f \in F,$$

then g is periodic on the nonvoid set $A = \bigcap \{g^n(X) : n \in N\}$.

Proof. Since g is continuous and X is a compact metric space, the set A is nonvoid and compact, and $g(A) = A$. Consequently, the restriction of g to A satisfies the hypothesis of the Lemma, and $\exists k$ in N such that for all $x \in A$, $d(x, g^k(x)) < \epsilon$. Since g^k is continuous on the compact set A, $\exists a \in A$ such that

$$d(x, g^k(x)) \leqq d(a, g^k(a)) < \epsilon \quad \text{for all } x \text{ in } A. \tag{1}$$

Now since $g(A)=A$, there is an element b in A such that $g(b)=a$. But $f(A)\subset A$ for each f in F, since g and f commute. Thus $f(b)\in A$ for all $f\in F$, and (1) therefore yields:

$$d(a, g^k(a)) = d(g(b), g^k(g(b))) \geqq d(f(b), g^k(f(b))) \quad \text{for all } f \in F.$$

Consequently, if $f\in F$,

$$d(g(b), g(g^k(b))) = d(g(b), g^k(g(b))) \geqq d(f(b), f(g^k(b)))$$

since f and g commute. Since $b\in A$, we have:

$$d(b, g^k(b)) < \epsilon, \quad \text{and} \quad d(g(b), g(g^k(b))) \geqq d(f(b), f(g^k(b))) \quad \text{for } all\ f \in F.$$

The hypothesis therefore demands that $b=g^k(b)$, and hence $a=g(b)=g^k(g(b))=g^k(a)$. In view of (1), the theorem is proved.

To appreciate the scope of Theorem 1, one should note that there is no requirement that distinct pairs x, y and a, b be served by the same f in F.

Clearly the set $\{g^n: n\in N_0\}\subset F$, where $N_0=N\cup\{0\}$ and $g^0=i$, the identity map. We thus have:

COROLLARY 1. *Let g be a c.e. map of a compact metric space (X, d) into itself. If there exists an $\epsilon>0$ such that*

$$0 < d(x, y) < \epsilon \text{ implies that } d(g(x), g(y)) < d(g^n(x), g^n(y))$$

for at least one n in N_0, then g is periodic on the nonvoid set A.

Since, as noted in the introduction, an ϵ-contractive map is c.e., the following is an immediate consequence of Corollary 1.

COROLLARY 2. (Edelstein) *Any ϵ-contractive map of a compact metric space into itself has a periodic point.*

Pseudo-expansive maps: motivation and definition. Now any continuous periodic map g of a compact metric space is c.e., but there do exist maps—e.g., rotations of a circle—which are c.e. but not necessarily periodic. We are thus led to ask: "What restrictions on g in conjunction with composition equi-continuity will yield a necessary and sufficient criterion for periodicity?" We conclude with an answer.

To motivate, let g be a homeomorphism of a metric space (X, d) onto itself. g is said to be *expansive* iff there is a positive real number c (called an "expansive constant") such that $x\neq y$ implies that $d(g^n(x), g^n(y))>c$ for some integer n. An expansive homeomorphism certainly need not be periodic. Indeed, it is known [3] that if (X, d) is compact and $g: X\to X$ is an expansive homeomorphism, then g has at most a countable number of periodic points. On the other hand, it is clear that no c.e. map—and hence, no periodic map—of an infinite compact metric space can be expansive.

Nevertheless, the first corollary to Theorem 1 suggests that a discrete modi-

fication of the expansive map concept should produce the required restriction. Hence, the following definition.

DEFINITION. *Let* (X, d) *be a bounded metric space and let* ρ *be the supremum metric for the set* F *of all maps of* X *into* X; *i.e., for* f *and* h *in* F,

$$\rho(f, h) = \sup \{d(f(x), h(x)) : x \in X\}.$$

We shall say that a map g *of* X *into itself is pseudo-expansive iff there is a real number* $c > 0$ *such that* $\rho(g^m, g^n) > c$ *whenever* $g^m \neq g^n (m, n \in N_0)$.

Clearly, any expansive homeomorphism g of a bounded metric space (X, d) onto itself is pseudo-expansive. For if g has expansive constant c and $g^m \neq g^n$, it is immediate that $\rho(g^m, g^n) > c$. In fact, Bryant [4] proved that in the event (X, d) is compact and infinite, $\rho(g^m, g^n) > c$ whenever $m \neq n$.

On the other hand, it is known [4] that no arc can carry an expansive homeomorphism. It is also known (and follows easily from the lemma) that any c.e. map of the unit interval onto itself is periodic. Consequently, we know immediately that the following homeomorphism is neither expansive nor c.e.; but it is pseudo-expansive.

Example 1. Let $g: I \to I$, where $g(x) = x^2$ for $x \in I = [0, 1]$. If $n, k \in N$ and $a = 2^{-1/2^n}$, then

$$| g^n(a) - g^{n+k}(a) | = 1/2 - 1/2^{2^k} \geqq 1/4.$$

Hence, $\rho(g^n, g^m) \geqq 1/4$ if $g^n \neq g^m$, so that g is pseudo-expansive.

Of course, a pseudo-expansive map may not even be one-one. Consider:

Example 2. Let $g: I \to I$, where $g(x) = 2x$ for $0 \leqq x \leqq 1/2$ and $g(x) = 1$ when $1/2 < x \leqq 1$. (We omit details.)

Final result. We now offer a characterization of periodic maps on compact metric spaces.

THEOREM 2. *Let* g *be a continuous map of a compact metric space* (X, d) *onto itself. Then* g *is periodic on* X *if and only if* g *is c.e. and pseudo-expansive.*

Proof (*Sufficiency*). Since g is pseudo-expansive, $\exists c > 0$ such that $g^m = g^n$ if $\rho(g^m, g^n) \leqq c$ and $m, n \in N_0$. But since g is c.e., the lemma yields $k \in N$ such that $\rho(g^0, g^k) = \rho(i, g^k) \leqq c$. Hence $i = g^k$, and $x = g^k(x)$ for all x in X.

(*Necessity*). Since g is continuous and periodic, g (as noted before) is c.e. (We assume that $g \neq i$, since the identity map is trivially pseudo-expansive.) To see that g is pseudo-expansive, let k be the least positive integer such that $i = g^k$, so that

$$g^m \in S = \{g^n : 0 \leqq n < k\} \quad \text{for each } m \text{ in } N_0. \tag{2}$$

But $S \times S$ is finite since S is finite, and so there exists $c > 0$ such that $c = \min \{\rho(f, h) : f, h \in S \text{ and } f \neq h\}$. Thus if $n, m \in N_0$ and $g^n \neq g^m$, we know by (2) that $\rho(g^m, g^n) > c/2$. g is therefore pseudo-expansive.

References

1. Michael Edelstein, On fixed points and periodic points under contractive mapping, J. London Math. Soc., 37 (1962) 74–79.

2. A. Edrei, On iteration of mappings of a metric space onto itself, J. London Math. Soc., 26 (1951) 96–103.

3. W. R. Utz, Unstable homeomorphisms, Proc. Amer. Math. Soc., 1 (1950) 769–774.

4. B. F. Bryant, Expansive self-homeomorphisms of a compact metric space, this MONTHLY, 69 (1962) 386–391.

A FIRST INTEGRAL OF THE GENERAL EULER DIFFERENTIAL EQUATION

C. J. BROOKES, The University of Aston in Birmingham, England

1. Introduction. A fundamental theorem of the Calculus of Variations may be stated as follows [1]:

THEOREM. *The integral*

$$\int_{x_1}^{x_2} F(x, y_1, y_1', \cdots, y_1^{(n_1)}, y_2, y_2', \cdots, y_2^{(n_2)}, \cdots, y_m, y_m', \cdots, y_m^{(n_m)})dx$$

whose end points are fixed, is stationary for weak variations when the y_r's satisfy the m equations

$$\frac{\partial F}{\partial y_r} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_r'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y_r''}\right) - \cdots + (-1)^{n_r}\frac{d^{n_r}}{dx^{n_r}}\left(\frac{\partial F}{\partial y_r^{(n_r)}}\right) = 0 \tag{1}$$

$(r=1, 2, \cdots, m)$ *where the variables y_r are assumed independent of each other and $y_r^{(n_r)} \equiv d^{n_r}y_r/dx^{n_r}$.*

If the function F does not contain x explicitly and $F \equiv F(y, y')$, a first integral of the appropriate Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

is known to be [2] $F = y'(\partial F/\partial y') + \text{constant}$.

For the case $F \equiv F(y, y', y'')$ it is easy to show that a first integral of the appropriate Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

has the form

$$F = y'\left\{\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right\} + y''\frac{\partial F}{\partial y''} + \text{constant}.$$

However, with x explicitly absent in the general case, i.e.

$$F \equiv F(y_1, y_1', \cdots, y_1^{(n_1)}, y_2, y_2', \cdots, y_2^{(n_2)}, \cdots, y_m, y_m', \cdots, y_m^{(n_m)}),$$

a first integral is apparently unknown.

The purpose of this paper is to obtain such a first integral.

2. A first integral. We begin by re-writing equations (1) in the form

$$(2)\qquad \begin{aligned} \frac{\partial F}{\partial y_r} &= \sum_{i=1}^{n_r} (-1)^{i-1} \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(i)}}\right), \qquad r = 1, 2, \cdots, m, \\ &= \frac{d}{dx} \sum_{i=0}^{n_r-1} (-1)^i \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(i+1)}}\right), \qquad r = 1, 2, \cdots, m. \end{aligned}$$

Also, we have

$$\begin{aligned} &\frac{d}{dx} F(y_1, y_1', \cdots, y_1^{(n_1)}, y_2, y_2', \cdots, y_2^{(n_2)}, \cdots, y_m, y_m', \cdots, y_m^{(n_m)}) \\ &= \sum_{r=1}^{m} \left\{ y_r' \frac{\partial F}{\partial y_r} + y_r'' \frac{\partial F}{\partial y_r'} + y_r''' \frac{\partial F}{\partial y_r''} + \cdots + y_r^{(n_r+1)} \frac{\partial F}{\partial y_r^{(n_r)}} \right\} \\ &= \sum_{r=1}^{m} y_r' \frac{d}{dx} \sum_{i=0}^{n_r-1} (-1)^i \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(i+1)}}\right) + \sum_{r=1}^{m}\sum_{i=1}^{n_r} y_r^{(i+1)} \frac{\partial F}{\partial y_r^{(i)}}, \quad \text{from (2)} \\ &= \sum_{r=1}^{m} \frac{d}{dx}\left\{ y_r' \sum_{i=0}^{n_r-1} (-1)^i \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(i+1)}}\right)\right\} + \sum_{r=1}^{m}\sum_{i=2}^{n_r} y_r^{(i+1)} \frac{\partial F}{\partial y_r^{(i)}} \\ &\quad + \sum_{r=1}^{m} y_r'' \sum_{i=1}^{n_r-1} (-1)^{i-1} \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(i+1)}}\right) \end{aligned}$$

$$(3)\qquad \begin{aligned} &= \frac{d}{dx} \sum_{r=1}^{m} \left\{ y_r' \sum_{i=0}^{n_r-1} (-1)^i \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(i+1)}}\right)\right\} + \sum_{r=1}^{m}\sum_{i=2}^{n_r} \left\{ y_r^{(i+1)} \frac{\partial F}{\partial y^{(i)}} \right. \\ &\quad \left. + (-1)^i y_r'' \frac{d^{i-1}}{dx^{i-1}}\left(\frac{\partial F}{\partial y_r^{(i)}}\right)\right\}. \end{aligned}$$

Considering now, the second double summation on the right-hand side of equation (3), it can be shown that

$i = 2$

$$y_r''' \frac{\partial F}{\partial y_r''} + y_r'' \frac{d}{dx}\left(\frac{\partial F}{\partial y_r''}\right) \equiv \frac{d}{dx}\left\{ y_r'' \frac{\partial F}{\partial y_r''}\right\}$$

$i = 3$

$$y_r^{(\mathrm{IV})} \frac{\partial F}{\partial y_r'''} - y_r'' \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y_r'''}\right) \equiv \frac{d}{dx}\left\{ y_r''' \frac{\partial F}{\partial y_r'''} - y_r'' \frac{d}{dx}\left(\frac{\partial F}{\partial y_r'''}\right)\right\}$$

$i = 4$

$$y_r^{(\mathrm{V})} \frac{\partial F}{\partial y_r^{(\mathrm{IV})}} + y_r'' \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y_r^{(\mathrm{IV})}}\right) \equiv \frac{d}{dx}\left\{y_r^{(\mathrm{IV})} \frac{\partial F}{\partial y_r^{(\mathrm{IV})}} - y_r''' \frac{d}{dx}\left(\frac{\partial F}{\partial y_r^{(\mathrm{IV})}}\right) + y_r'' \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y_r^{(\mathrm{IV})}}\right)\right\}$$

. . . .

$i = n_r$

$$y_r^{(n_r+1)} \frac{\partial F}{\partial y_r^{(n_r)}} + (-1)^{n_r} y_r'' \frac{d^{n_r-1}}{dx^{n_r-1}}\left(\frac{\partial F}{\partial y_r^{(n_r)}}\right)$$

$$\equiv \frac{d}{dx}\left\{y_r^{(n_r)} \frac{\partial F}{\partial y_r^{(n_r)}} - y_r^{(n_r-1)} \frac{d}{dx}\left(\frac{\partial F}{\partial y_r^{(n_r)}}\right) + \cdots + (-1)^{n_r} y_r'' \frac{d^{n_r-2}}{dx^{n_r-2}}\left(\frac{\partial F}{\partial y_r^{(n_r)}}\right)\right\}$$

$$\equiv \frac{d}{dx} \sum_{i=2}^{n_r} (-1)^i y_r^{(n_r-i+2)} \frac{d^{i-2}}{dx^{i-2}}\left(\frac{\partial F}{\partial y_r^{(n_r)}}\right),$$

so that addition gives

$$\sum_{i=2}^{n_r} \left\{y_r^{(i+1)} \frac{\partial F}{\partial y_r^{(i)}} + (-1)^i y_r'' \frac{d^{i-1}}{dx^{i-1}}\left(\frac{\partial F}{\partial y_r^{(i)}}\right)\right\} \equiv \frac{d}{dx} \sum_{j=0}^{n_r-2} \sum_{i=0}^{n_r-j-2} (-1)^i y_r^{(n_r-i-j)} \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(n_r-j)}}\right).$$

Integrating throughout equation (3) with respect to x now yields the required first integral, viz:

$$F = \sum_{r=1}^{m} \left\{y_r' \sum_{i=0}^{n_r-1} (-1)^i \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(i+1)}}\right) + \sum_{j=0}^{n_r-2} \sum_{i=0}^{n_r-j-2} (-1)^i y_r^{(n_r-i-j)} \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y_r^{(n_r-j)}}\right)\right\} + \text{constant}.$$

References

1. L. E. Elsgolc, Calculus of Variations, Pergamon Press, Long Island City, 1963, pp. 44–47.
2. O. Bolza, Lectures on the Calculus of Variations, University of Chicago Press, 1904, p. 27.

AN ELEMENTARY CONSTRUCTION IN SHEAF THEORY

HARRY GONSHOR, Rutgers, The State University

Since sheaf theory is commonly regarded as being high-powered and requiring a great deal of machinery, it may be of pedagogical interest to give an example of a proof which requires nothing but "mathematical maturity." Specifically we shall embed a presheaf satisfying the uniqueness property into a sheaf,

using neither *étale* spaces nor category theory. In fact, this note will be entirely self-contained using nothing but facility in working with sets of various types. Although, as is customary, we shall speak of a set function defined on the class of all open sets of a topological space, the role of the topology is entirely incidental. All that is really needed is a set with a class of subsets which is closed under finite intersection.

By definition, a presheaf is a set function F defined on the class of open sets of a topological space together with a map $G(V, U)$ from $F(U)$ into $F(V)$ for every pair U, V such that $U \supset V$ satisfying

(1) $G(U, U)$ is the identity,

(2) $G(W, V)\, G(V, U) = G(W, U)$ if $U \supset V \supset W$.

We consider here presheaves that satisfy the following uniqueness axiom:

If $\{U_\alpha\}$ *is a covering of* U *and* $G(U_\alpha, U)(a) = G(U_\alpha, U)(b)$ *for all* α *then* $a = b$.

We shall embed such a presheaf into a presheaf satisfying the above condition and the following existence axiom:

If $\{U_\alpha\}$ *is a covering of* U, $a_\alpha \in F(U_\alpha)$ *and for all pairs* (α, β) $G(U_\alpha \cap U_\beta, U_\alpha) \cdot (a_\alpha) = G(U_\alpha \cap U_\beta, U_\beta)(a_\beta)$ *then there exists* $a \in F(U)$ *such that* $G(U_\alpha, U)(a) = a_\alpha$ *for all* α.

A presheaf satisfying the two above axioms is called a sheaf. Many presheaves obtained in practice satisfy the uniqueness but not the existence axiom.

Let $F'(U)$ be the class each element of which is a set of pairs $[U_\alpha, a_\alpha]$ where $\{U_\alpha\}$ is a covering of U, $a_\alpha \in F(U_\alpha)$ and for all α, β $G(U_\alpha \cap U_\beta, U_\alpha)(a_\alpha) = G(U_\alpha \cap U_\beta, U_\beta)(a_\beta)$. We define an equivalence relation in $F'(U)$ as follows:

$[U_\alpha, a_\alpha] \sim [V_\beta, b_\beta]$ if and only if for all pairs α, β $G(U_\alpha \cap V_\beta, U_\alpha)(a_\alpha) = G(U_\alpha \cap V_\beta, V_\beta)(b_\beta)$.

The proof that this is an equivalence relation is straight-forward. Note that the uniqueness axiom is used to verify the transitive law.

We define $\overline{F}(U)$ as the set of equivalence classes obtained from the above equivalence relation.

Next we define $\overline{G}(V, U)$ from $\overline{F}(U)$ into $\overline{F}(V)$ as follows: Suppose $\bar{a} \in \overline{F}(U)$ and $[U_\alpha, a_\alpha]$ is a representation of $\bar{a}$. Then $\overline{G}(V, U)(\bar{a})$ is the class containing $[U_\alpha \cap V, G(U_\alpha \cap V, U_\alpha)(a_\alpha)]$. This is clearly in $F'(V)$.

We must show that the map is well defined. Let $[U_\alpha, a_\alpha] \sim [U'_\beta, a_\beta]$. Then $G(U_\alpha \cap U'_\beta, U_\alpha)(a_\alpha) = G(U_\alpha \cap U'_\beta, U'_\beta)(a_\beta)$. We are required to show that

$$[U_\alpha \cap V, G(U_\alpha \cap V, U_\alpha)(a_\alpha)] \sim [U'_\beta \cap V, G(U'_\beta \cap V, U'_\beta)(a_\beta)].$$

This follows from an easy computation.

It is obvious that $\overline{G}(U, U)$ is the identity and that $\overline{G}(W, V)\overline{G}(V, U) = \overline{G}(W, U)$. It is also easy to see that the uniqueness axiom is satisfied. In fact these follow immediately from the corresponding properties for G.

The alert reader (in fact this requires only semi-alertness) will anticipate the construction of the natural map from F into $\overline{F}$ and G into $\overline{G}$.

As a covering of U we take U itself (or pedantically, the unit set containing U). Then $a\in F(U)$ is mapped into the class containing $[U, a]$. Clearly $[U, a] \sim [U, b] \rightarrow a=b$ by property (1) for presheaves. It is also clear that this map commutes with the G's.

Before proving that the existence axiom is satisfied we state a trivial but important lemma.

LEMMA. *Let $\{U_\alpha\}$ be a covering of U and let $\{V_\beta\}$ be a refinement of U_α. Given β choose α so that $V_\beta \subset U_\alpha$. Then $G(V_\beta, U_\alpha)(a_\alpha)$ depends only on β and $[V_\beta, G(V_\beta, U_\alpha)(a_\alpha)] \sim [U_\alpha, a_\alpha]$. In particular, if V_β is a subcovering of U_α, the corresponding subset of $[U_\alpha, a_\alpha]$ is equivalent to $[U_\alpha, a_\alpha]$.*

To verify the existence axiom let $\{U_\alpha\}$ be a covering of U and suppose $\bar{a}_\alpha \in \bar{F}(U_\alpha)$ and $\bar{G}(U_\alpha \cap U_\beta, U_\alpha)(\bar{a}_\alpha) = \bar{G}(U_\alpha \cap U_\beta, U_\beta)(\bar{a}_\beta)$. As a representative of $\bar{a}_\alpha$ we have $[U_{\alpha\gamma}, a_{\alpha\gamma}]$ where $\{U_{\alpha\gamma}\}$ is a covering of U_α and $a_{\alpha\gamma} \in F(U_{\alpha\gamma})$. Furthermore

$$G(U_{\alpha\gamma} \cap U_{\alpha\delta}, U_{\alpha\delta})(a_{\alpha\delta}) = G(U_{\alpha\gamma} \cap U_{\alpha\delta}, U_{\alpha\gamma})(a_{\alpha\gamma}). \tag{1}$$

The next step is the most taxing on mathematical maturity. This involves translating the "compatibility" condition to a condition on the representatives. By definition of $\bar{G}$, as a representative of $\bar{G}(U_\alpha \cap U_\beta, U_\alpha)(\bar{a}_\alpha)$ we obtain $[U_{\alpha\gamma} \cap U_\beta, G(U_{\alpha\gamma} \cap U_\beta, U_{\alpha\gamma})(a_{\alpha\gamma})]$. Similarly using $U_{\beta\epsilon}$ as a typical set in the covering of U_β we obtain as a representative of $\bar{G}(U_\alpha \cap U_\beta, U_\beta)(\bar{a}_\beta)$ the set $[U_{\beta\epsilon} \cap U_\alpha, G(U_{\beta\epsilon} \cap U_\alpha, U_{\beta\epsilon})(a_{\beta\epsilon})]$. Translating the compatibility hypothesis we obtain:

$$\begin{aligned} G(U_{\alpha\gamma} \cap U_\beta \cap U_{\beta\epsilon} \cap U_\alpha, U_{\alpha\gamma} \cap U_\beta)\, G(U_{\alpha\gamma} \cap U_\beta, U_{\alpha\gamma})(a_{\alpha\gamma}) \\ = G(U_{\alpha\gamma} \cap U_\beta \cap U_{\beta\epsilon} \cap U_\alpha, U_{\beta\epsilon} \cap U_\alpha)\, G(U_{\beta\epsilon} \cap U_\alpha, U_{\beta\epsilon})(a_{\beta\epsilon}). \end{aligned}$$

This simplifies to

$$G(U_{\alpha\gamma} \cap U_{\beta\epsilon}, U_{\alpha\gamma})(a_{\alpha\gamma}) = G(U_{\alpha\gamma} \cap U_{\beta\epsilon}, U_{\beta\epsilon})(a_{\beta\epsilon}). \tag{2}$$

Now, clearly $\{U_{\alpha\gamma}\}$ is a covering of U if α is free to vary. As a representative of $\bar{a}$ we will take $[U_{\alpha\gamma}, a_{\alpha\gamma}]$. The two previous equalities (1) and (2) are just what are needed to show that $[U_{\alpha\gamma}, a_{\alpha\gamma}] \in F'(U)$. It remains to show that $\bar{G}(U_\alpha, U)(\bar{a}) = \bar{a}_\alpha$ for all α.

As a representative of $\bar{G}(U_\alpha, U)(\bar{a})$ we have $[U_\alpha \cap U_{\beta\epsilon}, G(U_\alpha \cap U_{\beta\epsilon}, U_{\beta\epsilon}) \cdot (a_{\beta\epsilon})]$, where β is free to vary (e.g., β may equal α). $U_\alpha \cap U_{\beta\epsilon}$ is of course a covering of U_α, and $U_\alpha \cap U_{\alpha\gamma}$ is clearly a subcovering. The last sentence of the previous lemma tells us that as another representative we may choose $[U_\alpha \cap U_{\alpha\gamma}, G(U_\alpha \cap U_{\alpha\gamma}, U_{\alpha\gamma})(a_{\alpha\gamma})]$. This is of course simply $(U_{\alpha\gamma}, a_{\alpha\gamma})$ which is nothing but our previously chosen representative of $\bar{a}_\alpha$!

We have proved that the presheaf we constructed is a sheaf. It is easy to see that in some sense this sheaf is the minimum sheaf containing the given presheaf. A precise statement of this fact and its proof is left to the reader.

A NOTE ON WELL-CHAINED SPACES

J. C. MATHEWS, Iowa State University

Introduction. A metric space (X, ρ) is well-chained iff for each $\epsilon>0$ and points $u, v \in X$ there is a finite subset $\{x_1, x_2, \cdots, x_n\}$ of X such that $x_1=u$, $x_n=v$, and $\rho(x_i, x_{i+1})<\epsilon$ for $i=1, 2, \cdots, n-1$. In this note we give several new characterizations of well-chained spaces.

The concept of well-chained is apparently due to M. Fréchet [1]. Fréchet and his contemporaries, and later Whyburn [2], made use of this concept in its role as a generalization of connectedness in the context of a metric space. It has also been found useful in several papers (see [3]–[5]) on various extensions of Banach's contraction mapping theorem.

Let us agree to call such a set as $\{x_1, x_2, \cdots, x_n\}$ an ϵ-chain from u to v. It is easy to prove that for an arbitrary metric space the set of points v such that there is an ϵ-chain from a given point u to v is both open and closed. It follows from this that each connected metric space is well-chained. Simple examples such as the set of rationals with their usual metric show that the converse is not true. In the presence of compactness, however, it is easily seen that the two concepts are equivalent. This result is also an easy corollary of the proposition that we prove.

I am indebted in what follows to D. W. Curtis for both the definition of closely separated and condition (4).

Characterizations of well-chained. We begin by recalling that (A, B) is a separation of a topological space X iff $A \cup B = X$, $A \cap B = \varnothing$, and neither A nor B is void nor contains a limit point of the other. We say that X is separated iff there is a separation of X. It is convenient to introduce the following definition: A separated metric space (X, ρ) is *closely separated* iff for each separation (A, B), $\rho(A, B)=0$.

PROPOSITION. *Let (X, ρ) be a metric space; then the following statements are equivalent:*

(1) *(X, ρ) is well-chained.*
(2) *Each uniformly continuous function $f: X \to R$ has a well-chained range.*
(3) *For $\epsilon>0$ and points $u, v \in X$, there are distinct components $C_1, C_2, \cdots, C_n$ such that $u \in C_1$, $v \in C_n$, and $\rho(C_i, C_{i+1})<\epsilon$ for $i=1, 2, \cdots, n-1$.*
(4) *(X, ρ) is either connected or closely separated.*
(5) *For each open uniform cover $\mathcal{C}$ and points $u, v \in X$, there is a finite subset $\{C_1, C_2, \cdots, C_n\}$ of $\mathcal{C}$ such that $u \in C_1$, $v \in C_n$, and $C_i \cap C_{i+1} \neq \varnothing$ for $i=1, 2, \cdots, n-1$.*

Proof. (1)$\Rightarrow$(2): Let $f(u), f(v) \in f(X)$ be distinct and let $\epsilon>0$ be given. There is $\theta>0$ and a θ-chain $\{u=x_1, x_2, \cdots, x_n=v\}$ from u to v such that $|f(x_i)-f(x_{i+1})|<\epsilon$ for $i=1, 2, \cdots, n-1$. Thus $\{f(x_1), f(x_2), \cdots, f(x_n)\}$ is an ϵ-chain from $f(u)$ to $f(v)$.

(2)$\Rightarrow$(3): We will assume that (3) is false; then there is an $\epsilon>0$ and distinct points u and v such that for no finite set of components $C_1, \cdots, C_n$, where $u\in C_1$ and $v\in C_n$, is it true that $\rho(C_i, C_{i+1})<\epsilon$ for each $i=1, 2, \cdots, n-1$. We define $H=\{w\in X$: there is an ϵ-chain from u to $w\}$ and observe that we must have $v\notin H$. Finally, if we define $f: X\rightarrow R$ by letting f be the characteristic function of H, it is clear that f is uniformly continuous and its range is not well-chained.

(3)$\Rightarrow$(4): Let (A, B) be a separation of X and choose $u\in A$ and $v\in B$. Since any component of X must be a subset of exactly one of A or B, it is a direct consequence of (3) that $\rho(A, B)=0$.

(4)$\Rightarrow$(5): We recall that a cover $\mathcal{K}$ of a metric space is a uniform cover iff there is $\theta>0$ such that each open ball of radius θ is contained in at least one member of $\mathcal{K}$. If (X, ρ) is connected, (5) follows from the result that a topological space is connected iff for each open cover $\mathcal{E}$ and points p and q, there is a finite subset $\{E_1, E_2, \cdots, E_n\}$ of $\mathcal{E}$ such that $p\in E_1$, $q\in E_n$, and $E_i\cap E_{i+1}\neq\emptyset$ for $i=1, 2, \cdots, n-1$. Thus suppose that (X, ρ) is closely separated, $\mathcal{C}$ is an open uniform cover, and u, v are distinct points of X. Let J be the set of points $w\in X$ such that there is a finite subset $\{C_1, C_2, \cdots, C_n\}$ of $\mathcal{C}$ such that $u\in C_1$, $w\in C_n$, and $C_i\cap C_{i+1}\neq\emptyset$ for $i=1, 2, \cdots, n-1$. It is easily seen that J is both open and closed. If $v\notin J$, then $(J, X-J)$ is a separation of X and consequently $\rho(J, X-J)=0$. This last condition together with the fact that $\mathcal{C}$ is a uniform cover immediately leads to a contradiction. Hence $v\in J$.

(5)$\Rightarrow$(1): This implication follows directly by considering open uniform covers consisting of all open balls of radius ϵ.

REMARKS. The example of the rationals given in the *Introduction* might raise the question of whether well-chained and completeness together imply connectedness. While this is true for subspaces of the line, the subspace of the plane consisting of the upper branch of the hyperbola $xy=1$ and the x-axis shows it is not true in general. One could ask for a characterization of those spaces for which this is true.

A second remark is that a metric space (X, ρ) may fail to be closely separated and yet there is a separation (A, B) such that $\rho(A, B)=0$.

Next we observe that the image of a well-chained space under a continuous function into a metric space may fail to be well-chained. If, however, the function is uniformly continuous, the image is well-chained. Thus the property of being well-chained is a uniform invariant.

As a final remark, the generalization of the concept of well-chained, as well as the several equivalent forms of it in the proposition, to uniform spaces is straight-forward. The definition of a well-chained uniform space $(X, \mathcal{U})$ is that for each $U\in\mathcal{U}$ and points $x, y\in X$, there is a positive integer n such that $(x, y)\in U\circ U\circ\cdots\circ U=U^n$.

References

1. M. Fréchet, Sur les ensembles abstraits, Ann. Éc. Norm. sup., 38 (1921) 341–387.
2. G. T. Whyburn, Topological analysis, Princeton Univ. Press, Princeton, N. J., 1958.

3. A. S. Davis, Fixpoint theorem for contraction of a well-chained topological space, Proc. Amer. Math. Soc., 14 (1963) 981–985.

4. M. Edelstein, On fixed and periodic points under contractive mappings, J. Lond. Math. Soc., 37 (1962) 74–79.

5. R. J. Knill, Fixed points of uniform contractions, J. Math. Anal. and Appl., 12 (1965) 449–455.

A NOTE ON FIRST ORDER SEMI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

I. I. KOLODNER AND R. N. PEDERSON, Carnegie-Mellon University

The theories of initial value problems for linear partial differential equations and for nonlinear ordinary differential equations have reached a fair degree of sophistication. By comparison, the corresponding theory for nonlinear partial differential equations is in a state of infancy. It is therefore of interest to classify those equations in the latter category for which one can obtain information by using properties of solutions of equations in the first two categories.

The authors [1] have recently obtained estimates for solutions of semi-linear parabolic equations

$$\frac{\partial u}{\partial t} - Lu = g(t, u), \qquad u(x, 0) = f(x), \tag{1}$$

where g is concave in u, by combining properties of solutions of the linear partial differential equation

$$\frac{\partial u}{\partial t} - Lu = 0, \qquad u(x, 0) = f(x), \tag{2}$$

and the ordinary differential equation

$$\frac{dv}{dt} = g(t, v), \qquad v(0) = q. \tag{3}$$

It was shown that, under mild regularity assumptions, if $u(x, t; f)$ and $v(t; q)$ are the solutions of (2) and (3), then the compositions $u(x, t; v(t; f(\cdot)))$ and $v(t; u(x, t; f))$ are pointwise lower and upper bounds to the solution of (1).

Perhaps a few remarks on the above notation are in order. The symbol $u(x, t; f)$ represents the value of a function which depends on the point (x, t) and on the function f; that is, it is a functional of f. The expression $v(t; f(\cdot))$, for fixed t, denotes a function, with the appropriate domain of definition, and therefore is a fit candidate for substitution into $u(x, t; \cdot)$. On the other hand, $v(t; q)$ is a function of the pair t and q. The substitution of $u(x, t; f)$ into $v(t; \cdot)$ is therefore a meaningful operation.

It is the purpose of this note to point out that the solutions of the semi-linear first order equations in n-space

(4)
$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} a_i(x, t) \frac{\partial u}{\partial x_i} = g(t, u), \qquad u(x, 0) = f(x),$$

are given exactly by either of the above indicated procedures. Furthermore, no assumption of concavity of g is needed; it is only required that solutions of (4) be unique. The proof makes use of the fact from the theory of characteristics, (see [2]) that the solution of

(5)
$$\frac{\partial w}{\partial t} + \sum_{i=1}^{n} a_i(x, t) \frac{\partial w}{\partial x_i} = 0, \qquad w(x, 0) = f(x),$$

$a_i, f \in C^1$, is given by $w(x, t; f) = f(\phi(x, t))$, where $\phi(x, t)$ is the function whose level surfaces satisfy the system of ordinary differential equations

$$\frac{dx_i}{dt} = a_i(x, t), \qquad \phi(x, 0) = x.$$

It is then easily seen that $w(x, t; v(t, f(\cdot)))$ and $v(t; w(x, t; f))$ are both equal to $v(t; f(\phi(x, t)))$. Indeed, the prescription for computing $w(x, t; v(t; f(\cdot)))$ is to substitute $\phi(x, t)$ into $v(t; f(\cdot))$, the result being $v(t; f(\phi(x, t)))$. A less subtle argument gives the same result for $v(t; w(x, t; f))$. That $v(t; f(\phi(x, t)))$ is the solution of (4) is a simple exercise in differentiation together with an application of the uniqueness theorem for solutions of (4).

Another way of looking at the result of the previous paragraph is that the operation of solving (5) with the solution of (3) as initial condition is identical with the operation of solving (3) with the solution of (5) as initial condition. That is, the indicated composition is commutative.

References

1. I. I. Kolodner, and R. N. Pederson, Pointwise bounds for solutions of semi-linear parabolic equations, Journal of Differential Equations, 4(1967)353–364.

2. R. Courant, and D. Hilbert, Methods of Mathematical Physics, Vol. 2, Interscience Publ., New York, 1960, 69–75.

GEODESICS ON SURFACES OF REVOLUTION

M. S. KLAMKIN, Ford Motor Company, Dearborn, Michigan

It is intuitive that the geodesic between two arbitrary points on a surface of revolution, which are coplanar with the axis and on the same side, is also coplanar with the axis. This can be easily established without recourse to variational methods. For the case when the two points lie in a plane perpendicular to the axis, it is shown using variational methods, that the geodesic will only lie in the perpendicular plane if the surface is a circular cylinder.

In spherical coordinates, we can take the equation of the surface of revolution as $r = F(\phi)$ (here the axis of revolution is the z-axis). The metric is

$$ds^2 = dr^2 + r^2 \sin^2 \phi d\theta^2 + r^2 d\phi^2.$$

Since the latter is invariant under rotations, we can assume without loss of generality that the two given points lie in the plane $\theta = 0$. Since we can represent a curve on the surface by the equations $r = F(\phi)$, $\theta = G(\phi)$, the distance along the curve between the points $\phi = \phi_1$ and $\phi = \phi_2$ is given by

$$L = \int_{\phi_1}^{\phi_2} \sqrt{F'^2(\phi) + F^2(\phi) + F^2(\phi)G'^2(\phi) \sin^2 \phi} \; d\phi.$$

The geodesic is the curve on the surface $r = F(\phi)$ which minimizes L. Since the minimum value of the integrand occurs for $\theta' = 0$ and $\theta = 0$ satisfies the boundary conditions, it is now obvious that L is minimized for this θ value.

In particular, the geodesics of a sphere are great circles. Also, since a plane is a surface of revolution about any axis perpendicular to the plane, it follows that the geodesics in the plane are all straight lines.

Many texts derive the above geodesics as illustrative examples in the calculus of variations as they are nice ones. However, it should be indicated that they could be derived more simply.

We now consider geodesics between two arbitrary points which lie in a plane perpendicular to the axis. For a cylinder, the geodesics are circles in the same plane. This follows immediately since the cylinder is developable. For a sphere, however, the latter result is not valid except for equatorial planes since the geodesics are great circles and not circles of latitude.

To show that the cylinder is the only surface of revolution with the latter property, assume that the two points are given by $\phi = \phi_0$, $\theta = \theta_1$ and $\phi = \phi_0$, $\theta = \theta_2$. Then the distance along the curve between them is given by

$$L = \int_{\theta_1}^{\theta_2} \sqrt{[F'^2(\phi) + F^2(\phi)]\phi'^2 + F^2(\phi) \sin^2 \phi} \; d\theta$$

(here $\phi' = d\phi/d\theta$). Since the corresponding Euler-Lagrange equation

$$\frac{d}{d\theta} \frac{\partial I}{\partial \phi'} - \frac{\partial I}{\partial \phi} = 0$$

must be satisfied identically for $\phi = \phi_0$ (ϕ_0-arbitrary), we obtain after differentiation that

$$F'(\phi) \sin \phi + F(\phi) \cos \phi = 0.$$

Integrating, yields $F = a/\sin \phi$, which implies that the surface is a right circular cylinder.

Note added in proof. A more elegant proof of the latter result has been suggested by G. D. Chakerian. Since we have two families of geodesics intersecting at right angles (the meridians and the circles orthogonal to them), the surface is developable [1]. It is known that the only developable surfaces of revolution are the right circular cone and cylinder. But obviously the cone is ruled out.

Alternatively, the more general result "If every geodesic on a surface of revolution cuts the

meridians at a constant angle, the surface is a right cylinder" appears as an exercise [1, p. 122] whose proof follows from Theorem 7 [1, p. 102].

Reference

1. C. E. Weatherburn, Differential Geometry of three Dimensions, Cambridge University Press, Cambridge, 1955.

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the Monthly. Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.

PYTHAGOREAN TRIPLES OVER GAUSSIAN DOMAINS WITH FUNDAMENTAL UNITS

N. E. SEXAUER, Northern Illinois University

Let A be a Gaussian domain with a fundamental unit w. Assume that A satisfies the General Hypotheses (see this Monthly, Vol. 73, No. 8, p. 829) relative to p and let $2=vp^n p_1^{n_1}\cdots p_m^{n_m}$ be a factorization of 2 into a unit and irreducible elements. Assume further that A and each p_i satisfy the General Hypotheses. Let $P=2/vp=dD$ and $P_1=2/vp^n=d_1D_1$ be complementary divisors of P and P_1. Let $D=p^k p_1^{k_1}\cdots p_m^{k_m}$ and $D_1=p_1^{k_1}\cdots p_m^{k_m}$.

THEOREM. *As $k, k_1, \cdots, k_m$ range over the integers such that $0\leqq k\leqq n-1$, $0\leqq k_i\leqq n_i$ with k even and k_i even if $k_i<n_i$, then the following formulas yield all primitive solutions of $x^2+y^2=z^2$,*

$x=(s^2-t^2)/D$	$x=-(s^2-t^2)/D$	$x=w(s^2-t^2)/D$	$x=-w(s^2-t^2)/D$
1. $y=2st/D$	2. $y=-2st/D$	3. $y=2wst/D$	4. $y=-2wst/D$
$z=(s^2+t^2)/D$	$z=-(s^2+t^2)/D$	$z=w(s^2+t^2)/D$	$z=-w(s^2+t^2)/D$
$x=d_1(s^2-t^2)/2$	$x=-d_1(s^2-t^2)/2$	$x=wd_1(s^2-t^2)/2$	$x=-wd_1(s^2-t^2)/2$
5. $y=d_1st$	6. $y=-d_1st$	7. $y=wd_1st$	8. $y=-wd_1st$
$z=d_1(s^2+t^2)/2$	$z=-d_1(s^2+t^2)/2$	$z=wd_1(s^2+t^2)/2$	$z=-wd_1(s^2+t^2)/2$

where in 1.2.3.4. if $k=0$, s and t range over the nonzero elements of A such that $(s, t)\sim 1$, $D\mid(s^2\pm t^2)$ and $(s^2+t^2)/D$ is odd with respect to p and each irreducible factor of d, while if $k>0$, $s=a_1p+1$, $t=a_2p+1$ and a_1, a_2 range over A such that $(s, t)\sim 1$, $(D/p^2)\mid(a_1^2\pm a_2^2)$, and $p^2(a_1^2+a_2^2)/D$ is odd with respect to p and each irreducible factor of d, and where in 5.6.7.8., $s=a_1p+1$, $t=a_2p+1$ and a_1, a_2 range over A such that $(s, t)\sim 1$, $p^{n-2}D_1\mid(a_1^2\pm a_2^2)$ and $(a_1^2+a_2^2)/p^{n-2}D_1$ is odd with respect to p and each irreducible factor of d_1. Formulas 1.2.3.4.(5.6.7.8.) give the primitive solutions in which x is odd and y is even (odd) with respect to p. All pairs of formulas are independent except possibly those obtained from

1.2.3.4.(5.6.7.8.) for the same $D(d_1)$. If in addition w generates an infinite number of units and A is a subdomain of the real numbers, then the set of formulas is independent.

Let m be a square free integer and let A_m denote the ring of integers of the extension field obtained by adjoining $\sqrt{m}$ to the rationals. Using the Theorem the solutions of the equation $x^2+y^2=z^2$ are listed for the fourteen domains A_m corresponding to $m=-7, -2, -1, 2, 3, 6, 7, 11, 17, 19, 33, 41, 57, 73$.

Mail address: Department of Mathematics, California State College at Long Beach, Long Beach, California 90804.

AN ELEMENTARY PROOF OF THE JORDAN-HÖLDER THEOREM

J. W. CANNON, University of Utah

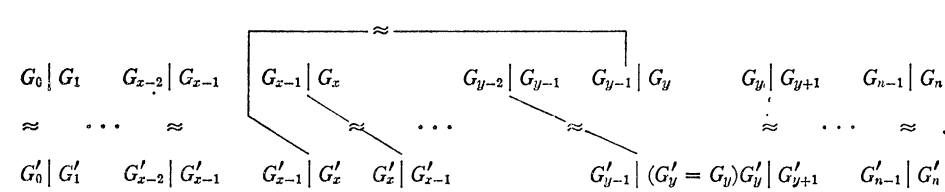
Let $S_1=\langle G=G_0, \cdots, G_n=E\rangle$ and $S_2=\langle G=H_0, \cdots, H_m=E\rangle$ denote distinct composition series of a group G. Hölder in his classic proof found induction on order and a simple lemma sufficient to show $S_1\simeq S_2$ for finite G.

LEMMA 0 (Hölder). *If A and B, $A\neq B$, are maximal proper normal subgroups of G, then* (1) $A\cup B=G$ *and* (2) $A\cap B$ *is maximal proper normal in A and B.*

In order to avoid the unnecessary hypothesis, $o(G)<\infty$, and yet employ only the same minimal machinery, one may apply Lemma 0 to transform S_1 to a series $S_1'=\langle G=G_0', \cdots, G_n'=E\rangle$ such that $S_1\simeq S_1'$ and $\max\{j \mid i\leqq j\Rightarrow G_i'=H_i\} > \max\{j \mid i\leqq j\Rightarrow G_i=H_i\}$. That $S_1\simeq S_2$ follows upon finite repetition of the process.

Let x be minimal such that $G_x\neq H_x$. Define $G_i'=G_i=H_i$ for $i=0, 1, \cdots, x-1$. Let $G_x'=H_x\neq G_x$. Let $G_i'=G_{i-1}\cap G_{i-1}'$ for $i>x$ and $G_{i-1}\neq G_{i-1}'$. Let $G_i'=G_i$ for $i>x$ and $G_{i-1}=G_{i-1}'$. One proves easily that $G_y'=G_y$ for some minimal $y>x$.

That S_1' is a composition series follows from Lemma 0. That $S_1\simeq S_1'$ follows from Lemma 0 and the basic isomorphism theorem for groups ($HN/N\simeq H/H\cap N$). The lattice with indicated isomorphisms is



Mail Address: Dept. of Math., University of Utah, Salt Lake City, Utah 84112.

ON THE PERIODICITY OF $f(g)$

F. GROSS, Bellcomm, Inc.

It was suggested in [1] that if f is an entire function, then ff is periodic if and only if f is. Here we discuss the more general problem: If $f(g)$ is periodic what can be said about g? The following partial answer is provided:

THEOREM 1. *Let f be a nonconstant meromorphic function and let g be entire with an infinite number of real a-points (a some complex constant). If $f(g)$ is periodic with real period then so is g.*

COROLLARY. *If f is entire and has an infinite number of real a-points, then $f(f)$ is periodic with real period if and only if f is.*

THEOREM 2. *Let f and g be two entire commuting functions (i.e., $f(g)=g(f)$). If there exists an unbounded sequence of complex numbers z_i such that the imaginary parts of both z_i and $f(z_i)$ are bounded, then $f(g)$ is periodic with real period if and only if either f or g is.*

Proof of Theorem 1. Using a simple variation of a well-known growth theorem of Polya [2], one proves that if f and h are nonconstant meromorphic functions, then an infinite family of entire solutions of $f(g)=h$ cannot have a common a-point. If τ is a period of $f(g)$ and ξ_i, $i=1, 2, \cdots$ are the real a-points of g, then one shows, using the above fact, that the set

$$S = \{\xi_i \,(\mathrm{mod}\ \tau),\ i = 1, 2, \cdots\}$$

is infinite unless g is periodic with real period. One easily verifies that if S is infinite, then f must be a constant and the theorem follows. A somewhat similar argument is used to prove Theorem 2.

Mail Adress: 1100 Seventeenth Street, N. W., Washington, D. C. 20036.

References

1. F. Gross, On factorization of meromorphic functions, Trans. Amer. Math. Soc., April, 1968.
2. G. Polya, On an integral function of an integral function, London Math. Soc. Journal 1, (1926) 12–15.

EXPANSION OF $\nabla\times(u\times v)$

CHARLES MCCRACKEN, Martin Company

One item that is not very well explained in books dealing with vector analysis is the formula

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}\nabla\cdot\boldsymbol{v} - \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\nabla\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are vector point functions. The following derivation is a reasonable approach:

Use the subscript notation $(\boldsymbol{u}\times\boldsymbol{v})_u$ to indicate that $\boldsymbol{u}$ is held constant when it appears inside the parentheses. Similarly for $\boldsymbol{v}$. Then by the standard rule for differentiating a product,

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = \nabla \times (\boldsymbol{u} \times \boldsymbol{v})_u + \nabla \times (\boldsymbol{u} \times \boldsymbol{v}). \tag{2}$$

Expanding the right hand terms formally as vector triple products:

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v})_u = \nabla\cdot\boldsymbol{v}\boldsymbol{u} - \nabla\cdot\boldsymbol{u}\boldsymbol{v}, \tag{3}$$

where $\boldsymbol{u}$ is held constant for the differentiations which occur in ∇, so that in the last term $\boldsymbol{u}$ may be put before the ∇:

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v})_u = \boldsymbol{u}\nabla\cdot\boldsymbol{v} - \boldsymbol{u}\cdot\nabla\boldsymbol{v}. \tag{4}$$

Similarly,

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v})_v = \boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{v}\nabla\cdot\boldsymbol{u}, \tag{5}$$

from which (1) follows immediately.

Mail Address: 9856 53rd Ave., St. Petersburg, FL 33708.

VIBRATING SYSTEMS WITH REPEATED FREQUENCIES

C. R. WYLIE, JR., University of Utah

Many texts on mechanical vibrations contain the incorrect statement, "A system with n degrees of freedom has n natural frequencies." A counterexample is provided by a frictionless system of three masses, M_1, M_2, M_3, guided to move in one direction, in which M_i is coupled to M_j by a spring of modulus k_{ij}, $i, j = 0, 1, 2, 3$, k_{0j} being understood to be a spring connecting M_j to 'ground.' Such a system has the following properties:

1. The frequency equation of the system will have a double root ω_2^2 if and only if

$$\omega_2^2 M_1 = k_{10} + k_{12} + k_{13} + \frac{k_{12}k_{13}}{k_{23}}, \qquad \omega_2^2 M_2 = k_{20} + k_{21} + k_{23} + \frac{k_{21}k_{23}}{k_{13}},$$

$$\omega_2^2 M_3 = k_{30} + k_{31} + k_{32} + \frac{k_{31}k_{32}}{k_{12}}.$$

2. If the frequency equation has a simple root ω_1^2 and a double root ω_2^2, then

$$\omega_2^2 - \omega_1^2 = \frac{k_{12}k_{13}}{k_{23}M_1} + \frac{k_{12}k_{23}}{k_{13}M_2} + \frac{k_{13}k_{23}}{k_{12}M_3}.$$

3. When the system is vibrating at the repeated frequency ω_2, the amplitudes, A_1, A_2, A_3, of M_1, M_2, M_3, satisfy the relation

$$\frac{A_1}{k_{23}} + \frac{A_2}{k_{13}} + \frac{A_3}{k_{12}} = 0.$$

4. When the frequency equation has a simple root ω_1^2 and a double root ω_2^2, and the system is vibrating at the frequency ω_1, the amplitudes A_1, A_2, A_3, of M_1, M_2, M_3, are proportional to

$$\frac{k_{12}k_{13}}{M_1}, \quad \frac{k_{12}k_{23}}{M_2}, \quad \frac{k_{13}k_{23}}{M_3}.$$

5. When the system is vibrating at the repeated frequency ω_2, the phase difference between successive masses need not be either 0° or 180°.

Mail Address: Dept. of Math., University of Utah, Salt Lake City, Utah 84112.

INNER ENDOMORPHISMS OF SEMIGROUPS

C. J. STUTH, University of Missouri

For a, b in a semigroup S let ${}_aT_b$ denote the mapping ${}_aT_b(x)=axb$ for all $x\in S$. An endomorphism T is said to be *inner* if $T={}_aT_b$ for some $a, b\in S$. In this note some of the results in [1] are sharpened.

THEOREM A. *If S has an identity, $aS\subset Sa$, $Sb\subset bS$, then ${}_aT_b$ is an endomorphism if and only if ab is an idempotent.*

THEOREM B. *If S is commutative with identity e, then ${}_aT_b$ is an endomorphism if and only if ab is idempotent. Moreover, the map ${}_aT_b\to ab$ is an isomorphism of the semigroup of all inner endomorphisms of S onto the semigroup of all idempotents of S.*

Proofs. The first part of B is a corollary of A. The rest is straightforward.

The following theorem strengthens Theorem 3 of [1], and Theorem 6 thereof is an immediate corollary since every inner endomorphism of a cancellative semigroup is a monomorphism.

THEOREM C. *${}_aT_b$ is a monomorphism if and only if S has an identity e and $ba=e$.*

Proof. Suppose ${}_aT_b$ is a monomorphism. $axbayb=axyb$, so ${}_aT_b(xbay)={}_aT_b(xy)$, which implies $xbay=xy$. In particular $abay=ay$ and $abayb=ayb$, so ${}_aT_b(bay)={}_aT_b(y)$, from which $bay=y$ for all $y\in S$. Likewise $xba=x$ for all $x\in S$. The converse is easy.

REMARK. Some insight into the last example of [1] is gained by observing that ${}_aT_b$ is an endomorphism if and only if ba is a middle unit between aS and Sb.

Mail Address: Dept. of Mathematics, Univ. of Missouri, Columbia, Mo. 65201.

Reference

1. M. L. Vitanza, Mappings of semigroups associated with ordered pairs, this MONTHLY, 73 (1966) 1078–1082.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT 06268.

REMARKS ON CERTAIN EULERIAN CONSTANTS

WALTER LEIGHTON, University of Missouri at Columbia

It is well known that although the harmonic series $\sum 1/n$ diverges, the sequence

$$\left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right\}_1^\infty$$

converges to Euler's constant $\gamma = 0.577$ (approx.). It is not known whether or not γ is irrational.

The present note indicates extensions of this idea. In an area boasting so voluminous a literature one would be rather foolhardy to assert novelty for the elementary ideas involved below. But a necessarily limited search of likely sources has not uncovered the results that follow.

Let $c_1, c_2, c_3, \cdots$ be a strictly decreasing sequence of positive constants such that $c_n \to 0$, as $n \to \infty$. Let $f_1, f_2, f_3, \cdots$ be any sequence of constants such that

$$c_{n+1} \leqq f_{n+1} - f_n \leqq c_n \qquad (n = 1, 2, \cdots). \tag{1}$$

It will be observed that the sequence $\{f_n\}$ is monotone increasing, but not necessarily positive. If α_n and β_n are defined by the equations

$$\alpha_n = c_1 + c_2 + \cdots + c_n - f_n,$$
$$\beta_n = \alpha_n - c_n \qquad (n = 1, 2, \cdots),$$

it is readily verified that $\alpha_{n+1} \leqq \alpha_n$ and $\beta_{n+1} \geqq \beta_n$. Inasmuch as $\beta_n - \alpha_n \to 0$, as $n \to \infty$, it follows that $\alpha_n \not\to -\infty$ and $\beta_n \not\to +\infty$, and, accordingly, α_n and β_n tend to a common finite limit Γ.

It is easy to see that there are infinitely many choices of f_n that satisfy (1) corresponding to each admissible sequence $\{c_n\}$. If, for example, $f_n = c_2 + c_3 + \cdots + c_n$, then $f_{n+1} - f_n = c_{n+1}$, and (1) is satisfied. In this instance $\alpha_n \to c_1$, of course. If $c(x)$ is any continuous monotone decreasing function on the interval $1 \leqq x < \infty$ with the property that $c(n) = c_n (n = 1, 2, \cdots)$, then

$$c_2 + c_3 + \cdots + c_n < \int_1^n c(x)dx < c_1 + c_2 + \cdots + c_{n-1}.$$

Further, if we set $f_n = \displaystyle\int_1^n c(x)dx$, the difference

$$f_{n+1} - f_n = \int_n^{n+1} c(x)\,dx$$

satisfies condition (1), and α_n tends to a finite limit, as $n \to \infty$.

One natural choice of $c(x)$ is the function whose graph is composed of straight line segments joining the points (i, c_i) $(i = 1, 2, \cdots)$. For this choice of $c(x)$

$$(2)\quad f_n = \int_1^n c(x)\,dx = \sum_{i=2}^{n} \tfrac{1}{2}(c_{i-1} + c_i) = \tfrac{1}{2}(c_1 + c_n) + c_2 + c_3 + \cdots + c_{n-1},$$

and

$$\alpha_n = \tfrac{1}{2}\,(c_1 + c_n),$$
$$f_{n+1} - f_n = \tfrac{1}{2}\,(c_n + c_{n+1}).$$

Then, $\alpha_n \to \frac{1}{2} c_1$ (which is rational, evidently, if and only if c_1 has this property).

When $c_n = 1/n$, Euler set $c(x) = 1/x$ and obtained the limit $\gamma = 0.577$ (approx.) referred to in the first paragraph above. In this case,

$$f_n = \int_1^n \frac{dx}{x},$$

and the difference

$$f_{n+1} - f_n = \log \frac{1+n}{n}$$

satisfies (1). Using the polygonal function (2) we have

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - f_n \to 0.5 \text{ (\textit{exactly})},$$

as $n \to \infty$.

Finally, note that extrema for the difference $f_{n+1} - f_n$, subject to (1) are c_{n+1} and c_n and that these values can be attained by choosing $f_n = c_2 + c_3 + \cdots + c_n (f_1 = 0)$ and $f_n = c_1 + c_2 + \cdots + c_{n-1}$, respectively. For these choices of f_n, $\alpha_n \to c_1$ and 0, respectively. And, equation (2) yields a choice of f_n for which $\alpha_n \to \frac{1}{2} c_1$, as we have seen. If in the analysis that led to (2) one replaces the line segment joining the points $(i-1, c_{i-1})$ and (i, c_i) $(i = 2, 3, \cdots)$ by a broken line segment composed of the line segment $y = c_{i-1}$ $(i-1 \leqq x \leqq i-1+\theta)$, where θ is a fixed number $0 < \theta < 1$, and the line segment joining the point $(i-1+\theta, c_{i-1})$ and (i, c_i) it is readily verified that this choice of $c(x)$ yields

$$f_n = \int_1^n c(x)\,dx = \sum_{i=2}^{n} \left[\frac{1}{2}\,(c_{i-1} + c_i) + \frac{\theta}{2}\,(c_{i-1} - c_i) \right],$$

and a little algebra shows that if λ is any number $0 < \lambda < 1/2$, we may set $\theta = 1 - 2\lambda$, and α_n will tend to λc_1. Similarly, if ϕ is any number such that $0 < \phi < 1$

and the line segment joining the points $(i-1, c_{i-1})$ and (i, c_i) is replaced by a broken line segment composed of the line segment joining the points $(i-1, c_{i-1})$ and $(i-1+\phi, c_i)$ and the line segment joining the point $(i-1+\phi, c_i)$ and (i, c_i), we find that this choice of $c(x)$ yields

$$f_n = \int_1^n c(x)dx = \sum_{i=2}^{n} \left[c_i + \frac{\phi}{2} \quad (c_{i-1} - c_i) \right].$$

Again, if λ is any number $1/2 < \lambda < 1$, we may set $\phi = 2 - 2\lambda$, and α_n will tend to λc_1.

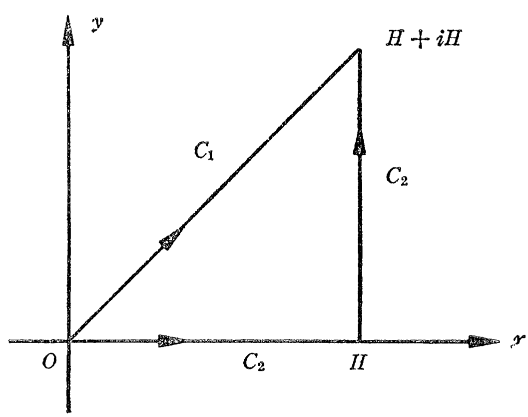
THE FRESNEL INTEGRALS

C. D. OLDS, San Jose State College

The Fresnel integrals

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \sqrt{2\pi}/4, \tag{1}$$

are intimately connected with diffraction phenomena at a slit and at a circular aperture. The following evaluation of these integrals is different from that usually found in elementary texts on complex variable theory.



The idea is simply to evaluate the line integral

$$\int_C e^{iz^2} dz \tag{2}$$

along the two different paths $C = C_1$ and $C = C_2$ as shown in the figure. Since $f(z) = e^{iz^2}$ is an entire function, the integral (2) is independent of the path taken

from (0, 0) to (H, H), hence the value of (2) along path C_1 can be equated to that along C_2. The Fresnel integrals (1) follow by letting H become infinite.

Along the path C_1 the integral (2) gives

$$\int_{C_1} e^{iz^2}dz = (1+i)\int_0^H e^{-2x^2}dx. \tag{3}$$

Similarly, along the "broken line" path C_2, we obtain

$$\int_{C_2} e^{iz^2}dz = \int_0^H e^{ix^2}dx + ie^{iH^2}\int_0^H e^{-2Hy-iy^2}dy. \tag{4}$$

Equating the right-hand sides of (3) and (4) we get

$$(1+i)\int_0^H e^{-2x^2}dx = \int_0^H e^{ix^2}dx + ie^{iH^2}\int_0^H e^{-2Hy-iy^2}dy. \tag{5}$$

If J stands for the second integral on the right of (5), then

$$|J| \leqq \int_0^H e^{-2Hy}dy < \int_0^\infty e^{-2Hy}dy = 1/2H,$$

so J approaches zero as H becomes infinite. Hence, letting H become infinite in (5), and using the fact that $\int_0^\infty e^{-2x^2}dx = \sqrt{2\pi}/4$, we get

$$(\sqrt{2\pi}/4)(1+i) = \int_0^\infty (\cos x^2 + i \sin x^2)dx, \tag{6}$$

and the integrals (1) follow immediately by equating real and imaginary components in (6).

APPROXIMATIONS OF *e*

M. T. Bird, San Jose State College

If one defines the natural logarithm of x by means of the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt,$$

one may define the number e as the solution of the equation

$$\int_1^x \frac{1}{t}\,dt = 1.$$

We seek to obtain approximations to e when e is defined in this way.

The reciprocal function f defined by $f(x) = 1/x$ is positive and monotonic decreasing for $x \geqq 1$. In the closed interval $[a, b]$, $1 \leqq a < b$, the function f has the maximum $f(a) = 1/a$ and the minimum $f(b) = 1/b$.

For the positive integer n we define the numbers r, s_n, and t_n by means of the equations

$$r = 1 + \frac{1}{n}, \qquad s_n = r^n, \qquad t_n = r^{n+1}.$$

The points $1, r, r^2, \cdots, r^n$ determine a partition of the interval $[1, s_n]$ on which the upper sum of the function f satisfies the equations

$$\sum_{i=1}^{n} f(r^{i-1})(r^i - r^{i-1}) = \sum_{i=1}^{n} \frac{1}{r^{i-1}} (r^i - r^{i-1}) = n(r-1) = 1.$$

This upper sum also satisfies the inequality

$$\sum_{i=1}^{n} \frac{1}{r^{i-1}} (r^i - r^{i-1}) > \int_1^{s_n} \frac{1}{t}\, dt.$$

From the foregoing equality and inequality together with the definition of e we conclude

$$e > s_n = \left(1 + \frac{1}{n}\right)^n.$$

The points $1, r, r^2, \cdots, r^n, r^{n+1}$ determine a partition of the interval $[1, t_n]$ on which the lower sum of the function f satisfies the equations

$$\sum_{i=1}^{n+1} f(r^i)(r^i - r^{i-1}) = \sum_{i=1}^{n+1} \frac{1}{r^i} (r^i - r^{i-1}) = (n+1)\left(1 - \frac{1}{r}\right)$$
$$= (n+1)\left(1 - \frac{n}{n+1}\right) = 1.$$

This lower sum also satisfies the inequality

$$\sum_{i=1}^{n+1} \frac{1}{r^i} (r^i - r^{i-1}) < \int_1^{t_n} \frac{1}{t}\, dt.$$

From the foregoing equality and inequality together with the definition of e we conclude

$$e < t_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

From the inequality obtained from the upper sum we have $e(1+(1/n)) > t_n$ so that we may conclude from the "squeeze" theorem that $e = \lim_{n\to\infty} t_n$. From this limit and the equality

$$t_n = \left(1 + \frac{1}{n}\right) s_n$$

we conclude $e=\lim_{n\to\infty} s_n$. We have obtained a sequence of upper and lower bounds to the value of e.

The monotonic character of the sequence $\{s_n\}$ may be established by using the fact that the geometric mean of a set of positive numbers never exceeds the arithmetic mean. In particular one might consider the set of $n+1$ numbers consisting of $n/(n+1)$ and n ones. (See [1], p. 563.)

Similarly the monotonic character of the sequence $\{t_n\}$ may be established by considering the set of $n+2$ numbers consisting of $(n+1)/n$ and $n+1$ ones.

Reference

1. N. S. Mendelsohn, An application of a famous inequality, this MONTHLY, 58 (1951) 563.

AN ELEMENTARY COMMUTATIVITY THEOREM FOR RINGS

E. C. JOHNSEN, D. L. OUTCALT, AND ADIL YAQUB, University of California at Santa Barbara

1. Introduction. It is well known that any group G with the property that $(xy)^2=x^2y^2$ for all x, y in G is necessarily commutative. This naturally gives rise to the question whether the ring-theoretic analogue of this group-theoretic result is valid. This question does not appear to have been fully considered in any of the standard texts. In this note, we give a simple and elementary proof of the theorem that any nonassociative (i.e., not necessarily associative) ring R *with identity* which satisfies $(xy)^2=x^2y^2$ for all x, y in R is necessarily commutative. We also show that the hypothesis of the existence of the identity is indeed essential. Moreover, we show that this result need not hold if "$(xy)^2=x^2y^2$" is replaced by "$(xy)^k=x^ky^k$" for *any* $k>2$.

2. Main result.

THEOREM 1. *Let R be any nonassociative ring with identity 1 such that $(xy)^2=x^2y^2$ for all x, y in R. Then R is commutative.*

Proof. Suppose $x\in R$, $y\in R$. Then $(xy)^2=x^2y^2$. Moreover,

$$\{x(y+1)\}^2 = x^2(y+1)^2 = x^2y^2 + 2x^2y + x^2.$$

But also $\{x(y+1)\}^2=(xy)^2+(xy)x+x(xy)+x^2$, hence

$$(xy)x + x(xy) = 2x^2y. \tag{1}$$

Now, repeating this argument for $x+1$ instead of x, we obtain

$$[(x+1)y](x+1) + (x+1)[(x+1)y] = 2(x+1)^2y.$$

This, in turn, reduces to $(xy)x+yx+x(xy)=2x^2y+xy$. By (1) this becomes $yx=xy$, and the theorem is proved.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS

COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

THE EXTENT OF IMPLEMENTATION OF LEVEL I AND LEVEL III CUPM RECOMMENDATIONS, PANEL ON TEACHER TRAINING

J. J. FISHER, Virginia Polytechnic Institute

This article presents a summary of a study [1] conducted by the writer to ascertain the extent of implementation of levels I and III recommendations of the Panel on Teacher Training, Committee on the Undergraduate Program in Mathematics. This survey investigates changes in the undergraduate mathematics programs for teacher education in colleges and universities of the United States since 1960, compared with the recommended minimum requirements of the CUPM.

Studies have indicated that elementary teachers are not adequately prepared to teach the mathematics required by recent curriculum revisions. At many institutions, appropriate courses are not available to them even as electives. Likewise, most studies indicate that, although there are exceptions, secondary school mathematics teachers have not had adequate preparation in mathematics.

A random sample of 117 colleges and universities was chosen from the 822 institutions listed in *Guide to Undergraduate Programs in Mathematics* [2] as offering programs for the preparation of secondary school mathematics teachers. The mathematics course requirements for both elementary and secondary school mathematics teacher education programs for the years 1960 and 1965 were determined in order to provide a comparison of requirements prior to and approximately five years after the publication of the CUPM recommendations. Eighty-seven appropriate replies were received, seventy-eight of which indicated a program for elementary teachers.

In the analysis of the elementary teacher education programs, the total number of semester hours of all mathematics content courses was used. For secondary school teacher programs, the data were compiled for total number of semester hours in (a) CUPM-type courses, (b) analysis, (c) abstract algebra, (d) geometry, and (e) probability and statistics. Conclusions were based on the results of independent, one-tailed t-tests of the difference between means using a .05 level of significance.

From seventy-eight institutions in 1960, well over half (41) of these required not a single course in mathematics for the preservice education of elementary teachers. In 1965, only thirteen of the seventy-eight did not require any mathematics. The rise in the mean number of semester hours during this period—from 1.97 to 4.15—indicates a trend favorable to the CUPM recommendations, which

TABLE I. Number of semester hours of mathematics required by institutions in the preparation program for secondary school mathematics teachers, 1960 and 1965 (Number of institutions in 1960: 66; Number of institutions in 1965: 87.)

		Total CUPM-Type	*Analysis*	*Abstract Algebra*	*Geometry*	*Probability-Statistics*
1960	Mean	16.05	9.66	1.94	4.35	0.35
	Median	15.44	8.92	0.94	4.04	0.00
1965	Mean	21.48	10.80	3.98	5.14	0.72
	Median	21.65	10.40	3.40	4.80	0.00
CUPM Recommendations		*27	6	6	9	6

* Including electives, 33.

TABLE II. Number of institutions and requirements in mathematics for the preparation of elementary teachers, 1960 and 1965, from random sample of 78 institutions in the United States

Semester Hours Required	*No. of Institutions 1960*	*No. of Institutions 1965*
9	0	1
8	1	4
7	0	3
6	13	27
5	0	2
4	4	10
3	13	14
2	6	4
1	0	0
0	41	13
Mean	1.97	4.15
Median	0.00	4.30

are twelve semester hours in selected mathematics courses. Indeed, the statistical analysis of these data showed that the difference between the means was significant at the level, $p < .001$. Data are given in Table II.

The CUPM Level I recommendations included two 3-semester-hour courses in the structure of the real number system, one in algebra and one in geometry. This survey finds that courses in real numbers are required most often, and in geometry least, with only one of the 128 required courses listed being in this area.

For prospective secondary school mathematics teachers there have been significantly increased course requirements since 1960 in CUPM-type mathematics, abstract algebra, geometry, and probability-statistics. In analysis, the increase was not statistically significant. Data are given in Table I.

While the survey indicates an increase in requirements in mathematics for prospective teachers, it is clearly evident that the complete implementation of CUPM Level III recommendations is far from accomplished. As might be expected, minimum requirements in calculus, on the average, exceed the CUPM recommended minima, while those in the other areas are below the CUPM standard.

So-called "Level I Conferences" have been held throughout the country during the past several years. These conferences, designed to give mathematicians, educators, administrators, classroom teachers, and representatives of state departments of education the opportunity to discuss the content and implementation of the CUPM recommendations for elementary teachers, seemed to have accomplished some of the desired effect. Indeed, it is possible that the full impact of these conferences has not yet reached the educational community. Plans should be made for similar conferences concerning the other levels of the recommendations, being certain to invite official representatives from such powerful organizations as the National Association of Secondary School Principals, the American Association of Colleges for Teacher Education, and the National Education Association's Commission on Teacher Education and Professional Standards.

References

1. John J. Fisher, A survey to determine the extent of implementation of CUPM Recommendations. Unpublished doctoral dissertation, University of Colorado, 1966.

2. United States Department of Health, Education and Welfare, *OE 56008, Guide to Undergraduate Programs in Mathematics*, 1962, United States Government Printing Office, Washington, D. C.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before July 31, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2065. *Proposed by R. R. Poole, University of Redlands, Cal.*

Find a solution for the recursion formula $x_n=(n-1)(x_{n-1}+x_{n-2})$, $n\geqq 4$, $x_2=1$, $x_3=2$.

E 2066. *Proposed by George Grossman, Board of Education, New York City*

Find the most general square root of the 3×3 identity matrix if the elements are to be (a) integers, (b) any real numbers, (c) complex numbers.

E 2067. *Proposed by Colonel Johnson, Jr., Southern University, Baton Rouge, La.*

We say that a group is near-simple if it has exactly one proper normal subgroup. Give a characterization of the near-simple Abelian groups.

E 2068. *Proposed by S. R. Conrad, Francis Lewis High School, Flushing, N. Y.*

The five-digit number (in decimal notation) $xy\,57z$ is divisible by 729. Find x, y, z.

E 2069. *Proposed by R. P. Sheets, University of Chicago*

Let it be defined that $x_0=1$, $x_{n+1}=x^{x_n}$, $n=0, 1, \cdots$. For $n=1, 2, \cdots, 8$ it seems that the derivative of x_n is given by

$$\frac{d}{dx}(x_n) = \frac{1}{x}\sum_{j=0}^{n-1}(\ln x)^j \prod_{i=n-j-2}^{n-1} x^{x_i},$$

where for $j=n-1$, the index i takes only the values $0, 1, \cdots, n-1$. Prove (or disprove) the formula in general.

E 2070. *Proposed by J. F. Burke, University of Vermont*

Let A be the matrix formed from the elements of the multiplication table of the multiplicative group of a Galois field $\mathrm{GF}(q)$, where $q=p^n$, p a prime and n a positive integer. Show that for $q\geqq 4$, $A^2=[\theta_{ij}]$, where $\theta_{ij}\equiv 0 \pmod{p}$.

E 2071. *Proposed by C. J. Mozzochi, University of Connecticut*

Let $P=\{e^{ix_1}, \cdots, e^{ix_n}\}$ be any partition of the unit circle $(0=x_0<x_1<\cdots<x_n=2\pi)$. Prove, or disprove: There exists a composite integer m such that in each cell of P there exists at least one primitive mth root of unity.

E 2072. *Proposed by Dorembus Leonard, Tel-Aviv University, Israel*

Find necessary and sufficient conditions for a $k\times n$ matrix $(k<n)$ with integral elements, in order that it be a submatrix of an integral $n\times n$ matrix with determinant 1.

E 2073. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, N. Y.*

Let I be the real line, and let S be the set of points each of whose distances from the origin is the sum of any two terms of the sequence $\{1/n\}$, $n=2, 3, \cdots$. Find the set of limit points of S in I.

E 2074. *Proposed by Erwin Just and John Furst, Bronx Community College, N.Y.*

Let m_i denote the slope of the line containing the points P_i and P_{i+1}, $(i=0, 1, \cdots, n)$ with $P_{n+1}=P_0$. If $f(x)$ is not a constant and f is defined for all real x, prove that a necessary and sufficient condition for the graph of $y=f(x)$ to be a parabola is that there exists a point P_0 on the graph of f such that for any set of n points $P_1, \cdots, P_n$ on the graph, $\sum_{i=0}^{n} (-1)^i m_i=0$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Sum of the Cubes of the Digits

E 1810 [1965, 781; 1967, 87]. *Proposed by John Harvey and John E. Wetzek, University of Illinois*

For each natural number n let $C(n)$ be the sum of the cubes of the decimal digits of n. Show that for every n, iterating C eventually leads to one of the following repeating cycles: 1, 55-250-133, 136-244, 153, 160-217-352, 370, 371, 407, 919-1459.

II. *Solution by Henry S. Cole, West Coast University, Los Angeles.* There appears to be an error in the previously published solution. The flaw concerns the assumption that q is one of the elements of B. But applying C to any n does not insure that the first iteration will produce a $q\in B$. Hence $C^p(n)$ must be tested for all p for each d. (Actually, it is not stated that $q\in B$, but $q\in S$. But then the solution proceeds as though $q\in B$.) The following avoids the difficulty.

Let $S(k)$ be the proposition that any n having k digits will eventually be reduced to an element of B by C. (B is the set whose members are the integers included in any of the stated cycles.) A Fortran program was run on an IBM 1130 computer and shows that S is true for $k=1, 2, 3, 4$. Now, if $k=5$, the first iteration of C will reduce the number to the case where $k\leqq 4$, since the "worst case," 99999, becomes

$$C(99999) = 5\cdot 9^3 = 3645.$$

Indeed, for $k>4$, the number of digits of $C(n)$ is always less than k, i.e.,

$$k\cdot 9^3 = k\cdot 729 < 10^{k-1}.$$

Hence $S(k)$ is true for all k.

Point with Irrational Distances from Three Given Points

E 1844 [1965, 1130; 1967, 445]. *Proposed by A. K. Austin, The University, Sheffield, England*

Do there exist points A, B, C such that for all points D in the plane of A, B, C, the distances AD, BD, CD are not all rational?

III. *Solution by David Singmaster, American University of Beirut, Lebanon.* The existence of such points can be shown by elementary countability and measure-theoretic arguments. Let A and B be any two distinct points in the plane. If DA and DB are both rational, then D must be one of the countably many points of intersection of circles of rational radius about A and B. Consider all circles of rational radius about such intersection points. There are countably many such, so they form a set of measure zero in the plane. If C is any point not on these circles, then DA, DB, DC cannot all be rational. A countability argument shows there are uncountably many such points, even on any given line or circle. Measure-theoretically, we see that almost every point C is suitable. Indeed, we have that almost every triple of points (A, B, C) has the desired property.

Clearly this argument can be given in n dimensions to yield that almost all $(n+1)$-tuples of points in n-space have the property that every point of n-space has at least one irrational distance from some point of the $(n+1)$-tuple. Further, one can replace "rational" by "algebraic" throughout.

Fibonacci Polynomials

E 1846 [1966, 81; 1967, 592]. *Proposed by M. N. S. Swamy, University of Saskatchewan*

Show that the Fibonacci polynomials defined by

$$f_n(x) = x\cdot f_{n-1}(x) + f_{n-2}(x), \qquad f_1 = 1, \qquad f_2 = x,$$

satisfy the inequality $f_n^2(x) \leqq (x^2+1)^2\cdot(x^2+2)^{n-3}$, $n>2$.

II. *Comment by Joel Pitcairn, Bryn Athyn, Pa.* It is surprising that no one has pointed out that the Fibonacci polynomials are Chebyshev polynomials:

$$i^{n-1}f_n(x) = U_{n-1}(\tfrac{1}{2}ix), \qquad U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}\cdot$$

Elementary Derivation of a Volume Formula

E 1898 [1966, 665]. *Proposed by T. L. Saaty, U. S. Arms Control and Disarmament Agency*

Give an elementary proof that the volume of the region in E_n enclosed by $\sum_{i=1}^{k} x_i \geqq 0$, $k=1, \cdots, n$, $|x_i| \leqq \frac{1}{2}$, $i=1, \cdots, n$ is given by $(-1)^n\binom{-1/2}{n}$.

Solution by Larry Shepp, Bell Telephone Laboratories, Murray Hill, N. J. The result can be obtained from probability considerations. Let $X_1, X_2, \cdots, X_n$ be continuous random variables, symmetric, independent and identically distributed. Let p be the number of positive terms in $X_1, X_1+X_2, \cdots, X_1+X_2+\cdots+X_n$. Then the distribution of p is independent of the X's and

$$\text{Prob}\,\{p = k\} = (-1)^n\binom{-1/2}{k}\binom{-1/2}{n-k}\cdot$$

The present problem corresponds to the case $k=n$, X_i uniform on $(-\frac{1}{2}, \frac{1}{2})$. A simple proof is given by W. Feller in his paper in the Harold Cramer Anniversary Volume. The original assertion is due to E. Sparre Anderson and is often stated in greater generality.

Also solved by Walter Weissblum & Henry Friedman. The proposer refers to a complicated solution due to Felix Pollaczek.

Sum of Digits of a Number

E 1926 [1966, 1016]. *Proposed by L. D. Yarbrough, Harvard Computing Center*

Express in terms of N and b the sum of the digits of the integer N as written in radix b notation. (This is a generalization of the rule of "casting out nines," and for $b=2$ the formula yields the number of 1's in the binary representation of N, which is a measure of the multiplication speed of certain digital computers.)

Solution by Stanley Rabinowitz, Far Rockaway, N. Y. Suppose $N=\sum_{k=0}^{n} a_k b^k$. Then

$$a_j = \left[\frac{N}{b^j}\right] - b\left[\frac{N}{b^{j+1}}\right],$$

so

$$\begin{aligned}\sum_{j=0}^{n} a_j &= \sum_{j=0}^{\infty}\left(\left[\frac{N}{b^j}\right] - b\left[\frac{N}{b^{j+1}}\right]\right)\\ &= \sum_{j=0}^{\infty}\left[\frac{N}{b^j}\right] - b\sum_{j=1}^{\infty}\left[\frac{N}{b^j}\right] = N-(b-1)\sum_{j=1}^{\infty}\left[\frac{N}{b^j}\right].\end{aligned}$$

Also solved by Marcia Ascher, Jack Dix, Neal Felsinger, E. S. Langford, Donald Jeffords, M. J. Merscher, D. C. B. Marsh, Andrzej Makowski (Poland), L. J. Marx, S. F. Robinson, P. A. Scheinok, W. J. Sonsin, and the proposer.

Factorization of a Polynomial

E 1927 [1966, 1016]. *Proposed by D. W. Burns, Woodstock, Ill.*

Given any integer $m(>6)$ which possesses primitive roots, let r be a primitive root of m. Put $a_j \equiv r^j \pmod{m}$, where a_j is the least positive residue. Now form

$$F(x) = \sum_{j=1}^{\phi(m)} a_j x^{\phi(m)-j}.$$

Prove that $F(x)$ is the product of $t+1$ polynomials

$$F(x) = f_1(x)\cdot f_2(x)\cdots f_t(x)\cdot g(x),$$

where every coefficient of $f_i(x)$ is 1, and the sum of the coefficients of $g(x)$ is m.

Here t is the number of distinct prime divisors of $\frac{1}{2}\phi(m)$. The result is still true if the coefficients of $F(x)$ are permuted cyclically. (The result is true for $m=5$ and, trivially, for $m=3, 4, 6$.)

Solution by E. P. Starke, Plainfield, N. J. Put $q=\frac{1}{2}\phi(m)$ and take

$$f'(x) = \sum_{k=0}^{q-1} x^k, \qquad h(x) = a_1x^q + 1 + \sum_{j=1}^{q-1} (a_{j+1} - a_j)x^{q-j}.$$

Noting that $r^q \equiv m-1 \pmod{m}$ so that, $a_i+a_{q+i}=m(i=0, 1, \cdots, q)$, we see easily that the product $f'(x)\cdot h(x)$ is $F(x)$.

Note also that the set $\{a_j\}$ is the set $\{a \mid 0<a<m, (a, m)=1\}$. Therefore $\sum_{j=1}^{2q} a_j=mq$ (for there are $2q$ members of $\{a_j\}$, and for every a_j in the set there is an a_k such that $a_j+a_k=m$). So $F(1)=mq$. Also $f'(1)=q$, whence $h(1)$ must equal m, i.e., the sum of the coefficients of $h(x)$ is m.

Let q have the prime factorization

$$q = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_t^{\alpha_t}.$$

Then $f'(x)=(x^q-1)/(x-1)$ admits the factors

$$\frac{x^{p_1^{\alpha_1}}-1}{x-1}, \frac{x^{p_2^{\alpha_2}}-1}{x-1}, \cdots, \frac{x^{p_t^{\alpha_t}}-1}{x-1}$$

the sums of whose coefficients are $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $\cdots$, respectively. If $f''(x)$ is the product of all factors of $f'(x)$ not listed above, we can put

$$f_1(x) = (x^{p_1^{\alpha_1}} - 1)/(x - 1), \qquad f_2(x) = (x^{p_2^{\alpha_2}} - 1)/(x - 1),$$

and so on. Then $f'(x)=f_1(x)\cdot f_2(x) \cdots f_t(x)\cdot f''(x)$ where

$$f'(1) = p_1^{\alpha_1}\cdot p_2^{\alpha_2} \cdots p_q^{\alpha_q}\cdot f''(1) = q.$$

Hence $f''(1)=1$ and we can put $g(x)=f''(x)\cdot h(x)$ where $g(1)=f''(1)\cdot h(1)=m$. Thus the sum of the coefficients of $g(x)$ is m, and, from the definition, all coefficients of $f_i(x)$ are ones.

Also solved by Leonard Carlitz, M. G. Greening, and Donald Jeffords.

Carlitz raises the question concerning the cases when $g(x)$ is reducible. The proposer asks: suppose m does not possess a primitive root and we consider the set $A=\{a \mid 0<a<m, (a, m)=1\}$; can A always be so ordered, $A=\{a_i\}$, $i=1, 2, \cdots, \phi(m)$, that $F(x)=f_1(x) \cdots g(x)$, with properties as above? For example:

$m=15$: $x^7+7x^6+11x^5+13x^4+14x^3+8x^2+4x+2=(x^3+x^2+x+1)(x^4+6x^3+4x^2+2x+2)$;

$m=28$: $x^{11}+9x^{10}+13x^9+3x^8+5x^7+11x^6+27x^5+19x^4+15x^3+25x^2+23x+17=(x+1)(x^2+x+1)$
$(x^8+7x^7-3x^6-6x^5+16x^4-6x^3+13x^2-11x+17)$.

Minors of a Bidiagonal Matrix

E 1928 [1966, 1017]. *Proposed by D. Ž. Djoković, University of Belgrade, Yugoslavia*

Let $A_n=(a_{ij})$ be an $n\times n$ matrix such that $a_{ii}=a_i$, $i=1, 2, \cdots, n$; $a_{i,i+1}=b_i$, $i=1, 2, \cdots, n-1$; $a_{ij}=0$ otherwise. Let M be the minor of $\det A_n$ obtained by deleting the rows $i_1, i_2, \cdots, i_k (1\leqq i_1<i_2<\cdots<i_k\leqq n)$ and the columns $j_1, j_2, \cdots, j_k (1\leqq j_1<j_2<\cdots<j_k\leqq n)$. Prove that

$$\begin{aligned} M = {} & (a_1a_2\cdots a_{j_1-1})(b_{j_1}b_{j_1+1}\cdots b_{i_1-1})(a_{i_1+1}a_{i_1+2}\cdots a_{j_2-1}) \\ & \cdot(b_{j_2}b_{j_2+1}\cdots b_{i_2-1})(a_{i_2+1}a_{i_2+2}\cdots a_{j_3-1})\cdots(b_{j_k}b_{j_k+1}\cdots b_{i_k-1}) \\ & \cdot(a_{i_k+1}a_{i_k+2}\cdots a_n) \end{aligned}$$

if $1\leqq j_1\leqq i_1<j_2\leqq i_2<j_3\leqq i_3<\cdots<j_k\leqq i_k\leqq n$; $M=0$ otherwise. We take $(a_ra_{r+1}\cdots a_s)=1$ whenever $s<r$.

Solution by M. S. Klamkin, Ford Scientific Laboratory. We employ induction. Assume that the result holds for all matrices A_r, $r=1, 2, \cdots, n$ and for all $k=1, 2, \cdots, r$. Now consider A_{n+1}. If $n+1=j_k>i_k$, then $A_{n+1}=0$. If $n+1=i_k>j_k$, delete row i_k and column j_k and expand by minors using the last column, giving b_nA_{n-1}. If $i_k, j_k<n+1$, then $A_{n+1}=a_{n+1}A_n$. By the inductive hypothesis the result also holds for A_{n+1}. Since it clearly holds for A_2, it is valid for all A_n.

Also solved by W. D. Markel.

Properties of a Tournament

E 1929 [1966, 1017]. *Proposed by D. M. Bloom, Brooklyn College*

Each of the n teams of a baseball league plays each other team exactly r times during the season. (There are no tied games.) Let $\|a_{ij}\|$ $(i=1, \cdots, n; j=1, 2)$ be the $n\times 2$ matrix representing the final won-and-lost records of the teams, arranged in the usual order. Then it is easily seen that

$$a_{i1}+a_{i2}=r(n-1) \quad (\text{all } i), \tag{1}$$

$$\sum_{i=1}^{n} a_{i1}=\sum_{i=1}^{n} a_{i2}, \tag{2}$$

$$\text{if } i<j, \quad \text{then} \quad a_{i1}\geqq a_{j1} \quad \text{and} \quad a_{i2}\leqq a_{j2}, \tag{3}$$

$$\sum_{i=1}^{k} a_{i2}\geqq rk(k-1)/2 \qquad (k=1, \cdots, n). \tag{4}$$

Prove that any $n\times 2$ matrix $\|a_{ij}\|$ satisfying conditions (1) through (4) represents a possible outcome of the baseball season.

Editorial Comment. No correct solutions were received. However, the result is known. See F. Harary, R. Z. Norman and D. Cartwright, *Structural Models*, Wiley, 1965, pp. 290–292. The special case $r=1$ corresponds to Theorem 11.2 and is ascribed to H. G. Landau, *On dominance relations and the structure of animal societies*: III, *The condition for a score sequence*, Bull. Math. Biophysics, 15 (1953) 114–148. The general case can be treated similarly.

Two Triangle Inequalities

E 1930 [1966, 1017]. *Proposed by Simeon Reich, Haifa, Israel*

Let a, b, c be the sides of an acute triangle, r its inradius, and r_a, r_b, r_c its exradii. Deduce:

$$(1) \qquad (\sum a)^3 \leqq 5\sum a^2b - 3abc,$$

$$(2) \qquad 9r(\sum r_a)^2 + 9r^3 \geqq 32r_ar_br_c - 14r^2\sum r_a,$$

with equality if and only if the triangle is equilateral.

Solution by Roberto W. Frucht, Universidad Tecnica Federico Santa Maria, Valparaíso, Chile. Both (1) and (2) are new forms of known inequalities. This may be seen upon expressing the proposed inequalities in terms of R, r, s, where these symbols represent respectively the circumradius, the inradius and the semiperimeter of the triangle. The following identities will be helpful:

$$\sum a^2b = 2s(s^2 - 2Rr + r^2),$$
$$abc = 4Rrs, \qquad r_ar_br_c = rs^2,$$
$$\sum r_a = 4R + r, \qquad \sum a^2 = 2(s^2 - 4Rr - r^2).$$

(These are easily deduced, or can be found in W. J. Blundon, *On certain polynomials associated with the triangle*, Mathematics Magazine, 36 (1963) 247–248.)

Hence (1) is equivalent to $8s^3 \leqq 10s(s^2-2Rr+r^2)-12Rrs$, or

$$(1') \qquad r(16R - 5r) \leqq s^2,$$

and this, except for a factor 4, is the left-hand part of inequality (15) in J. Steinig, *Inequalities concerning the inradius and circumradius of a triangle*, Elem. der Mathematik, XVIII (1963) 127–131.

Similarly (2) is equivalent to $9r(4R+r)^2+9r^3 \geqq 32rs^2-14r^2(4R+r)$ and this, by obvious reductions, is equivalent to

$$(2') \qquad 9R^2 \geqq \sum a^2,$$

a well-known inequality that Steinig (*loc. cit.*) ascribes to Kubota, Tohoku Math. J., 25 (1925) 122–126.

It follows also that we have equality if and only if the triangle is equilateral.

Also solved by A. N. Aheart, Andrzej Makowski (Poland), and the proposer.

On Stable Sequences

E 1931 [1966, 1018]. *Proposed by D. W. Burns, Woodstock, Ill.* Given a finite sequence of nonnegative integers, consider the process of replacing each of the integers by the number of integers to its right which are smaller. Repeat the process in turn on each of the sequences obtained. Show that one will finally obtain a sequence such that repetition of the process makes no further change, i.e., each integer equals the number of smaller integers to its right.

I. *Solution by E. S. Langford, U. S. Naval Postgraduate School.* A n-sequence $(x_1, x_2, \cdots, x_n)$ is *stable* if further iteration (as described in the problem) produces no change in the sequence. We show that every finite sequence is ultimately stable. An inductive argument is suggested since the stability of the $(n-1)$-sequence $(x_2, x_3, \cdots, x_n)$ is not affected by the value of x_1.

The ultimate stability of 1-sequences is trivial, so suppose that every k-sequence is ultimately stable, and consider the $(k+1)$-sequence $(x_1, x_2, \cdots, x_{k+1})$. Iterate until the k-sequence $(x_2, x_3, \cdots, x_{k+1})$ is stable; suppose that the sequence becomes $(x_1', x_2', \cdots, x_{k+1}')$. Iterate again to get $(x_1'', x_2', \cdots, x_{k+1}')$. Suppose that $x_1'' \geqq x_1'$. Iterate again to get $(x_1''', x_2', \cdots, x_{k+1}')$; evidently $x_1''' \geqq x_1''$, hence the sequence $(x_1', x_1'', x_1''', \cdots)$ is nondecreasing. But this sequence is bounded above (by k), so it must be ultimately constant. That is, the original $(k+1)$-sequence must be ultimately stable. A similar argument holds if $x_1'' < x_1'$, since the sequence $(x_1', x_1'', x_1''', \cdots)$ is bounded below by 0.

An associated combinatorial problem would be to enumerate all stable k-sequences.

A similar problem suggests itself when we replace "smaller" by "smaller than or equal to" in the statement of the problem. In this case we can give the following more complete answer: Every n-sequence terminates ultimately in the following particular stable sequence $(n-1, n-2, \cdots, 0)$ after at most n iterations. The proof is by induction on the following statement: Let the initial n-sequence be $(x_1(0), x_2(0), \cdots, x_n(0))$. If after k iterations $(1 \leqq k \leqq n)$ we reach the sequence $(x_1(k), x_2(k), \cdots, x_n(k))$, then

$$x_1(k) \geqq k-1, \qquad x_2(k) \geqq k-1, \cdots, x_{n-k}(k) \geqq k-1,$$
$$x_{n-k+1}(k) = k-1, \qquad x_{n-k+2}(k) = k-2, \cdots, x_n(k) = 0.$$

II. *Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.* We generalize the problem as follows: Given a finite sequence of objects from a set S, consider the process of replacing each object by an object in S depending only on the collection of objects to its right, the number of times an object occurs in the collection being taken into account. Repeat the process, etc.

The empty set is such a collection, and the first object (counting right to left) in the second sequence is thus determined and fixed thereafter. In general the nth object in the final sequence is determined by at most the nth sequence after the first.

It is not necessary that the process be applied to each object of the sequence, provided that some change results at each application. Furthermore, the arrangement of the objects in the sequence could be changed, if the same change is applied to all the sequences.

Also solved by Jack Dix, F. M. Eccles, Donald Jeffords, P. S. Kornya, E. L. Magnuson, L. J. Marx, Norman Miller, L. J. Pratte, D. A. Profitt, G. F. Schumm, D. L. Silverman, Stephen Spindler, and D. R. Wilder.

Editorial Note. Miller and Pratte note that at most $n-1$ iterations are necessary for a sequence of n terms to achieve stability, and they exhibit a sequence requiring that number, viz. $2, 1, n, n-1, \cdots, 4, 3$.

A number of solutions were incomplete because of assertions made without proof. Some of the inductive proofs were incomplete. For example, if $x_2, x_3, \cdots, x_n$ is stable, it was assumed that $x_1, x_2, x_3, \cdots, x_n$ would be stable under one more iteration. A counterexample is 1, 0, 4, 3, 2, 1, 0.

A Mean Value Property

E 1932 [1966, 1018]. *Proposed by C. S. Ogilvy, Hamilton College*

For any point $P(x, y)$ on the curve $y=f(x)$, the slope of the line OP is equal to $f'(x_0)$, where $x_0=n^{1/(1-n)}x$. Find $f(x)$.

Solution by H. J. Fletcher, Brigham Young University. Assuming $n>0$ and that $f(x)$ has a power expansion about $x=0$, we have

$$\frac{F(x)}{x} = \sum_{r=0}^{\infty} a_r x^{r-1} = F'(\alpha x) = \sum_{r=0}^{\infty} r a_r (\alpha x)^{r-1},$$

where $\alpha=n^{1/(1-n)}$. On equating like coefficients $a_0=0$, a_1 is arbitrary and $1=r\alpha^{r-1}$ or, equivalently,

$$e^{\log r/(1-r)} = e^{\log n/(1-n)}.$$

Since $e^{\log x/(1-x)}$ is monotonically decreasing for $x>0$, the only other solution for r is n. Whence $F(x)=a_1x+a_nx^n$, n a positive integer, a_1 and a_n arbitrary.

Also solved by R. M. Gasper, Michael Goldberg, M. S. Kaplan, P. G. Kirmser, Norman Miller, P. H. Young, and the proposer.

Editorial Note. All the solutions essentially assume a power expansion. However, even if we assume a Fröbenius expansion, $F(x)=\sum_0^\infty a_r x^{r+c}$, we would end up with the same result. Note that this does not preclude other solutions which do not have $x=0$ in their domains. To obtain other solutions, let $\alpha=e^\lambda$, $x=\exp(te^{-\lambda})$, $F(x)=H(t-\lambda e^\lambda)$ which transforms $F(x)=xF'(\alpha x)$ into $H'(t)=H(t-\lambda e^\lambda)$, a differential-difference equation. Since here λ is negative, we can assume H to be an arbitrary continuous function in $\lambda e^\lambda \leqq t<0$ and then determine H in the interval $(2\lambda e^\lambda, \lambda e^\lambda)$, and so forth. For more details see R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963, pp. 45–46.

A Generalization of Cochran's Theorem

E 1933 [1966, 1018]. *Proposed by H. Kestelman, University College, London, England*

Let $M_1, \cdots, M_k$ be $n\times n$ matrices whose sum is (δ_{rs}), and such that the sum of the ranks of the matrices is n. Prove that $M_rM_s=\delta_{rs}M_r (1\leqq r, s\leqq k)$.

I. *Solution by D. Ž. Djoković, University of Belgrade, Yugoslavia.* It is well known that a $p\times q$ matrix A of rank r has a representation $A=BC$, where B is a $p\times r$ matrix of rank r and C is a $r\times q$ matrix of rank r. Hence, if n_i is the rank of M_i, then

$$(1) \qquad M_i = A_iB_i \qquad (i=1, \cdots, k),$$

where A_i is an $n \times n_i$ matrix of rank n_i and B_i is an $n_i \times n$ matrix of rank n_i. We suppose that $n_i > 0$ for all i (if not, the assertion is false). We have $A_1B_1 + A_2B_2 + \cdots + A_kB_k = I$,

$$[A_1, \cdots, A_k]\begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix} = I, \qquad \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix}[A_1, \cdots, A_k] = I,$$

which implies

$$(2) \qquad B_iA_i = I_{n_i}, \qquad B_iA_j = 0 \qquad (i \neq j).$$

Using (1) and (2) we obtain

$$M_i^2 = A_iB_iA_iB_i = A_iI_{n_i}B_i = A_iB_i = M_i,$$
$$M_iM_j = A_iB_iA_jB_j = A_iOB_j = 0 \qquad (i \neq j).$$

The assertion of this problem is implicitly contained in Lovass-Nagy Viktor, *Matrixszâmítás*, Müszaki matematikai gyakorlatok, C.IV, Budapest 1956.

II. *Solution by E. S. Langford, U. S. Naval Postgraduate School.* Suppose that the vectors $x(i, 1), x(i, 2), \cdots, x(i, n_i)$ are a basis for the range of M_i. By assumption $n_1 + n_2 + \cdots + n_k = n$, so that the collection of vectors $X = \{x(i, j) : i = 1, 2, \cdots, k, \text{ and } j = 1, 2, \cdots, n_i\}$ consists of precisely n vectors. But for any vector x, $x = Ix = M_1x + M_2x + \cdots + M_kx$, so that X spans the space. (I represents the identity matrix (transformation).) Therefore X is a linearly independent set and a basis. It follows that the ranges of the M_i form a direct sum decomposition of the space and are disjoint; that is, the range of M_i and the range of M_j have only the zero-vector in common if $i \neq j$. In terms of matrices (transformations), $M_iM_j = 0$ if $i \neq j$. The idempotence follows from the fact that for any x,

$$M_ix = M_iIx = M_i\left(\sum_{j=1}^{k} M_jx\right) = \sum_{j=1}^{k} M_iM_jx = M_i^2x.$$

Also solved by J. P. Celenza, C. M. Joiner, Jr. & S. J. Pierce, Jack Mettauer, Olga Taussky, R. C. Thompson, and the proposer.

Editorial Note. O. Taussky and the proposer note that the case where the M_i are real symmetric matrices is known as Cochran's theorem and arises in statistics: see, e.g., a recent paper by N. Y. Luther, Annals Math. Stat., 36 (1965) 683–690, and a forthcoming paper by O. Taussky in Monatshefte f. Math. & Phys.

It would be of interest to prove whether or not the requirement of symmetry is necessary in the following two related problems taken from R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960, p. 68:

34. If each A_i is symmetric and $\sum_{i=1}^{m} A_i = I$, then the three following conditions are equivalent:

(a) Each A_i is idempotent. (b) $A_iA_j = 0$, $i \neq j$.

(c) $\sum_i n_i = N$, where n_i is the rank of A_i and N the dimension of the A_i.

35. Let A_i be a collection of $N\times N$ symmetric matrices where the rank of A_i is p_i. Let $A=\sum_i A_i$ have rank p. Consider the four conditions

C1. Each A_i is idempotent. C2. $A_iA_j = 0 \quad i \neq j.$

C3. A is idempotent. C4. $p = \sum_{i=1}^{m} p_i.$

Then

(1) Any two of the three conditions C1, C2, C3 imply all four of the conditions C1, C2, C3, C4.

(2) C3 and C4 imply C1 and C2.

For a proof of the foregoing and some applications see F. A. Graybill and G. Marsaglia, *Idempotent Matrices and Quadratic Forms in the General Linear Hypothesis*, Ann. Math. Stat., 28 (1957) 678–686.

R. C. Thompson uses induction on k to prove the following stronger result: If $M_1+\cdots+M_k=I_n$ and $\rho(M_1)+\cdots+\rho(M_k)=n$, *where ρ denotes rank, then after a simultaneous similarity of $M_1,\cdots,M_k$, the matrices $M_1,\cdots,M_k$ assume the form*

$$\begin{aligned} M_1 &= \operatorname{diag}(I_{\alpha_1}, O_{\alpha_2}, O_{\alpha_3}, \cdots, O_{\alpha_k}),\\ M_2 &= \operatorname{diag}(O_{\alpha_1}, I_{\alpha_2}, O_{\alpha_3}, \cdots, O_{\alpha_k}), \cdots,\\ M_k &= \operatorname{diag}(O_{\alpha_1}, O_{\alpha_2}, \quad \cdots, O_{\alpha_{k-1}}, I_{\alpha_k}). \end{aligned}$$

Here O_α is the α-square zero matrix and I_α is the α-square identity matrix. The result $M_rM_s=\delta_{rs}M_r$ now follows immediately.

A Bounded Newtonian Sequence

E 1934 [1966, 1018]. *Proposed by Gregory Dropkin and Brian Schmidt, Washington, D. C.*

Apply Newton's method for approximating the roots of a function [root $=\lim_{n\to\infty} x_n$, $x_{i+1}=x_i-f(x_i)/f'(x_i)$] to the polynomial $y=1+x^2$. What is the nature of the sequence $\{x_n\}$? For what choices, if any, of x_0 is the sequence bounded?

Solution by Robert Breusch, Amherst College. The problem implies that $x_{i+1}=(x_i^2-1)/2x_i$. This gives, with $x_i=\cot(\alpha_i)$, that $\cot(\alpha_{i+1})=\cot(2\alpha_i)$, whence

$$\alpha_i = 2^i\alpha_0 \qquad (0 < \alpha_0 < \pi).$$

It follows that the sequence $\{x_i\}$ will be bounded if and only if $|2^i\alpha_0-k\pi|$ has a positive lower bound (i, k are nonnegative integers). This, in turn, is the case *if and only if in the binary representation of α_0/π, the lengths of sequences of consecutive zeros as well as the lengths of sequences of consecutive ones are bounded.*

For instance, if α_0/π is the irrational number

$$0.100101001010100101010100\cdots$$

then $|2^i\alpha_0-k\pi|>0.001\pi$, thus $|x_i|<\cot(\pi/8)$; but if

$$\alpha_0/\pi = 0.101001000100001\cdots,$$

the $\{x_i\}$ are unbounded. For rational α_0/π, $\alpha_0/\pi=p/q$, $(p, q)=1$, $\{x_i\}$ is bounded if and only if $q\neq 2^n$.

Also solved by Marcia Ascher, E. J. F. Primrose (England), and the proposer.

Editorial Comment. For the case of complex sequences,

$$x_n = \frac{\tan 2^n a\cdot \operatorname{sech}^2 2^n b - i(\tanh 2^n b\cdot \sec^2 2^n a)}{\tan^2 2^n a + \tanh^2 2^n b},$$

where $\alpha_0=a+ib$. Thus, $\operatorname{Re}(x_n)$ is bounded and approaches 0, $\operatorname{Im}(x_n)$ is bounded and approaches $\pm i$ according as $b<0$ or >0, these being the roots of $x^2+1=0$.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before September 30, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5570. *Proposed by Kwangil Koh, North Carolina State University*

Let R be a ring such that $xRy=0$ implies $x=0$ or $y=0$. Let $x^\perp=\{y \mid xy=0\}$. Let M be the set of all x such that $x^\perp\neq R$, and $x^\perp\subseteq y^\perp$ implies $y^\perp=x^\perp$ or $y^\perp=R$. Prove that if M is finite and not empty then R is finite.

5571. *Proposed by R. Wiegandt, Mathematical Institute, Hungarian Academy of Sciences, Budapest*

Prove that a compact topological ring with no proper closed left ideals is a finite field or a zero-ring of prime order.

5572. *Proposed by Howard Kleiman, Queensborough Community College, New York*

In an Artinian ring A which is commutative and has an identity element, let S be a multiplicatively closed subset of A such that $S\cap(\bigcup_{i=1}^n m_i)=\varnothing$, where the m_i are the maximal ideals of A. Prove that every ideal of A is a contracted ideal of $S^{-1}A$ in A, where $S^{-1}A$ is the ring of fractions of A with respect to S.

5573. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let R be the group of units of a Noetherian ring A which is commutative with an identity. Then every ideal of A is a contracted ideal of $R^{-1}A$, where $R^{-1}A$ is the ring of fractions of A with respect to R.

5574. *Proposed by K. Drabek and L. Beran, CVUT, Prague, Czechoslovakia*

Give a simple formula for the determinant of the matrix Q partitioned into submatrices $B_{ij}=\delta_{ij}B'+b_{ij}I$, $i, j=1, 2, \cdots, n$, where δ_{ij} is the Kronecker delta,

$B'=(b_{ji})$ the transpose of the matrix B, I the identity matrix, b_{ij} elements of a commutative field R.

5575. *Proposed by Alexandru Lupas, Cluj, Rumania*

Let $Q(0, a)$ be the set of functions which are defined and bounded on $[0, \infty)$ and which are continuous on the interval $[0, a]$, $a>0$. On $Q(0, a)$ we define, for a positive integer n, the linear positive operator L_n by

$$(*)\qquad (L_nf)(x) = \sum_{\nu=0}^{\infty}\binom{n+\nu-1}{\nu}\Delta_{1/n}^{\nu}f(0)x^{\nu},$$

where $\Delta_{1/n}^{\nu}$ is the difference of order ν corresponding to the increment $1/n$.

(1) Prove that $\|L_nf-f\|\to 0$ where $\|f\|$ is the maximum of $|f(x)|$ for $x\in[0, a]$.

(2) We say that a function f is P-convex of order s on the reals if the divided differences of order $s+2$, $[x_1, x_2, \cdots, x_{s+2}; f]$, are positive for every real x_i, $i=1, 2, \cdots, s+2$. Show that if f is convex of order s then L_nf is also a convex function of order s.

(3) The sequence of operators $\{L_n\}$, defined by (*) is decreasing on the class of convex functions of order 1, and satisfies $L_{n+1}-L_n=0$, for sufficiently large n, on the class of linear functions.

5576. *Proposed by F. L. Sandomierski, University of Wisconsin*

Show that an Abelian group D is divisible if and only if every nonzero homomorphic image of it is not finitely-generated.

5577. *Proposed by J. L. Ercolano and F. G. Gustavson, IBM, Yorktown Heights, N. Y.*

Let $A=\{a_{ij}\}$ be an "X" matrix, i.e. a matrix with nonzero elements only on the main or alternate diagonals. Determine conditions under which A is similar to a diagonal matrix over the field of complex numbers.

5578. *Proposed by John H. Smith, Massachusetts Institute of Technology*

Show that a real $n\times n$ matrix with the property that $a_{ij}=0$ for $|i-j|\geqq 2$ and $a_{i,i+1}\cdot a_{i+1,i}>0$, for all i, has distinct eigenvalues.

5579. *Proposed by David Cohoon, Purdue University*

Let X denote an arbitrary topological space. Then a homeomorphism $f:X\to X$ is said to be *mixing* provided that for every pair of nonempty open sets U and V contained in X, there exists an integer $N>0$ such that $n>N$ implies $f^n(U)\cap V\neq\emptyset$. It is easy to see that any Tychonoff product X^s, with s infinite, admits a topologically mixing homeomorphism.

Define the *mixing codimension* of X to be $\inf\{$cardinality of s: X^s admits a topologically mixing homeomorphism$\}$. Prove or disprove the conjecture that the mixing codimension of the unit circle, $\{\exp(i\theta): 0\leqq\theta\leqq 2\pi\}$, with the topology induced by the plane E^2, is two.

SOLUTIONS OF ADVANCED PROBLEMS

Fields in Z_m

5469 [1967, 208]. *Proposed by Charles Green, University of Maine*

Let Z_m be the ring of integers modulo the composite integer m, and let k be a proper factor of m. Then $I=\{x\in Z_m: \bar{k}x=\bar{0}\}$ is an ideal in Z_m. Find necessary and sufficient conditions that I be a field; if it is, characterize the identity.

Solution by L. D. Crowson, Birmingham-Southern College. The required condition is that k be a prime and that k^2 does not divide m.

Proof: necessity: If $k=pq$ and $m=pqn$, then $\bar{p}\bar{n}$ and $\bar{q}\bar{n}$ are both nonzero elements of I, but $\bar{p}\bar{n}\cdot\bar{q}\bar{n}=\bar{p}\bar{q}\bar{n}^2=\bar{0}$. If k^2 divides m, then $m=k^2n$ and $\bar{k}\bar{n}$ is a nonzero element of I such that $\bar{k}\bar{n}\cdot\bar{k}\bar{n}=\bar{k}^2\bar{n}^2=\bar{0}$.

Sufficiency: If k is a prime such that k^2 does not divide m, then $x=m/k$ and k are relatively prime. If $y\in I$, then $y=\bar{p}\bar{x}$ for some integer p. Obviously $a\equiv b(\text{mod } k)$ implies $ax\equiv bx(\text{mod } m)$. Hence the distinct elements of I are $\bar{p}\bar{x}$ for $p=0, 1, \cdots, k-1$. An element $\bar{p}\bar{x}$ of I is 0 if and only if k divides p. Hence, if $\bar{a}\bar{x}$ and $\bar{b}\bar{x}$ are nonzero elements of I, then $\bar{a}\bar{x}\cdot\bar{b}\bar{x}=\bar{a}\bar{b}\bar{x}^2$ is nonzero, since k, being prime, cannot divide abx because it does not divide a, b, or x. Hence I is an integral domain and, since it is finite, it is a field.

To characterize the identity of this field, we consider x as an element of $Z_{(k)}$, the field of integers modulo k. Since x is not divisible by k, there exists a nonzero integer p such that $px\equiv 1(\text{mod } k)$ and we can choose p to be an element of the set $\{1, 2, \cdots, k-1\}$. In that case $\bar{p}\bar{x}$ is a nonzero element of I, and $\bar{p}\bar{x}\cdot\bar{p}\bar{x}=\overline{p^2x^2}=\overline{p^2xx}$. Now since $px\equiv 1$ (mod k), $p^2x\equiv p$ (mod k) and $p^2xx\equiv px$ (mod m). Hence $\bar{p}\bar{x}\cdot\bar{p}\bar{x}=\bar{p}\bar{x}$ and, since $\bar{p}\bar{x}$ is nonzero, it is the identity of I.

Also solved by C. B. Baytop & J. E. Joseph, Kenneth Dahlberg, E. R. Gentile (Argentina), M. G. Greening (Australia), G. A. Heuer, D. S. Lawrence, P. L. Montgomery, Barbara L. Osofsky, Simeon Reich (Israel), Charles Riley, Ira Rosenholtz, D. P. Sumner, Stephen Tice, Philip Trauber, Necdet Üçoluk, Steven Weintraub, D. G. Wilson, Kenneth Yanosko, and the proposer.

Density in the Complementary Class $\mathfrak{L}_q$

5470 [1967, 328]. *Proposed by Edwin Hewitt, University of Washington*

Let $(X, \mathfrak{A}, \mu)$ be a measure space such that if $A\in\mathfrak{A}$ and $\mu(A)=\infty$, then there is a set $B\in\mathfrak{A}$ such that $B\subset A$ and $0<\mu(B)<\infty$. Let p be a real number such that $p>1$ and let f be a complex-valued $\mathfrak{A}$-measurable function on X such that $f\notin\mathfrak{L}_p(X, \mathfrak{A}, \mu)$. Let $p'=p/(p-1)$. Prove that the set $\{\phi\in\mathfrak{L}_{p'}(X, \mathfrak{A}, \mu): f\bar{\phi}\in\mathfrak{L}_1, \int_X f\bar{\phi}d\mu=0\}$ is dense in $\mathfrak{L}_{p'}(X, \mathfrak{A}, \mu)$.

I. *Solution by P. R. Chernoff and W. C. Waterhouse, Harvard University.* The subspace $E=\{\phi\in\mathfrak{L}_{p'}: f\bar{\phi}\in\mathfrak{L}_1\}$ is dense in $\mathfrak{L}_{p'}$, since it includes characteristic functions of all sets A with $\mu(A)<\infty$ and $A\subseteq\{x \mid n\leqq|f(x)|<n+1\}$. If the linear functional $T(\phi)=\int_X\bar{f}\phi d\mu$ on E is continuous, it extends to $\mathfrak{L}_{p'}$, and hence there is a $g\in\mathfrak{L}_p$ such that $\int\bar{f}\phi=\int\bar{g}\phi$ for all $\phi\in E$. This, however, implies $f=g$ almost

everywhere, or $f \in \mathfrak{L}_p$. This contradiction shows that T cannot be continuous on E; its kernel therefore is dense in E, and hence in $\mathfrak{L}_{p'}$.

II. *Solution by L. J. Wallen, Stevens Institute of Technology.* We must suppose $|f| < \infty$ almost everywhere. The hypothesis on μ implies that if $\int_B |f|^p d\mu = \infty$, then there exists $C \subset B$ such that $1 \leqq \int_C |f|^p d\mu < \infty$. If not, there would be $D \subset B$ with $\int_E |f|^p d\mu = 0$ or ∞ for each measurable $E \subset D$, and $\int_D |f|^p d\mu = \infty$. But D contains a set of finite nonzero measure on which f satisfies $0 < m \leqq |f| \leqq M < \infty$, a contradiction.

It suffices to approximate the indicator function ψ of a set A_0 of finite measure on which f is essentially bounded. Use the above remark to find mutually disjoint sets A_n outside A_0 with $1 \leqq \int_{A_n} |f|^p d\mu < \infty$. Next find numbers ξ_n satisfying $\sum_{n=1}^{\infty} |\xi_n| < \infty$, $\sum |\xi_n|^q < \epsilon$ (using q for p') and $\sum_1^{\infty} \xi_n = -\int_{A_0} f d\mu = \lambda$. The ξ_n's may, for instance, be defined by $\xi_n = \lambda(n^r \sum_{k=1}^{\infty} k^{-r})^{-1}$ with r close to 1. Now define

$$\phi(x) = \begin{cases} 1 & \text{for } x \in A_0 \\ \xi_n \overline{\operatorname{sgn} f(x)} \, |f|^{p-1}(x) \Big/ \displaystyle\int_{A_n} |f|^p d\mu & \text{for } x \in A_n \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\int_X |\psi - \phi|^q d\mu = \sum |\xi_n|^q (\int_{A_n} |f|^p d\mu)^{1-q} < \epsilon$ and

$$\int_X f\phi = -\lambda + \sum \xi_n = 0.$$

Also solved by the proposer, who notes that this is a generalization of Problem 5237 [1965, 1034]. This solution contains a small error: the set E_n should be defined as $\{x: x \in [0, 1], |F(x)| \leqq n\}$.

Some Solvable Groups

5471 [1967, 328]. *Proposed by W. A. McWorter, University of British Columbia*

Let G be a group and let σ be a nontrivial automorphism of G, such that for each $x \in G$, $\sigma(x) = x$ or $\sigma(x) = x^{-1}$. Prove that G is solvable.

Solution by A. D. Sands, Queen's College, Dundee, Scotland. Let $H = \{x \in G \mid \sigma(x) = x\}$. Then clearly H is a proper subgroup of G. Let $h \in H$, $a \notin H$. $\sigma(ah) = \sigma(a)\sigma(h) = a^{-1}h$. $\sigma(ah) = ah$ or $(ah)^{-1}$. If $ah = a^{-1}h$, then $a = a^{-1}$ and $a \in H$; thus it must be true that $a^{-1}h = (ah)^{-1}$, i.e. $a^{-1}ha = h^{-1}$. Therefore H is normal. Let h, $k \in H$, $a \notin H$. As above, $h^{-1} = (ak)^{-1}h(ak) = k^{-1}a^{-1}hak = k^{-1}h^{-1}k$. Thus $hk = kh$ and H is Abelian.

If a, b, $ab \notin H$, then $\sigma(a^{-1}b^{-1}) = (a^{-1}b^{-1})^{-1} = \sigma(a^{-1})\sigma(b^{-1})$, i.e. $ba = ab$. If $a, b \notin H$, $ab \in H$, then $aHbH = H = bHaH$. It follows that G/H is Abelian.

Thus G is metabelian and so solvable.

All such groups may be classified. Clearly G is Abelian if and only if $h=h^{-1}$ for each $h\in H$ and in this case $\sigma(x)=x^{-1}$ for all $x\in G$. If there exists an element $h\in H$, $h^2\neq e$, and a, $b\notin H$, then $(ab)^{-1}hab=b^{-1}a^{-1}hab=h$. Thus $ab\in H$ and H has index 2 in G. In particular, $a^2\in H$, $a^2\neq e$, as $\sigma(a)=a^{-1}\neq a$, but $a^4=e$, as $a^{-1}a^2a=a^{-2}$. Conversely, given any Abelian group H containing an element b of order 2, H may be extended by an automorphism α where $\alpha h=h^{-1}$ and $\alpha^2=b$, to give a group G with an automorphism σ of the required type, where $\sigma(h)=h$, $\sigma(h\alpha)=hb\alpha$.

Also solved by Christine Ayoub (Germany), William Bonney, Warren Brisley (England), P. R. Chernoff & W. C. Waterhouse, Roy O. Davies (England), Peter Dembowski (Germany), John P. Dixon (Australia), D. Ž. Djoković, (Yugoslavia), M. A. Ettrick, M. F. Friedell, M. G. Greening (Australia), C. M. Joiner, Jr. & S. J. Pierce, Ka Menehune, M. S. Osborne, Barbara L. Osofsky, David Promiscow, P. R. Sanders (Kenya), W. R. Scott, Bhama Srinivasan (India), R. A. Stroud, C. J. Stuth, Hugo Sun, Kenneth Yanosko, and the proposer.

The First Zero of $J_n'(x)$

5472 [1967, 328]. *Proposed by James Duemmel, University of Montana*

Let $J_n(x)$ be the Bessel function of the first kind of order n, $n\geqq 0$. Let $J_n'(\lambda)=0$, $\lambda>0$. Prove that $\lambda>n$.

I. *Solution by Sidney Spital, California State Polytechnic College, Pomona.* By an application of one of Lommel's integrals,

$$I\equiv\int_0^\lambda xJ_n(x)^2dx=\tfrac{1}{2}\lambda^2J_n'(\lambda)^2+\tfrac{1}{2}(\lambda^2-n^2)J_n(\lambda)^2.$$

Since (for $J_n'(\lambda)=0$, $\lambda>0$) $I>0$ and $J_n(\lambda)\neq 0$, it follows that $\lambda^2>n^2$, as required.

II. *Solution Richard Kowalski, Baltimore, Md.* The statement is trivially true when $n=0$. When $n>0$, we need only show that $\beta>n$ where β is the least positive value of λ for which $J_n'(\lambda)=0$. The function $J_n(x)$ satisfies the equation

$$x^2J_n''(x)+xJ_n'(x)+(x^2-n^2)J_n(x)=0, \tag{1}$$

which may be rewritten as

$$(xJ_n'(x))'=\left(\frac{n^2-x^2}{x}\right)J_n(x).$$

It follows that

$$J_n'(x)=\frac{1}{x}\int_0^x\left(\frac{n^2-y^2}{y}\right)J_n(y)dy. \tag{2}$$

Since $J_n(x)$ is differentiable on $(0,\infty)$, $J_n(0)=0$ for $n>0$, and $J_n(x)$ is positive for small x, therefore the function $J_n(x)$ has the same sign on the interval $(0,\beta)$. Set $x=\beta$ in (2) to obtain

$$(3) \qquad 0 = J_n'(\beta) = \frac{1}{\beta}\int_0^\beta \left(\frac{n^2 - y^2}{y}\right) J_n(y)dy.$$

If $\beta \leqq n$ the integrand in (3) does not change sign on the interval $(0, \beta)$ and, hence, the integral cannot equal zero. Hence $\beta > n$ as required.

Also solved by C. T. Beers, L. Carlitz, P. R. Chernoff & W. C. Waterhouse, A. E. Danese, D. Ž. Djoković (Yugoslavia), M. A. Ettrick, C. W. Haines, D. A. Hejhal, M. E. Muldoon, G. H. Ryder, P. van der Driessche, J. E. Wilkins, Jr., Students in Professor Vayo's Class in Mathematics 475, and the proposer.

The problem has been found in Watson, *Theory of Bessel Functions*, 1952, pp. 485–486, and in Birkhoff and Rota, *Ordinary Differential Equations*, p. 263.

An Inequality for Inner Products

5473 [1967, 328]. *Proposed by A. A. Goldstein and J. V. Ryff, University of Washington*

Let $F(r) = r^2 + 1$. Let $[,]$ represent an inner product, with $||x||^2 = [x, x]$. Prove that for any real number r, and any x, y with $||x|| \geqq ||y||$,

$$||x||^{2r} + ||y||^{2r} - 2||x||^r||y||^r \frac{[x, y]}{||x|| \cdot ||y||} \leqq \begin{cases} ||x||^{2r-2}||x - y||^2 F(r), & r \geqq 1, \\ ||y||^{2r-2}||x - y||^2 F(r), & r < 1. \end{cases}$$

Solution by L. E. Clarke, University of East Anglia, Norwich, U.K. For a real-valued inner product and positive r it will be shown that the inequality holds with $F(r) = \max(1, r^2)$, and that this value of $F(r)$ is the best possible.

Write $\xi = ||x||$ and $\eta = ||y||$. Then $\xi \geqq \eta > 0$.

Case (i): $r > 1$ and $F(r) = r^2$. The given inequality is equivalent to

$$\xi^{2r} + \eta^{2r} - 2\xi^{r-1}\eta^{r-1}[x, y] \leqq r^2\xi^{2r-2}(\xi^2 + \eta^2 - 2[x, y]),$$

i.e. to

$$(1) \qquad 2\xi^{r-1}[x, y](r^2\xi^{r-1} - \eta^{r-1}) \leqq (r^2 - 1)\xi^{2r} + r^2\xi^{2r-2}\eta^2 - \eta^{2r}.$$

Since $r^2\xi^{r-1} - \eta^{r-1} > 0$ and $[x, y] \leqq \xi\eta$, it suffices to prove that

$$2\xi^r\eta(r^2\xi^{r-1} - \eta^{r-1}) \leqq (r^2 - 1)\xi^{2r} + r^2\xi^{2r-2}\eta^2 - \eta^{2r}$$

or that

$$(2) \qquad (r^2 - 1)\zeta^{2r} - 2r^2\zeta^{2r-1} + r^2\zeta^{2r-2} + 2\zeta^r - 1 \geqq 0,$$

where $\zeta = \xi/\eta \geqq 1$. The inequality (2) is equivalent to

$$r^2\zeta^{2r-2}(\zeta - 1)^2 \geqq (\zeta^r - 1)^2.$$

This is trivial if $\zeta = 1$, and holds with strict inequality if $\zeta > 1$ since then, by the mean-value theorem, $(\zeta^r - 1)/(\zeta - 1) = r\zeta_1^{r-1}$ for some $\zeta_1 \in (1, \zeta)$.

Case (ii): $r < 1$ and $F(r) = 1$. A similar argument shows that the given inequality is equivalent to

$$(1') \qquad 2\eta^{r-1}[x, y](\eta^{r-1} - \xi^{r-1}) \leqq \xi^2(\eta^{2r-2} - \xi^{2r-2}),$$

and that it suffices to prove that

$$(2') \qquad \zeta^{2r} - 2\zeta^r - \zeta^2 + 2\zeta \leqq 0.$$

The inequality (2') is equivalent to $(\zeta^r-1)^2 \leqq (\zeta-1)^2$. This is trivial if $\zeta=1$, and holds with strict inequality if $\zeta>1$.

It will be noted that there is equality if and only if

(a) $r = 1$; or

(b) $r > 1$ and $\xi = \eta$, $[x, y] = \xi\eta$, i.e. $x = y$; or

(c) $r < 1$ and $\xi = \eta$.

To show that the inequalities are best possible, suppose that $x=\lambda y$. Then for $r>1$ and $\lambda>1$, this yields

$$F(r) \geqq (\lambda^r - 1)^2/\{\lambda^{2r-2}(\lambda - 1)^2\} \to r^2 \quad \text{as } \lambda \to 1.$$

Similarly, for $r<1$ and $\lambda<-1$, it yields

$$F(r) \geqq (|\lambda|^r + 1)^2/(|\lambda| + 1)^2 \to 1 \quad \text{as } \lambda \to -1.$$

Also solved by J. Chaudhuri (Scotland), Mary R. Embry, L. Kuipers, E. A. Memmott, and the proposers.

Dr. Embry, in her solution, notes a pair of slightly improved inequalities in case r is negative, by using the corresponding results for $-r$ and then multiplying by $\|x\|^{2r}\|y\|^{2r}$. She also notes the possibility of a complex inner product $[x, y]$, by replacing $[x, y]$ by its real part in the statements of the inequalities.

A Generalized Binomial Theorem

5474 [1967, 329]. *Proposed by Benjamin Volk, Yeshiva University*

Given an integer x and a sequence $\{\lambda_n\}$, $\lambda_0=1$, $\lambda_k\neq 0$. Define the $\{\lambda_n\}$ kth power of x, $x^{(k)}_{\{\lambda_n\}}=x^{(k)}$ by

$$\frac{(x+1)^{(k)}}{\lambda_0\lambda_1\cdots\lambda_k} = \sum_{j=0}^{k} \frac{x^{(k-j)}}{\lambda_0\lambda_1\cdots\lambda_{k-j}}\cdot\frac{1^{(j)}}{\lambda_0\lambda_1\cdots\lambda_j}, \qquad 0^{(k)} = \begin{cases}1, & k = 0,\\ 0, & k > 0.\end{cases}$$

Then prove

$$\frac{(x+y)^{(k)}}{\lambda_0\lambda_1\cdots\lambda_k} = \sum_{j=0}^{k} \frac{x^{(k-j)}}{\lambda_0\lambda_1\cdots\lambda_{k-j}}\cdot\frac{y^{(j)}}{\lambda_0\lambda_1\cdots\lambda_j}.$$

This generalizes a result in Gloria Olive, *Generalized Powers*, this MONTHLY, 72 (1965) 619–627.

Solution by P. R. Chernoff and W. C. Waterhouse, Harvard University. Let $\phi_k(x)=x^{(k)}/\lambda_0\lambda_1\cdots\lambda_k$. Set $f(x, t)=\sum_{k=0}^{\infty}\phi_k(x)t^k$; the result to be proved is equivalent to the identity

$$f(x+y, t) = f(x, t)\cdot f(y, t).$$

But by definition $f(x+1, t)=f(x, t)\cdot f(1, t)$ whence, by induction, $f(n, t)=f(1, t)^n$. The identity is now immediate.

Also solved by L. Carlitz, Roy O. Davies (England), D. A. Hejhal, Gloria Olive, and M. S. Osborne.

Davies observes that the given conditions do not determine $1^{(k)}$. Carlitz notes that the expression $[f(1, t)]^n$ may be considered with n complex and thus $x^{(k)}$ may be defined for arbitrary x.

Abridged Power Sets

5475 [1967, 329]. *Proposed by Louis Comtet, Viroflay, France*

Let s be an integer >0, E an at most countable set, and $\mathcal{V}$ a system of subsets of E which has the property that each member of E is a member of at most s sets of $\mathcal{V}$. What can be said about Card $\mathcal{V}$?

Solution by Robert E. Johnson, University of Mississippi. Consider first the case where E is infinite. In this case the elements of E can be written in a sequence, $\{x_n\}$. Let S_n denote the subset of $\mathcal{V}$ such that each member of S_n contains x_n. Each S_n contains at most s elements. Thus $\bigcup_{n=1}^{\infty} S_n$ is at most countable. Suppose $A\in\mathcal{V}$ then A contains a member of E. Suppose $x_k\in A$, then $A\in S_k$ and hence $A\in\bigcup_{n=1}^{\infty} S_n$. Thus $\mathcal{V}\subset\bigcup_{n=1}^{\infty} S_n$. This shows $\mathcal{V}$ is at most countable.

Now consider the case where E is finite. If E contains n elements then the power set, P, of E contains 2^n elements. Partition the set P into 2^{n-1} disjoint subsets $P_1, \cdots, P_k$ $(k=2^{n-1})$ such that each P_i contains two elements of P which are complementary (i.e. their union is E). Each element of E occurs in 2^{n-1} elements of P. Hence if $s=2^{n-1}$ then $\mathcal{V}$ contains at most $2^n=2s$ elements. If $s=2^{n-1}-1$, then $\mathcal{V}$ contains at most $2^n-1=2s+1$ elements, since to reduce s by 1 requires the removal of each element of E from at least one element of P. This can be accomplished by removing the set E from P. If $s=2^{n-1}-2$, then $\mathcal{V}$ contains at most $2^n-3=2s+1$ elements since to reduce s by 1 again requires the removal of another P_i which contains two sets. In general, if $s=2^{n-1}-r$, then Card $\mathcal{V}\leqq 2^n-1-2(r-1)=2s+1$. In summary, if $s<2^{n-1}$ then there are at most $2s+1$ elements of $\mathcal{V}$; if $s=2^{n-1}$ there are at most $2s$ elements of $\mathcal{V}$.

Also solved by K. A. Bowen, P. R. Chernoff & W. C. Waterhouse, Roy O. Davies (England), G. F. Schumm, Hugo Sun, and the proposer.

Note a related problem, E 1753 [1966, 200].

Subalgebra of Square Matrices

5476 [1967, 329]. *Proposed by W. A. McWorter, University of British Columbia*

Let $I=\{1, \cdots, n\}$. Let T be a subset of $I\times I$. Define M_T to be the set of all $n\times n$ matrices $A=(a_{ij})$ over a field F such that (i, j) in $I\times I\backslash T$ implies $a_{ij}=0$ in F. Characterize all T such that M_T is a subalgebra of the complete $n\times n$ matrix algebra over F.

Solution by Ka Menehune, University of Hawaii. Note that $M_T = \{A \mid a_{ij} \neq 0$ implies $(i, j) \in T\}$ and that M_T is linearly closed; hence M_T is a subalgebra if and only if M_T is multiplicatively closed.

If T contains two pairs (i, r), (r, j) then the standard matrix units E_{ir} and E_{rj} belong to M_T, as must the product $E_{ir}E_{rj} = E_{ij}$. Hence $(i, r) \in T$, $(r, j) \in T$ implies $(i, j) \in T$ is a necessary condition if M_T is to be a subalgebra. This condition is also sufficient: Let A, $B \in M_T$. If $AB = 0$, then $AB \in M_T$. If $AB \neq 0$, let (i, j) be such that $(AB)_{ij} \neq 0$. Now

$$(AB)_{ij} = \sum_{r \in I} a_{ir}b_{rj} = \sum_{r \in R} a_{ir}b_{rj},$$

where $R = \{r \in I \mid a_{ir} \neq 0,\ b_{rj} \neq 0\} = \{r \in I \mid (i, r) \in T,\ (r, j) \in T\} \neq \varnothing$. It follows that $(i, j) \in T$, and hence that $AB \in M_T$.

If we regard T as a relation on I to I, then $T \circ T$ is defined and the result is conveniently expressed as follows: M_T is a subalgebra if and only if $T \circ T \subset T$.

Also solved by P. R. Chernoff & W. C. Waterhouse, M. D. Mavinkurve (India), and the proposer.

Bounded Variation Solution of a Functional Equation

5477 [1967, 329]. *Proposed by D. J. Newman, Yeshiva University*

Let $|\alpha| = 1$. Show that there is a nontrivial bounded variation function $f(x)$ on $[0, 1]$ which satisfies $\alpha f(x) = f(x/2) + f((x+1)/2)$.

Solution by the proposer. For $\alpha = 1$ or indeed any root of unity, the desired $f(x)$ can be explicitly given in terms of simple step functions. For general α no such simple solution seems available and we must rely on a pure existence proof.

We introduce the well-known Banach space C of continuous (complex valued) functions with period 1, under the usual uniform norm, and prove first that the set of functions $\alpha g(x) - g(2x)$, $g(x) \in C$, is not dense in C. To establish this we apply a special case of a theorem of Szidon [see A. Zygmund, *Trigonometric Series*, 2nd ed., New York (1952), pp. 139–140]:

$$\text{(1)} \qquad \operatorname*{Max}_{|z|=1} \left| \sum a_n z^{2^n} \right| \geq \tfrac{1}{4} \sum |a_n| .$$

This enables us to prove that the function $e(x) = e^{2\pi i x}$ cannot be closer than $\frac{1}{4}$ to any $\alpha g(x) - g(2x)$. For, if we write

$$\alpha g(x) - g(2x) = e(x) + \delta(x)$$

then, by iteration we get

$$\alpha^N g(x) - g(2^N x) = \sum_{n=1}^{N} \alpha^{N-n} e(2^{n-1}x) + \sum_{n=1}^{N} \alpha^{N-n} \delta(2^{n-1}x).$$

It follows from (1) that

$$2\|g\| \geqq \left| \sum_{n=1}^{N} \alpha^{N-n} e(2^{n-1}x) \right| - N\|\delta\| \geqq N(\tfrac{1}{4} - \|\delta\|).$$

Therefore $\|\delta\| \geqq \frac{1}{4} - 2\|g\|/N$; letting $N \to \infty$ shows that $\|\delta\| \geqq \frac{1}{4}$ as required.

From the Riesz representation theorem we can now find a nonconstant function of bounded variation, $f(x)$, such that

$$\int_0^1 [\alpha g(x) - g(2x)]df(x) = 0$$

for all $g \in C$. Since

$$\int_0^1 [\alpha g(x) - g(2x)]df(x) \equiv \int_0^1 g(x)d\{\alpha f(x) - f(x/2) - f((x+1)/2)\},$$

we conclude that $\alpha f(x) - f(x/2) - f((x+1)/2)$ is a constant. Hence, $f(x)$, modified if necessary by a constant, satisfies the given functional equation.

We note finally that the result is true by a simpler proof if $|\alpha| < 1$ and is false if $|\alpha| > 1$.

A Converse Convergent Theorem

5478 [1967, 329]. *Proposed by Benjamin Volk, Yeshiva University*

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic for $|z| < 1$, and let $\{F(x_n)\}_{n=0}^{\infty}$ form a convergent sequence for some sequence of nonnegative reals $\{x_n\}$ converging to 1. If $\{a_n\}$ lies in some central sector of the closed unit disk of angle less than π, prove that $\sum_{n=0}^{\infty} |a_n| < \infty$.

Solution by D. A. Hejhal, University of Chicago. It is enough to assume that $\{F(x_n)\}$ is bounded on some sequence $x_n \to 1$. Multiplication of $F(z)$ by a constant shows that we may take $a_n = R_n e^{i\phi_n}$ with $|\phi_n| \leqq \frac{1}{2}\pi - \delta$, $\delta > 0$. If also $a_n = c_n + id_n$, then $R_n \geqq c_n = R_n \cos \phi_n \geqq R_n \sin \delta > 0$ and $\sum R_n$, $\sum c_n$ converge or diverge together.

Now if $F(x_m)$ is bounded, the same must be true for Re $F(x_m) = \sum_{n=0}^{\infty} c_n x_m^n$, $x_m \to 1$, $c_n > 0$, from which the boundedness and convergence of $\sum c_n$ is an immediate consequence.

Also solved by P. R. Chernoff and W. C. Waterhouse, Roy O. Davies (England), M. D. Mavinkurve (India), Chang Sung-sheng (Taiwan), and Laurence Zalcman.

Zalcman notes that the given condition on a_n is sufficient but is not necessary, as evidenced by the series $\sum (-1)^n z^n$. Generalizations of the problem are possible both in terms of the conditions on a_n and in terms of the sequence $x_m \to 1$, along lines similar to extensions of Tauber's original theorem. See, for example, E. C. Titchmarsh, *Theory of Functions*, 2nd ed., pp. 231, ff., and the treatise on *Divergent Series* by G. H. Hardy.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto, 5, Canada. Correspondence about reviews is welcomed.

Lectures on Functional Equations and their Applications. By J. Aczel. Translated by Scripta Technica Inc. Supplemented by the Author. Edited by Hansjorg Oser. Academic Press, New York, 1966. xix+510 pp. $19.50.

In his concluding remarks the author states, "The principal task, however, would be to create a *systematic theory* of functional equations, whose outlines still appear rather nebulous with our present knowledge. It seems that great difficulties exist, merely in drawing from the structure of a functional equation conclusions concerning the structure of its solutions, . . ." This remark indicates what the structure of a book of this nature must be. It is an examination of many special cases of functional equations with some applications to geometry, physics, probability, and group theory. In his foreword to the German edition the author points out that in order to limit the scope of the book functional equations are so defined as not to include differential, integral, and integro-differential equations. To limit the techniques used, equations for operators and functionals are not considered. A perusal of the table of contents shows that this leaves an adequate field to study.

The techniques used in this book are elementary in nature relying mainly on the calculus, a small knowledge of measure theory, and mostly ingenuity. The bibliography is extensive (pp. 383–508) and is listed by the years in which the papers appeared (1747–1965). The book is certainly an excellent reference bringing under one cover material which is scattered throughout the literature. It is not meant as a text and there are no exercises. In the concluding remarks and in the body of the text various unsolved problems are mentioned.

FRANK HAHN, Yale University

Elements of Probability Theory. By J. Bass. Translated by W. McKay. Translation Editor A. Jeffrey. Foreword by E. W. Barankin. Academic Press, New York, 1966.

This is a fine book. It would serve as an excellent text for science or engineering students who have had a very solid background in mathematics and have some 'mathematical maturity'. However, the reviewer does not feel that the book is a suitable text for a class composed of average junior engineering students. The book is well written and has many interesting examples. The author's explanations are clear and the book is an enjoyable one to read. One criticism of the book is that there are too few exercises. It would also help the American student if the bibliography contained more references in English.

MELVIN KATZ, University of New Mexico

Topology. By Gustave Choquet. Translated by Amiel Feinstein. Academic Press, New York and London, 1966. x+337 pp. $12.50.

The title of this book is a misnomer. It is an introduction to functional analysis well suited for a good undergraduate curriculum. I agree with the publishers that "this introduction to topology and analysis is distinguished by its author's facility for selecting and rendering easily understandable the concepts that are truly fundamental, and by his ability to present this subject in a simple and straightforward manner." The book is further distinguished by its large collection of problems (370 in all) and its interesting examples.

The first chapter covers the bare essentials of topological spaces and metric spaces.

Function spaces are discussed and the theorem of Ascoli is proved for compact metric spaces. The second chapter deals with assorted properties and theorems for functions valued in the extended real line $\overline{R}$. The Stone-Weierstrass theorem is proved. A particularly interesting section is devoted to means relative to a monotone function. As a corollary to this theory, it is shown very neatly that harmonic mean $\leq$geometric mean $\leq$arithmetic mean $\leq$quadratic mean. The third and final chapter is a fairly condensed introduction to locally convex topological and normed vector spaces.

Occasionally, the book suffers from being the second of three parts of a course of analysis, as there are some references to the missing parts. At points, the style is succinct to the point of being bald and the book may, therefore, not be too well suited for self-study. A further disadvantage that it shares with many books is that there is a lack of explicit connection with other parts of the mathematical literature. This, to be sure, is mitigated by a short bibliography at the end of each chapter. However, the faults of the book are relatively minor and it is a very welcome addition to the mathematical literature in English.

J. C. TAYLOR, McGill University

Theory and Examples of Point-Set Topology. By John Greever. Brooks/Cole, Belmont, California, 1967. x+130 pp. $6.95.

This book belongs to the growing family of texts which set out to teach point set topology to the undergraduate. In such a new undertaking, the points of view of these texts vary considerably. This one is rather ambitious. In two chapters, the first on topological spaces, and the second on metric spaces, it covers the basic topics which one would expect to find in a first course in topology with well chosen examples, intuitive discussions, and an excellent historical account of the main theorems. But besides all this, there is Hewitt's example of a regular but not completely regular space, Stone's theorem that every metric space is paracompact, and other topics which have rarely strayed outside the graduate course. In chapter zero are collected the topics from algebra and set theory which will be needed later, including Zorn's lemma and the ordinals obtained from the reals by well ordering.

The proofs of many theorems are left as exercises for the student, in the tradition of R. L. Moore, so that he may learn to prove theorems for himself. The book is intended for a student with a first course in calculus and no background in algebra. The thought that such a student, even in an honors class, could supply a proof of Baire's category theorem on p. 116 would be only a fond hope at the reviewer's institution, and probably at a good many others as well.

J. V. WHITTAKER, University of British Columbia

Plane Geometry and its Groups. By Heinrich W. Guggenheimer. Holden-Day, San Francisco, 1967. xii+288 pp. $9.50.

The title of this book is not quite accurate. Its main function is the development of euclidean geometry and of the group of the euclidean motions following Bachmann and using the reflections as building stones. The elementary geometry of the euclidean plane and the group of the similitudes are adequately dealt with. Conformal geometry is approached via euclidean geometry and the conformal model of the hyperbolic plane is discussed. Neither affine nor projective geometry are treated. Instead, attractive sections on crystallographic groups and on geometric inequalities are included. (The fundamental theorems of projective geometry are dealt with by euclidean methods. The reviewer believes that, presented this way, they lose their "importance for a general education in geometry"; cf. pp. 98 f.)

This book contains enough interesting material for a course for mathematics minors and future high school teachers. (In the reviewer's opinion, the geometric education of

these groups should stress affine rather than euclidean geometry.) A mathematics senior or a graduate student interested in geometry might prefer the weightier works of Artin, Bachmann, and Coxeter. Through them he can experience the beauty of a geometric theory.

PETER SCHERK, University of Toronto

A Short Course in Automorphic Functions. By Joseph Lehner. Holt, Rinehart and Winston, New York, 1966. vii+144 pp. $5.95.

This book is a greatly abridged version of the author's stimulating treatise *Discontinuous Groups and Automorphic Functions* (Math. Surveys 8, Amer. Math. Soc.). Chapter I covers some basic facts about Fuchsian groups in the upper half plane. Chapter II discusses automorphic functions and forms via Poincaré series. Chapter III applies results from Riemann surface theory, most of which are not proved, to compute the dimension of certain spaces of automorphic forms. This book may be a useful supplement to the usual chapter on elliptic functions in the first course on complex variables; it requires only a minimum of preparation.

A. MARDEN, University of Minnesota

Algebraic Extensions of Fields. By Paul J. McCarthy. Blaisdell, Waltham, Mass., 1966. ix+166 pp. $8.50.

Until the appearance of Jacobson's *Lectures in Abstract Algebra*, Volume III, no adequate text was available for the study of Galois theory beyond the material in Artin's classic work. McCarthy's excellent text is another fine addition to the literature.

The text begins with the Artin approach—standard for everyone except Jacobson. Infinite Galois theory is included and presumes some familiarity with topology, although inverse limits are fully discussed. There are chapters on valuation theory and prolongations of valuations, and the final chapters provide applications of these ideas to local fields and fields of number theory. The material is clearly presented in the style of a graduate text, with some examples, exercises with each section, and a magnificent set of over two hundred problems which include new ideas and powerful applications. This is not a book for beginners. The text and the problems presume some topology and a solid background in abstract algebra.

Although this is a small book it is an important one; it provides an excellent foundation for further study in algebraic number theory.

NEIL GRABOIS, Williams College

Quick Calculus. By Daniel Kleppner and Norman Ramsey. Wiley, New York, 1965. iii+294 pp. $2.95.

Reviewing the work before us is like passing on the merits of the latest Howard Johnson's. A paperback by two Harvard physicists, it is more or less equivalent to the first few chapters of those texts which open with a bird's eye view of calculus. It is "programmed" for self-study in an elementary way—no branching, just skipping (over the solution, or the next problem, if you got the first one right). The "quickness" is achieved by omitting hypotheses and proofs, and working far fewer problems than the average text. A long appendix offers a summary, proofs, and more problems. Instant calculus, anyone?

I don't care if physicists—or chimpanzees, for that matter—write mathematics books, but they ought at least to bring an honest point of view to the subject. Apparently all functions have definite integrals: fine, if that's their experience. But they must be kidding then in the beginning when they define and discuss continuous functions, solemnly going through the whole δ–ϵ bit on limits. There is hardly a mention of the standard heuristic use of differentials by physicists (and mathematicians). It's as if they were afraid some

mathematician were going to look at their book. In case one does, I recommend their treatment of negative area and backwards integration as being particularly humorous.

ARTHUR MATTUCK, Massachusetts Institute of Technology

Foundations of Linear Algebra. By A. I. Mal'cev. Translated by T. C. Brown and edited by J. B. Roberts. Freeman, San Francisco, 1963. xi+304 pp. $8.00.

The following topics are among those included in this book: (1) Matrix theory; (2) Linear spaces; (3) Unitary and Euclidean spaces; (4) Quadratic forms; (5) Bilinear-metric spaces; (6) Tensor and exterior algebra. There are quite a large number of problems, some routine, others difficult.

As the title indicates, this book provides the foundations of various branches of linear algebra; since the edition here translated was intended as a teaching aid it is very useful to the instructor but less so to the undergraduate. Those wishing to adopt this book as a text should be cautioned that the author presupposes knowledge of any facts concerning determinants, polynomials and fields that he requires. The presentation is on the whole terse, but clear and comprehensive. The author includes pertinent and illuminating discussions to motivate and give direction to the material.

Unfortunately the text is marred by a large number of annoying misprints which in some cases seriously obscure the meaning. In addition, the translator has not always chosen the most widely-used terms. For instance, *linear function* and *permute* have been used instead of *linear functional* and *commute*, although in the latter case not consistently. It is to be hoped that in any future printing the excellent content of this book will be freed of these flaws.

R. WESTWICK, University of British Columbia

Three-Dimensional Problems of the Theory of Elasticity. By A. I. Lur'e. Edited by J. R. M. Radok. Translated from the Russian by D. B. McVean. Interscience, New York, 1964. xii+493 pp. $17.50.

This is a translation of a rather specialized book by one of the most prominent contemporary Russian elasticians. The title is completely accurate. The first two chapters develop the tools to be used later, while each of the remaining chapters is devoted to a single important problem in three-dimensional elasticity. In each a general problem is stated and solved and then several particular cases of that problem considered. There is no discussion of general theorems such as is found in Love's Chapter 7 or in Sokolnikoff's Chapters 3, 6 and 7, nothing on mathematical questions such as existence and uniqueness of solutions or the problem of representation of solutions by stress functions, and very little on the physical aspects of elasticity. All has been done deliberately in order to present in a limited space a cohesive and readable account of the way in which Russian elasticians have in recent years employed the stress function approach to the solution of certain three-dimensional problems in elasticity.

One of the most informative features of Lur'e's book is the portion entitled "Notes and Bibliography" at the end of each chapter. This consists of a short historical sketch of the problem considered in that chapter which concludes with remarks about the contemporary Russian work on which almost the entire book is based. Indicative of the changed political climate in the USSR is the generous referencing of non-Russian work, not merely the classical, but also the work of Mindlin, Sadowsky, Sternberg, Green, and others over the past two decades.

This book is a valuable and important addition to the English language literature on the mathematical theory of elasticity.

W. H. PELL, National Bureau of Standards

Lines and Surfaces in Three-dimensional Affine Space. By L. K. Tutaev. Israel Program for Scientific Translations, Jerusalem, 1964. ix+92 pp. (Published in the U.S.A. by D. Davey & Co. Inc., 257 Park Avenue S., New York), \$3.50.

This book was written originally in Russian and then translated into English. The topic covered is the differential geometry of submanifolds of affine 3-space, treated as a generalization of the Serret-Frenet theory of curves in euclidean 3-space. The principal tool for this approach is the "repère mobile" of E. Cartan without the orthogonality conditions which arise from a metric structure.

A *frame* in affine n-space is an $(n+1)$-tuple of vectors $(P, e_1, \cdots, e_n)$ where the e_i are based at P and linearly independent. Displacements of the frame are described by $dP = w^i e_i$, $de_i = w_i^j e_j$, where the w^i and w_i^j are the so-called structural 1-forms. To investigate submanifolds, one chooses frames which are adapted to the special situation. The exterior differential calculus of Cartan applied to these canonical frames gives geometric quantities which have affine invariance properties. The study becomes an exercise in choosing suitable canonical frames and differentiating everything in sight. This book covers the possibilities fairly exhaustively for $n \leqq 3$.

The North American reader may be a little startled at the descriptive style. The word "curve" is never used, all one-dimensional submanifolds being referred to as lines (even in the title). The text abounds with phrases such as "infinitesimal displacements" and "lines having four successive points in common." These anachronisms are probably attributable to the author. Other stylistic gaffes, of which there are many, probably stem from the translation. The pages seem to have been typed and photographically reduced. At any rate formulas involving subscripts on subscripts are often almost illegible because of blurring and small size.

J. R. VANSTONE, University of Toronto and the Summer Research Institute of the Canadian Mathematical Congress

Topological Methods in the Theory of Nonlinear Integral Equations. By M. A. Krasnosel'skii. Translated by A. H. Armstrong. Translation edited by J. Burlak. Pergamon, New York, 1964. xi+395 pp. \$10.00.

The original Russian edition of this important book appeared in 1956. It deals with a theory of nonlinear equations

$$f(x) = 0 \tag{1}$$

in Banach spaces with special applications to nonlinear integral equations.

The central notion of the theory is the concept of what the author calls the rotation of a vector field f and was called the characteristic of f by previous authors (see e.g. Alexandroff-Hopf, *Topologie*, p. 478, I. Springer 1935). Briefly, the meaning and significance of this concept may be described as follows: let f be a continuous map defined on the closure $\bar{D}$ of an open bounded domain $D \subset E^n$ (real Euclidean n-space) into E^n. Then the Brouwer mapping degree $A(y, D, f)$ is defined for any point $y \in E^n$ which is not in the image of the boundary $\dot{D}$ of D.

For smooth enough f this number gives the algebraically counted number of preimages of y. The degree depends only on the values of f taken on $\dot{D}$. Consequently if f is defined on $\dot{D}$ but not necessarily on D the "order $u(y, \dot{D}, f)$" of y with respect to $f(\dot{D})$ is well defined by setting $u(y, \dot{D}, f) = A(y, D, \bar{f})$ where $\bar{f}$ is a continuous extension of f to $\bar{D}$. For $y = \theta$, the origin of E^n, we write $u(\dot{D}, f) = u(\theta, \dot{D}, f)$. If $\| \quad \|$ denotes the Euclidean norm in E^n, then the normalized map $f_1 = f/\|f\|$ maps $\dot{D}$ into the unit sphere $\|x\| = 1$, and it turns out that $u(\dot{D}, f) = u(\dot{D}, f_1)$. Frequently one thinks of $f(x)$ as the vector which is attached to the point $x \in \dot{D}$ and whose arrow point is $x + f(x)$. With this geometric interpretation the characteristic $\chi(f)$ of the vector field f is defined by

$$\chi(f) = \chi(\dot{D}, f) = u(\dot{D}, f_1) = u(\dot{D}, f) = A(\theta, D, \bar{f}). \tag{2}$$

The author motivates the central role played by this number as follows: he calls the problem (1) well posed if its solutions (in some sense) depend continuously on f. But by a well-known theorem by H. Hopf, two vector fields have the same characteristic if and only if they are homotopic (in $E^n-\theta$). Moreover $\chi(\dot{D},f)\neq 0$ implies the existence of a root of (1) in D.

If in particular (1) has only isolated roots $x_1,\cdots,x_r$ in D then with $S_r^i=\{x\mid \|x-x_i\|=r\}$ and r sufficiently small, the number $\chi(S_r^i,f)$ does not depend on r. It is called the index j_i of the root x_i and

$$\chi(\dot{D},f)=\sum_{r=1}^{r} j_i. \tag{3}$$

In 1934 Leray and Schauder generalized the definition and theory of the Brouwer degree $A(y,D,f)$ to the case where D is an open bounded domain in a Banach space E, provided that f is of the form

$$f(x)=x-\lambda A(x)\quad(\lambda \text{ real}) \tag{4}$$

with A completely continuous (i.e. A maps $\overline{D}$ onto a set whose closure is compact). Subsequently various mathematicians defined for the Banach space case the other notions introduced above and proved that all theorems mentioned above are still valid. In this development the author played a prominent role as he also did in the development of most of the other topics treated in this book.

The theory of the characteristic $\chi(f)$ is given in Chapter II. In Chapter III the contracting principle is introduced which asserts the existence of a unique fixed point for a contracting (but not necessarily completely continuous) map, i.e. a map which shortens each line segment (uniformly). Various existence theorems are proved using this principle alone but the main emphasis is on combining it with the theory of Chapter II.

In Chapter III the nonlinear eigenvalue problem is treated: Suppose f in (1) is of the form (4) with $A(\theta)=\theta$ such that $x=\theta$ is a solution of (1) for all λ. Then, as in the linear case, λ is an eigenvalue of A if (1) has a solution $x\neq\theta$, called an eigenelement of A, and the set of all such λ is the spectrum of A. While in the linear case (for A completely continuous) the spectrum is discrete and has no accumulation point $\lambda_0\neq 0$, it may very well be continuous in the nonlinear case. Another nonlinear phenomenon is the possible existence of bifurcation and branching points.

It is well known from the classical theory that while a completely continuous linear operator does not necessarily have an eigenvalue, the existence of an eigenvalue is assured for two important classes of such operators, namely, the positive and the symmetric ones. Chapters V and VI, the last two chapters of the book, deal with the generalization to the nonlinear case of these two classes, viz. the class of mappings leaving a "cone" invariant, and the class of potential or gradient mappings.

The numerous applications of the abstract theory to nonlinear integral equations are given at the appropriate places throughout chapters III, IV and V. The prerequisites concerning the properties of integral operators needed for the application of the general theory are collected in chapter I.

E. H. ROTHE, University of Michigan

Stability and Asymptotic Behavior of Differential Equations. By W. A. Coppel. Heath, Boston, 1965. 166 pp. $9.00.

This book is a monograph on stability and asymptotic behavior of solutions of ordinary differential equations whose independent variable is real. With the addition of two topics, this book would furnish a good background for prospective research workers in the area of stability theory and would furnish those interested primarily in applications with a very readable account of the basics of stability theory. Liapunov's direct method

is not treated; however, there are several books which deal with this topic so this is no real hardship. The topic which this reviewer would have liked to have seen in this book is the problem of global stability. Liapunov's direct method is frequently used to study this problem, but there are some interesting theorems on this subject that are analogous to results presented here.

The five chapters of the book are titled Initial Value Problems, Linear Differential Equations, Stability, Asymptotic Behavior, and Boundedness. In addition, there is a nice appendix on the Routh-Hurwitz Problem. The first two chapters are a (good and somewhat different) treatment of the standard material on ordinary differential equations which is necessary to discuss stability and asymptotic behavior. The Schauder-Tychonov theorem and the Contraction Principle are introduced early (Chapter I) and are used throughout the book. Chapter III is especially noteworthy; it develops carefully the parallelism which exists among five types of stability—stability, asymptotic stability, uniform stability, uniform asymptotic stability, and strong stability. A good treatment is also given of conditional and orbital stability.

This is a very well written book which contains few errors (although, since it is a monograph it contains no exercises). This reviewer heartily recommends this book to any person who is interested in learning about stability and asymptotic behavior of solutions of ordinary differential equations. While it may not always contain the most general (and most complicated) theorems known, it does provide a firm and readily understandable foundation on which to build.

F. S. VAN VLECK, University of Kansas

Problems of Mathematical Physics. By N. N. Lebedev, I. P. Skalskaya, Y. S. Uflyand. Translated from the Russian by R. A. Silverman with supplement by E. L. Reiss. Selected Russian Publications in the Mathematical Sciences, Prentice-Hall, Englewood Cliffs, N. J., 1965, xi+429 pp. $13.50.

As pointed out by the translator, the terms "applied physics and engineering" might be more appropriate than "mathematical physics" in English usage. The main part of the volume is the presentation of a large number of problems drawn from mechanics, the theory of heat conduction, and the theory of electricity and magnetism. Hints and answers are given, and in the second part of the book some of these problems are solved in detail. The problems are typical of those encountered in advanced undergraduate courses in physics and engineering and are restricted to those which can be solved explicitly. Methods employed include Green's function, conformal mapping, Fourier series, eigenfunction expansions, integral transforms, curvilinear coordinates, and integral equations.

A very fine supplement by E. L. Riess gives methods of obtaining approximate solutions based on variational principles. It is on a slightly higher level than the main text and includes such topics as the Ritz method, Kantorovich's method, Galerkin's method, the collocation method, and the method of least squares. Complete solutions are not given.

MARTIN SCHECHTER, Institute for Advanced Study

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)–18 (second graduate year). A boldface star (★) marks a notable book that might be overlooked.

Algebra

Lattice Theory. By Garrett Birkhoff (Harvard Univ.) Third (New) edition. Colloquium Publications, Vol. 25. Amer. Math. Soc., 1967. vi+414 pp. $11.80. The first edition

of 1940 and the revised edition of 1948 are well known. In this rewritten edition the author includes "current developments" and "various unpublished ideas of my own." T (17–18), S, P, L.

The Algebraic Theory of Semigroups. Vol. II. By A. H. Clifford (Tulane Univ.) and G. B. Preston (Monash Univ.). Math. Surveys 7. Amer. Math. Soc., 1967. xv+350 pp. $13.70. This volume completes a survey of the theory of semigroups. P.

Topics in Local Algebra. By Jean Dieudonné (Univ. of Nice). Edited and supplemented by Mario Borelli (Univ. of Notre Dame). Univ. of Notre Dame Press, 1967. 122 pp. $2.25. These are notes of lectures given at the University of Notre Dame in the fall of 1966. They are intended as an introduction to modern algebraic geometry in general and in particular to the properties of an algebraic variety at a point—including such concepts as dimension, depth, regularity, normality and completeness. S, P.

Characters of Finite Groups. By Walter Feit (Yale Univ.). Benjamin, New York, 1967. viii+186 pp. $9.50 (cloth), $4.95 (paper). These are lecture notes intended to be used in part of a second graduate course and to include material not otherwise available in book form. After developing the general theory of characters of finite groups and some material about the Brauer characterization of characters, splitting fields and Schur indices, the rest of the book is devoted to applications to structure of finite groups. T (18), S.

Multilinear Algebra. By W. H. Greub (Univ. of Toronto). Grundlehren der mathematischen Wissenschaften, Vol. 136. Springer-Verlag, New York, 1967. x+224 pp. $8.00. This volume is intended to follow the third edition of the author's *Linear Algebra,* of which a telegraphic review appeared in this MONTHLY in January 1968. Topics include tensor algebra, exterior algebra, mixed exterior algebra, and multilinear functions. The two volumes, totalling nearly 700 pages offer one of the most complete and authoritative treatments available. T (15–17), S, P, L.

Tensor Analysis. By Edward Nelson. Princeton Univ. Press, Princeton, N. J. and Univ. of Tokyo Press (in Japan only), 1967. iv+127 pp. $2.00. This is one of a series of publications called "Preliminary Informal Notes of University Courses and Seminars in Mathematics." The author describes it as "an expository account of the formal algebraic aspects of tensor analysis using both modern and classical notations." T (16–17), S, P.

Linear Representation of Finite Groups. By P. Ribenboim. Queen's Papers in Pure and Applied Mathematics No. 5. Queen's University, Kingston, Ont. 1966. iii+378 pp. $5.00. Notes to the second part of a fourth year undergraduate honors course for students already acquainted with finite groups, linear algebra, commutative theory and Galois theory. T (16), S.

Education

Goals for Mathematical Education of Elementary School Teachers. A Report of the Cambridge Conference on Teacher Training. Houghton Mifflin, Boston, 1967. 138 pp. $1.80. Like *Goals for School Mathematics* (the report of the 1963 Cambridge Conference on School Mathematics) this is the work of a distinguished group of mathematicians, and will undoubtedly be widely discussed and play a significant role in mathematics education. The Cambridge Conference on Teacher Training took place June 13 through July 8, 1966 and was organized by Educational Services Incorporated, now merged with Institute for Educational Innovation under the new name Education Development Center, Inc. S, P, L.

New Trends in Mathematics Teaching. Prepared by International Commission of Mathematical Instruction (ICMI). Series: The Teaching of Basic Sciences. United Nations Educational, Scientific and Cultural Organization, Paris, 1967. 438 pp. $6.00. After an introduction by A. Lichnerowicz there are five parts: A collection of papers presented to congresses, meetings and seminars during the previous two years; original articles and reprints; a description of congresses, meetings and seminars relating to mathematical education; a list of centers of reform in mathematical education in each country with addresses; and a list of periodicals in mathematical education in each country. This volume is an extraordinarily useful source, and UNESCO plans to publish a similar volume every two years. TT, S, T, L.

Geometry in the Secondary School. A compendium of papers presented in Houston, Texas, January 29, 1967 at a joint session of The Mathematical Association of America and the National Council of Teachers of Mathematics. N.C.T.M., Washington, D. C., 1967. 54 pp. 50¢ (paper). Discounts on quantity orders. Contributors are B. E. Meserve, H. S. M. Coxeter, Andrew Elliott, G. P. Johnson, Herbert E. Vaughan, Seymour Schuster, E. E. Moise, G. S. Young, C. E. Springer. TT, P, L.

Education in Applied Mathematics. Proceedings of a Conference sponsored by Society for Industrial and Applied Mathematics under a grant of the National Science Foundation with University of Denver as host. Aspen, May 24–27, 1966. Reprinted from SIAM Review, April 1967. 125 pp. $2.40. In a preface added to the reprinted materials, H. J. Greenberg says the purpose of the conference was "to initiate nationally a broad discussion of the problems of education of applied mathematicians, with the hope of contributing to the solution of these problems and, by circulating widely the proceedings of this conference to provide information and ideas to those interested in applied mathematics education." To this end a distinguished group of participants came from universities, industry and government. There were six half-day sessions, each reflected in these proceedings by the paper of the principle speaker, the contributions of four prepared discussants, and a report on general discussion. Major speakers were C. C. Lin, P. D. Lax, E. W. Montroll, G. F. Carrier, J. B. Rosser, R. E. Gomory. There is a summary of the conference by F. J. Weyl. Many different points of view are represented. Perhaps the only one rather slighted is the opinion that the current curricular reform movement is a good thing for all applications. TT, P, L.

Teaching General Mathematics. By Max A. Sobel (Montclair State College). Prentice-Hall, Englewood Cliffs, N. J., 1967. xiii+93 pp. $3.95. Relevant if you are involved in teacher training or are curious as to how the mathematically underprivileged live. TT, P.

Modern Elementary Mathematics: a programmed introduction. By Robert M. Todd (Boston Univ.) and Cecil W. McDermott (Hendrix College). Allyn and Bacon, Boston, 1967. xii+292 pp. $3.95 (paper). There are now on the market a number of programmed books on elementary mathematics and calculus that can be used very effectively as optional or assigned supplementary workbooks. This one introduces the student to some of the ideas of "modern mathematics," such as sets and the number system. An interesting feature is an effort to program the mastery of a proof (the irrationality of the square root of 2) by giving the proof and providing over 50 questions for the student to answer in order to help him understand the proof itself and what lies behind it. The effectiveness of these new materials can only be judged by classroom use, but many of them deserve trial. S (13).

Mathematical Quickies. By Charles W. Trigg (Los Angeles City College). McGraw-Hill, New York, 1967. xi+210 pp. $7.50. Two hundred and fifty intriguing problems with quick solutions (if they occur to you) selected from the author's collection of over sixteen thousand. No recommendation is needed for those who are familiar with the quickies published by Trigg in the *Mathematics Magazine* while he edited the Problem Department. Although they involve no advanced mathematics, these problems can be useful to a teacher to enliven and involve the class. P, L.

Geometry

Intrinsic Geometry of Surfaces. By A. D. Aleksandrov and V. A. Zalgaller. Translations of Math. Monographs 15. Amer. Math. Soc., 1967. vi+336 pp. $11.80. A translation by J. M. Danskin of a book published by the Steklov Mathematical Institute in 1962 under the title *Two-Dimensional Manifolds of Bounded Curvature*. The surfaces are nonregular, in contrast to the regular surfaces of the intrinsic geometry of Gauss. P, L.

Theory and Problems of Projective Geometry. By Frank Ayres, Jr. (Dickinson College). Schaum, New York, 1967. 243 pp. $3.50 (paper). Designed to be used as a textbook or supplement in a formal course in real projective geometry, this paperback must compete with a number of classical treatises and brief textbooks available in inexpensive editions. Projective space is obtained by adjoining ideal elements. The final chapter is on projective, affine and Euclidean geometry. There is an appendix on matrix algebra, and both synthetic and analytic methods are used throughout the book. T, S.

Convex Polytopes. By Branko Grunbaum (Univ. of Washington), with the cooperation of Victor Klee (Univ. of Washington), M. A. Perles (Univ. of California, Los Angeles) and G. C. Shephard. Interscience, New York, 1967. xiv+456 pp. $18.75. Meant to serve as a textbook, reference, guide to the literature, and stimulant to further research on this topic in which interest reached a peak in the 19th Century and then waned before reviving under the stimulation of linear programming, it includes recent results over a broad range. There is a substantial bibliography. T, S, P, L.

Informal Geometry. By Lawrence A. Ringenberg (Eastern Illinois University). Wiley, New York, 1967. xi+151 pp. $5.50. The book is a "geometry textbook for prospective teachers who are preparing to teach kindergarten through grade 6 in a modern style of combining arithmetic, algebra and geometry." In line with CUPM recommendations, it is intended to follow a course on number systems and elementary algebra and can be adapted to various combinations of courses. TT.

Geometry Revisited. By H. S. M. Coxeter (Univ. of Toronto) and S. L. Greitzer (Rutgers Univ.) New Mathematical Library 19. Random House, New York, 1967. School Edition from Singer. xiv+193 pp. $1.95 (paper). The topics in this pamphlet, which is written for high school students but can also be useful at the university level, are: points and lines connected with a triangle, properties of circles, collinearities and concurrence, transformations, inversive geometry, and projective geometry. There are exercises, references and a glossary. S, P, L.

Probability and Statistics

Theory of Rank Tests. By Jaroslav Hajek (Charles University, Prague) and Zbynek Sidak, (Math. Inst., Czechoslovak Acad. of Sci.). Academic Press, New York, 1967. 297 pp. $8.00. This is a specialized book designed for teachers, advanced students,

and workers in statistics who already have substantial training in the field. It develops the ideas of LeCam and the senior author. S, T.

Random Matrices and the Statistical Theory of Energy Levels. By M. L. Mehta (Univ. of Delhi). Academic Press, New York, 1967. x+259 pp. $12.00. The random matrices referred to in the title are Hermitian matrices whose elements are random variables, and the theory is related to nuclear and atomic spectra. P.

Joint Statistical Papers of J. Neyman and E. S. Pearson. University of California Press, Berkeley. 299 pp. $7.00. This is the second of two volumes issued by the Biometrika Trustees to celebrate the long service by Pearson as managing editor. The first consisted of articles by Pearson alone. This one consists of ten joint papers showing the development of the Neyman-Pearson theory of testing statistical hypothesis. There is also a paper by Neyman alone printed as an appendix. Two of the papers have appeared previously only in Polish Journals. A volume of collected papers by Neyman alone will be published later. S, P.

Probability Measures on Metric Spaces. By K. R. Parthasarathy (Univ. of Sheffield). Academic Press, New York, 1967. xi+276 pp. $12.00. Intended to be a self-contained and complete treatment of probability distributions and limit theorems in metric spaces, this book is written for graduate students and specialists with a knowledge of measure theory and general topology. It includes original work and emphasizes the results of the Indian school of probabilists. P.

Statistical Methods. By George W. Snedecor (Iowa State Univ.) and William G. Cochran (Harvard Univ.). 6th ed. Iowa State University Press, Ames, Iowa, 1967. xiv+593 pp. $9.95. This classic text requiring only elementary traditional algebra, has gone through five editions and eighteen printings since the first edition in 1937. Illustrations are mostly biological. In this edition there are some rearrangements and the addition of several new minor topics, including linear regression when the independent variable is subject to error and a short introduction to probability. T, S (for applications), P (for non-mathematicians).

FILMS

The Committee on Educational Media of the MAA has produced seven films on arithmetic, designed for training elementary teachers at the CUPM Level I. All are available from Modern Learning Aids (1212 Avenue of the Americas, New York, N. Y. 10036, and branch offices). The reviews below of four of these films are based on comments sent to the editor by Professor Carol Kipps of the University of California at Los Angeles, who showed them to her methods class and obtained student reaction. The films not reviewed, but for which we hereby solicit reviews, are *Mr. Simplex Saves the Aspidistra* (33 minutes, color, $210.00, an introductory and motivational film with Leon Henkin and others, on odd and even numbers), *Addition and Subtraction* (8 minutes, color, $60.00, on the relation between set union and the operation), and *Multiplication and Division* (7 minutes, color, $60.00, on the relation between Cartesian products and multiplication). The prices are for purchase. Films may also be rented or leased to buy. Preview prints may be obtained.

What is a Set? Two parts, seven minutes each, color, $110.00. Standard notation for sets, membership, set-builder, null set and intersection. Students found the film artistically attractive both visually and musically. They thought the motivation insufficient and suggested a list of discussion questions for use by busy teachers.

One-to-one Correspondence. 10 minutes, color, $75.00. Equivalence of sets is defined and methods of establishing the relations are indicated. Students felt that the film was well organized with good examples, but that the illustrations involved unnecessarily elaborate elements. They suggested that these films should involve more questions, problems and answers. This film and the following one are closely linked and should be used together and in sequence.

Counting. 9 minutes, color, $75.00. The intuitive idea of cardinal and its relation to one-to-one correspondence. Methods of counting related to the number line are considered. As in the other films, students were pleased with the animation, and they particularly liked the voice in this film. They felt that the film should have said more about the role of zero and the significance of the variable "P" which was used in the film. Illustrations other than the number line might have been helpful.

Sets: Union and Intersection. 6 minutes, color, $45.00. The film covers union, intersection, disjoint sets, multiple union, and the commutative property. It leads into the films on arithmetical operation mentioned above. This film does not offer much new information after the previous ones, but does give a quick survey and review. Perhaps it could be used before the other films as a preview and again afterwards. Students felt that the illustration of intersection did not emphasize enough that elements in the intersection possess the defining properties of both sets.

All these films are mathematically sound, as one might expect from the sponsorship and the authors. Artistically they are quite pleasant. However, they cannot do the entire teaching job and are not intended to. They can be used to good effect as a part of the teaching process for introduction, motivation, and review. We have considered the usefulness of these films entirely for teacher training. Of course they may also be suitable for appropriate uses in the schools themselves at the elementary level, but we leave this question to other journals.

It would be very helpful to have information about the extent to which students learn from these films. We hope that some users in the future will give some pre- and post-tests to their teacher training classes in order to find out the extent to which knowledge and understanding have been increased.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor D. W. Hall, SUNY at Binghamton, represented the Association at the inauguration of President W. A. Brown of SUNY Agricultural and Technical College at Cobleskill on October 21, 1967.

Professor C. W. Huff, Winthrop College, represented the Association at the inauguration of President J. E. Smylie of Queens College on October 25, 1967.

Professor B. W. Jones, University of Colorado, represented the Association at the inauguration of Chancellor M. B. Mitchell of the University of Denver on October 20, 1967.

Professor D. A. Penner, Phillips Academy, represented the Association at the inauguration of President B. C. Hallowell of Tufts University on September 24, 1967.

Professor F. A. Valentine, University of California, Los Angeles, represented the Association at the inauguration of President R. C. Kramer of California State Polytechnic College, Kellogg-Voorhis on October 24, 1967.

University of Calgary: Dr. E. C. Milner, University of Reading, has been appointed Professor; Assistant Professor A. S. B. Holland has been promoted to Associate Professor.

California State College-Hayward: Assistant Professor Mary W. Gray has been promoted to Associate Professor; Associate Professor J. W. Summers has been promoted to Professor; Assistant Professor A. K. Charnow, Mt. Holyoke College, has been appointed Associate Professor; Associate Professor Sidney Spital, California State Polytechnic College, Pomona, has been appointed Associate Professor.

Carleton University: Dr. G. K. Zelmer, University of British Columbia, has been appointed Assistant Professor; Assistant Professor B. M. Puttaswamaiah has been promoted to Associate Professor.

City College of CUNY: Associate Professor W. L. Zlot, Yeshiva University, has been appointed Associate Professor; Assistant Professor Alvin Hausner has been promoted to Associate Professor; Associate Professor Gerald Freilich has been promoted to Professor; Assistant Professor Bernard Sohmer has been appointed Assistant Dean of Curricular Guidance.

Drexel Institute of Technology: Associate Professor W. F. Trench has been promoted to Professor; Drs. Frederick Hoffman, Institute for Defense Analyses, H. J. Biesterfeldt, University of Massachusetts, W. R. Boland, University of New Mexico, and Vivian Y. Kraines, University of New Hampshire, have been appointed Assistant Professors; Associate Professor W. J. Pervin, University of Wisconsin, has been appointed Professor.

Fresno State College: Mr. D. J. Lyons, Carnegie Institute of Technology, has been appointed Assistant Professor; Associate Professor T. C. Kipps has been promoted to Professor.

University of Massachusetts-Amherst: Dr. M. H. Stone, University of Chicago, has been appointed to the David Birkhoff Chair of Mathematics, effective September 1968; Professor Haskell Cohen, Louisiana State University, has been appointed Professor; Associate Professors D. R. Hayes, University of Tennessee, S. S. Holland, Boston College, and M. F. Janowitz, Western Michigan University, have been appointed Associate Professors; Drs. J. T. Buckley, Dartmouth College, T. A. Cook, Florida State University, and R. T. Douglass, University of Kansas, have been appointed Assistant Professors; Assistant Professor S. I. Allen has been promoted to Associate Professor.

McGill University: Associate Professor W. G. Brown, University of British Columbia, has been appointed Associate Professor; Assistant Professor Michael Herschorn has been promoted to Associate Professor.

University of Oklahoma: Drs. S. B. Eliason, University of Nebraska, Tom Hill, Columbia University, and C. H. Scanlon, University of Texas, have been appointed Assistant Professors.

Washington and Jefferson College: Dr. M. G. Zabetakis, Health and Safety Research and Testing Center, Pittsburgh, has been appointed Professor and Chairman of the Mathematics Department; Mr. Felix Magnotta has been promoted to Assistant Professor.

Sister M. Raimonda Allard, Ph.D., Rosary College, has been promoted to Assistant Professor and appointed Chairman of the Mathematics Department.

Mrs. Margaret E. Asprey, Espanola Public School System, has been appointed a programmer in the Theoretical Division of the Los Alamos Scientific Laboratory.

Dr. A. G. Azpeitia, Puerto Rico, has been appointed Professor at Texas Christian University.

Visiting Tallman Professor Mahedev Dutta, Bowdoin College, has been appointed Professor at the Indian Institute of Technology, Bombay.

Dr. M. M. Fraser, University of Illinois, has been appointed Assistant Professor at Albion College.

Assistant Professor Suzanne Glass, Roanoke College, has been promoted to Associate Professor.

Dr. Rufus Isaacs, Center for Naval Analyses, has been appointed Professor in the Departments of Operations Research and Electrical Engineering at Johns Hopkins University.

Dr. Jane A. Slezak, Rensselaer Polytechnic Institute, has been appointed Research Associate in the Chemistry Department at the University of Pittsburgh.

Assistant Professor M. E. Waddill, Wake Forest University, has been promoted to Associate Professor.

Associate Professor F. L. Wolf, Carleton College, has been promoted to Professor.

Dr. D. G. Zill, Iowa State University, has been appointed Assistant Professor at Loras College.

Professor J. W. Cell, North Carolina State University, died on November 9, 1967. He was a member of the Association for thirty-seven years.

Mr. Eleuterio De La Garza, wholesale grocer of Brownsville, Texas, died on August 25, 1967 at the age of 84. He was a Charter Member of the Association.

Mr. W. R. Evans, Jr., William Penn Charter School, Pennsylvania, died on June 22, 1967. He was a member of the Association for six years.

Professor Emeritus P. H. Linehan, CUNY College, died on September 20, 1967. He was a Charter Member of the Association.

Professor Emeritus Mayme I. Logsdon, University of Chicago and University of Miami, Coral Gables, died on July 4, 1967. She was a Charter Member of the Association.

Dr. Morris Ostrofsky, Westinghouse Defense & Space Center, died on September 24, 1967. He was a member of the Association for nineteen years.

Major B. W. Rose, Jr., U. S. Military Academy, West Point, died on November 4, 1967. He was a member of the Association for two years.

NEW AMS MATHEMATICAL OFFPRINT SERVICE

The American Mathematical Society, with partial support from the National Science Foundation, has established a new type of service for subscribers—the Mathematical Offprint Service (MOS). This is an entirely new concept in the distribution of mathematical information. Through MOS, the individual mathematican will be able to specify categories of journal articles in which he is interested and obtain articles within those categories simultaneously with or shortly after publication.

MOS will operate in the following way. After a subscriber orders a MOS subscription he will receive an interest profile form to complete and return to the AMS. On this form he will specify his interests: authors he specifically wishes to receive or to exclude; fields of primary and secondary interest; languages in which he wishes to receive articles; and journals from which he does not wish to receive articles. Fields of interest will be specified according to a subject classification scheme which the subscriber will receive for reference with his profile form. He may specify in his original subscription order whether he wishes to receive the subject classification scheme in English, French, German, Italian, or Russian. He will also receive with his profile form a list of participating jour-

nals to use in ordering the exclusion of journals from which he does not wish to receive articles.

Approximately sixty journals chosen from the list of highest priority journals reviewed in MATHEMATICAL REVIEWS will be asked to participate in the service. Galley proofs of articles will be sent from these journals to the AMS, and a complete profile corresponding to a subscriber profile will be set up for each article.

Each month, a subscriber will receive offprints of articles which fall into the category specified as desirable on his profile, together with a statement listing titles of articles which satisfied one or more requirements on the profile but did not satisfy all of them. Each title listing will include the reason the subscriber did not receive an offprint of that article.

One subscription to MOS will consist of 100 offprints (10 title listings will equal one offprint) and will cost $30. Shortly before each subscription is completed the subscriber will be billed automatically for the next subscription.

The AMS welcomes subscription orders now. It is expected that MOS will be fully operative in early spring 1968, or as soon as a minimum number of journals agree to participate and the service has a sufficient number of subscribers. With each order, the subscriber should indicate whether he wishes to receive the subject classification in English, French, German, Italian, or Russian. Shortly after his order is received, the subscriber may expect to receive an interest profile form, a list of participating journals, and the classification scheme.

For a MOS subscription, write to the American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02904.

DONATIONS OF BOOKS ARE NEEDED FOR DISTRIBUTION IN ASIA

The Asia Foundation is continuing for the fourteenth year its program of collecting useful books for distribution to institutions in Asia. Needed are books in excellent condition, in all college and adult level disciplines. Titles in the physical sciences should carry publication date of 1955 or later, in the social sciences and humanities, 1950 or later. Literary classics and anthologies of any date are welcome. Small quantities may be mailed by 'Special 4th Class Rate—Books' directly to BOOKS FOR ASIAN STUDENTS, 451 Sixth St., San Francisco, Calif. 94103. Book donations to The Asia Foundation are tax deductible.

TWO NEW AMS TRANSLATION JOURNALS

The American Mathematical Society has established two new translation journals: Mathematics of the USSR—Sbornik and Mathematics of the USSR—Izvestija. Sbornik is a continuing, cover-to-cover translation into English of Matematičeskiĭ Sbornik (New Series), published monthly by the Moscow Mathematical Society and the Academy of Sciences of the USSR. The translation will be published monthly.

Izvestija is a continuing, cover-to-cover translation of Izvestija Matematičeskaja Serija. It is a bimonthly publication of the Academy of Sciences of the USSR, and the translation will also be published bimonthly.

Izvestija and Sbornik both present contemporary research results in pure mathematics of some of the best mathematicians in the Soviet Union. They are of unquestionable value to the research mathematician. The AMS has previously translated a significant portion of these journals in Selected Translations Series I and II and in Selected Translations in Mathematical Statistics and Probability; it now feels that the importance of these journals justifies the complete, cover-to-cover translation of each.

The first issue of Izvestija will be available by April 1, 1968, and the first issue of Sbornik will be published shortly thereafter. The AMS is now welcoming subscription orders.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The fiftieth annual meeting of the Rocky Mountain Section of the MAA was held at Western State College, Gunnison, Colorado, on Friday and Saturday, May 12 and 13, 1967.

There were 95 people registered for the meeting including Professor F. M. Stein of Colorado State University, Sectional Governor, and Professor W. E. Dorgan of Western State College, Section Chairman.

The invited address was delivered on Friday afternoon by Dr. John Gary, National Center for Atmospheric Research, Boulder, Colorado. He spoke on "Large Scale Numerical Simulation of Atmospheric Processes."

Professor W. E. Dorgan presided at the banquet Friday night. The Section was welcomed by Dean D. H. Cummins, Dean of Faculties of Western State. Following the banquet mathematical movies were shown.

The following 16 papers were presented at the meeting:

1. *Semi-continuity and linear transformation*, by J. A. Jensen, University of Wyoming.

This paper presents several theorems which show that, with appropriate conditions on the topological spaces, if a function is semicontinuous, then it is continuous.

2. *Some equations in a finite field*, by A. D. Porter, University of Wyoming.

3. *On a cyclide and its associated circular cubic*, by N. X. Vinh, University of Colorado.

This paper discusses the cyclide defined by the equation $y(x^2+y^2+z^2)=a^2x$ and the circular cubic, intersection of the surface and the plane $z=0$. The cyclide contains three real and two imaginary families of circles. They are the intersections of the surface and the families of spheres having their centers on the xy plane, and on two paraboloids of revolution. Properties of the circular cubic were derived and its connections with the lemniscate of Bernoulli and the rectangular hyperbola were shown.

4. *A note on a class of generating functions*, by J. E. Faulkner, Brigham Young University.

5. *Some results on the Mikusiński convolution ring*, by Daryl Kreiling and S. Johnson, University of Wyoming.

It is shown that the Mikusiński convolution ring C is a Jacobson radical ring in which the descending chain condition on ideals does not hold. It is also shown that C can be expressed as a subring of the direct sum of some family of rings and that C has a family of ideals whose intersection is the zero ideal. Such a representation and family of ideals is given.

6. *Some results on the number of rings of order n*, by Clyde Martin and A. D. Porter, University of Wyoming.

It is shown that the number of rings of order n may be obtained by determining the number of rings of prime power order. An upper bound for rings of order p^a, p prime, is then obtained. A second approach to the problem shows that finding the number of rings of order p^a is equivalent to determining the number of solutions to a certain set of n^3 congruences.

7. *Similarity and orthogonal similarity in a finite field*, by John Adams and A. D. Porter, University of Wyoming.

Classical results for similarity, orthogonal similarity, and unitary similarity of square matrices over the real and complex fields are paralleled in finite fields. Necessary and sufficient conditions are given for these relations to hold between certain classes of square matrices and diagonal matrices.

8. *Matrix notations for the Taylor series expansion*, by C. A. Halijak, University of Denver.

9. *The mathematics program at the USAF Academy*, by Lt. Colonel R. C. Rounding, USAF Academy.

10. *Evaluation and placement system at the USAF Academy*, by Colonel Archie Higdon, USAF Academy.

11. *A survey course for business majors*, by H. H. Frisinger, Colorado State University.

12. *Criteria for positive definiteness of large multi-diagonal matrices*, by E. L. Allgower, Colorado State University.

The approach is to seek conditions on the elements such that sequences of symmetric multi-diagonal matrices obtained by bordering are positive definite independent of the size. Such sequences of matrices are also characterized in terms of monotonicity of the sequences of the corresponding matrices having 1's down the main diagonal. This result is related to the nature of the solutions of the recursion or difference equations which describe the determinants of multi-diagonal type matrices.

13. *Robust estimation of location I*, by E. L. Crow, Environmental Science Services Administration, Boulder, Colorado, and M. M. Siddiqui, Colorado State University.

14. *Robust estimation of location II*, by M. M. Siddiqui, Colorado State University, and E. L. Crow, Environmental Science Services Administration, Boulder, Colorado.

15. *Recurrency in the integer solutions of quadratic equations*, by E. I. Emerson, Boulder, Colorado.

16. *Some investigations on partially balanced arrays*, by D. V. Chopra, Southern Colorado State College.

At the business meeting, which was held on Saturday morning, May 13, with Professor Dorgan presiding, the following officers were elected for 1967/68:

Chairman—Kenneth Noble, University of Denver; Vice-Chairman—Jerrold Bebernes, University of Colorado; Secretary-Treasurer—C. R. Wylie, Jr., University of Utah.

W. N. SMITH, *Secretary-Treasurer*

OCTOBER MEETING OF THE OHIO SECTION

A special meeting of the Ohio Section of the MAA was held on October 20–21, 1967, at Stouffer's University Inn, Columbus, Ohio. This was a joint meeting of the Ohio Section and the Committee on the Undergraduate Program in Mathematics (CUPM) and was entitled "Conference on Collegiate Mathematics in Ohio." Professor Daniel Finkbeiner, Chairman of the Section, and Professors Arnold Ross and H. D. Lipsich presided at the general sessions. There were two hundred and eight registered in attendance including one hundred fifty-four members of the Association.

The following program was presented:

1. *A Brief Survey of CUPM Activities*, by R. D. Anderson, Louisiana State University, Chairman, CUPM.

This talk discussed the current status of CUPM's activities, particularly in the areas of the Teacher Training Panel, the College Teacher Preparation Panel, the Panel on Two Year Colleges, and the three panels on Applications of Mathematics: the Panel on Mathematics in the Life Sciences, the Panel on Computing, and the Panel on Statistics.

2. *A General Curriculum in Mathematics for Colleges*, by G. B. Price, University of Kansas.

3. *Preparation for Graduate Study in Mathematics*, by D. W. Bushaw, Washington State University.

A brief description of, and commentary on, the recommendations of the late Pregraduate Panel of CUPM.

4. *Preparation for Graduate Study in Analysis*, by G. L. Weiss, Washington University.

The speaker is in substantial agreement with the recommendations made in the CUPM booklets, *A General Curriculum in Mathematics for Colleges* and *Pre-Graduate Preparations of Research Mathematicians*. A special plea is made, however, that many classical notions not be overlooked in the preparation of students who want to do graduate work in analysis. The value of abstraction and generalization is recognized and encouraged; nevertheless, the direction in which this abstraction and generalization should go can best be found if the basic classical situation is understood.

5. *Preparation for Graduate Study in Algebra*, by H. J. Zassenhaus, The Ohio State University.

There are two sides to the question. What is the college teacher supposed to do or not to do in helping his colleagues in the graduate school? What does one suggest to the student who wants to develop his understanding and grasp of algebraic concepts? Regarding the first question a good training in linguistic skills including studies in two foreign languages (Russian, German, or French) helps greatly, furthermore a good course in mathematical logics, a knowledge of the history of science and finally a competent basic education in linear algebra would form a good foundation. Regarding the second, we are looking for the student who has done some studies of his own in number theory, or in coding, or in elementary algebra, or in some other discipline leading to modern algebra, and who gives in to his natural curiosity about concepts.

6. *Preparation for Graduate Study in Geometry*, by Walter Prenowitz, Brooklyn College.

7. *The CUPM Qualification Report*, by Alex Rosenberg, Cornell University.

The report "Qualifications for a College Faculty in Mathematics" issued by CUPM was discussed. It was pointed out that the need for such a report arose after the publication of the GCMC, since it then became necessary to describe the preparation of teachers who can implement the GCMC. The report suggests four levels of preparation for college teachers of mathematics which were described in some detail. The current status of the graduate schools with regard to making training for these levels available was discussed and the kind of teaching people at different levels could do was mentioned. It was pointed out that probably the two most important aspects of the report were that it states that a Ph.D. is not always necessary to become a fully qualified faculty member of a college mathematics department, and that it emphasizes the need for faculty members at all levels to keep mathematically alive.

FOSTER BROOKS, *Secretary-Treasurer*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The forty-second annual meeting of the Philadelphia Section of the MAA was held at the University of Delaware, Newark, on November 18, 1967. Due to the illness of the Chairman, Professor Emil Amelotti, the Secretary-Treasurer, Professor V. V. Latshaw, presided at the meeting. The meeting was attended by 121 persons including 90 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor Emil Amelotti, Villanova University; Secretary-Treasurer, Professor A. E. Filano, West Chester State College; Member of the Executive Committee, Professor W. E. Baxter, University of Delaware. The three top performers (all from Swarthmore College) on the 1966 Putnam Competition were recognized and each awarded an annual membership in MAA.

The following papers were presented:

1. *Counting finite graphs*, by H. S. Wilf, University of Pennsylvania.

2. *Equivalence of matrices and modules over Dedekind domains*, by J. O. Brooks, Villanova University.

3. *Algebraic topology for undergraduates*, by W. J. Pervin, Drexel Institute of Technology.

4. Film: *Fixed points, a lecture, with Solomon Lefschetz.*

V. V. LATSHAW, *Secretary-Treasurer*

NOVEMBER MEETING OF THE UPPER NEW YORK STATE SECTION

The annual fall meeting of the Upper New York State Section of the MAA was held at State University College, Buffalo, on November 4, 1967. One hundred thirty-seven persons attended including one hundred seventeen members of the Association.

A short business meeting was held. A resolution was passed that the Secretary write to Professor D. E. Kibbey, expressing the appreciation of this Section for his services to the Association.

Chairman D. W. Hall presided at the morning session and Vice-Chairman F. R. Olson presided at the afternoon session.

The following papers were presented:

1. *Unit intervals in arbitrary fields*, by Michael Gemignani, SUNY at Buffalo.

A subset I of an arbitrary field F is said to be a unit interval for F if I satisfies six algebraic axioms. Given a field F with unit interval I and a vector space V over F, I can be used to define a geometry and topology on V. In the case of R^n, where R is the field of real numbers, the geometry relative to the usual unit interval is the usual geometry, but the topology is strictly finer than the usual topology. The topology is the finest topology which gives lines the order subspace topology.

2. *Approximation theory*, by R. E. Dowds, State University College, Fredonia.

Approximation theory (a.t.), the application of functional analysis to problems of numerical analysis, provides a point of view that has a number of advantages over the classical numerical analysis. A.t. is a unifying concept. Thus, problems of interpolation, numerical differentiation and integration, Fourier analysis and numerical solution of differential equations are not unrelated. Indeed, from the viewpoint of a.t., they are seen to be merely different aspects of the same problem. Moreover, a.t. provides new and interesting results. Finally, in many concrete situations, the a.t. solution is more elegant and elementary than the one provided by classical numerical analysis.

3. *Colors and queens*, by F. D. Parker, St. Lawrence University, Canton.

Two classical problems, the problem of coloring a "politically" divided area so that adjacent areas have different colors and the problem of placing queens on a chessboard so that no queen can capture another, have a high degree of similarity. Each problem involves a graph, and each graph is represented by a 0, 1 matrix. Each problem can be stated as a matrix condition, and these conditions are formally identical.

4. *Experimental work in elementary and secondary mathematics in the USSR*, by Nura D. Turner, SUNY at Albany.

To satisfy modern demands of social and economic needs, new material has been introduced into the curriculum necessitating revamping, reshuffling, and changing. There has been an increase in the level of rigor as well as a provision for an understanding of the applications of mathematics to every-day life, industry, science, technology and agriculture. Planning the curriculum is approached from a much more sensible viewpoint than in the U.S.A.

5. *The combinatorial problem of the counting of corridors*, by G. J van der Maas and A. Thuswaldner, Northern Electric Laboratories, Ottawa, Canada.

A solution of the combinatorial problem of the counting of "corridors" mentioned in the article by B. Penkov and Bl. Sendov [*Hausdorffsche Metrik und Approximationen*, Numerische Mathematik, 9 (1966) 214–226] on Hausdorff Metric and Approximations is presented. A compatibility matrix is introduced and the solution is expressed in terms of the elements of this matrix.

6. *Almost convergent and almost summable sequences*, by Paul Schaefer, State University College, Geneseo.

A bounded sequence is said to be almost convergent if and only if all of its Banach limits coincide. Lorentz, *A contribution to the theory of divergent series*, Acta Math., 80 (1948) 167–190; Petersen, *Almost convergence and the Buck-Pollard property*, Proc. Amer. Math. Soc., 17 (1966) 1219–1225, have investigated relationships between almost convergent sequences and summability methods defined by infinite matrices.

In this paper, a brief exposition about almost convergent sequences and some results of the above-mentioned authors was given. It was shown that, with suitable modifications, Theorems 2 and 7 of Lorentz can be extended to the case of almost regular matrices.

7. *Unitary near-rings on generalized quaternion group*, by C. J. Maxson, State University College, Fredonia.

We consider the problem of defining multiplications, $\cdot$, on a generalized quaternion group $\langle G, +\rangle$ in such a manner that $\langle G, +, \cdot\rangle$ is a near-ring with identity (unitary near-ring). The major results are: THEOREM A: If $\langle G, +, \cdot\rangle$ is a near-ring defined on a generalized quaternion group $\langle G, +, \cdot\rangle$ and $0\in G$ is not a two-sided zero then $a\cdot b=b$, $a, b\in G$; THEOREM B: If $\langle G, +, \cdot\rangle$ is a near-ring defined on a generalized quaternion group then $\langle G, +, \cdot\rangle$ has no multiplicative identity; COROLLARY C: If $\langle H, +\rangle$ is a p-group with only one subgroup of order p and if $\langle H, +, \cdot\rangle$ is a unitary near-ring then $\langle H, +, \cdot\rangle$ is a ring. (The above results are included in a paper *The near-rings with identities on generalized quaternion groups*, written jointly with J. R. Clay, University of Arizona.)

MARY E. WILLIAMS, *Secretary-Treasurer*

NEW MAA PUBLICATIONS

Two new books and a revised edition of another will be published by the Association in the spring of 1968.

Carus Monograph 15 is entitled *Non-Commutative Rings*. The author is Professor I. N. Herstein of the University of Chicago. The book is based in part upon lecture notes from the MAA Cooperative Summer Seminar of 1965. Chapter titles are: The Jacobson Radical, Semisimple Rings, Commutativity Theorems, Simple Algebras, Representations of Finite Groups, Polynomial Identities, Goldie's Theorem, The Golod-Shafarevitch Theorem.

One copy of each Carus Monograph may be purchased by each MAA member for $3 per copy. Orders accompanied by payment should be sent to the MAA Buffalo office. Additional copies of Carus Monograph 15 and copies for non-members may be purchased for $6 per copy from John Wiley and Sons, 605 Third Avenue, New York, New York 10016.

MAA Studies 5 is entitled *Studies in Modern Topology* and is edited by Professor P. J. Hilton. It contains a comprehensive introduction by the editor and chapters by Professors G. T. Whyburn, Wolfgang Haken, V. K. A. M. Gugenheim, Eldon Dyer, and Valentin Poénaru.

One copy of each volume in the MAA Studies series may be purchased by each MAA member for $3 per copy. Orders accompanied by payment should be sent to the MAA Buffalo office. Additional copies and copies for non-members may be purchased for $6 per copy from Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07631.

The third edition of the *Guidebook* to Departments in the Mathematical Sciences in the United States and Canada has already been published. Copies at fifty cents each are available from the MAA Buffalo office. In this pamphlet, every attempt has been made to give complete coverage of all educational institutions. The Guidebook provides information in summary form about location, size, staff, library facilities, and course offerings for both undergraduate and graduate programs in the mathematical sciences. Its purpose is to assist prospective students in obtaining objective information about many institutions so that the selection of a proposed place of study may be narrowed down to a few from which more detailed information may then be sought.

CALENDAR OF FUTURE MEETINGS

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

Fifty-second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Indiana University of Pennsylvania, Indiana, April 27, 1968.

FLORIDA

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA, Ball State University, Muncie, May 4, 1968.

IOWA, Wartburg College, Waverly, April 19, 1968.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April 27, 1968.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Old Dominion College, Norfolk, Virginia, April 27, 1968.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, College of St. Teresa, Winona, May 4, 1968.

MISSOURI, Lindenwood College, St. Charles, April 27, 1968.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY, Rider College, Trenton, May 4, 1968.

NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.

NORTHERN CALIFORNIA, University of Santa Clara, February 8, 1969.

OHIO, Miami University, Oxford, April 26–27, 1968.

OKLAHOMA-ARKANSAS

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.

ROCKY MOUNTAIN, University of Denver, Colorado, May 10–11, 1968.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, New Mexico State University, University Park, April 12–13, 1968.

TEXAS, Texas A and M University, College Station, April 19–20, 1968.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 11, 1968.

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

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AMERICAN MATHEMATICAL SOCIETY, University of Wisconsin, Madison, August 27–30, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

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INSTITUTE OF MATHEMATICAL STATISTICS, University of Wisconsin, Madison, August 27–28, 1968.

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SHAPE RECOGNITION, PRAIRIE FIRES, CONVEX DEFICIENCIES AND SKELETONS

L. CALABI AND W. E. HARTNETT, Parke Mathematical Laboratories

1. Introduction. Imagine an ideally homogeneous prairie, dry and ready to burn except for a very wet area A, with total absence of disturbing effects, as those of winds and slopes. Assume now that the grass at the edge of A is set afire, all at once, and observe how the fire develops. Since the shape of A is the only factor that we have not excluded, it and it alone will influence the spreading of the fire.

For instance, if A is a full circular region (a disc), each point outside A is eventually burned by fire moving radially away from A and continuing on, unchecked, in the same direction. If we now take for A two separate discs A_1, A_2, again the fire will eventually reach every point outside A, moving radially away from A. But now there are special points at which the radial spreading is checked: at these points the fire quenches itself. Assuming for simplicity that A_1 and A_2 have equal radii and centers O_1, O_2, the quench points are those equidistant from O_1 and O_2: in fact, at such points x the radial motion of the fire is stopped, since the rays O_1x and O_2x enter regions already burned.

For a last example, assume A to be the perimeter of a square field. The fire now will burn both the field and the prairie outside the fence; and it will quench itself at the points of the diagonals of the square, with the exception of the end-points.

Other experiments with such prairie fires (and we mean *Gedanken Experimenten*) should readily suggest that the presence of quench points is equivalent to the nonconvexity of the region A at whose edge the fire starts; further, that the set of quench points reflects some shape attributes of A.

Blum [2] had used the prairie fire as a model for a phase of the neurophysiological process of visual perception and suggested that a mathematical study of the correspondence between A and the set S of quench points, together with the function q expressing the time at which the fire reaches them, be undertaken.

In carrying out this study we have obtained results which clearly show that the pair (S, q) is a natural tool for the study of closed, nonconvex sets A in the Euclidean plane. Our work should thus be of interest to mathematicians even more perhaps than to firemen and neurophysiologists.

We present here, without proofs, the more important theorems, inviting the interested reader to obtain copies of the detailed treatment appearing in [**3**], [**4**], [**5**] and [**8**]. In Section 2 we introduce some of our basic tools and use a well-known theorem of Motzkin to identify closed convex sets as those for which there are no quench points. We define S and q in Section 3; and after an heuristic discussion of some examples, we characterize the family of sets A having a given pair (S, q). This theorem is an extension of Motzkin's theorem and identifies that property of A, called its convex deficiency, which is determined by the pair (S, q). The section terminates by discussing some situations in which

S and q uniquely determine A. In Section 4 we treat some intuitive properties of S and q and their interpretations in terms of A. In the paper we indicate some of the more obvious outstanding questions.

2. **Preliminaries.** We begin by recalling a number of standard notions about the Euclidean plane and some of its subsets. Throughout the rest of the paper A will always denote a nonvoid closed subset.

For each point x in the plane, the *distance* $r(x)$ *from* x *to* A is defined by $r(x) = \text{glb}\{d(x, y): y \in A\}$ where d is the usual Euclidean metric. Because A is closed, there always exists at least one point y of A "nearest" x such that $r(x) = d(x, y)$ and so the set $\pi x = \{y: y \in A,\ r(x) = d(x, y)\}$ is never void. Hence we have a function π defined on the plane with subsets of A as values; π is called the *projection onto* A, and πx the *projection of* x *onto* A. Notice that $\pi x = \{x\}$ if and only if $x \in A$.

Such set-valued functions have been studied by a number of authors (see [1], for example) and one may introduce the concepts of upper and lower semi-continuity as well as that of continuity. The projection π onto A is upper semi-continuous in the plane and is continuous on the set of points x such that πx is a singleton [7].

For each x we let $B(x)$ denote the closed ball of radius $r(x)$ and let $B^0(x)$ denote its interior. It is easy to see that $B^0(x)$ is nonvoid if and only if x belongs to the complement $-A$ of A. If $x \notin A$, we shall call $B(x)$ the *ball of support of* A *at* x. Observe that

$$\pi x = A \cap B(x) = \text{bd}(A) \cap B(x)$$

and $A \cap B^0(x) = \emptyset$, where $\text{bd}(A)$ denotes the boundary of A.

The function r has the property [6] that for x, x' in the plane

$$|\, r(x) - r(x') \,| \leqq d(x, x'),$$

that is, r is a Lipschitzian function with constant 1. If $x \notin A$, $y \in \pi x$, and $l = [y, x \rightarrow)$ is the closed ray through x with endpoint y, then it is easy to observe that $r(x) = r(x') + d(x, x')$ if $x' \in [y, x]$. This last equality is equivalent to the fact that $B(x') \subset B(x)$ with $\{y\} = A \cap B(x')$; that is, one ball of support is contained in the other and $y \in \pi x \cap \pi x'$. In terms of prairie fires this means that the fire burns along the ray l from y through x' and on to x. The natural question arises: is there a last point on the ray beyond which the fire will not spread? If so, then that point must be a quench point.

Before we try to answer the general question we look at a special case. Suppose A is a convex set, $x \notin A$, $y \in \pi x$, $l = [y, x \rightarrow)$ and x is the last point beyond which the fire on l will not spread. Pick a point x' beyond x on l, let $k = d(x', y)$ and construct a ball $B'(x')$ with center x' and radius k. Then y is on the boundary $B'(x')$ but the ball is not a ball of support. Hence there exists a point $y' \in A$ in the interior of $B'(x')$ such that $y' \in \pi x'$. But A is convex and so the closed interval $[y, y']$ belongs to A, which is impossible because $[y, y']$ contains points of $B^0(x)$ and $B(x)$ is a ball of support. Therefore, no last point exists.

Notice further that for each $x' \in l$, $\pi x' = \{y\}$ and so πx is a singleton for each x in the plane. Hence if $x' \in l$ there always exists other balls of support which contain $B(x')$ and so there are no balls of support which are maximal with respect to inclusion.

Our first theorem affirms that the converse implications also hold, as expected:

THEOREM 1. *The following statements are equivalent*:
(a) *A is a convex set*;
(b) *πx is a singleton for each x*;
(c) *There exist no balls of support that are maximal with respect to inclusion.*

As mentioned in the introduction, we could use a result of [**10**] proving the equivalence of (a) and (b). This equivalence can also be obtained as an immediate corollary of Theorem 2 below; and may be interpreted as the equivalence between the convexity of A and the absence of quench points. This interpretation will become more obvious after the definition of the set S in the next section.

3. The correspondence between A and (S, q). We are now ready to formalize the notion of quench point as the elements of what we shall call the skeleton of A. Recalling our earlier comments about the behaviour of the function r, we define the ***skeleton* S *of* A** as the set of those points $x \notin A$ for which

$$\begin{aligned} r(x) &= r(x') + d(x, x') && \text{if } x' \in [y, x], \\ r(x') &< r(x) + d(x, x') && \text{if } x' \in l - [y, x], \end{aligned}$$

where $y \in \pi x$ and $l = [y, x \rightarrow)$.

The last inequality simply expresses the fact that for points x' beyond x on l there are points of A closer to x' than y. In the language of prairie fires, then, the burning reaches x' via another ray before it can reach it along l. Thus x is the last point of l burned by fire coming from y; that is, x is a quench point.

In terms of balls of support, it is easy to recognize that S is the set of those points $x \notin A$ at which $B(x)$ is maximal, that is not contained in any other ball of support. Both characterizations are useful in the development of the theory.

It might be worthwhile to observe that the first definition may be used to define the skeleton of any Lipschitzian function f with constant 1. We have only to replace r by f and to interpret the relation $y \in \pi x$ to mean that y belongs to the zero-set of f and to the boundary of the ball around x of radius $f(x)$. On the other hand, the second characterization of S may be used to define the skeleton of subsets A of metric spaces more general than the Euclidean plane.

Returning now to our subset A of the plane, observe that, by Theorem 1, S is empty if and only if A is convex. Thus if A is not convex, there are points in S. More specifically, if A is not convex there are points x at which πx contains at least two points (again by Theorem 1); and such points x belong to S because no ball of support can contain $B(x)$ unless it coincides with it. Those may not be all the points of S; however, as already shown [**11**], they are at least everywhere dense in S.

As already indicated in the introduction, besides its skeleton S we associate to A also the restriction q of r to S, called the *quench function* of A. The pair (S, q) will then be called the *skeletal pair* of A. For example, if A is convex then S is empty and so is q (we call empty the only function defined on the empty set); thus $(\emptyset, \emptyset)$ is the skeletal pair associated to each convex set A. Some examples with A nonconvex, chosen primarily to suggest and illustrate the results of this section on the correspondence $A \to (S, q)$ are given in Fig. 1. In the drawings, S is represented by dashed lines while A is drawn with heavy points, continuous lines and shaded areas. The value of q at some points x, x' of S is the length of the corresponding dotted and labelled segments; if $[y, x]$ is such a segment, then $y \in \pi x$ by the definition of q.

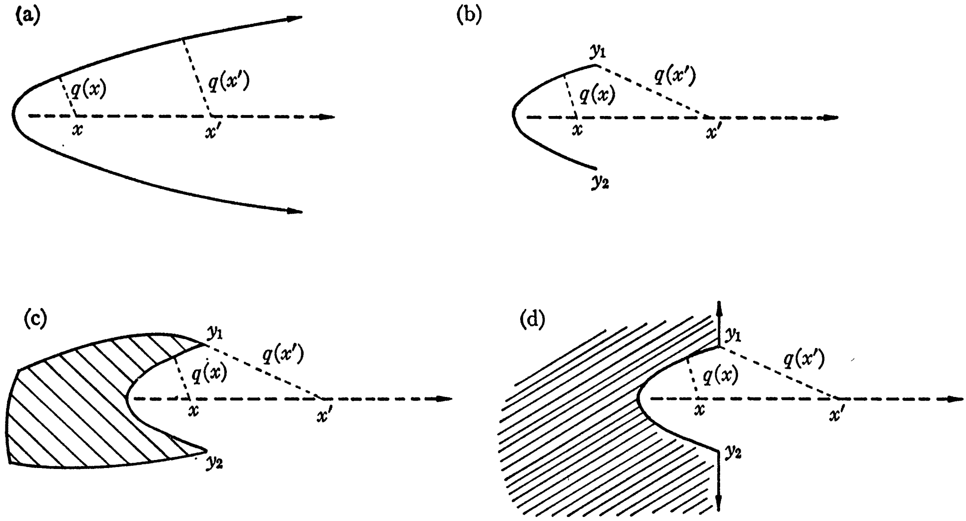

Fig. 1

The sets A illustrated in Fig. 1 are all constructed around the arc of the parabola y_1, y_2 of example (b), and all have the same skeleton S, the axis of parabola from the center of curvature at the vertex on. While examples (a) and (b) clearly have different quench functions q, the reader will readily realize that, as indicated, the last three examples do have the same quench function. Thus different sets A may have the same skeletal pairs, even if they are not convex.

Those examples, as well as the entire study so far, suggest that S and q could be determined by the manner in which A fails to be convex. In order to formulate this thought in a more precise manner, let us denote by C the closed convex hull of A and let us call the set $D = C \cap (-A) = C - A$ *the convex deficiency of* A. For the example (a) of Fig. 1, D is the region interior to the parabola; while for each of the other three examples D is part of the same set but does not extend beyond the open segment (y_1, y_2). Our examples then support the conjecture that the skeletal pair is more closely related to the convex deficiency D than to

the set A itself. That this be indeed true is established in our central result, which may be considered as an extension of Motzkin's characterization of convex sets and of a theorem of Valentine [**15**, p. 96]:

THEOREM 2. *Two closed sets have the same skeletal pair if and only if they have the same convex deficiency.*

Thus, in suggestive language, not only the skeletal pair is determined by the "holes" and the "dents" of A, but also these, in turn, are determined by (S, q). In the more detailed studies [**4**] and [**5**] it has been shown how the holes of A correspond to connected components of S, and the dents of A to unbounded components of $S-C$.

Since it is possible to describe all the sets A having a given set D for their convex deficiency, we obtain a complete description of the correspondence $A \to (S, q)$ by way of the decomposition $A \to D \to (S, q)$. We do not describe these results here, preferring to emphasize the following corollaries of Theorem 2.

COROLLARY 3. *A closed set is uniquely determined by its convex hull and its skeletal pair.*

Our examples are good illustrations of this last result; in the light of it, example (b) suggests that there is but one set A having a given skeletal pair and satisfying $A^0 = \varnothing$. This is actually not true if A is contained in a line; but if $C^0 \neq 0$, then such a set A is the smallest having (S, q) as skeletal pair and is thus unique:

COROLLARY 4. *Assume $C^0 \neq \varnothing$ and $A^0 = \varnothing$. Then A is uniquely determined by its skeletal pair.*

This result is particularly important for the application of our theory to the visual recognition process: it establishes the adequacy of our tools for the characterization of "line figures," that is, of contours, the prime aim of Blum's investigation [2].

Corollary 4 may be obtained independently of Theorem 2 in the following way. We first show that the assumptions $C^0 \neq \varnothing$ and $A^0 = \varnothing$ imply $\overline{\pi S} = \mathrm{bd}(A) = A$, where $\overline{\pi S}$ denotes the closure of the union of πx for $x \in S$. We then establish that the restriction of π to S, and hence $\overline{\pi S}$, is explicitly determined by S and q: for $x \in S$, $\pi x = \{y: d(x, y) = q(x) \text{ and } d(x', y) \geqq q(x') \text{ for } x' \in S\}$.

In this context we might also quote the following more general statement:

COROLLARY 5. *Among the sets A having (S, q) as skeletal pair there is at most one for which* $\mathrm{bd}(A) = \overline{\pi S}$. *Thus, if there is one, it is uniquely determined by (S, q).*

We terminate this section by mentioning two open questions concerning the correspondence $A \to (S, q)$. The first is: what is the range of that mapping? In other words, if X is a subset of the plane and if f is a real valued function defined on it, under what conditions is (X, f) the skeletal pair of some set A? We know several rather stringent necessary but not sufficient conditions, some

discussed below in Section 4; and we have obtained some sufficient but not necessary conditions. A satisfactory solution seems still far away.

The second and still wide-open problem may be introduced as follows.

If (S, q) can be used to recognize A (up to its convex hull C), "similar" sets A should have "neighboring" skeletal pairs. In more technical terms, let $\mathcal{A}$, $\mathcal{S}$, $\mathcal{S}^*$ be, respectively, the family of closed, nonconvex sets A of the plane, the family of their skeletal pairs (S, q) and the family of the triplets (S, q, C). Can we define topologies on these families such that the mapping $A \rightarrow (S, q, C)$ of $\mathcal{A}$ onto $\mathcal{S}^*$ be an homeomorphism and the induced mapping $A \rightarrow (S, q)$ of $\mathcal{A}$ onto $\mathcal{S}$ be continuous? Clearly the answer is yes: but can we do it in a natural manner? We feel that here "natural" is a very strict though still vague requirement, since it refers at once to mathematical and to visual criteria.

4. Two properties of S. In order to eliminate some of the pathology that can occur because of our weak assumptions on A, we shall restrict ourselves in this section to sets A such that $(\bar{S}-S) \subset A$. For this section only we say that such a set is a *good set*. All of the examples given so far have been good. An example of a nongood set A is a convergent sequence of points on a circle together with the limit point y. Each point on the open segment from the center x to y belongs then to $\bar{S}-S$ but clearly not to A.

In this example, the point x belongs to S but has no compact neighborhoods in S. Thus, if A is nongood, S may fail to be locally compact. It is easy to prove, on the other hand, that S is always "thin," at least insofar as $S^0 = \emptyset$. When A is good the situation is much nicer. Recall [16] that a *dendrite* is a compact, connected, and locally connected set which contains no simple closed curves; essentially, a dendrite is a nice union of simple arcs. We have then:

THEOREM 6. *Suppose A is good and $x \in S$. Then there exists a ball $B_0(x)$ with center x such that $S \cap B_0(x)$ is a dendrite.*

Thus S is not only locally compact and "thin," but even "graphlike." Actually we can say a bit more. Recall [16] that *the order $o(x)$ of x in S* is the smallest cardinal number, if it exists, such that each neighborhood of x in S contains another neighborhood whose boundary consists of $o(x)$ points. In a graph, $o(x)$ is defined for each x and is the number of branches issuing from x. Our theorem enables us to prove that the closure $\bar{S}$ of S is a graph (not necessarily bounded) when all of its points have finite order and when, say, $o(x) \neq 2$ for a finite number of points $x \in \bar{S}$. In all our examples, except for our nongood set, $\bar{S}$ is a graph. In the example of the square field, the vertices have order 1, the center has order 4, and all other points have order 2.

It is quite interesting to observe that the order $o(x)$ carries some information on what we could call the local shape of A. Denoting by $n(X)$ the number of connected components of a set X, one can show that, without any special assumption, $n(\pi x)$ never exceeds $o(x)$ (when defined) for x in S. Under our present assumptions we have more precisely:

THEOREM 7. *Suppose that A is good and $x \in S$. Then there exists a ball B around x such that $n(\pi x)$ equals $n((S \cap B) - \{x\})$. Moreover, if $o(x)$ is defined, then $n(\pi x) = o(x)$ whenever either of these numbers is finite.*

To interpret this theorem picturesquely, imagine that we stand at a point $x \in S$ whose order is finite. Then, in the ball B of the theorem, S consists of $o(x)$ branches leading into x. Because $n(\pi x) - o(x)$, the $n(\pi x)$ components of πx must occur "between" the branches of S (consider the example of the square with x the center). Hence, if we could look around us but only see points at exactly distance $q(x)$ away, then we would be able to see the $n(\pi x)$ connected subsets of $\mathrm{bd}(A) \cap B(x)$ but no other points of A. Indeed, A must "bend away" from the circumference of $B(x)$ by having $o(x)$ "dents" or "cuts" as seen from x.

5. Concluding remarks. In presenting the material above we have considered only nonvoid closed sets in the plane with the usual metric. There are two obvious modifications possible. One may specialize and study sets with additional properties or one may generalize and investigate the situation in the setting of different spaces and/or different metrics.

The first path has been taken by Riley [13] and by Ting [14]. Riley limits himself to sets A which are graphs whose vertices have no points of accumulation and whose edges have a differentiable curvature with only finitely many changes of sign. Under these assumptions he shows in particular that $\bar{S}$ is itself a graph.

The work of Ting contains results that, in our context, suggest that the result of Riley may be strengthened. It appears that one may relax some of Riley's assumptions and still prove not only that $\bar{S}$ is a graph but that the edges are themselves differentiable.

The second path, already started by Motzkin and continued, for example, in [9] and [12] in connection with convex sets, seems also quite fruitful. The consideration of non-Euclidean metrics in the plane would moreover enhance the applicability of our studies to the neurophysiological process considered by Blum.

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References

1. C. Berge, Topological Spaces, Macmillan, New York, 1963, Chp. VI.

2. H. Blum, A transformation for extracting new descriptors of shape, Symposium on Models for the Perception of Speech and Visual Form, MIT Press, 1967.

3. L. Calabi, On the shape of plane figures, PML No. 60429, SR-1, March 1965.

4. ———, A study of the skeleton of plane figures, PML No. 60429, SR-2, June 1965.

5. ———, The skeletal pair determines the convex deficiency, PML No. 5711, TM-1, January 1966.

6. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960, p. 32.

7. H. Federer, Curvature measures, Trans. Amer. Math. Soc., 93 (1959) 418–491.

8. W. E. Hartnett, Plane figures: their skeletons and quench functions, PML No. 60429, *SR*-3, October, 1965.

9. V. L. Klee, A characterization of convex sets, this MONTHLY, 56 (1949) 247–249.

10. Th. Motzkin, Sur quelques propriétés caractéristiques des ensembles convexes, Atti. Acad. Naz. Lincei, 21 (1935) 562–567.

11. ———, Sur quelques propriétés caractéristiques des ensembles bornés non convexes, *ibid.*, 21 (1935) 773–779.

12. R. R. Phelps, Convex sets and nearest points, Proc. Amer. Math. Soc., 8 (1957) 790–797.

13. J. Riley, Plane graphs and their skeletons, PML No. 60429, TM-1, March, 1965.

14. T. W. Ting, The ridge of a Jordan domain and completely plastic torsion, J. Math. and Mech., 15 (1) (1966) 15–47.

15. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

16. G. T. Whyburn, Analytic Topology, AMS Colloq. Publ. Vol. XXIII, 1942.

ON MÜNTZ' THEOREM AND COMPLETELY MONOTONE FUNCTIONS

WILLIAM FELLER, Princeton University

1. Introduction. Weierstrass' theorem asserts that every continuous (real) function on $[0, 1]$ is the uniform limit of an appropriate sequence of polynomials. In a famous paper [7] Ch. Müntz considered the possibility of a similar approximation when not all powers of the coordinate variable are admitted, that is, an approximation by polynomials of the form

$$\sum_{k=1}^{n} c_k x^{\rho_k} \tag{1.1}$$

with prescribed exponents ρ_k. In this formulation it is not necessary that the ρ_k be integers, but the exponent 0 plays a special role being indispensable for approximations to the constant function. It is therefore preferable to consider only functions vanishing at the origin, and we denote the Banach space of such functions by $C_0[0, 1]$. As usual we put $\|f\| = \max|f(x)|$. Given a sequence of numbers ρ_k satisfying

$$0 < \rho_0 < \rho_1 < \cdots, \qquad \rho_k \to \infty, \tag{1.2}$$

Müntz' result may be stated as follows

THEOREM 1. *In order that the linear combinations of the form* (1.1) *be dense in* $C_0[0, 1]$ *it is necessary and sufficient that*

$$\sum \rho_k^{-1} = \infty. \tag{1.3}$$

In the case of nondenseness there exists a bounded linear functional annihilating all the powers x^{ρ_k}. But linear functionals on C_0 can be represented by signed measures on the half-open interval $(0, 1]$. Given such a (finite) measure μ we put

$$f(t) = \int_0^1 x^t \mu(dx). \tag{1.4}$$

Theorem 1 is therefore equivalent to

THEOREM 2. *A signed measure μ on* (0, 1] *such that $f(\rho_k)=0$ for all k exists if, and only if,*

$$\sum \rho_k^{-1} < \infty. \tag{1.5}$$

In other words, the divergence of the series is necessary and sufficient in order that a measure be uniquely determined by its moments of order ρ_k.

Several proofs of this beautiful theorem are available, but they are all rather intricate and computational. Probably the simplest proof is still the original proof by Müntz based on Fourier series and the evaluation of certain determinants. For proofs more in line with general approximation theory see Kaczmarz and Steinhaus [5] or Natanson [8]. An alternative approach based on a generalization of the Bernstein polynomials was devised by Hirschman and Widder and was further developed by Gelfond [3]. (A short outline is contained in the book by Lorentz [6].) Gelfond used this approach to investigate the degree of approximation, and also to generalize Müntz' result to systems of functions $x^{\rho_k} \log^p x$ with $p=0, 1, \cdots, \nu_k$. This amounts to taking the point ρ_k with multiplicity ν_k. Our method covers this situation if the divided differences are interpreted by the obvious passage to the limit in terms of the derivatives of f. The purpose of this paper is to give a simple direct proof, which reveals the nature of the theorem and clarifies its connection with other facts.

In the case of convergence we exhibit the required measure of Theorem 2 explicitly and show that it is absolutely continuous its density being square integrable. (In this way we actually cover the analogues to Müntz' theorem for the L and L_2 spaces, but we shall not stress this point.) The construction depends on elementary properties of Laplace transforms, and the remainder of this paper is independent of section 2.

In the case of divergence we prove the asserted uniqueness by showing that a function of the form (1.4) may be represented by the Newton interpolation series (4.1) based on the values $f(\rho_k)$. What we require of Newton's method is so elementary that it was simpler to give a complete derivation rather than to refer to texts in which emphasis is on deeper matters [Section 3].

A pleasing feature of this approach is that it leads to further results. We may drop the first N terms of the sequence $\{\rho_k\}$ and represent the function f by the interpolation series involving only the values $f(\rho_N), f(\rho_{N+1}), \cdots$. The latter is related to the generating function of a measure μ_N and it turns out that $\mu_N \to \mu$ as $N\to\infty$. In this way we obtain an explicit representation for the measure μ defining the function f. In particular, if $f(t)=\alpha^t$ with $0<\alpha<1$, the measure μ is concentrated at the point α and in this case the integral with respect to μ_N of any continuous function u tends to the value $u(\alpha)$. This leads to an explicit form for the desired approximation of $u(\alpha)$ by linear combinations of α^{ρ_k}. Thus the proof of Theorem 2 leads us automatically to the approximation described in Theorem 1. This reverses the approach which starts from an *ad hoc* constructed analogue

to the Bernstein polynomials. In the case of equally spaced points ($\rho_k = kh$) our measures are related to the Poisson distribution just as the ordinary Bernstein polynomials are related to the binomial distribution; cf. [1] Chapter VII. (See section 5.)

In the proof it suffices to consider nonnegative measures μ, and such measures have the property that for the derivatives $f^{(n)}$

$$(1.6) \qquad (-1)^n f^{(n)}(t) \geqq 0, \qquad t > 0.$$

A function on $(0, \infty)$ with this property is called *completely monotone*. It may be unbounded at the origin. If it is continuous at the origin we say that it is *completely monotone* on $[0, \infty)$.

Now our main argument does not depend on f being of the form (1.4), but only on the fact that f is completely monotone. As a byproduct of our proof in the case of divergence we obtain therefore the following celebrated characterization of completely monotone functions due to S. Bernstein. (For modern proofs see Feller [1] or Widder [9].)

THEOREM 3. *A function f on* $(0, \infty)$ *is completely monotone if, and only if, it is of the form* (1.4) *with μ a nonnegative measure on* $(0, 1]$.

Incidentally, in section 6 it is shown that complete monotonicity may be defined in terms of the values $f(\rho_k)$, and the obvious interpolation problem is solved.

To conform with standard usage we introduce in (1.4) the change of variables $x = e^{-y}$ and write f in the form

$$(1.7) \qquad f(t) = \int_0^\infty e^{-ty} m(dy)$$

fully equivalent to (1.4). Here m is a signed measure on $[0, \infty)$, and f is called its *Laplace transform*.

2. The case of convergence. In this section we suppose that (1.5) holds We have to prove that this implies the existence of a finite signed measure m whose Laplace transform (1.7) satisfies the conditions $f(\rho_k) = 0$ for $k = 0, 1, \cdots$. Our measure will be absolutely continuous so that f will be of the form

$$(2.1) \qquad f(t) = \int_0^\infty e^{-ty} u(y) dy, \qquad t \geqq 0.$$

A plausible candidate is given by the infinite product

$$(2.2) \qquad f(t) = \frac{1}{(1 + \eta + t)^2} \prod_{k=0}^{\infty} \frac{\rho_k - t}{\rho_k + 2\eta + t}$$

which obviously converges.

LEMMA 1. *If* $\eta \geqq 0$ *the function* f *of* (2.2) *is of the form* (2.1) *with* $e^{\epsilon y}|u(y)|$ *integrable over* $(0, \infty)$ *for all* $\epsilon < \eta$.

Proof. Offhand it would seem that the proof could be conducted by a partial fraction expansion of f. Unfortunately the coefficients of the partial fraction expansion of f_k do not remain bounded as $k \to \infty$ and all attempts to prove directly that the total variation of u_k remains bounded were futile. The partial products in (2.2) may be defined recursively by

$$f_0(t) = \frac{1}{(1 + \eta + t)^2}, \qquad f_k(t) = \frac{\rho_k - t}{\rho_k + 2\eta + t} f_{k-1}(t). \tag{2.3}$$

We begin by showing that f_k is the transform (of the form (2.1)) of a continuously differentiable function u_k vanishing at 0 and at ∞. This is true for $k=0$ with $u_0(y) = ye^{-(1+\eta)y}$. Assume by induction the assertion to be true for $k-1$. A trite integration by parts then shows that $tf_{k-1}(t)$ is the transform of the derivative u_{k-1}', and so the recursive definition of f_k amounts to saying that f_k is the transform of the solution of the differential equation

$$u_k' + u_{k-1}' = \rho_k u_{k-1} - (\rho_k + 2\eta)u_k \tag{2.4}$$

with the initial condition $u_k(0) = 0$. It is readily seen that $u_k(y) \to 0$ as $y \to \infty$, and so the induction hypothesis is satisfied.

Multiplying (2.4) by $u_k + u_{k-1}$ we find

$$\begin{aligned} \frac{1}{2}\frac{d}{dy}[u_k + u_{k-1}]^2 &= \rho_k u_{k-1}^2 - (\rho_k + 2\eta)u_k^2 - 2\eta u_{k-1}u_k \\ &\leqq (\rho_k + \eta)(u_{k-1}^2 - u_k^2). \end{aligned} \tag{2.5}$$

Integrate over $(0, x)$ and let $x \to \infty$. The left side goes to zero and we conclude that

$$\int_0^\infty u_k^2(y)dy \leqq \int_0^\infty u_{k-1}^2(y)dy. \tag{2.6}$$

Thus the integral of u_k^2 exists and has a bound independent of k.

We have now shown that f_k is the transform (1.7) of a signed measure m_k with square integrable density u_k. From Schwarz' inequality it is clear that the total variation of u_k in $(0, x)$ is $\leqq x$. Now $f_k \to f$ and by the continuity theorem for Laplace transforms (see for example [1] chapter XIII) this implies that the measures m_k converge weakly to a measure m with Laplace transform f. From the weak sequential compactness of square integrable functions it follows that m has a square integrable density u, that is, f is of the form (2.1) with u^2 integrable.

This result would suffice for the Hilbert space analogue of Theorem 2, but it remains to establish the integrability of $|u|$. Fortunately it suffices to notice that the function f^* defined by $f^*(t) = f(t - \eta)$ is of the form (2.2) except that ρ_k is replaced by $\rho_k + \eta$ and η by 0. But f^* is the transform of the function u^* defined

by $u^*(y)=e^{\eta y}u(y)$. It follows that u^* is square integrable and hence $e^{\epsilon y}|u(y)|$ s integrable for all $\epsilon<\eta$.

3. Newton's interpolation formula. Let $\{\rho_k\}$ be a (not necessarily monotone) sequence of distinct points in $[0, \infty)$ and let there to each ρ_k correspond a number f_k. The *divided differences* are defined recursively by

$$(3.1)\qquad [f_k]=f_k,\quad [f_{k_0},\cdots,f_{k_n}]=\frac{[f_{k_1},\cdots,f_{k_n}]-[f_{k_0},\cdots,f_{k_{n-1}}]}{\rho_{k_n}-\rho_{k_0}}.$$

These symbols are meaningless unless the correspondence between f_k and ρ_k is kept in mind. If $\rho_k=kh$ the expression in (3.1) reduces to the familiar difference ratio $\Delta^n f_0/h^n$.

For each n we define a polynomial of degree n by

$$(3.2)\qquad P_n(t)=\sum_{k=0}^{n}[f_0,\cdots,f_k](t-\rho_0)\cdots(t-\rho_{k-1}).$$

LEMMA 2. *If f is a function on $[0, \infty)$ then identically*

$$(3.3)\qquad f(t)=P_n(t)+R_n(t),$$

where $f_k=f(\rho_k)$ and

$$(3.4)\qquad R_n(t)=[f(t),f_0,\cdots,f_n](t-\rho_0)\cdots(t-\rho_n).$$

(Here it is understood that the point t with corresponding value $f(t)$ has been added to the sequence $\{\rho_k\}$.)

Proof. The assertion is trivially true for $n=0$ and follows generally by induction upon noticing that the identity

$$(3.5)\qquad P_{n+1}-P_n=R_{n+1}-R_n$$

reduces to the definition of $[f(t),f_0,\cdots,f_{n+1}]$.

COROLLARY 1. *In general P_n represents the unique polynomial of degree n which at $\rho_0,\cdots,\rho_n$ assumes the values $f_0,\cdots,f_n$, respectively.*

This follows at once upon identifying f in (3.3) with the polynomial in question.

A permutation of $\rho_0,\cdots,\rho_n$ leads to a different representation of the same polynomial, but does not affect the coefficient of t^n. Hence we have

COROLLARY 2. *The divided difference $[f_0,\cdots,f_n]$ is a symmetric function of its arguments.*

COROLLARY 3. *If f is n times continuously differentiable then*

$$(3.6)\qquad [f_0,\cdots,f_n]=\frac{1}{n!}f^{(n)}(\tau),$$

where τ is a point in the smallest interval containing $\rho_0,\cdots,\rho_n$.

Indeed, since $f-P_n$ vanishes at $\rho_0, \cdots, \rho_n$ the mean value theorem guarantees the existence of a point at which the nth derivatives of f and P_n are equal.

4. The case of divergence. In this section we suppose that (1.3) holds. Given a finite signed measure m on $[0, \infty)$ we must show that m is uniquely determined by the values $f(\rho_k)$ of its Laplace transform (1.7). Obviously it suffices to consider nonnegative measures, and their transforms are completely monotone, that is, (1.6) holds. To complete the proof of Theorems 1–2 it suffices therefore to establish the following

LEMMA 3. *If f is completely monotone on $[0, \infty)$ then for $t \geqq 0$*

$$f(t) = \sum_{n=0}^{\infty} [f_0, \cdots, f_n](t-\rho_0) \cdots (t-\rho_{n-1}) \tag{4.1}$$

the series converging absolutely. Here $f_k = f(\rho_k)$ and (1.2) *and* (1.3) *are assumed.*

Proof. Comparing (1.6) and (3.6) it is seen that

$$(-1)^n[f_0, \cdots, f_n] \geqq 0. \tag{4.2}$$

Consider first only values t in the interval $[0, \rho_0)$. For such values all terms of the series in (4.1) are positive, and the same is true of the remainder term R_n in (3.3). It follows that

$$P_0(t) \leqq P_1(t) \leqq \cdots \leqq f(t) \tag{4.3}$$

and

$$R_n(t) \downarrow \alpha(t) \quad \text{as} \quad n \to \infty. \tag{4.4}$$

We must show that $\alpha(t)=0$. But for $0<\tau<\rho_0$ we have from (3.4)

$$0 \leqq R_n(\tau) \leqq [f(\tau), f_0, \cdots, f_n](-1)^{n+1}\rho_0 \cdots \rho_n \tag{4.5}$$

and hence

$$\frac{\tau}{\rho_n}\alpha(\tau) \leqq \frac{\tau}{\rho_n} R_n(\tau) \leqq [f(\tau), f_0, \cdots, f_n](-1)^{n+1}\tau\rho_0 \cdots \rho_{n-1}. \tag{4.6}$$

The right side represents the general term of the series (4.1), evaluated at the point $t=0$, when the point τ is added to the sequence $\rho_0, \rho_1, \cdots$. In view of (4.3) therefore the right side is the term of a convergent series, and since $\sum \rho_n^{-1}$ diverges, this implies $\alpha(\tau)=0$.

We have thus justified the expansion (4.1) for $0 \leqq t < \rho_0$. The same argument holds for $\rho_{2k-1} < t < \rho_{2k}$ except that finitely many terms of the series are negative and so the relations (4.3) and (4.4) can be asserted only for $n \geqq 2k$. In intervals of the form $\rho_{2k} < t < \rho_{2k+1}$ almost all terms are negative, and the inequalities are reversed.

5. The Bernstein theorem. Let us now apply Lemma 3 replacing the whole sequence $\rho_0, \rho_1, \cdots$ by the subsequence $\rho_N, \rho_{N+1}, \cdots$. Then (4.1) takes on

the form

$$(5.1)\qquad \sum_{n=0}^{\infty}(-1)^n[f_N, f_{N+1}, \cdots, f_{N+n}](\rho_N - t)\cdots(\rho_{N+n-1} - t) = f(t).$$

For $t=0$ all terms of the series are positive, and we may therefore define an *atomic measure m_N attributing weight*

$$(-1)^n[f_N, \cdots, f_{N+n}]\rho_N\cdots\rho_{N+n-1}$$

to the point

$$(5.2)\qquad \tau_{n,N} = \frac{1}{\rho_N} + \cdots + \frac{1}{\rho_{N+n}}.$$

The total mass of m_N equals $f(0)$, and its Laplace transform is given by

$$(5.3)\qquad f_N(t) = \sum_{n=0}^{\infty}(-1)^n[f_N, \cdots, f_{N+n}]\rho_N\cdots\rho_{N+n-1}\exp(-t\tau_{n,N}).$$

Now for fixed n as $N\to\infty$ the ratio of $1-t/\rho_{N+n}$ to $\exp(-t/\rho_{N+n})$ approaches unity uniformly in finite t-intervals, and the same is true of the ratio of the nth terms of the series (5.1) and (5.3). It follows that $f_N(t)\to f(t)$ for all $t\geqq 0$. This convergence of the Laplace transforms implies that the measures m_N converge to a finite measure m whose Laplace transform (1.7) coincides with f. We have thus proved that every completely monotone f with $f(0)<\infty$ is the Laplace transform of a finite measure. The condition $f(0)<\infty$ is easily removed if one replaces $f(t)$ by $f_\epsilon(t)=f(t+\epsilon)$. This f_ϵ is the transform of a finite measure $m^{(\epsilon)}$, and one concludes trivially that f is the transform of the measure $m(dy)=e^{\epsilon y}m^{(\epsilon)}(dy)$, which is independent of ϵ and finite on finite intervals. This concludes the proof of Bernstein's Theorem 3.

The fact that $m_N\to m$ has further consequences. Suppose again that $f(0)<\infty$ so that m is finite. The integral with respect to m_N of a continuous function u vanishing at infinity tends to the corresponding integral with respect to m, that is

$$(5.4)\qquad \sum_{n=0}^{\infty}(-1)^n[f_N, \cdots, f_{N+n}]\rho_N\cdots\rho_{N+n-1}u(\tau_{n,N}) \to \int_0^{\infty}u(y)m(dy).$$

We have thus proved

LEMMA 3. *If f is completely monotone on* $[0, \infty)$ *then* (5.4) *holds for every continuous function u vanishing at infinity.*

Consider, in particular, the function $f(t)=e^{-\lambda t}$. The corresponding measure m attributes unit mass to the point λ, and so the right side in (5.4) reduces to $u(\lambda)$. The coefficient of $u(\tau_{n,N})$ on the left is a linear combination of $e^{-\lambda\rho_N}$, $\cdots$, $e^{-\lambda\rho_{N+n}}$, and so (5.4) *exhibits $u(\lambda)$ as the pointwise limit of finite linear com-*

binations of functions $e^{-\rho_k\lambda}$. It is not difficult to see that the limit is uniform, and in terms of the original variable $x=e^{-\lambda}$ we obtain an explicit form for the approximation of functions in $C_0[0, 1]$ by means of linear combinations of x^{ρ_k}.

6. The interpolation problem. We indicate briefly how the argument of section 4 yields an easy answer to the following problem: Given values $f_0, f_1, \cdots$, does there exist a completely monotone function on $(0, \infty)$ such that $f(\rho_k)=f_k$? It is obviously necessary that

$$(-1)^n[f_{k_0}, \cdots, f_{k_n}] \geqq 0 \tag{6.1}$$

for all possible k_j. The next lemma states that these conditions are essentially sufficient.

LEMMA 4. *Suppose* $0=\rho_0<\rho_1<\cdots$ *and* $\rho_1^{-1}+\cdots=\infty$. *To each* ρ_k *let there correspond a value* f_k *such that* (6.1) *holds. There exists then a unique completely monotone function* f *such that* $f(\rho_k)=f_k$ *for* $k=1, 2, \cdots$ *and* $f(0)\leqq f_0$. *For it the Newton expansion* (5.1) *holds for* $N=1, 2, \cdots$.

NOTE. It cannot be asserted that $f(0)=f_0$ because the inequalities (6.1) remain valid when f_0 is replaced by an arbitrary value $f_0+\eta>f_0$.

Proof. For $t=\rho_1, \cdots, \rho_{N-1}$ the terms of the series in (5.1) are nonnegative and as in (4.3) it is seen trivially that at $t=\rho_k$ the sum is $\leqq f_k$. The further argument of section 4 does not apply to $\rho_0=0$, but for $k=1, \cdots, N-1$ it shows that at $t=\rho_k$ the sum *equals* f_k. Again, in the interval $t<\rho_N$ all terms of the series are positive and less than the terms for $t=0$. It follows that for $t<\rho_N$ the series in (5.1) converges to a sum which we denote by $F_N(t)$. The same argument shows that (for $t<\rho_N$) the derivative f_N' is given by a series with negative terms, and generally, that F_N is completely monotone in $(0, \rho_N)$. By the selection theorem for monotone functions it is possible to let N run through a sequence $N_1, N_2, \cdots\to\infty$ such that F_{N_k} tends pointwise to a limit f. Now it is easily seen that a pointwise limit of completely monotone functions is again completely monotone, and so this is true of the limit f. By construction $f(\rho_k)=f_k$ or $k\geqq 1$ and $f(0)\leqq f_0$, so f solves our interpolation problem. From the uniqueness theorem in section 4 it follows that all the functions F_N are identical. (This was verified directly in [2], but the present proof is simpler.)

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References

1. W. Feller, An introduction to probability theory and its applications, vol. II. Wiley, New York, 1966.

2. W. Feller, Completely monotone functions and sequences, Duke Math. J., 5 (1939) 661–674.

3. A. O. Gelfond, Differenzenrechnung, Berlin, 1958. (Translated from the Russian.)

4. I. I. Hirschman and D. V. Widder, Generalized Bernstein polynomials, Duke Math. J., 16 (1949) 433–438.

5. S. Kaczmarcz and H. Steinhaus, Theorie der Orthogonalreihen, Warsaw, 1935.

6. G. G. Lorentz, Bernstein polynomials, Mathematical Expositions no. 8, Toronto, 1953.

7. Ch. H. Müntz, Über den Approximationssatz von Weierstrass, Math. Abhandlungen H. A. Schwarz zu seinem 50. Doktorjubiläum gewidmet, Berlin 1914, pp. 303–312.
8. I. P. Natanson, Constructive function theory, vol. II. New York, 1965. (Translated from the Russian.)
9. D. V. Widder, The Laplace transform, Princeton, 1946.

DECOMPOSABILITY OF POSITIVE FUNCTIONS ON R^n

W. G. DOTSON, JR., North Carolina State University

1. Introduction. In differential equations and in statistics, one is sometimes interested in writing a positive function of several variables $G(x_1, \cdots, x_n)$ as a product of functions of single variables: $P_1(x_1) \cdots P_n(x_n)$. Our first result is a necessary and sufficient condition (stated in terms of the partial derivatives of G) for the existence of such a decomposition. Some specific applications are then made in the areas mentioned above. In the event that such a decomposition is not possible, it is sometimes of interest to know whether or not a product type decomposition can be effected by means of a nonsingular linear transformation of the variables $x_1, \cdots, x_n$. For example, it is well known that the bivariate normal probability density function, $\exp(-\frac{1}{2}(ax^2+2bxy+cy^2))$, can be decomposed as

$$\exp(-\tfrac{1}{2}au^2)\cdot\exp(-\tfrac{1}{2}((ac-b^2)/a)v^2)$$

by means of the nonsingular linear transformation $x=u-(b/a)v$, $y=v$. One may also use an orthogonal transformation (viz., rotation of coordinate axes) to accomplish this type of decomposition as the product of a function of u and a function of v. Our second main result is a characterization of positive functions on R^n which can be decomposed (as products of functions of single variables) by means of nonsingular linear transformations. Some special cases and applications are then considered.

2. The fundamental theorem. We consider R^n as the space of all n-tuples of real numbers, with the Euclidean metric topology. $R=R^1$, and R^+ is the set of positive real numbers considered as a subspace of R. If D is a region (i.e., a *convex** open set) in R^n, then D_i denotes the image of D under the projection $(u_1, \cdots, u_n)\rightarrow u_i$ of R^n onto the u_i-axis, $i=1, \cdots, n$. Since the projection of R^n onto each coordinate axis is a continuous open mapping, each D_i is a (possibly infinite) open interval on the u_i-axis.

THEOREM 1. *Suppose D is a region in R^n and G is a function from D to R^+. Then the following statements are equivalent:*

(1) *There exist continuously differentiable functions P_i: $D_i\rightarrow R^+$, $i=1, \cdots, n$, such that $G(u_1, \cdots, u_n)=\prod_{i=1}^n P_i(u_i)$ for all points $(u_1, \cdots, u_n)$ in D.*

(2) *The partial derivatives* G_{u_i}, $i=1, \cdots, n$, *and* $G_{u_iu_j}$, $1 \leqq i<j \leqq n$, *are continuous on* D *and* $G_{u_i}G_{u_j}-GG_{u_iu_j}=0$ *identically on* D, $1 \leqq i<j \leqq n$.

Proof. Assuming (1), then for $i \neq j$ we have at once

$$\begin{aligned} G_{u_i}G_{u_j} &= \{P_i'(u_i) \prod_{k\neq i} P_k(u_k)\} \cdot \{P_j'(u_j) \prod_{k \neq j} P_k(u_k)\} \\ &= \{P_i'(u_i)P_j'(u_j) \prod_{k\neq i,j} P_k(u_k)\} \cdot \prod_{k=1}^{n} P_k(u_k) = G_{u_iu_j}G; \end{aligned}$$

and, as products of continuous functions, G_{u_i} and $G_{u_iu_j}$ are continuous on D. We turn now to the proof that (2) implies (1). Notice first that the continuity of the G_{u_i} implies the differentiability, and hence also the continuity, of G on D. Thus if we assume merely the existence of $G_{u_iu_j}$, $1 \leqq i<j \leqq n$, then the continuity of these $G_{u_iu_j}$ follows from the relations

$$G_{u_iu_j} = G_{u_i}G_{u_j}/G, \quad 1 \leqq i < j \leqq n;$$

and from the continuity of $G_{u_iu_j}$, $i<j$, it follows that $G_{u_ju_i}$ exists and equals $G_{u_iu_j}$ on D. Hence, we have $G_{u_i}G_{u_j}-GG_{u_iu_j}=0$ identically on D for all i, $j=1, \cdots, n$ with $i \neq j$. For any fixed i it follows that for all $j \neq i$

$$\frac{\partial}{\partial u_j}\left(\frac{G_{u_i}}{G}\right) = \frac{GG_{u_iu_j} - G_{u_i}G_{u_j}}{G^2} = 0 \quad \text{identically on } D;$$

and so, since D is a region, there exist functions H_i: $D_i \rightarrow R$, $i=1, \cdots, n$, such that $G_{u_i}/G = H_i(u_i)$ identically on D, $i=1, \cdots, n$. Continuity of the H_i follows from that of the G_{u_i} and G. We now observe that

$$(*) \qquad d(\ln G) = \sum_{i=1}^{n} \left(\frac{G_{u_i}}{G}\right) du_i = \sum_{i=1}^{n} H_i(u_i)du_i.$$

Let $a=(a_1, \cdots, a_n)$ be a point in D. If $u=(u_1, \cdots, u_n)$ is any point in D then there is a polygonal path L from a to u which lies in D. Taking the line integral of the exact differential form (*) along L gives us

$$\ln G(u_1, \cdots, u_n) - \ln G(a_1, \cdots, a_n) = \sum_{i=1}^{n} \int_{a_i}^{u_i} H_i(t_i)dt_i$$

from which it follows that $G(u_1, \cdots, u_n) = \prod_{i=1}^{n} P_i(u_i)$, where the functions P_i: $D_i \rightarrow R^+$, $i=1, \cdots, n$, are defined by

$$P_i(u_i) = \sqrt[n]{G(a_1, \cdots, a_n)} \exp\left(\int_{a_i}^{u_i} H_i(t_i)dt_i\right)$$

for all $u_i \in D_i$. Since the functions H_i are continuous, it is clear that the functions P_i are continuously differentiable. This completes the proof of the theorem.

An obvious but rather interesting corollary is that any function G satisfying

the conditions of statement (2) of the theorem can be extended from D to the n-dimensional rectangular parallelepiped $D_1 \times D_2 \times \cdots \times D_n$ in such a way that the partial derivatives G_{u_i}, $i=1, \cdots, n$, and $G_{u_iu_j}$, $i, j=1, \cdots, n$, $i \neq j$, remain continuous on this larger domain and the equations $G_{u_i}G_{u_j}-GG_{u_iu_j}=0$, $i \neq j$, also hold on this larger domain.

We note that no use of the hypothesis $G>0$ was made in the proof that (1) implies (2); thus, if R^+ were replaced by R in statement (1), then (2) would still be implied by (1). Also, a simple corollary to the fact that (2) implies (1) is that any nonvanishing function $G: D \rightarrow R$ which has its partial derivatives G_{u_i}, $G_{u_iu_j}$ satisfying the conditions of statement (2) can be expressed as a product of nonvanishing continuously differentiable functions $P_i: D_i \rightarrow R$. For, since G is continuous on the connected set D, either $G>0$ on D or $-G>0$ on D, and so, since

$$(-G)_{u_i}(-G)_{u_j} - (-G)(-G)_{u_iu_j} = G_{u_i}G_{u_j} - GG_{u_iu_j} = 0,$$

either G or $-G$ is a product of positive P_i's. The following example shows that the nonvanishing of G is essential in order to obtain a product decomposition from the conditions of statement (2): $G(u, v)=0$ if $u>0$ and $v>0$; $G(u, v)=u^2v^2$ for all other points (u, v) in R^2. Here the partial derivatives G_u, G_v, G_{uv} are continuous on R^2 and satisfy $G_uG_v-GG_{uv}=0$ identically on R^2; but there do not exist functions $P(u)$, $Q(v)$ such that $G(u, v)=P(u)Q(v)$ identically on R^2.

Next let us observe that a product decomposition of a nonvanishing function $G(u_1, \cdots, u_n)$ is essentially unique (i.e., unique to within multiplication of the factors by constants), since if $G(u_1, \cdots, u_n)=P_1(u_1) \cdots P_n(u_n)$ on D and if $(a_1, \cdots, a_n) \in D$ then

$$P_k(u_k) = G(a_1, \cdots, a_{k-1}, u_k, a_{k+1}, \cdots, a_n) \Big/ \prod_{i \neq k} P_i(a_i), \qquad k = 1, \cdots, n.$$

Finally we have the following important corollary to Theorem 1.

COROLLARY 1.1. *Suppose D is a region in R^n and $G: D \rightarrow R$ is a nonvanishing function with first partial derivatives G_{u_i} continuous on D. Suppose that for each pair u_i, u_j $(1 \leqq i<j \leqq n)$ there exist functions $Q_{ij}: D \rightarrow R$, $R_{ij}: D \rightarrow R$ such that $(Q_{ij})_{u_i}$ and $(R_{ij})_{u_j}$ exist on D, $(Q_{ij})_{u_j}=(R_{ij})_{u_i}=0$ identically on D, and $G=Q_{ij}R_{ij}$ identically on D. Then there exist continuously differentiable nonvanishing functions $P_i: D_i \rightarrow R$ such that $G(u_1, \cdots, u_n)=\prod_{i=1}^n P_i(u_i)$ identically on D.*

Proof. Given u_i, u_j $(1 \leqq i<j \leqq n)$ we have

$$G_{u_i} = (Q_{ij})(R_{ij})_{u_i} + (R_{ij})(Q_{ij})_{u_i} = (R_{ij})(Q_{ij})_{u_i}$$

and similarly $G_{u_j}=(Q_{ij})(R_{ij})_{u_j}$. Since $(Q_{ij})_{u_j}=0$, $(Q_{ij})_{u_ju_i}=\{(Q_{ij})_{u_j}\}_{u_i}=0$, and so, since the constant 0 function is continuous, $(Q_{ij})_{u_iu_j}=(Q_{ij})_{u_ju_i}=0$ on D. Hence $G_{u_iu_j}=\{(R_{ij})(Q_{ij})_{u_i}\}_{u_j}=(R_{ij})_{u_j}(Q_{ij})_{u_i}$. It now follows that $G_{u_i}G_{u_j}-GG_{u_iu_j}=0$ identically on D. Continuity of $G_{u_iu_j}$ follows from $G_{u_iu_j}=G_{u_i}G_{u_j}/G$. Hence, Theorem 1 $\{(2)\Rightarrow(1)\}$ can now be applied to G or $-G$, whichever is positive on the connected set D.

Thus, for nonvanishing functions with continuous first partial derivatives on a region in R^n, pairwise decomposability implies decomposability. The following example shows that if the nonvanishing of the function is not required, then pairwise decomposability is not sufficient to insure decomposability: $F(u, v, w) = u^2v^2w^2$ if $u>0$, $v>0$, and $w>0$, or if $u<0$, $v<0$, and $w<0$; and let $F(u, v, w)=0$ for all other points (u, v, w) in R^3. F_u, F_v, F_w are continuous on R^3, and F is pairwise decomposable. For example, $F(u, v, w)=Q(u, v, w)\,R(u, v, w)$ where

$$Q(u, v, w) = \begin{cases} u^2w & \text{if } w > 0 \text{ and } u > 0, \text{ or if } w \leqq 0 \text{ and } u < 0 \\ 0 & \text{if } w > 0 \text{ and } u \leqq 0, \text{ or if } w \leqq 0 \text{ and } u \geqq 0 \end{cases}$$

$$R(u, v, w) = \begin{cases} v^2w & \text{if } w > 0 \text{ and } v > 0, \text{ or if } w \leqq 0 \text{ and } v < 0 \\ 0 & \text{if } w > 0 \text{ and } v \leqq 0, \text{ or if } w \leqq 0 \text{ and } v \geqq 0. \end{cases}$$

One checks that Q_u and R_v exist everywhere in R^3, and $Q_v=R_u=0$ identically in R^3. Similar pairwise decompositions split u from w and v from w .But there do not exist functions $P_1(u)$, $P_2(v)$, $P_3(w)$ such that $F(u, v, w)=P_1(u)P_2(v)P_3(w)$ identically in R^3.

3. Applications. We shall mention three applications of the above results.

(1) Criterion for solvability of ordinary first order differential equations by separation of the variables: If $F(x, y)$ is nonvanishing in the region of interest, and if F_x, F_y are continuous and F_{xy} exists, then the differential equation $y'=F(x, y)$ will be solvable by separation of the variables if and only if $F_xF_y - FF_{xy}=0$ identically in the region. If the equation $y'=F(x, y)$ is written in the form $Mdx+Ndy=0$, then $F=-M/N$, and the condition $F_xF_y-FF_{xy}=0$ becomes $N^2(M_xM_y-MM_{xy})=M^2(N_xN_y-NN_{xy})$. The necessity of this condition holds under the weaker hypothesis that F_x, F_y, F_{xy} exist (with F not necessarily nonvanishing). Since this is not immediately obvious, we will supply a proof. We take $F(x, y)$ to be defined on a region $D\subset R^2$ having projections D_1, D_2 on the x and y axes, respectively. We assume that F_x, F_y, F_{xy} exist on D, and that there exist functions $P: D_1\to R$, $Q: D_2\to R$ such that $F(x, y)=P(x)Q(y)$ identically on D. To carry out the integration of $y'=F=PQ$, one would want P and $1/Q$ to be integrable, but we will make no assumptions concerning P and Q. Let $(x_0, y_0)\in D$. There exist open intervals I_1, I_2 such that $x_0\in I_1\subset D_1$, $y_0\in I_2\subset D_2$, and $I_1\times I_2\subset D$. Suppose first that $F(x_0, y_0)\neq 0$. Then $P(x_0)\neq 0$, and so $Q(y) = F(x_0, y)/P(x_0)$ for all $y\in I_2$. Since F_y exists on D, $Q'(y)=F_y(x_0, y)/P(x_0)$ exists for all $y\in I_2$. Similarly, since $Q(y_0)\neq 0$ and F_x exists on D,

$$P'(x) = F_x(x, y_0)/Q(y_0)$$

exists for all $x\in I_1$. It now follows that for all $(x, y)\in I_1\times I_2$, $F_x(x, y)=P'(x)Q(y)$, $F_y(x, y)=P(x)Q'(y)$, and $F_{xy}(x, y)=P'(x)Q'(y)$. In particular, at (x_0, y_0) we now have $F_xF_y-FF_{xy}=(P'Q)(P\,Q')-(PQ)(P'Q')=0$. Now suppose $F(x_0, y_0)=0$. Then $P(x_0)=0$ or $Q(y_0)=0$. Suppose $P(x_0)=0$. Then $F(x_0, y)=P(x_0)Q(y)=0$ for all $y\in I_2$. Hence $F_y(x_0, y_0)=0$, and so at (x_0, y_0) we have $F_xF_y-FF_{xy}=(F_x)(0)$

$-(0)(F_{xy})=0$. Similarly, if $Q(y_0)=0$ then $F_x(x_0, y_0)=0$ and so $F_xF_y-FF_{xy}=0$ at (x_0, y_0).

(2) Restrictions introduced in solving the Laplace equation $U_{xx}+U_{yy}+U_{zz}=0$ by making the assumption $U(x, y, z)=P(x)Q(y)R(z)$: If U is a solution of the Laplace equation in a region $D\subset R^3$ and if U has the form $U=PQR$ where P, Q, R are differentiable functions of x, y, z, respectively, then U is actually a simultaneous solution of a system of four differential equations in D, viz. $U_{xx}+U_{yy}+U_{zz}=0$, $U_xU_y-UU_{xy}=0$, $U_xU_z-UU_{xz}=0$, $U_yU_z-UU_{yz}=0$. Conversely, if U is a simultaneous solution of this system in a region D, then U is, of course, a solution of the Laplace equation in D, and U will have the form $U=PQR$ in any subregion of D in which U is nonvanishing and U_x, U_y, U_z are continuous.

(3) Criteria for determining when a positive n-variate probability density function can be factored as a product of marginal densities, so that the random variables involved are statistically independent: Both Theorem 1 and Corollary 1.1 provide such criteria, and their application would seem to require no comment.

4. Functions decomposable by linear transformations. We shall confine our discussion to positive functions, since, as we have seen, no essential gain in generality is obtained by discussing nonvanishing continuous functions with connected domains. If $A=[a_{ij}]$ is a $n\times n$ matrix of real numbers, then $T_A: R^n\to R^n$ is the linear transformation defined by $T_A(u_1, \cdots, u_n)=(x_1, \cdots, x_n)$ where $x_i=\sum_{j=1}^n a_{ij}u_j$, $i=1, \cdots, n$. T_A and A are said to be nonsingular if $\det A\neq 0$; and if T_A is nonsingular then, since T_A and T_A^{-1} are continuous open mappings, both T_A and T_A^{-1} map regions onto regions. If D is a region in R^n then $C^2(D)$ denotes the set of all functions $F: D\to R$ which have continuous first and second order partial derivatives on D.

DEFINITION. Suppose D is a region in R^n. A function $F: D\to R^+$ which belongs to $C^2(D)$ is said to be *decomposable by linear transformation* (abbr. DLT) on D provided there exists a nonsingular linear transformation $T_A: R^n\to R^n$ such that the function $G: T_A^{-1}[D]\to R^+$ defined by $G(u_1, \cdots, u_n)=F(\sum_{j=1}^n a_{1j}u_j, \cdots, \sum_{j=1}^n a_{nj}u_j)$, for all $(u_1, \cdots, u_n)\in T_A^{-1}[D]$, is decomposable in the sense that there exist continuously differentiable functions $P_i: T_A^{-1}[D]_i\to R^+$, $i=1, \cdots, n$, such that $G(u_1, \cdots, u_n)=\prod_{i=1}^n P_i(u_i)$ for all $(u_1, \cdots, u_n)\in T_A^{-1}[D]$.

If D is a region in R^n and $B=[B_{ip}]$ is an $n\times n$ matrix whose elements B_{ip} are functions from D to R, then B is said to be *congruent to a diagonal matrix on* D provided there exists an $n\times n$ diagonal matrix Δ, whose diagonal elements Δ_{ii} are functions from D to R, and a nonsingular $n\times n$ matrix $A=[a_{ij}]$, whose elements are real numbers, such that $A'BA=\Delta$ identically on D (where A' denotes the transpose of A).

THEOREM 2. *Suppose D is a region in R^n and $F: D\to R^+$ belongs to $C^2(D)$. Let the functions $B_{ip}: D\to R$ be defined by $B_{ip}=F_{x_i}F_{x_p}-FF_{x_ix_p}$, $i, p=1, \cdots, n$. Then*

F is DLT *on D if and only if the symmetric matrix $B=[B_{ip}]$ is congruent to a diagonal matrix on D.*

Proof. If F is DLT on D then there exists a nonsingular linear transformation T_A, with $A=[a_{ij}]$, such that the function $G\colon T_A^{-1}[D]\to R^+$ defined by $G(u_1,\cdots,u_n)=F(\sum_{j=1}^n a_{1j}u_j,\cdots,\sum_{j=1}^n a_{nj}u_j)$ is decomposable as a product of continuously differentiable functions $P_i(u_i)$. By Theorem 1 ((1)$\Rightarrow$(2)) it follows that $G_{u_j}G_{u_k}-GG_{u_ju_k}=0$ identically on $T_A^{-1}[D]$ for all $j,k=1,\cdots,n$ with $j\neq k$. Calculating with the chain rule for partial differentiation, one finds that these relations on $T_A^{-1}[D]$ are equivalent to the relations $\sum_{i,p=1}^n a_{ij}B_{ip}a_{pk}=0$ identically on D for all $j,k=1,\cdots,n$ with $j\neq k$. These latter relations say precisely that $A'BA$ is a diagonal matrix (the diagonal elements of which are functions from D to R). The proof of necessity is thus completed, and we note that the proof of necessity does not actually require that F be positive or nonvanishing. Conversely, suppose there exists a nonsingular matrix $A=[a_{ij}]$ such that $A'BA$ is diagonal, with diagonal elements functions from D to R. Then A determines a nonsingular linear transformation T_A. Since $A'BA$ is diagonal, we obtain the relations $\sum_{i,p=1}^n a_{ij}B_{ip}a_{pk}=0$ on D (all $j\neq k$) which are equivalent to the relations $G_{u_j}G_{u_k}-GG_{u_ju_k}=0$ on $T_A^{-1}[D]$ (all $j\neq k$), provided G is defined on $T_A^{-1}[D]$ by $G(u_1,\cdots,u_n)=F(\sum_{j=1}^n a_{1j}u_j,\cdots,\sum_{j=1}^n a_{nj}u_j)$. Since F is positive, it now follows from Theorem 1 $\{(2)\Rightarrow(1)\}$ that there exist continuously differentiable functions $P_i\colon T_A^{-1}[D]_i\to R^+$, $i=1,\cdots,n$, such that $G(u_1,\cdots,u_n)=\prod_{i=1}^n P_i(u_i)$ for all $(u_1,\cdots,u_n)\in T_A^{-1}[D]$. This completes the proof of Theorem 2.

A simple example seems to be in order. We consider $F\colon R^2\to R$ defined by $F(x,y)=x^2-y^2$. The nonsingular linear transformation $T_A\colon x=\frac{1}{2}u+\frac{1}{2}v$, $y=\frac{1}{2}u-\frac{1}{2}v$ maps R^2 onto R^2, and we have $G(u,v)=F(\frac{1}{2}u+\frac{1}{2}v,\ \frac{1}{2}u-\frac{1}{2}v)=uv$ for all $(u,v)\in R^2$. Hence, the matrix B should be congruent to a diagonal matrix on R^2. We have $B_{11}=F_x^2-FF_{xx}=2x^2+2y^2$, $B_{22}=F_y^2-FF_{yy}=2x^2+2y^2$, $B_{12}=B_{21}=F_xF_y-FF_{xy}=-4xy$. Hence

$$B=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}=\begin{pmatrix}2x^2+2y^2 & -4xy\\ -4xy & 2x^2+2y^2\end{pmatrix}.$$

Also

$$A=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}=A',$$

and we compute

$$A'BA=\begin{pmatrix}(x-y)^2 & 0\\ 0 & (x+y)^2\end{pmatrix}.$$

A special case of considerable interest is the n-variate normal probability density function defined on R^n by

$$F(x_1, \cdots, x_n) = K \cdot \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} c_{ij}x_ix_j\right)$$

where $C = [c_{ij}]$ is a positive definite real symmetric $n \times n$ matrix and $K = (1/2\pi)^{n/2} \cdot (\det C)^{1/2}$. A simple calculation shows that $B_{ip} = F_{x_i}F_{x_p} - FF_{x_ix_p} = F^2c_{ip}$, and so $B = [B_{ip}] = F^2C$. Since C is real and symmetric, there exists a real orthogonal $n \times n$ matrix A such that $A'CA = A^{-1}CA =$ a diagonal matrix with diagonal elements the characteristic roots of C. It follows that $A'BA = A'F^2CA = F^2(A'CA)$ $=$ a diagonal matrix with diagonal elements given by F^2 times the characteristic roots of C. Since $F > 0$ on R^n, it now follows from Theorem 2 that F is DLT on R^n, and, moreover, the decomposition can be effected by means of an orthogonal linear transformation. Thus, as a special case of Theorem 2, we have the well-known result that n random variables with an n-variate normal joint probability density can be transformed into n statistically independent random variables by means of an orthogonal linear transformation. Consequently, Theorem 2 can be thought of as providing a generalization of this well-known result. We have, in fact, the following corollary to Theorem 2:

COROLLARY 2.1. *If $F(x_1, \cdots, x_n)$ is a positive probability density function which belongs to $C^2(R^n)$, then the random variables $x_1, \cdots, x_n$ can be transformed, by means of a nonsingular linear transformation, into statistically independent random variables $u_1, \cdots, u_n$ if and only if the matrix $B = [F_{x_i}F_{x_p} - FF_{x_ix_p}]$ is congruent to a diagonal matrix on R^n.*

For the proof, one has only to note that if $T_A: x_i = \sum_{j=1}^{n} a_{ij}u_j$, $i = 1, \cdots, n$, is a nonsingular linear transformation, then the joint probability density of the variables $u_1, \cdots, u_n$ is given by

$$G(u_1, \cdots, u_n) = F\left(\sum_{j=1}^{n} a_{1j}u_j, \cdots, \sum_{j=1}^{n} a_{nj}u_j\right) \cdot \det [a_{ij}].$$

Our final result is a rather interesting specialization to the case $n = 2$.

COROLLARY 2.2. *If $F(x, y)$ is a positive function belonging to $C^2(R^2)$, then $F(x, y)$ is* DLT *on R^2 if and only if there exist real numbers p, q, r such that $q^2 - 4pr > 0$ and $pB_{11} + qB_{12} + rB_{22} = 0$ identically on R^2.*

Proof. If F is DLT on R^2, there is a nonsingular linear transformation $T_A: x = a_{11}u + a_{12}v$, $y = a_{21}u + a_{22}v$, such that $G(u, v) = F(a_{11}u + a_{12}v, a_{21}u + a_{22}v)$ satisfies $G_uG_v - GG_{uv} = 0$ identically on R^2. But, via the chain rule as in the proof of Theorem 2, this is equivalent to $pB_{11} + qB_{12} + rB_{22} = 0$ identically on R^2, where $B_{11} = F_x^2 - FF_{xx}$, $B_{12} = F_xF_y - FF_{xy}$, $B_{22} = F_y^2 - FF_{yy}$, and $p = a_{11}a_{12}$, $q = a_{11}a_{22} + a_{12}a_{21}$, $r = a_{21}a_{22}$, so that $q^2 - 4pr = (a_{11}a_{22} - a_{12}a_{21})^2 > 0$ since T_A is nonsingular. Conversely, if there exist real numbers p, q, r, with $q^2 - 4pr > 0$ and $pB_{11} + qB_{12} + rB_{22} = 0$ on R^2, then it is easily shown that there exist real numbers a_{11}, a_{12}, a_{21}, a_{22} such that $p = a_{11}a_{12}$, $q = a_{11}a_{22} + a_{12}a_{21}$, $r = a_{21}a_{22}$. One can, in fact, take $a_{11} = 1$,

$a_{12}=p$, $a_{21}=$ a root of the equation $pa_{21}^2-qa_{21}+r=0$, and $a_{22}=q-pa_{21}$. Using these a_{ij} to define T_A, we have immediately that T_A is nonsingular and T_A decomposes F on R^2.

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* Connectedness is not sufficient for the success of our proofs. On the other hand, convexity is a little stronger than is needed in section 2. It is sufficient to have a region defined as a connected open set D such that if i is a positive integer $\leqq n$ and $a \in D_i$, then the intersection of the $(n-1)$-dimensional hyperplane $u_i=a$ with D is connected. In section 4, however, convexity does seem to be necessary (since the property just described is not preserved under linear mappings).

INVERT SETS IN POLYHEDRA

J. G. HOCKING, Michigan State University

This work is an extension of P. H. Doyle's investigation of the invert set $I(P)$ of a polyhedron P reported in this journal (see [1]). In particular, the continuous invert set $CI(P)$ is shown to be identical to $I(P)$ in every case in which $I(P)$ is not a zero-sphere. This leads to some interesting remarks concerning the relation between invert sets and some current problems in combinatorial topology.

$\mathfrak{H}(X)$ denotes the group of all homeomorphisms of a topological space X onto itself and $\mathfrak{G}(X)$ denotes the subset of $\mathfrak{H}(X)$ consisting of all maps in $\mathfrak{H}(X)$ which are isotopic (via a family in $\mathfrak{H}(X)$) to the identity map on X. For compact Hausdorff spaces, at least, $\mathfrak{G}(X)$ is a subgroup of $\mathfrak{H}(X)$.

The following concepts were introduced in [5]: A point $p \in X$ is a [*continuous*] *invert point* of the space X provided that for each open neighborhood U of p there exists $h \in \mathfrak{H}(X)$ [$h \in \mathfrak{G}(X)$] such that $h(X-U) \subset U$. The map h is called an *inverting map* for U. The set of all [continuous] invert points of X is the [*continuous*] *invert set* of X and is denoted by $I(X)$ [$CI(X)$]. Clearly, we have $CI(X) \subset I(X)$. If $I(X)=X$ [$CI(X)=X$], then X is said to be [*continuously*] *invertible.*

Theorem 2 of [5] states that for a T_1 space X, $I(X)$ contains 0, 1, 2 or infinitely many points and Theorem 25 of [5] implies that for a Hausdorff space X, $CI(X)$ contains 0, 1 or uncountably many points. Theorems 3, 4, 19 and 20 of [5] combine to say that $I(X)$ [$CI(X)$] is a closed, [continuously] invertible subspace of X.

The proof of the following result is trivial:

LEMMA 1. *Let K be a (finite) geometric complex. Then the interior of any simplex of K is homogeneous under the action of $\mathfrak{G}(K)$.*

In [1] Doyle obtains the following two results which are re-stated here to include mention of the continuous invert set. His proofs apply directly, using Lemma 1.

THEOREM 2. *Let P be a polyhedron. Then both $I(P)$ and $CI(P)$ carry subcomplexes at each triangulation of P.*

THEOREM 3. *Let P be a polyhedron. If $I(P)$ is nondegenerate, then $I(P)$ is a sphere. If $CI(P)$ is nondegenerate, then $CI(P)$ is an n-sphere, $n>0$.*

Note that if $I(P)$ is a 0-sphere, $p\cup q$, then $CI(P)$ is necessarily empty. To see this we notice that $CI(P)\subset I(P)$ and $CI(P)$ cannot contain precisely two points. Thus if $CI(P)$ were not empty, we would have $CI(P)=p$, say. Choose an open neighborhood U of p not containing q. Then there would be an isotopy moving q into U. But every point on the isotopy path of q would be an invert point of P, contradicting the assumption that $I(P)=p\cup q$.

THEOREM 4. *Let P be a polyhedron. Then $I(P)$ is homogeneous under the action of $\mathfrak{H}(P)$ and $CI(P)$ is continuously homogeneous under the action of $\mathfrak{G}(P)$.*

Proof. If $I(P)$ contains 0 or 1 point, the result is trivial. If $I(P)$ contains precisely two points p and q, choose an open neighborhood U of p not containing q. An inverting map h for U must throw q into U and, since $I(P)$ is invariant under $\mathfrak{H}(P)$, $h(q)=p$.

If $I(P)$ contains infinitely many points then $I(P)$ is an n-sphere, $n>0$. Let x and y be any two points on $I(P)$. Choose a triangulation K of P sufficiently fine so that there exists a simplex A of maximal dimension n in the sphere $I(P)$ such that $x, y\notin \mathrm{St}(A, K)$. Each point in $\mathrm{Int}\, A$ is in $I(P)$ so there exists an inverting map $h\in\mathfrak{H}(P)$ for the open set $\mathrm{Int}\,\mathrm{St}(A, K)$. Clearly, we have $h(x)$, $h(y)\in\mathrm{Int}\, A$. By Lemma 1 there exists $g\in\mathfrak{G}(P)$ with $gh(x)=h(y)$ whence $h^{-1}gh(x)=y$ and surely $h^{-1}gh\in\mathfrak{H}(P)$.

For $CI(P)$ we mimic the proof above, choosing the inverting map h from $\mathfrak{G}(P)$.

LEMMA 5. *Let P be a polyhedron with a nondegenerate [continuous] invert set. If $x, y\in I(P)$ [$x, y\in CI(P)$] and if U is an open neighborhood of x not containing y, then there is an inverting map $h\in\mathfrak{H}(P)$ [$h\in\mathfrak{G}(P)$] for U with $h(y)=x$.*

Proof. By Theorem 4 it suffices to prove this result in the case where x is an interior point of a principal simplex on $I(P)$ [$CI(P)$] in some triangulation K of P. We may choose K fine enough so that $y\notin A$. An inverting map h for Int St. (A, K) certainly carries y into $\mathrm{Int}\, A$ whence by Lemma 1 there exists $g\in\mathfrak{G}(P)$ such that $gh(y)=x$. Obviously, we can also require that $g|P-\mathrm{Int}\,\mathrm{St}(A, K)$ be the identity map from which it follows that gh is also an inverting map for $\mathrm{Int}\,\mathrm{St}(A, K)$. By choosing K sufficiently fine that we have $\mathrm{Int}\,\mathrm{St}(A, K)\subset U$, the proof is complete.

The above lemma permits an easy proof of a result obtained by Klassen in his doctoral dissertation (V.P.I., 1965).

THEOREM 6. *The polyhedron P is a suspension if and only if the invert set $I(P)$ contains a zero-sphere.*

Proof. If P is a suspension, then the suspension points are in $I(P)$ (see Theorem 8 of [5]).

Assume that $I(P)$ contains two points v and w and let K be a triangulation of P. By starring K at v and w, if need be, we may assume both points to be vertices of K and by taking a subdivision of K, if need be, we may assume that $w \notin \mathrm{St}(v, K)$.

Now take successive barycentric subdivisions $K^{(i)}$ of K. The open stars, $\mathrm{Int\ St}(w, K^{(i)})$, constitute a monotone decreasing sequence of open cone neighborhoods of w. Since $w \in I(P)$, Lemma 5 applies to choose an inverting map $h_i \in \mathfrak{H}(P)$ for each $\mathrm{Int\ St}(w, K^{(i)})$ such that $h_i(v) = w$. Then h_i^{-1} $(\mathrm{Int\ St}(w, K^{(i)})$ is an open cone neighborhood U_i of v for each i. Furthermore, U_i contains $P - \mathrm{Int\ St}(w, K^{(i)})$. In view of Theorem 3 of [6], then, $P - w$ is an open cone, whence P is a suspension.

LEMMA 7. *Let P be a polyhedron and assume that $I(P)$ has positive dimension. If $CI(P)$ is not empty, then $CI(P) = I(P)$.*

Proof. Let K be a triangulation of P and let L be the subcomplex of K carried by $I(P)$. Let v be a point of $CI(P)$. By starring K at v we may assume that v is a vertex of K and hence of L. Let A be a principal simplex of L in $\mathrm{St}(v, K)$. Note that $\dim A = \dim I(P)$.

Consider any principal simplex B of L and choose a point $x \in \mathrm{Int}\, B$. By Theorem 4 there exists $h \in \mathfrak{H}(P)$ such that $h(x) = v$. Now take a derived subdivision K' of K such that x is still an interior point of a principal simplex B' of L' and choose K' fine enough so that $h(\mathrm{Int\ St}(B', K')) \subset \mathrm{Int\ St}(v, K)$. Choose a point y in the set $\mathrm{Int}\, A \cap h(\mathrm{Int\ St}(B', K'))$. By Lemma 1 there is an isotopy f_t of P onto itself such that $f_t | P - \mathrm{Int\ St}(B', K')$ is the identity map while $f_1(x) = h^{-1}(y)$. Define an isotopy g_t of P onto itself by

$$g_t \mid P - h(\mathrm{Int\ St}\ (B', K')) = \text{identity}$$
$$g_t \mid h(\mathrm{St}(B', K')) = hf_t h^{-1}.$$

On the closed set $\mathrm{BdSt}(B', K')$ the map g_t is the identity under both definitions so g_t is well-defined and continuous. Also we have

$$g_0 \mid h(\mathrm{St}(B', K')) = h(f_0 \mid \mathrm{St}(B', K'))h^{-1} = hh^{-1} = \text{identity},$$

whence $g_1 \in \mathfrak{G}(P)$. Finally $g_1(v) = hf_1h^{-1}(v) = hf_1(x) = hh^{-1}(y) = y$ so y is also a point of $CI(P)$. This implies that $\mathrm{Int}\, A \subset CI(P)$ and suffices to prove the lemma.

THEOREM 8. *If the invert set $I(P)$ of a polyhedron P contains more than two points, then $CI(P) = I(P)$.*

Proof. By Theorem 6 we may consider P to be the suspension $S(Q)$ of a polyhedron Q. Let v and w be the suspension points of $S(Q)$. Refer $S(Q)$ to "coordinates" (q, t) where $q \in Q$, $-1 \leqq t \leqq 1$ and where $(q, 1) = v$, $(q, -1) = w$ for all $q \in Q$. Any point (q, t), $-1 < t < 1$, can be moved by an isotopy of the form $g_s(q, t) = (q, f_s(t))$, where f_s is an isotopy of the interval $[-1, 1]$ leaving endpoints fixed, onto the point $(q, 0)$. It follows that $Q \cap I(S(Q))$ is not empty.

Let $(q, 0)$ be a point in $Q \cap I(S(Q))$ and, without loss of generality, choose a triangulation K of $S(Q)$ so that $(q, 0)$ is interior to a principal simplex A on $I(S(Q))$. Also choose K fine enough so that neither v nor w is in $\mathrm{St}(A, K)$. Let $h \in \mathfrak{H}(S(Q))$ be an inverting map for the open neighborhood $U = \mathrm{Int}\ \mathrm{St}(A, K)$ of the point $(g, 0)$. Then both $h(v)$ and $h(w)$ lie in $\mathrm{Int}\ A$. By Lemma 1 there is an isotopy g_s of P onto itself which is the identity outside of U and is such that $g_1 h(v) = h(w)$. Clearly, $h^{-1} g_s h$ is again an isotopy of P onto itself and we have $h^{-1} g_1 h(v) = h^{-1} h(w) = w$. Thus there is an isotopy f_s of P onto itself such that $f_1(v) = w$.

Now let U be any open neighborhood of w whence $f_1^{-1}(U)$ is an open neighborhood of v. There exists $\epsilon > 0$ such that the open set $\{(q, t) \mid q \in Q, -1 \leqq t < -1 + \epsilon\}$ lies in U and the open set $\{(q, t) \mid q \in Q, 1 - \epsilon < t \leqq 1\}$ lies in $f_1^{-1}(U)$. Define an isotopy h_s of P onto itself of the form $h_s(q, t) = (q, k_s(t))$ where k_s is an isotopy of $[-1, 1]$ onto itself leaving endpoints fixed and carrying the point $-1 + \epsilon$ to the point $1 - \epsilon$. Then the composition $f_1 h_1$ is an element of $\mathfrak{G}(P)$ throwing $P - U$ into U, whence $w \in CI(P)$. In view of Lemma 7, then, $CI(P) = I(P)$.

Klassen (V.P.I. thesis, 1965) has shown that for polyhedra with exactly one invert point we also have $CI(P) = I(P)$. Thus the only polyhedra for which inverting maps cannot be selected from $\mathfrak{G}(P)$ are those with precisely two invert points. An $(n+1)$-fold suspension (the join of a polyhedron Q with a triangulated n-sphere) provides a polyhedron in which $CI(P) = I(P)$ contains an n-sphere. That is, we have the following generalization of Theorem 7 of [1]:

THEOREM 9. *Let $P = Q \circ S^n$ be the join of a polyhedron Q and a triangulated n-sphere S^n, $n > 0$. Then $S^n \subset CI(P)$.*

Proof. Consider P as a suspension $S(Q \circ S^{n-1})$ with suspension points v and w. We know that v and w are in $I(P)$ and, since S^n is homogeneously imbedded in P, we have $S^n \subset I(P)$. Then Theorem 8 applies.

In one special case at least there is a converse to Theorem 9. I have not been able to carry this any farther.

THEOREM 10. *Let P be an n-dimensional polyhedron and assume that $CI(P)$ contains an $(n-1)$-sphere. Then P is the n-fold suspension of a finite number of points.*

Proof. The case where $n = 1$ is trivial so assume $n > 1$. Let K be a triangulation of P with the subcomplex L carried by $CI(P)$. If A is an $(n-1)$-simplex in L, then A is the common face of a finite numbers of n-simplexes, $B_1, \cdots, B_k$, in K. We consider the number k of these n-simplexes.

If $k=1$, then every point of P-$CI(P)$ either lies in Int B_1 or can be carried into Int B_1 by an isotopy. Thus each point of P-$CI(P)$ has an open n-cell neighborhood. Clearly, too, points of $CI(P)$ have neighborhoods whose closures are closed n-cells. Thus P is an n-manifold with boundary $CI(P)$. Theorem 6 of [5] then applies and we conclude that P is itself a closed n-cell which is the n-fold suspension of a single point.

Next suppose that $k=2$. The same argument as above shows now that P is an n-manifold without boundary. Since P has a nonempty invert set, Theorem 7 of [3] implies that P is an n-sphere, the n-fold suspension of two points.

Let $k>2$ and take U_i to be that component of $K-L$ which contains Int B_i for each $i=1, 2, \cdots, k$. Each point of U_i can be carried by an isotopy g_t into Int St(A, K) and, of course, we have $g_t(U_i)\cap CI(P)$ empty for all t. Two facts follow immediately:

(1) $g_1(U_i-\text{Int } B_i)\subset \text{Int } B_i$ and

(2) each point of U_i has an open n-cell neighborhood in P.

Therefore the closure $\overline{U}_i$ is an n-manifold with boundary $CI(P)$. Restricting the maps in $\mathfrak{G}(P)$ to $\overline{U}_i$, we easily see that $CI(\overline{U}_i)$ is not empty. Hence Theorem 6 of [5] applies again to give the conclusion that each $\overline{U}_i$ is a closed n-cell with boundary $CI(P)$. That is, P is the union of k closed n-cells all sharing the same boundary $CI(P)$ and hence is the n-fold suspension of k points.

REMARKS: 1. Let $P=S(Q)$ be the suspension of a polyhedron Q and suppose that dim $I(P)>0$. Then we have the inclusion

$$I(Q) \subset Q \cap I(P).$$

To see this, let x be a point in $I(Q)$ and y be a point in $Q\cap I(P)$. Let U be any open neighborhood of x in P with $y\notin U$. By assumption, there exists $g\in\mathfrak{H}(Q)$ with $g(y)\in U\cap Q$. By suspending the map g we obtain a map $h\in\mathfrak{H}(P)$ with $h(y)\in U$. It follows that x is a limit point of $I(P)$ and since $I(P)$ is closed, we have $x\in I(P)$.

2. If $P=Q \circ S^n$, $n>0$, is the join of a compact triangulated manifold Q with a triangulated n-sphere S^n, then we have either $CI(P)=S^n$ or P is a sphere. To prove this we need only note that P is necessarily locally euclidean at points away from Q and S^n. So if $CI(P)$ meets $P-Q\cup S^n$, then P is invertible at a point with an open cell neighborhood. In such a case Theorem 1 of [2] applies to conclude that P is a sphere.

3. In view of Remark 1 above it seems natural to conjecture that for every suspension $S(Q)$ we have

$$I(Q) = Q \cap I(S(Q)).$$

But suppose this is true and that $S(Q)$ is a sphere. Then we have $I(S(Q))=S(Q)$ whence $I(Q)=Q\cap I(S(Q))=Q\cap S(Q)=Q$. But if $I(Q)=Q$, then Q is also a sphere. An easy induction shows that if $P=Q^k \circ S^{n-k-1}$, then P is an n-sphere if and only if Q^k is a k-sphere. Now it is known (Hirsch) that the suspension of a polyhedral homotopy 4-sphere is a real 5-sphere. Hence the equality $I(Q)$

$=Q\cap I(S(Q))$ implies the 4-dimensional Poincaré conjecture! In view of a recent result of Rice [7] this same equality implies that every triangulated n-manifold is combinatorial! In short, then, this seemingly innocuous equality is actually a very strong assumption. It remains to be seen whether these ideas can be carried farther.

Added in proof: The converse of Theorem 9 can also be established. Indeed, a polyhedron P is an m-fold suspension if and only if $CI(P)$ contains an $(m-1)$-sphere.

References

1. P. H. Doyle, Symmetry in geometric complexes, this MONTHLY, 73 (1966) 625–628.

2. P. H. Doyle and J. G. Hocking, A characterization of euclidean space, Mich. Math. J., 7(1960) 199–200.

3. ———, Invertible spaces, this MONTHLY, 68 (1961) 959–965.

4. ———, Continuously invertible spaces, Pacific J. Math., 12 (1962) 499–503.

5. P. H. Doyle, J. G. Hocking and R. P. Osborne, Local invertibility, Fund. Math., LIV (1964) 15–25.

6. K. W. Kwun, Uniqueness of the open cone neighborhood, Proc. Amer. Math. Soc., 15 (1964) 476–479.

7. P. M. Rice, The Hauptvermutung and the polyhedral Schoenflies theorem, Bull. Amer. Math. Soc., 71 (1965) 521–522.

ON THE ITERATION OF LINEAR FRACTIONAL TRANSFORMATIONS

NATHAN ELJOSEPH, Tel-Aviv University, Israel

1. Introduction. This paper is a sequel to the author's paper [2].

We shall deal with the properties of the iterations of the transformations,

$$w=\frac{az+b}{cz+d},\qquad ad-bc\neq 0 \tag{1}$$

and

$$w=\frac{a\bar{z}+b}{c\bar{z}+d},\qquad ad-bc\neq 0 \tag{2}$$

in the complex plane, where a, b, c, d are complex numbers.

2. Iterations of (1). The number of fixed points of (1) is one or two, unless (1) is of the form

$$w=z. \tag{3}$$

If (1) has one fixed point, assume it to be the point at infinity. Then (1) will be of the form $w=z+f, f\neq 0$. The iteration of nth order is $w=z+nf$.

If (1) has two fixed points, then by a linear fractional transformation, (1) can be reduced to the form $w=Az$, $A\neq 0$, where

$$A=\frac{a+d+\sqrt{D}}{a+d-\sqrt{D}},\qquad D=(a+d)^2-4\,(ad-bc)\neq 0,$$

(Caratheodory [1]). The iteration of nth order is $w=A^nz$. A necessary and sufficient condition for (1) to have one fixed point is $D=0$, hence $A=1$.

Put:

$$B=a+d,\qquad C=ad-bc\neq 0,\qquad \cos x=B/2\sqrt{C} \tag{4}$$

(x is not necessarily real), then $A=e^{2ix}$. Hence:

THEOREM 1. (See Knopp [3].) *A necessary and sufficient condition for some iteration of* (1), *but not* (1) *itself to be of the form* (3) *is that A be a root of unity, but $A\neq 1$, or equivalently that* arc cos $B/2\sqrt{C}$ *be a rational but not an integral multiple of π. (This implies that $B/2\sqrt{C}$ is real and* $|B/2\sqrt{C}|<1$.)

REMARK. For an involutory transformation, $n=2$, $x=\pi/2$, $B=0$. Hence, $a=-d$.

A second proof of Theorem 1 may be found using matrices. Put

$$M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

and write (1) as $w=M(z)$. If $z_1=M(z_0)$ and $z_2=N(z_1)$ then $z_2=P(z_0)$, where $P=NM$ (matrix multiplication). Thus the nth iteration is $w=M^n(z)$. The necessary and sufficient condition for an iteration of (1) to be of the form (3) is that there exists an n for which $M^n=\mu I$, where I is the identity matrix. By determinants, $\mu=\pm(ad-bc)^{n/2}\neq 0$.

Let S be any nonsingular matrix of order 2. Then $S^{-1}M^nS=\mu I$ or $(S^{-1}MS)^n=\mu I$. Denote the eigenvalues of M by λ_1 and λ_2. They solve the equation

$$\lambda^2-(a+d)\lambda+(ad-bc)=0. \tag{5}$$

If $\lambda_1=\lambda_2=\lambda$, there exists an S for which

$$S^{-1}MS=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}.$$

Here $\lambda\neq 0$ since $ad-bc\neq 0$. By induction,

$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^n=\begin{pmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{pmatrix}$$

and this is never of the form μI.

If $\lambda_1\neq\lambda_2$, there exists an S for which

$$S^{-1}MS=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}.$$

Hence $S^{-1}M^nS$ will be of the form μI if and only if $\lambda_1^n=\lambda_2^n$. As $\lambda_1\neq\lambda_2$ this is equivalent to $Q_n=0$, where

$$Q_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

Here $Q_0=0$, $Q_1=1$, $Q_2=a+d$. Substituting λ_1 and λ_2 into (5), after multiplying this equation by λ^{n-2}, we have

$$Q_n - BQ_{n-1} + CQ_{n-2} = 0. \quad \text{(See (4).)}$$

Hence,

$$Q_n = B^{n-1} - \binom{n-2}{1} B^{n-3}C + \binom{n-3}{1} B^{n-5}C^2 \mp \cdots.$$

(See Perron [4]). Let $B\neq 0$. Then

$$Q_n = B^{n-1}\left[1 - \binom{n-2}{1}\frac{C}{B^2} + \cdots\right].$$

The expression in brackets is a Tchebycheff polynomial of the second kind if $\cos x = B/2\sqrt{C}$. (See (4).) Hence $Q_n = B^{n-1}(\sin nx/\sin x)$. Now $Q_n=0$ if and only if $\sin nx=0$ but $\sin x\neq 0$. This is equivalent to Theorem 1. If $B=0$ (or $a=-d$) we have $Q_2=0$.

3. Iterations of (2). Iterations of odd order of (2) are never of the form (3). Put $n=2k$ in this section.

Write (2) as $w=M(\bar{z})$, where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $z_1=M(\bar{z}_0)$ and $z_2=N(\bar{z}_1)$, then z_2 is a Möbius transformation of z_0 with the matrix $N\overline{M}$. The matrix corresponding to the nth iteration is $(M\overline{M})^k$.

If (2) has one fixed point, assume it to be the point at infinity. Hence (2) will be of the form $w=\lambda\bar{z}+\mu$, $\lambda\neq 0$. Here $\mu\neq 0$, for otherwise $z=0$ is a second fixed point. Likewise $|\lambda|=1$, or $z_0=(\lambda\bar{\mu}+\mu)/(1-|\lambda|^2)$ is a fixed point.

The matrix corresponding to one iteration is

$$\begin{pmatrix} \lambda & \mu \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{\lambda} & \bar{\mu} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad g = \lambda\bar{\mu} + \mu.$$

If $g=0$ then (2) is involutory, and (2) having one fixed point has an infinite number of fixed points (Eljoseph [2]). Hence $g\neq 0$. As

$$\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & kg \\ 0 & 1 \end{pmatrix}$$

we have proved the first part of

THEOREM 2. *If* (2) *has one or two fixed points no iteration of* (2) *is of the form* (3).

To conclude the proof of Theorem 2, assume that the two fixed points are 0 and ∞. Now (2) will be of the form $w=\lambda\bar{z}$, $\lambda\neq 0$. The nth iteration is $w=|\lambda|^n z$, (n is even). If $|\lambda|\neq 1$, this is never of the form (3). If $|\lambda|=1$ then $z=\sqrt{\lambda}$ is a third fixed point of $w=\lambda\bar{z}$.

If (2) has three fixed points it is involutory (see Eljoseph [2]), and an iteration of second order is of the form (3).

Suppose that (2) is noninvolutory and has no fixed points. We shall show that there are two points z_0 for which $z_0=M\overline{M}(z_0)$. (Such points will be called *semi-fixed points.*) Write (2) in the form $w=(\alpha\bar{z}+\beta)/(\bar{z}+\delta)$ for $c\neq 0$ as $z=\infty$ is not a fixed point. We are to solve the equation

$$z=\frac{(\alpha\bar{\alpha}+\beta)z+(\alpha\bar{\beta}+\beta\bar{\delta})}{(\bar{\alpha}+\delta)z+(\bar{\beta}+\delta\bar{\delta})}\,.$$

If $\bar{\alpha}+\delta\neq 0$ this is a quadratic equation and has at least one solution. Suppose this to be $z=\infty$. *Now* $\bar{\alpha}+\delta=0$. Substituting we have

$$z=\frac{(\alpha\bar{\alpha}+\beta)z+\alpha(\bar{\beta}-\beta)}{\alpha\bar{\alpha}+\bar{\beta}}\,.$$

Here $\beta\neq\bar{\beta}$, for otherwise (2) is involutory. Our equation has exactly one more solution, $z=\alpha$.

Assume the semi-fixed points to be 0 and ∞. Then (2) takes the form $w=h/\bar{z}$, $h\neq\bar{h}$. Its nth iteration is $w=(h/\bar{h})^k z$. Hence,

THEOREM 3. *A necessary and sufficient condition for* (2) *to have a noninvolutory iteration of the form* (3) *is that* arg $h/\bar{h}$ *be a rational, but not an integral, multiple of* 2π.

REMARK. If z_0 and z_1 are the semi-fixed points, $h=(\alpha-z_0)/(\alpha-z_1)$.

4. Iterations of a given point z_1. Given a point z_1, denote its iterations by $z_2, z_3, \cdots$. By the remarks preceding and immediately after Theorem 1, we have:

THEOREM 4. (See Knopp [3]). (a) *If* (1) *has one fixed point,* $\lim_{n\to\infty} z_n$ *exists and equals the fixed point. The* $\{z_n\}$ *lie on a generalized circle* (*i.e. a Euclidean circle or a straight line*) *through the fixed point.*

(b) *If* (1) *has two fixed points and* $|A|\neq 1$, $\lim_{n\to\infty} z_n$ *exists and equals one of the fixed points. If* arg $A=2\pi(p/q)$ (*where p and q are coprime positive integers and $p<q$*) *then the* $\{z_n\}$ *lie on q generalized circular arcs through the fixed points, one of them through z_1, and the angle between two adjacent arcs is $2\pi p/q$. If q is*

even, these arcs comprise of $\frac{1}{2}q$ circles. If $\arg A$ *is not a rational multiple of* 2π, *the* $\{z_n\}$ *lie on a spiral through* z_1 *and the fixed points.*

(c) *If* $A \neq 1$ *is a root of unity,* $\{z_n\}$ *is a finite set on the generalized circle through* z_1 *orthogonal to the pencil of circles through the fixed points.*

(d) *In the remaining case, if* $A \neq 1$, $\{z_n\}$ *is a set densely covering the circle mentioned in* (c).

By the remarks preceding Theorem 2 and 3 we have:

THEOREM 5. (a) *If* (2) *has one fixed point,* $\lim_{n\to\infty} z_n$ *exists and equals the fixed point. The* $\{z_n\}$ *lie in general on two different generalized circles tangent at the fixed point, one through* z_1 *and the other through* z_2. *The* z_n *with even affix lie on one circle and the others on the second.*

(b) *If* (2) *has two fixed points,* $\lim_{n\to\infty} z_n$ *exists and equals one of the fixed points. The* $\{z_n\}$ *lie on two different generalized circles through the fixed points. The* $\{z_n\}$ *with even affix lie on one circle and the others on the second.*

(c) *If* (2) *is noninvolutory and has no fixed points the* $\{z_n\}$ *lie in general on two generalized circles orthogonal to the pencil of circles through the semi-fixed points. For odd* n *the circle passes through* z_1 *and for even through* z_2. *The density of the* $\{z_n\}$ *on the circles depends in an obvious manner on* $h/\bar{h}$.

References

1. C. Caratheodory, Conformal Representations, Cambridge Tracts, 1941, p. 14.
2. N. Eljoseph, On the fixed points of $w=(a\bar{z}+b)/(c\bar{z}+d)$, this MONTHLY, 72 (1965) 759.
3. K. Knopp, Problem Book in the Theory of Functions, Vol. 1, Dover, 1948, p. 117, Problem 22.
4. O. Perron, Kettenbrüche, Chelsea, New York, 1950, p. 10.

MATHEMATICAL NOTES

THE CLASSIFICATION OF REAL DIVISION ALGEBRAS

R. S. PALAIS, Brandeis University

Let D be a finite dimensional division algebra over the field $\mathbf{R}$ of real numbers. One way of stating the fundamental theorem of algebra is to say that if D is commutative (i.e. a field) then D is isomorphic over $\mathbf{R}$ to either $\mathbf{R}$ or the field $\mathbf{C}$ of complex numbers. A famous theorem of Frobenius asserts that if we allow D to be noncommutative then there is only one new possibility: D can be isomorphic over $\mathbf{R}$ to the quaternion algebra of Hamilton. This is an algebra $\mathbf{H}$ of dimension four generated as a vector space by basis elements 1, i, j, k which satisfy the multiplication table

$$i^2 = j^2 = k^2 = -1; \quad ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j.$$

The proofs of Frobenius' theorem in the literature seem to be of two types.

Either they are elementary, but rather computational, e.g. [2], or else they deduce the theorem from sophisticated general results about division algebras, e.g. [1]. We wish to give here a short, self-contained proof which seems both elementary and conceptual. Besides the inevitable use of the fundamental theorem of algebra we use only the simplest facts about the eigenvalues of linear transformations.

Before starting the proof we note that the two-dimensional subspace of $\mathbf{H}$ generated by 1 and i is isomorphic to the complex numbers. If we denote it by $\mathbf{C}$ then $\mathbf{H}$ becomes a vector space over $\mathbf{C}$ (using left multiplication for the scalar operations). Moreover $\mathbf{C}$ is clearly $\{x\in D \mid ix=xi\}$, while the complementary two-dimensional space spanned by j and k is just $\{x\in D \mid ix=-xi\}$. It is this observation which motivates the proof.

Let 1 denote the unit of D. As usual we can think of $\mathbf{R}$ as embedded in D via the map $x\to x\cdot 1$. We may assume $D\neq\mathbf{R}$. Let d be any element of D not in $\mathbf{R}$ and let $\mathbf{R}\langle d\rangle$ denote the two-dimensional subspace $\mathbf{R}+\mathbf{R}d$ spanned by 1 and d. We claim:

(1) $\mathbf{R}\langle d\rangle$ *is a maximal commutative subset of* D, *consisting of all the elements of* D *which commute with* d. *Moreover it is a field isomorphic to* $\mathbf{C}$.

Proof. Choose a subspace F of D of maximal dimension which includes $\mathbf{R}\langle d\rangle$ and is commutative. If $x\in D$ commutes with everything in F then $F+\mathbf{R}x$ is commutative and so must equal F, so $x\in F$ proving that F is a maximal commutative subset of D. In particular, if $x\neq 0$ is in F then x^{-1} commutes with everything in F (because $xy=yx\Rightarrow yx^{-1}=x^{-1}y$) so $x^{-1}\in F$ and F is a field. By the fundamental theorem of algebra F is isomorphic over $\mathbf{R}$ to $\mathbf{C}$. In particular, F has dimension two so that $F=\mathbf{R}\langle d\rangle$. Finally, if $x\in D$ commutes with d it commutes with everything in $\mathbf{R}\langle d\rangle=F$, hence belongs to F.

According to (1) we can select an element $i\in D$ such that $i^2=-1$ and we may identify $\mathbf{R}\langle i\rangle$ with $\mathbf{C}$. We can now view D not merely as a vector space over $\mathbf{R}$, but also as a vector space (of half the dimension) over $\mathbf{C}$ as well, the scalar operations of $\mathbf{C}$ on D being given by multiplication on the left. On the other hand multiplication on the *right* by i can then be interpreted as a (complex) linear transformation T on the (complex) vector space D; i.e., we define

(2) $Tx\equiv xi$.

Since $T^2=-$(identity), the only possible eigenvalues of T are $+i$ and $-i$; denote by D^+ and D^- the corresponding eigenspaces:

(3) $D^+=\{x\in D \mid xi=ix\}$, $D^-=\{x\in D \mid xi=-ix\}$.

Of course $D^+\cap D^-=\{0\}$. We claim moreover

(4) $D=D^+\oplus D^-$.

This follows immediately from the decomposition $x=\frac{1}{2}(x-ixi)+\frac{1}{2}(x+ixi)$ for all $x\in D$, the two summands being respectively in D^+ and D^- as one checks by (3). We next note that

(5) $D^+=\mathbf{C}$ and $x,\ y\in D^-\Rightarrow xy\in D^+$.

The first statement is immediate from (1), the second from (3).

If $D^- = 0$ then by (4) and (5) we have $D = \mathbf{C}$, so let us assume $D^- \neq 0$ and show that D must be isomorphic to $\mathbf{H}$. First of all the real dimension of D must be four, i.e., its complex dimension must be two. This follows from (4), (5) and

(6) $\dim \mathbf{C} D^- = 1$.

Proof. Select any nonzero $\alpha \in D^-$. Then right multiplication by α gives a complex linear transformation on D which is nonsingular (its inverse is right multiplication by α^{-1}), and it interchanges D^+ and D^- by (5) so $\dim_{\mathbf{C}} D^- = \dim_{\mathbf{C}} D^+ = 1$. Moreover

(7) $\alpha^2 \in \mathbf{R}$ and $\alpha^2 < 0$.

Proof. Since by (1) $\mathbf{R}\langle\alpha\rangle$ is a field it contains α^2. But also $\alpha^2 \in \mathbf{C}$ by (5) and therefore $\alpha^2 \in \mathbf{C} \cap \mathbf{R}\langle\alpha\rangle = \mathbf{R}$. If $\alpha^2 > 0$ it would have two square roots in $\mathbf{R}$ hence three square roots in the field $\mathbf{R}\langle\alpha\rangle$ which is impossible by field theory (or more concretely here, because $\mathbf{R}\langle\alpha\rangle \simeq \mathbf{C}$).

By (7) a suitable positive multiple of α is an element j of D^- satisfying $j^2 = -1$. Define $k = ij$, so that by (6), j and k form a basis for D^- over $\mathbf{R}$, and hence by (4) the elements $1, i, j, k$ form a basis for D over $\mathbf{R}$. Since $j, k \in D^-$, they anticommute with i. This together with $i^2 = j^2 = -1$ and $k = ij$ show that $1, i, j, k$ satisfy the multiplication table given above for the quaternions.

References

1. A. Albert, The Structure of Algebras, AMS Colloquium Publication, 1939.
2. I. N. Herstein, Topics in Algebra, Blaisdell, New York, 1964 p. 326.

APPROXIMATE FUNCTION VALUES AND HYPERPLANES

W. R. HARE, JR. AND J. W. KENELLY, Clemson University

By using an appropriate Helly-type theorem [1] it is possible to give a characterization of hyperplanes through a condition involving approximate function values on arbitrary finite point sets. The development gives a result that generalizes a theorem of Karlin and Shapley [2] in that it removes a condition on boundedness from the hypothesis.

1. Preliminaries. General geometric terminology of n-dimensional euclidean space, R^n, is used, and particular concepts are noted for clarity. When a function F, of n variables, is specified to be between two given functions I and S, over a domain D, we say that *approximate values* of the function are given. That is, for all $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in D, $I(\mathbf{x}) \leqq F(\mathbf{x}) \leqq S(\mathbf{x})$. For convenience, call the closed segment from the point $(\mathbf{x}, I(\mathbf{x}))$ to the point $(\mathbf{x}, S(\mathbf{x}))$, in R^{n+1} an *approximate point*

$$\mathcal{P}(\mathbf{x}) = \{(\mathbf{x}, z) : I(\mathbf{x}) \leqq z \leqq S(\mathbf{x})\}.$$

Also for a specified class $\Phi = \{\phi_1, \phi_2, \cdots, \phi_k\}$ of linearly independent functions, term $L(A, \mathbf{x}) = \sum_{i=1}^{k} a_i \phi_i(\mathbf{x})$ an (linear) *approximating function* from the Φ class established by the parameter set $A = \{a_1, a_2, \cdots, a_k\}$. In case a particular

$L(A_0, \mathbf{x})$ is such that $I(\mathbf{x}) \leqq L(A_0, \mathbf{x}) \leqq S(\mathbf{x})$ over D, then we say that this $L(A_0, \mathbf{x})$ *approximates* $F(\mathbf{x})$ over the domain D within the bounds $I(\mathbf{x})$ and $S(\mathbf{x})$. Since the graph of $L(A_0, \mathbf{x})$ intersects every member of the set of approximate points, established by D, I, and S, we call the graph a *transverse* of the set of approximate points. A *closed slab* in R^n is the set of points which lie between two parallel hyperplanes. The following Helly-type theorem from Valentine [1] is stated for reference.

THEOREM 1. *Let $\{S_i: i \in \lambda\}$, where λ is a nonempty index set, be a collection of closed slabs in R^n. Then a necessary and sufficient condition that $\bigcap_{i\in\lambda} S_i$ be nonempty is that $\bigcap_{i\in\mu} S_i$ be nonempty for all subsets μ consisting of $n+1$ members of λ.*

2. Particular results.

THEOREM 2. *A collection of approximate points has a transverse that is the graph of an $L(A, \mathbf{x})$ if and only if each subfamily of $k+1$ approximate points has a transverse that is the graph of an $L(A, \mathbf{x})$. (Here k is the number of elements in the Φ class.)*

Proof. For each approximate point $\mathcal{P}(\mathbf{x})$, let $C_{\mathbf{x}}$ be the closed slab in R^k defined by

$$C_{\mathbf{x}} = \left\{(a_1, a_2, \cdots, a_k): I(\mathbf{x}) \leqq \sum_{i=1}^{k} a_i\phi_i(\mathbf{x}) \leqq S(\mathbf{x})\right\}.$$

By hypothesis, each $k+1$ of these $C_{\mathbf{x}}$'s have a nonempty intersection; so by Theorem 1, all have a point in common, say $(b_1, b_2, \cdots, b_k)$. Then $\sum_{i=1}^{k} b_i\phi_i(\mathbf{x})$ has a graph that is a transverse of the collection of approximate points. The trivial converse completes the proof.

It should be noted that the bounds of the function $F(\mathbf{x})$ may vary over the domain and that no hypotheses are required about the nature of the domain. But, the closed nature of the approximate points is essential. A counter example in regard to one of the intervals being open at one endpoint is given by $I(x) = 1 - |x-1|$ and $S(x) = 2 - |x-1|$. In particular, assume that $I(x) \leqq F(x) \leqq S(x)$ for $0 < x < 2$ and $0 \leqq F(2) < 1$ and specify the domain to be $D = \{x: 0 < x \leqq 2\}$. It is a simple matter to show that these parallel intervals are such that any three have a common transversal, but there does not exist a line that intersects the complete collection.

We now note a known corollary that is a consequence of Theorem 2.

COROLLARY 1. *Let F and G be two real valued functions of one variable defined over a closed interval $[a, b]$, such that $F(x) \leqq G(x)$ on $[a, b]$. Then there is a polynomial of degree n, P, satisfying $F(x) \leqq P(x) \leqq G(x)$ on $[a, b]$ if and only if for every set T of $n+2$ distinct points from $[a, b]$, there is a polynomial of degree n, P_T satisfying $F(x_t) \leqq P_T(x_t) \leqq G(x_t)$, $x_t \in T$.*

Proof. This is the case in which $k = n+1$, and $\phi_i(x) = x^{i-1}$, $i = 1, 2, \cdots, n+1$.

THEOREM 3. *If F is a real valued function defined on R^k, such that for any $\epsilon>0$ and each $k+2$ points $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{k+2}$ in R^k there exists a hyperplane that intersects the closed segments*

$$[(\mathbf{x}_i, F(\mathbf{x}_i) - \epsilon), (\mathbf{x}_i, F(\mathbf{x}_i) + \epsilon)] \; i = 1, 2, \cdots, k+2 \text{ in } R^{k+1};$$

then the graph of F is a hyperplane.

Proof. For $\epsilon=\frac{1}{4}m$, Theorem 2 gives a hyperplane described by $F(A_m, \mathbf{x}) \equiv a_{0m}+a_{1m}x_1+ \cdots +a_{km}x_k$. The hyperplane, in R^{k+1}, is contained in the set $S\{(\mathbf{x}, y)\colon F(\mathbf{x})-\epsilon\leqq y\leqq F(\mathbf{x})+\epsilon\}$. So we have for $n>m$, and any point $\mathbf{x}$,

$$\begin{aligned} | F(A_n, \mathbf{x}) - F(A_m, \mathbf{x}) | &\leqq | F(A_n, \mathbf{x}) - F(\mathbf{x}) | + | F(A_m, \mathbf{x}) - F(\mathbf{x}) | \\ &\leqq \tfrac{1}{4}m + \tfrac{1}{4}n < \tfrac{1}{2}m. \end{aligned}$$

Evaluation at the origin of R^k gives $|a_{0n}-a_{0m}| < \frac{1}{2}m$. Likewise at the jth unit point of R^k, we have

$$| a_{jn} - a_{jm} | \leqq | a_{jn} - a_{jm} + a_{0n} - a_{0m} | + | a_{0n} - a_{0m} | < 1/m.$$

The Cauchy sequences $\{a_{ji}\}$, $j=0, 1, 2, \cdots, k$ converge respectively to a_j^*. These values establish $F(A^*, \mathbf{x})\equiv a_0^*+a_1^*x_1+ \cdots +a_k^*x_k$, i.e., the R^{k+1} hyperplane $\{(\mathbf{x}, y)\colon y=F(A^*, \mathbf{x})\}$. The $F(A^*, \mathbf{x})$ hyperplane is identical with the graph of F, since at any point $\mathbf{x}$,

$$| F(A^*, \mathbf{x}) - F(\mathbf{x}) | \leqq | F(A^*, \mathbf{x}) - F(A_n, \mathbf{x}) | + | F(A_n, \mathbf{x}) - F(\mathbf{x}) | < \epsilon$$

for any $\epsilon>0$ and n sufficiently large.

References

1. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.
2. S. Karlin and L. S. Shapley, Some applications of a theorem on convex functions, Ann. of Math., 53 (1950) 148–153.

MAGNITUDE OF THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL $F_{pqr}(x)$

SISTER MARION BEITER, Rosary Hill College

1. Introduction. It is the purpose of this paper to prove that $\frac{1}{2}(p+1)$ is an upper bound for the magnitude of the coefficients in the cyclotomic polynomial $F_{pqr}(x)$ where p, q, r are odd primes, $p<q<r$, and either q or $r=kp\pm 1$. This is an improvement on the bound $p-1$ established by Bang [1], for coefficients in F_{pqr}.

The cyclotomic polynomial $F_m(x)$ is the polynomial whose zeroes are the primitive mth roots of unity. We have the relation

$$F_m(x) = \prod_{d|m} (x^d - 1)^{\mu(m/d)},$$

where $\mu(n)$ is the Moebius function. (By definition, $\mu(n)=0$ if n has a square factor; if n is the product of k distinct primes, $\mu(n)=(-1)^k$; $\mu(1)=1$.)

It follows [2, p. 32], that

$$F_{pqr}(x) = \sum (-1)^{\delta_1+\delta_2} x^n \tag{1}$$

summed over n such that $0 \leqq n \leqq \phi(pqr)$, where $\phi(m)$ is the Euler function, and

$$n = a + \alpha pq + \beta pr + \gamma qr + \delta_1 q + \delta_2 r, \tag{2}$$

$\delta_1, \delta_2 = 0, 1$; $0 \leqq a < p$; $0 \leqq \alpha < r$; $0 \leqq \beta < q$; $0 \leqq \alpha < p-1$.

2. Value of coefficient c_n. If we let

$$F_{pqr}(x) = \sum_{n=0}^{\phi(pqr)} c_n x^n,$$

then $c_n = 0$ when n has no partition of the form (2). Otherwise the value of c_n will be determined by the number of such partitions of n and by the values of δ_1 and δ_2 in these partitions. Since the cyclotomic polynomial is symmetric, it is necessary to examine c_n only for $0 \leqq n \leqq \frac{1}{2}\phi(pqr)$.

We use the following notation for the possible partitions of n in the form (2):

$$\left.\begin{array}{ll} \delta_1=\delta_2=0, & P_{1i}(n)=a_{1i}+\alpha_{1i}pq+\beta_{1i}pr+iqr \\ \delta_1=\delta_2=1, & P_{2i}(n)=a_{2i}+\alpha_{2i}pq+\beta_{2i}pr+iqr+q+r \\ \delta_1=1, \delta_2=0, & P_{3i}(n)=a_{3i}+\alpha_{3i}pq+\beta_{3i}pr+iqr+q \\ \delta_1=0, \delta_2=1, & P_{4i}(n)=a_{4i}+\alpha_{4i}pq+\beta_{4i}pr+iqr+r \end{array}\right| \quad i=0, 1, \cdots, \tfrac{1}{2}(p-3).$$

3. Conditions for max $|c_n|$. From (1) we see that the value of $|c_n|$ is $p-1$ iff there exist either (a) $\frac{1}{2}(p-1)$ partitions of n in each of the forms P_{1i} and P_{2i} but none in the forms P_{3i} and P_{4i}, or (b) $\frac{1}{2}(p-1)$ partitions of n in each of the forms P_{3i} and P_{4i} but none in the forms P_{1i} and P_{2i}.

4. q (or r) $=kp\pm 1$. We show that for either q or $r=kp\pm 1$, $p>3$, neither (a) nor (b) above can be satisfied.

Assume that for $q=kp+1$, there exist all possible partitions P_{1i} and P_{2i}. Then $P_{1(i+1)}$ implies P_{4i} since

$$\begin{aligned} P_{1(i+1)}(n) &= a_{1(i+1)} + \alpha_{1(i+1)}pq + \beta_{1(i+1)}pr + (i+1)qr \\ &= a_{1(i+1)} + \alpha_{1(i+1)}pq + \beta_{1(i+1)}pr + iqr + (kp+1)r \\ &= a_{1(i+1)} + \alpha_{1(i+1)}pq + (\beta_{1(i+1)} + k)pr + iqr + r \\ &= P_{4i}(n). \end{aligned}$$

Similarly, for $r=kp+1$, $P_{1(i+1)}$ implies P_{3i}.

Assume that for $q=kp-1$, there exist all possible partitions P_{1i} and P_{2i}. Then $P_{2(i+1)}$ implies P_{3i} since

$$\begin{aligned} P_{2(i+1)}(n) &= a_{2(i+1)} + \alpha_{2(i+1)}pq + \beta_{2(i+1)}pr + (i+1)qr + q + r \\ &= a_{2(i+1)} + \alpha_{2(i+1)}pq + (\beta_{2(i+1)} + k)pr + iqr + q \\ &= P_{3i}(n). \end{aligned}$$

Similarly, for $r=kp-1$, $P_{2(i+1)}$ implies P_{4i}.

The reader can verify that by assuming the existence of all possible partitions P_{3i} and P_{4i}, we find: For $q=kp\pm 1$, $P_{3(i+1)}$ implies P_{2i}, $P_{4(i+1)}$ implies P_{1i}, respectively, for $r=kp\pm 1$, $P_{4(i+1)}$ implies P_{2i}, $P_{3(i+1)}$ implies P_{1i}, respectively. Thus, whenever there exist all $p-1$ partitions of n such that $(-1)^{\delta_1+\delta_2}=1$ in (1), i.e., all P_{1i} and all P_{2i} exist, then there must also exist $\frac{1}{2}(p-3)$ partitions such that $(-1)^{\delta_1+\delta_2}=-1$. Hence $\max c_n=(p-1)-\frac{1}{2}(p-3)=\frac{1}{2}(p+1)$. Similarly, whenever there exist all $p-1$ partitions of n such that $(-1)^{\delta_1+\delta_2}=-1$, i.e., all P_{3i} and all P_{4i} exist, then there must also exist $\frac{1}{2}(p-3)$ partitions such that $(-1)^{\delta_1+\delta_2}=1$ and $\max |c_n|=|-(p-1)+\frac{1}{2}(p-3)|=|-\frac{1}{2}(p+1)|$. There follows the

THEOREM. *In the monic cyclotomic polynomial*

$$F_{pqr}(x) = \sum_{n=0}^{\phi(pqr)} c_n x^n,$$

with p, q, r odd primes, $3<p<q<r$, and either q or $r=kp\pm 1$, $|c_n|\leqq\frac{1}{2}(p+1)$.

5. Remarks. It is interesting to note that $F_{3qr}(x)$ satisfies the theorem trivially. For every prime p greater than 3 we have $p\equiv\pm 1 \pmod 3$. On the other hand, γ in (2) can equal only zero when $n\leqq\frac{1}{2}\phi(3qr)$. Hence it is possible to have all allowable partitions P_{1i} and P_{2i} and still not imply the existence of either P_{30} or P_{40}. In fact, in $F_{105}(x)$, $|c_7|=2$. In this case $p-1=\frac{1}{2}(p+1)$.

The theorem says, for instance, that it is impossible to find $|c_n|=4$ in F_m for $m=5qr$ with either q or $r=5k\pm 1$.

It is a conjecture of the author that for all primes q and r, $\frac{1}{2}(p+1)$ is an upper bound for c_n in $F_{pqr}(x)$.

References

1. A. S. Bang, Om Ligningen $\phi_n(x)=0$, Nyt Tidsskrift for Mathematik, 6 (1895) 6–12.

2. M. Beiter, Coefficients in the Cyclotomic Polynomial for Numbers with at Most Three Distinct Odd Primes in Their Factorization, The Catholic University of America Press, Washington, 1960.

ON THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIALS

D. M. BLOOM, Brooklyn College

1. Introduction. The cyclotomic polynomial $F_n(x)$ is the polynomial whose roots are the primitive nth roots of unity. It is well known that the coefficients c_k of $F_n(x)$ are integers. However, an explicit algorithm for computing the c_k, using only operations that are defined in $Z[x]$ (Z denoting the integers), is not so familiar. We give such an algorithm in Section 2 below. (The method is not new; [1] gives the general idea.)

In the remainder of the paper, the aforementioned algorithm is used to investigate the existence of bounds on the *size* of the coefficients. Let A_n be the

maximum value of $|c_k|$, where $F_n(x)=\sum c_k x^k$, and let S_n denote the set of distinct odd prime factors of n. The following is essentially proved by Bang [1]:

THEOREM 1. *If $S_n=\{p, q, r\}$ with $p<q<r$, then $A_n \leqq p-1$. If n has fewer than three odd prime factors, then $A_n=1$.*

In Section 3 we give Bang's proof, and also show that for certain special residues of q and r (mod p) the case $A_n=p-1$ cannot occur. In Section 4 we sketch a proof, using a similar method, of the following result for four primes:

THEOREM 2. *If $S_n=\{p, q, r, s\}$ with $p<q<r<s$, then $A_n \leqq p(p-1)(pq-1)$.*

Results such as Theorems 1 and 2 raise the question: is A_n, in general, bounded by a function of a proper subset of S_n? (Without the word "proper," the question is trivial since A_n depends only on S_n, cf. Section 2 below.) Specifically, if n has the distinct odd prime factors $p_1< \cdots <p_r$, is A_n bounded by a function of $p_1, \cdots, p_i$ with $i<r$? Theorems 1 and 2 show that we can take $i=r-2$ for $r=2, 3, 4$, and the same can be proved for $r=5$ by a similar method. However, the method of proof seems to break down for $r>5$, and the question remains unresolved.

2. An alogrithm in $Z[x]$ for $F_n(x)$. Let W be the ring of formal power series over Z; then $Z[x]$ is a subring of W. Let $p_1, \cdots, p_r$ be fixed distinct primes, and, for $1 \leqq k \leqq r$, let

$$P_k = \prod (1 - x^e),$$

where the product extends over all possible exponents e such that

$$e = p_{i_1}p_{i_2} \cdots p_{i_k}; \qquad 1 \leqq i_1 < \cdots < i_k \leqq r. \tag{1}$$

(We let $P_0=1-x$.) According to [3, p. 158, equation (3)], if $n=p_1p_2 \cdots p_r$ then

$$F_n(x) = (P_rP_{r-2} \cdots)/(P_{r-1}P_{r-3} \cdots) \tag{2}$$

(in the notation of [3], P_k is written as $\prod_{r-k}$). Since the inverse of $(1-x^e)$ in the ring W is $(1+x^e+x^{2e}+ \cdots)$, it follows easily that the inverse of P_k is $Q_k=\sum_\alpha x^\alpha$, where α runs over all possible sums $a_1e_1+a_2e_2+ \cdots$, the a_i being nonnegative integers and $e_1, e_2, \cdots$ being the different values of e in (1). Equation (2) thus becomes

$$F_n(x) = P_rQ_{r-1}P_{r-2} \cdots$$

with the last factor being P_0 or Q_0 according to whether r is even or odd. Since $F_n(x)$ has finite degree $\phi(n)$, all terms of degree greater than $\phi(n)$ in each P_i and Q_i can be ignored if we also ignore them in the final product. This gives us, as promised, an algorithm strictly within $Z[x]$. It also shows that the initial factor P_r can be ignored, since $P_r=1-x^n$ and $n>\phi(n)$. A further simplification is obtained when r is odd by observing that

$$P_1Q_0 = P_1/(1 - x) = (1 + x + \cdots + x^{p_1-1})(1 - x^{p_2}) \cdots (1 - x^{p_r}).$$

The above discussion covers the special case where n is a product of distinct primes. However, the general case reduces to the special, for if p is a prime dividing m then

$$F_{pm}(x) = F_m(x^p)$$

(proof: both sides have the same roots). Moreover, only odd primes need be considered, since if $m>1$ is odd then $F_{2m}(x)=F_m(-x)$ (same proof). Thus the absolute values of the coefficients of $F_n(x)$ (hence also the number A_n) depend only on the set of primes S_n.

3. Discussion of Theorem 1. We prove the case $S_n=\{p, q, r\}$ of Theorem 1 (the proof for fewer than three primes is similar but easier). As shown above, we lose no generality in taking $n=pqr$. Let

$$N = (\deg F_n(x))/2 = (p-1)(q-1)(r-1)/2.$$

Since $F_n(x)=\sum c_k x^k$ reads the same backwards as forwards, it is enough to consider those cofficients c_k for which $k \leqq N$. As shown in Section 2, c_k is the coefficient of x^k in the expansion of

$$(1 + x + \cdots + x^{p-1})(1 - x^q)(1 - x^r)\sum x^\alpha,$$

the last sum extending over all expressions of the form

$$\alpha = apq + bpr + cqr; \qquad a, b, c \geqq 0. \tag{3}$$

An easy argument (given in [1]) shows that an integer $\alpha \leqq N$ has at most one expression of the form (3). Hence $|c_k| \leqq 2b_k$ where b_k is the maximum number of integers, taken from among p consecutive integers $\leqq k$, which are expressible in the form (3). Since any two of these b_k integers are noncongruent (mod p), they correspond to different values of c in (3). Since

$$0 \leqq c \leqq \alpha/qr \leqq k/qr \leqq N/qr < \tfrac{1}{2}(p-1),$$

it follows that $b_k \leqq \frac{1}{2}(p-1)$ and hence $A_n \leqq p-1$, as we wished to show.

When can equality occur ($A_n=p-1$)? This seems to be a difficult question in general. However, the following result shows that the equation $A_n=p-1$ imposes at least some restrictions on the values of the primes:

THEOREM 3. *Suppose that $A_n=p-1$ in Theorem* 1, *with $p \geqq 5$. Then*:

(1) *Neither q nor r is congruent to* ± 1 (mod p);

(2) *For one of the two values $\delta=\pm 1$, the following is true: for all integers m such that* $|m| \leqq \min(3, \frac{1}{2}(p-3))$, *$mqr+\delta q+r$ is not divisible by p.*

Outline of proof. As before, it suffices to take $n=pqr$, and to consider coefficients c_k such that $k \leqq N$ (with N as above). Examining the preceding proof, we see that each contribution of $+1$ to the value of c_k corresponds to an equation

of one of the following forms, with all coefficients nonnegative:

(4A) $k = a_i pq + b_i pr + (i-1)qr + \lambda_i \qquad (1 \leqq i \leqq \frac{1}{2}(p-1); 0 \leqq \lambda_i < p)$

(4B) $k = a_j' pq + b_j' pr + (j-1)qr + q + r + \mu_j \quad (1 \leqq j \leqq \frac{1}{2}(p-1); 0 \leqq \mu_j < p)$.

Similarly, a contribution of -1 to c_k corresponds to an equation of one of the following forms:

(5A) $k = a_i pq + b_i pr + (i-1)qr + q + \lambda_i$

(5B) $k = a_j' pq + b_j' pr + (j-1)qr + r + \mu_j$

(i, j, λ_i, μ_j as above).

For definiteness, suppose $c_k = p-1$. Then, for this value of k, all of the $(p-1)$ equations (4) hold, but none of the equations (5). If $r \equiv 1 \pmod p$ and we write $r = pd+1$, then equation (4A) with $i=2$ has the same form as equation (5A) with $i=1$ (and with a_i replaced by a_i+d), contradiction. The assumption $r \equiv -1 \pmod p$ leads to a contradiction by the same type of argument; and similarly for $q \equiv \pm 1$. This proves Theorem 3(1). Also, since $k \leqq N$ we must have

(6)
$$b_i p \leqq [\tfrac{1}{2}(p+1) - i]q - \tfrac{1}{2}(p+1) \qquad (\text{all } b_i \text{ in (4A)})$$
$$b_i' p \leqq [\tfrac{1}{2}(p+1) - j]q - \tfrac{1}{2}(p+1) \qquad (\text{all } b_j' \text{ in (4B)}).$$

Still assuming $c_k = +(p-1)$, we show that Theorem 3(2) holds with $\delta = 1$. If instead this is false for some $m \geqq 0$, consider equation (4B) with $j = \frac{1}{2}(p-1)$ and equation (4A) with $i = \frac{1}{2}(p-1) - m$. Comparing these two equations modulo p, we see that $\lambda_i \equiv \mu_j \pmod p$, and hence $\lambda_i = \mu_j$. Now comparing the same two equations modulo q, we see that q divides

$$b_j' pr + r - b_i pr$$

and hence q divides $(b_j' - b_i)p + 1$, so that

(7)
$$(b_i - b_j')p = \beta q + 1 \qquad (\beta \in Z).$$

Clearly $\beta \neq 0$, and $|\beta| \neq 1$ by Theorem 3(1). Hence $|\beta| \geqq 2$. Since $j = \frac{1}{2}(p-1)$, (6) and (7) imply that $b_i > b_j'$, $\beta \geqq 2$. Subtracting (4B) from (4A) and dividing by q, we obtain

$$0 = (a_i - a_j')p + (\beta + i - j)r - 1$$

which implies (applying Theorem 3(1) to r) that $|\beta + i - j| \geqq 2$. Since $\beta \geqq 2$ and we have $j - i = m \leqq 3$, this implies that $\beta + i - j \geqq 2$, so that

$$(b_i - b_j')p > (2 + j - i)q > [\tfrac{1}{2}(p+1) - i]q > b_i p,$$

contradiction. The proof for $m < 0$ is similar. The argument for $c_k = -(p-1)$ goes the same way, with $\delta = -1$, using equations (5) rather than (4).

COROLLARY. *If $p = 5$ in Theorem* 1, *then $A_n \leqq 3$. If $p = 7$ with $A_n = 6$, then $q \equiv \pm 3$ and $r \equiv \pm 3$* (mod 7).

We remark that for $p=3$ and $p=5$, the results of Theorem 1 and the above corollary are "best possible," since $F_{3\cdot5\cdot7}(x)$ has the coefficient $c_7=-2$ and $F_{5\cdot7\cdot11}(x)$ has the coefficient $c_{120}=-3$. It would be interesting to know if the case $p=7$, $A_n=6$ ever occurs.

4. Sketch of the proof of Theorem 2. Here we may assume that $n=pqrs$. As usual, let $F_n(x)=\sum c_k x^k$. We may restrict our attention to coefficients c_k for which $k\leqq(\deg F_n)/2$. Applying the formulas of Section 2 to both F_n and F_{pq}, and observing that

$$(1-x^{pr})(1-x^{ps})\sum_{c,d\geqq 0} x^{cr+ds}=\sum_{\lambda=0}^{p-1}\sum_{\mu=0}^{p-1} x^{\lambda r+\mu s},$$

we find that c_k is the coefficient of x^k in the expansion of

$$F_{pq}(x)(1-x^{qr})(1-x^{qs})(1-x^{rs})\sum_{\lambda,\mu=0}^{p-1} x^{\lambda r+\mu s}\sum_{\alpha} x^{\alpha} \tag{8}$$

where α runs over all expressions of the form

$$\alpha = apqr+bpqs+cprs+dqrs;\qquad a,b,c,d\geqq 0. \tag{9}$$

Hence each contribution of $+1$ to c_k corresponds to an equation of the form

$$k=\sigma+\alpha+\epsilon_1 qr+\epsilon_2 qs+\epsilon_3 rs+\lambda r+\mu s, \tag{10}$$

where α is as in (9), λ and μ are as in (8), $F_{pq}(x)$ has a term $\epsilon x^\sigma=\pm x^\sigma$, and $\epsilon_i=\pm1$ with $\epsilon_1\epsilon_2\epsilon_3=\epsilon$. There are $\frac{1}{2}(p-1)$ choices for the coefficient d in (9) (same argument as for the coefficient c in (3)), $\theta(pq)$ choices for σ (where $\theta(pq)$ is the number of nonzero terms in $F_{pq}(x)$), and then four choices for the triple $(\epsilon_1,\epsilon_2,\epsilon_3)$. The number $\lambda r+\mu s$ in (10) is then determined uniquely modulo p (giving p choices), and the coefficients a, b, c in (9) are then uniquely determined since $k<pqrs$. It follows that

$$c_k\leqq 2p(p-1)\theta(pq) \tag{11}$$

and a similar argument shows that $-c_k$ has the same upper bound. Carlitz [2] has shown that $\theta(pq)=1+2u(pq-uq-1)/p$, with $0<u<p$. Using calculus to find the maximum as u varies, one easily deduces that

$$\theta(pq)\leqq\tfrac{1}{2}(pq-1)$$

(we omit details). Substituting the latter into (11), one obtains Theorem 2.

We remark that since u depends on the residue of q, modulo p (in fact, $qu\equiv-1$, mod p), certain residues of q (mod p) will give substantial improvements on the result of Theorem 2.

5. Acknowledgment. I am indebted to Martin Klein, a student at Brooklyn College, for writing and then running a computer program providing numerical computations of $F_n(x)$ for many values of n (cf. last paragraph of Section 3).

References

1. A. S. Bang, Om Ligningen $\phi_n(x)=0$, Nyt Tidsskrift for Mathematik, Afdeling B, 6 (1895) 6–12.
2. L. Carlitz, The number of terms in the cyclotomic polynomial $F_{pq}(x)$, this MONTHLY, 73 (1966) 979–981.
3. T. Nagell, Introduction to number theory, Wiley, New York (Almqvist and Wiksell, Stockholm, 1951).

A NOTE ON 0-DIMENSIONAL DECOMPOSITIONS OF E^3

R. B. SHER AND W. R. ALFORD, University of Georgia

1. H. W. Lambert and R. B. Sher have shown in [3] that point-like 0-dimensional decompositions of E^3 are definable by cubes with handles. The purpose of this note is to show that this result holds if each element of the decomposition only possesses a point-like imbedding in E^3.

2. If G is an upper semicontinuous decomposition of E^3, we use H_G to denote the union of the nondegenerate elements of G, E^3/G to denote the decomposition space of G, and P to denote the natural map from E^3 onto E^3/G.

The decomposition G of E^3 is *0-dimensional* if $\mathrm{Cl}\, P(H_G)$ is compact and 0-dimensional. The continuum M in E^3 is *point-like* if E^3-M is homeomorphic with the complement of a point in E^3. The continuum M in E^3 is *like a point* relative to E^3 if there is a homeomorphism $h\colon M\to E^3$ such that $h(M)$ is point-like. Note that being point-like in E^3 depends on an imbedding, while the property of being like a point relative to E^3 is a topological property.

For definitions not included here, one may refer to [1].

3. We begin this section with a pair of lemmas that give us information on imbeddings in E^3 of continua that are like a point relative to E^3.

LEMMA 1. *If M is a continuum in E^3 that is like a point relative to E^3 and U is a neighborhood of M, then M is inessentially imbedded in U.*

Proof. Let N be a homeomorphic image of M which is point-like in E^3 and let $h\colon N\to M$ be a homeomorphism. Since N is point-like, there is a sequence $C_1, C_2, C_3, \cdots$ of 3-cells such that for each positive integer i, $C_{i+1}\subset \mathrm{Int}\, C_i$, and $N=\bigcap_{i=1}^{\infty} C_i$. Since N is compact, by the Tietze extension theorem, there is a map $h_*\colon C_1\to E^3$ such that $h_*|N=h$. Let $V=U\cap h_*(C_1)$. Then $h_*^{-1}(V)$ is open relative to C_1 and contains N. Hence, there is a positive integer j such that $C_j\subset h_*^{-1}(V)$. The homeomorphism h^{-1} is null-homotopic in C_j, so there exists a mapping $f\colon M\times[0,1]\to C_j$ such that $f|M\times\{0\}=h^{-1}$ and $f|M\times\{1\}$ is a constant map. The mapping $h_*f\colon M\times[0,1]\to U$ shows that the inclusion: $M\to U$ is null-homotopic in U.

LEMMA 2. *If M is a continuum in E^3, U is a neighborhood of M, and M is inessentially imbedded in U, then some neighborhood of M is inessentially imbedded in U.*

Proof. Let $V_1 \supset V_2 \supset V_3 \supset \cdots$ be a nested sequence of bounded open sets whose intersection is M. By hypothesis, there is a mapping $f: M\times[0, 1]\to U$ such that $f|M\times\{0\}$ is the inclusion and $p=f(M\times\{1\})$ is a point of U. Define the mapping g on $(V_1\times\{0\})\cup(M\times[0, 1])\cup(V_1\times\{1\})\subset V_1\times[0, 1]$ by letting g be the inclusion on $V_1\times\{0\}$, letting g agree with f on $M\times[0, 1]$, and letting g map $V_1\times\{1\}$ onto p. We may use the Tietze extension theorem to extend g to $V_1\times[0, 1]$ and we denote the extension map by g_*. Then $g_*^{-1}(U)$ is open and contains $M\times[0, 1]$. Hence, there is a positive integer j such that $V_j\times[0, 1]\subset g_*^{-1}(U)$. The mapping $g_*|V_j\times[0, 1]$ shows that the inclusion: $V_j\to U$ is null-homotopic in U.

THEOREM 1. *If G is a 0-dimensional decomposition of E^3 and each element of G is like a point relative to E^3, then G is definable by cubes with handles.*

Theorem 1 follows from Lemmas 1 and 2 and the proof of Theorem 1 of [**3**].

THEOREM 2. *If G is a 0-dimensional decomposition of E^3, each element of G is like a point relative to E^3, and $E^3/G=E^3$, then each element of G is point-like.*

Theorem 2 follows for Theorem 1 and the result of [2].

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References

1. S. Armentrout, Monotone decompositions of E^3, Wisconsin Topology Seminar, Summer 1965, Annals of Math Studies, 60 (1966) 1–25.

2. R. J. Bean, Decompositions of E^3 which yield E^3, Pacific J. Math., 20 (1967) 411–413.

3. H. W. Lambert and R. B. Sher, Point-like 0-dimensional decompositions of S^3, to appear in Pacific J. Math.

SOME THEOREMS ON GENERALIZED INVERTIBLE SPACES

DANG XUAN HONG, University of Saigon and Vanderbilt University

Let X be a topological space, U an open set, h a homeomorphism of X onto X such that for every $x\in X$, there is an integer n satisfying $h^n(x)\in U$. The pair (U, h) is called an inverting pair for X. The author has proved [3] that if U is a T_i $(i=0, 1)$ subspace then so is X. Furthermore, suppose $U\subset A$, where A is closed. If A is a T_2, T_3, paracompact, metrizable subspace then so is X respectively. The purpose of this paper is to prove corresponding results in the cases where A is a T_4-, uniformizable, Moore and even Cantor subspace.

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Throughout the remainder of the paper, we suppose that (U, h) is an inverting pair for X. In view of Theorem 9 of [3] it is necessary to consider in Theorems 1 and 2 only the case where A and O are normal subspaces, and in Theorems 3 and 4, completely regular subspaces.

THEOREM 1. *Suppose $U\subset A$, where A is closed. If A is a normal (T_4-) sub-*

space, then X is a normal (T_4-) space.

Proof. Suppose that A is a normal subspace. Let F_1 and F_2 be two disjoint closed sets of X. For each n, $A\cap h^n(F_1)$ and $A\cap h^n(F_2)$ are two disjoint closed sets of A, therefore they have two disjoint relatively open neighborhoods $V_n\cap A$, $W_n\cap A$ where V_n and W_n are open sets of X. Let $P_n=h^{-n}(V_n\cap A\cap U)=h^{-n}(V_n\cap U)$, $Q_n=h^{-n}(W_n\cap U)$, then P_n and Q_n satisfy the following conditions:

(1) P_n, Q_n are open sets of X,

(2) $\overline{P}_n \cap F_2 = \varnothing = F_1 \cap \overline{Q}_n$,

(3) $P_n \cap Q_n = \varnothing$,

(4) The family $\{P_n \mid n \in N\}$ is a countable open cover of F_1, and

(5) The family $\{Q_n \mid n \in N\}$ is a countable open cover of F_2 (because (U, h) is an inverting pair for X).

We can index again $\{P_n\}$, $\{Q_n\}$ so that $n=0, 1, 2, \cdots$ and P_n, Q_n still keep these four preceding properties. With these new indices, we construct the two following families $\{R_n\}$, $\{S_n\}$:

$$R_n = P_n - \bigcup_0^{n-1} \overline{Q}_j, \qquad S_n = Q_n - \bigcup_0^{n-1} \overline{P}_j.$$

Then $\bigcup_0^\infty R_n$, and $\bigcup_0^\infty S_n$ are easily proved to be disjoint open neighborhoods respectively of F_1, F_2.

THEOREM 2 (Levine [4]). *Let X be an invertible space. If X contains a nonempty open set O such that O is a normal (T_4-) subspace, then X is a normal (T_4-) space.*

Proof. Suppose that O is a normal subspace. Then, by Theorem 9 of [**3**], X is T_3. Therefore there exists a nonempty open set U such that $\overline{U}\subset O$. Since a closed subset of a normal space is normal, $\overline{U}$ is normal. Also, there exists a homeomorphism h such that (U, h) is an inverting pair. Thus, by the above theorem, X is a normal space.

THEOREM 3. *Suppose $U\subset A$, where A is closed. If A is a completely regular (Tychonoff) subspace, then X is completely regular (Tychonoff).*

Proof. Suppose that A is a completely regular subspace. Let x be a point of X and O an open neighborhood of x. There exists an integer n such that $h^n(x)\in U$. Since $O'=h^n(O)\cap U$ is an open neighborhood of $h^n(x)$, there exists a continuous function f of A into $[0, 1]$ such that $f[h^n(x)]=0$ and $f[(A-O')]=1$. Let $F: X\to[0, 1]$ be defined as follows:

$$F[X - h^{-n}(A)] = 1$$
$$F[h^{-n}(a)] = f(a), \qquad \forall\, a \in A.$$

Then we easily verify that $F(x)=0$, $F(X-O)=1$, $F(X)\subset[0, 1]$. Finally, we have to prove that F is continuous. Since F is constant on the open set $X-h^{-n}(A)$, it is continuous on $X-h^{-n}(A)$. Also, F is continuous at every point of $h^{-n}(O')$ because f is continuous on O'. To prove that F is continuous at every point $h^{-n}(A)-h^{-n}(O')$ we proceed as follow: let $z\in h^{-n}(A)-h^{-n}(O')$ we have $F(z)=f(h^n(z))\in f(A-O')=1$. Take any neighborhood W of $1=F(z)$. Since f is continuous at $h^n(z)$, there exists a relatively open set $V'\cap A$ of A where V' is open in X, such that $f(V'\cap A)\subset W$. Take $V=h^{-n}(V')$; then $F[h^{-n}(V')]=F[h^{-n}(V'\cap A)\cup h^{-n}(V'-A)]\subset W\cup\{1\}=W$. In conclusion, F is the desired function.

THEOREM 4. *Let X be an invertible space. If X contains a nonempty open set O such that O is a completely regular (Tychonoff) subspace then X is a completely regular (Tychonoff) space.*

Proof. O is completely regular, hence T_3. Then by Theorem 9 of [3], X is T_3. Therefore there exists a nonempty open set U such that $\overline{U}\subset O$. Also, there exists a homeomorphism h such that (U, h) is an inverting pair of X. Thus, by the above theorem, X is completely regular.

A necessary and sufficient condition for a space to be uniformizable is that it is completely regular [1]. Hence, as a consequence, we have the following theorems:

THEOREM 5. *Suppose $U\subset A$, where A is closed. If A is uniformizable then X is uniformizable.*

THEOREM 6. *Let X be an invertible space. If X contains a nonempty open set O such that O is uniformizable, then X is uniformizable.*

A Moore space X is a regular Hausdorff space (i.e. T_3-space) such that there exists a countable family $\mathfrak{F}$ of open covers of X, $\{F_i\}$, satisfying the following condtion M: If F is a closed subset of X and $p\in X-F$, then there exists a F_i of $\mathfrak{F}$ such that no element of F_i intersects both p and F. The family $\mathfrak{F}$ is called a development of X.

THEOREM 7. *Suppose $U\subset A$ where A is closed. If A is a Moore subspace, then X is a Moore space.*

Proof. A is a T_3-subspace, hence X is a T_3-space by Theorem 9 of [3]. Let $\mathfrak{F}=\{F_i \mid i \text{ a positive integer}\}$ be a development of A. Then $F_i=\{O_{\alpha i}\cap A\}$, where each $O_{\alpha i}$ is an open set of X, is an open cover of X. We construct a countable family $\mathcal{G}$ of open covers of X as follows: $\mathcal{G}=\{G_i \mid i \text{ a positive integer}\}$ where

$$G_i = \{h^{-n}(O_{\alpha i}\cap U) \mid O_{\alpha i}\cap A\in F_i,\ n \text{ an integer}\}.$$

Since (U, h) is an inverting pair for X, one can show that for each i, G_i is an open cover of X. We shall show that the family $\mathcal{G}$ satisfied the condition M. Suppose that there exists a closed set F of X and a point p, $p\in X-F$, such that

for each i, there is an element $h^{-n}(O_{\alpha i}\cap U)$ of G_i which intersects both p and F. Since $(O_{\alpha i}\cap U)\subset A$, $h^n(p)$ and $h^n(F)\cap A$ are in A, and for each i there exists an element $O_{\alpha i}\cap A$ of F_i which intersects both $h^n(p)$ and $h^n(F)\cap A$.

Thus, the family $\mathfrak{F}$ does not satisfy the condition M. We have a contradiction. X is therefore a Moore space.

THEOREM 8. *Let X be an inverting space. If X contains a nonempty open set O such that O is a Moore subspace, then X is a Moore space.*

Proof. U is T_3-subspace, hence X is a T_3-space by Theorem 9 of [**3**]. Therefore there exists a nonempty open set U such that $\overline{U}\subset O$. Since a closed subset of a Moore space is a Moore subspace, $\overline{U}$ is a Moore subspace. Also, there exists a homeomorphism h such that (U, h) is an inverting pair for X. Thus, by the above theorem, X is a Moore space.

REMARK. In the most important steps on the way to "metrizability" such as T_2, T_3, T_4, paracompact, uniformizable, Moore space and even metrizability itself, the condition that the open set U lies in the closed set having one of these properties implies that the entire space X has the corresponding property. As on "connectivity" one can not always get similar results. Let X be the integers, U is a single point, and h the shift homeomorphism; then (U, h) is an inverting pair for X, but the connectedness of U does not imply that X is connected. However, in some important cases such as when U is locally connected or a Cantor space, we still have positive answers. For the first case the proof is quite evident. We shall give proof for the second case in the following theorem.

THEOREM 9. *Suppose $U\subset A$, where A is a closed set. If A is a Cantor subspace and there exists N such that for each x, $h^n(x)\in U$ for some n with $|n|<N$, then X is a Cantor space.*

Proof. A Cantor space is a compact, metric, totally disconnected, perfect space. First, one can easily show that X is perfect (i.e. every point is a limit point). Secondly, by Theorem 12 and 13 of [**3**], X is a compact metric space. Finally, let $C(x)$ be a component of X containing a point $x\in X$. We shall show that $C(x)=\{x\}$. Let n be an integer satisfying $h^n(x)\in U$. Since U is a neighborhood of $h^n(x)$ in the compact metric totally disconnected space A, U contains a relatively open and closed neighborhood V of x in A. Since $V\subset U\subset A$, V is also an open and closed set of X. Thus, $h^n(C(x))\cap V$ is a relatively open and closed nonempty set of $h^n(C(x))$. But $h^n(C(x))$ is a maximal connected set of X because $C(x)$ is; therefore $h^n(C(x))=h^n(C(x))\cap V(\subset A)$. Thus $h^n(C(x))$ is a component of A containing $h^n(x)$. Hence $h^n(C(x))=\{h^n(x)\}$ and $C(x)=\{x\}$. In conclusion, X is a Cantor space.

THEOREM 10. *Let X be an invertible space. If X contains a nonempty open set U such that $\overline{U}$ is a Cantor subspace, then X is a Cantor space.*

Proof. Take $N=2$ and $A=\overline{U}$ in Theorem 9.

COROLLARY. *Let X be an invertible space. If X contains a nonempty open set U such that U is a Cantor subspace, then X is a Cantor subspace.*

Proof. U is a Cantor subspace, hence compact and metrizable. Thus, X is metrizable by the Corollary of Theorem 12 of [3], and U is closed in X. Hence, by the above theorem, X is a Cantor space.

References

1. M. Choquet, Structures topologiques—structures uniformes—espaces de fonctions, Centre de Documentations Universitaires, Paris 1964.
2. William J. Gray, On the metrizability of invertible spaces, this MONTHLY, 71 (1964) 533–534.
3. Dang-Xuan-Hong, Generalized invertible spaces, this MONTHLY, 73(1966) 150–154.
4. Norman Levine, Some remarks on invertible spaces, this MONTHLY, 70(1963) 181–183.

REPETITIONS

N. L. JOHNSON, University of North Carolina at Chapel Hill

1. Introduction. Consider a sequence of independent trials in each of which just one of two events—'success' (S) or 'failure' (F)—can occur, with constant probabilities p and q $(=1-p)$ respectively. If the sequence is stopped at a randomly chosen trial, the expected number of trials needed to repeat the event occurring at the last trial is $p\cdot p^{-1}+q\cdot q^{-1}=2$, which is an integer not depending on p. It is the purpose of this note to show that this is also true of the expected number of trials needed to repeat the sequence of results ('pattern') of the last N trials, for any N.

2. Demonstration. Following the approach of Bizley [1], we note that the expected number of trials needed to produce any specified pattern—e.g. $SSFSSSF$—can be expressed in terms of p, and the numbers a_i, b_i of S's and F's respectively up to the ith 'critical point' of the pattern $(i=1, 2, \cdots, k)$. A critical point is defined as a position between two trials, such that the subsequence up to that position is identical with the subsequence of the same length concluding the pattern. The position following the final trial of the pattern is always a critical point. For example, the pattern $SSFSSSF$ has two critical points, with $a_1=2$, $b_1=1$ and $a_2=5$, $b_2=2$; the pattern $SFSFS$ has three critical points, with $a_1=1$, $b_1=0$ and $a_2=2$, $b_2=1$ and $a_3=3$, $b_3=2$.

The expected number of trials is $\sum_{i=1}^{k} p^{-a_i}q^{-b_i}$. The length of the pattern is (a_k+b_k) and the probability that the last $N(=a_k+b_k)$ trials give this pattern is $p^{a_k}q^{b_k}$. Hence the overall expected number of trials needed for repetition is

$$E_N = \sum^{*} p^{a_k}q^{b_k} \sum_{i=1}^{k} p^{-a_i}q^{-b_i}$$

where $\sum^*$ denotes summation over all possible 2^N patterns of length N.

We consider separately the probabilities that any particular position is a critical point. First, and simplest, we note that the position following the last

trial is always a critical point, and contributes $p^{a_k}q^{b_k}(p^{-a_k}q^{-b_k})=1$ to E_N for each pattern, giving a total contribution of 2^N.

Positions following the jth trial of the pattern, for $j=1, 2, \cdots, [\frac{1}{2}N]$ will be critical points provided only that the last j trials of the pattern form a subsequence identical with the first j trials. The total contribution from the jth position is then the sum of terms $p^\alpha q^\beta$ with $\alpha+\beta=N-j$, over all possible orderings of p's and q's in the first $(N-j)$ trials of the pattern, and is equal to $(p+q)^{N-j}=1$.

If j is greater than $[\frac{1}{2}N]$ then for the position to be a critical point the pattern must consist of a cyclic repetition of the first subsequence of length $(N-j)$. This is evident on considering that the last j trials start with the $(N-j+1)$-th trial, and this must be the same as the first trial, and so on. The complete pattern is thus defined by the pattern of the first subsequence of length $(N-j)$ which can be chosen arbitrarily. The total contribution to E_N from the jth position is again the sum of terms $p^\alpha q^\beta$ with $\alpha+\beta=N-j$, and as before is equal to $(p+q)^{N-j}=1$.

Hence each of the $(N-1)$ internal positions of the pattern contribute 1 to E_N, and combining these contributions with 2^N from the position following the final trial, we obtain

$$E_N = 2^N + N - 1.$$

3. Extension. If there are k mutually exclusive possible outcomes at each trial, with constant probabilities $p_1, p_2, \cdots, p_m$ ($\sum_{j=1}^m p_j=1$, $p_j>0$), then

$$E_1 = \sum_{j=1}^{m} p_j p_j^{-1} = m.$$

The contribution to E_N from the terminal position is again 1 for each pattern. As there are now m^N different patterns, the total contribution to E_N is m^N. For each of the other $(N-1)$ positions the total contribution is now a sum of terms like $\prod_{i=1}^m p_i^{\alpha_i}$ with $\sum_{i=1}^m \alpha_i=N-j$, equal to $(\sum_{i=1}^m p_i)^{N-j}=1$, as before. Hence, in this more general case

$$E_N = m^N + N - 1.$$

4. Conclusion. While the expected value of the number of trials needed for repetition does not depend on $p_1, p_2, \cdots, p_m$, the variance does depend on these parameters. For $N=1$, for example, the variance is

$$2\sum_{j=1}^{m} p_j^{-1} - m(m+1).$$

The least possible value (corresponding to $p_1=p_2=\cdots=p_m=m^{-1}$) is $m(m-1)$.

Reference

1. M. T. L. Bizley, Patterns in repeated trials, J. Inst. Actu., 88 (1962) 360–366.

THE PAN-4-AGONAL MAGIC TESSARACT

J. R. Hendricks, Maritimes Weather Office, Halifax, Canada

By means of a previous paper [Canad. Math. Bull., vol. 5, no. 2, May 1962] and using a Latin Tessaract, not shown here, it is possible to construct a magic tessaract which is not only additive to the magic sum 514 in the four kinds of rows, and 8 four-dimensional diagonals, but also in the pandiagonals which are parallel to these four-dimensional diagonals.

53	176	201	84	131	214	127	42	252	97	8	157	78	27	178	231
90	3	166	255	237	124	17	136	151	206	107	50	36	181	224	73
200	93	60	161	114	39	142	219	9	148	245	112	191	234	67	22
171	242	87	14	32	137	228	117	102	63	154	195	209	72	45	188

210	123	46	135	108	13	152	241	31	182	227	74	165	196	89	64
144	229	116	25	55	146	203	110	65	44	189	216	250	95	6	163
35	138	223	118	153	256	101	4	238	71	18	187	88	49	172	205
125	24	129	236	198	99	58	159	180	217	80	37	11	174	247	82

12	145	248	109	190	235	66	23	197	96	57	164	115	38	143	218
103	62	155	194	212	69	48	185	170	243	86	15	29	140	225	120
249	100	5	160	79	26	179	230	56	173	204	81	130	215	126	43
150	207	106	51	33	184	221	76	91	2	167	254	240	121	20	133

239	70	19	186	85	52	169	208	34	139	222	119	156	253	104	1
177	220	77	40	10	175	246	83	128	21	132	233	199	98	59	158
30	183	226	75	168	193	92	61	211	122	47	134	105	16	149	244
68	41	192	213	251	94	7	162	141	232	113	28	54	147	202	111

PROPERTIES OF SET-VALUED ADDITIVE FUNCTIONS

Dagmar Henney, The George Washington University

The object of this paper is to examine certain properties of set-valued additive functions which are defined on the positive cone in Euclidean space E_m.

Let E_m denote the m-dimensional Euclidean space and let $D=\{x\in E_m: x=(x_1, \cdots, x_m), x_i \geqq 0 \text{ for } i=1, \cdots, m\}$. Let X be a linear topological space and C the family of nonempty convex sets of the space X. The family C under the operation of algebraic addition of sets and multiplication of a set by a scalar forms a semi-linear space. One introduces a uniform topology on C by means of the uniformity

$$\{(B_1, B_2) \in C \times C: B_1 \subset B_2 + V \text{ and } B_2 \subset B_1 + V\},$$

where V denotes any symmetric neighborhood of zero in X. Let A be an additive function from D into C.

THEOREM 1. *If A is continuous at the point $x=0$ then A is uniformly continuous on the positive cone D.*

Proof. Notice that $A(0)=\{0\}$, that is, $A(0)$ is the additive identity of the space C. Let V_0 denote any symmetric neighborhood of zero in the space X and let V be a symmetric neighborhood of zero in X satisfying $V+V\subset V_0$ in the space X. Then there exists a $\delta>0$ such that $A(x)\subset V$ if $x=(x_1, \cdots, x_m)\in D$ and $|x_i|\leqq\delta$ for $i=1, \cdots, m$.

Let $x, y\in D$ and $x=(x_1, \cdots, x_m)$ and $y=(y_1, \cdots, y_m)$ be such that $|x_i-y_i|<\delta$ and let $z_i=\sup\{x_i, y_i\}$ for $i=1, \cdots, m$. Now $0\leqq z_i-x_i<\delta$ and $0\leqq z_i-y_i<\delta$ for $i=1, \cdots, m$ implies that $A(z-x)\subset V$ and $A(z-y)\subset V$. Since $A(z)=A(z-x)+A(x)$ we have $A(z)\subset A(x)+V$ and $A(x)\subset A(z)-A(z-x)$ implies $A(x)\subset A(z)+V$. Similarly, one obtains $A(z)\subset A(y)+V$ and $A(y)\subset A(z)+V$. Thus

$$A(x) \subset A(y) + V + V \subset A(y) + V_0 \quad \text{and} \quad A(y) \subset A(x) + V + V \subset A(x) + V_0$$

and this is equivalent to the fact that $(A(x), A(y))$ is contained within a neighborhood of the diagonal of $C\times C$ if $x, y\in D$ and $|x_i-y_i|\leqq\delta$.

This proves uniform continuity of A on the positive cone D.

THEOREM 2. *Let D_0 be a subset of E_m (or more generally one may assume D_0 to be any linear space) which is closed under addition and has the property that $(1/n)x\in D_0$ for all $x\in D_0$ and any positive integer n. Let A again be an additive function defined on D_0 and with values in C. If the sets $A(x_1)$ and $A(x_2)$ are disjoint whenever $x_1\neq x_2$ then $A(x)$ has void interior for every $x\in D_0$.*

Proof. It will be proven first that A is rationally homogeneous, that is, if $r>0$ is a rational number then $A(rx)=rA(x)$ for all $x\in D_0$.

By induction one obtains that $A(nx')=nA(x')$ for $n=1, 2, \cdots$, and all $x'\in D_0$. Since this is true for all elements of the set D_0 it must also be true for the element x/n (which is in D_0 by hypothesis of the theorem). Hence it follows that $(1/n)A(x)=A(x/n)$. Now it is true that

$$A(rx) = A(n/mx) = nA(x/m) = (n/m)A(x) = rA(x)$$

and A is therefore rationally homogeneous.

Suppose now that the interior of the set $A(x)$ is not void. Then there exists an element x_0 in $A(x)$ and a neighborhood V of zero in the space X such that $x_0+V\subset A(x)$.

Now consider the function $f(u)=ux_0$ for any real number u and $x_0\in A(x)$. Since f is continuous this implies that there exists a positive δ such that $f(u)\in V+f(0)=V$ for all $|u|<\delta$. Take any rational number r greater than 1 such that $|r-1|<\delta$. Then $(r-1)x_0 = f(r-1)\in V$, that is, $rx_0\in x_0+V\subset A(x)$. Thus $rx_0\in A(x)$.

But $rx_0\in rA(x)=A(rx)$ and one has $A(x)\cap A(rx)\neq\emptyset$ whereby $x\neq rx$. This is a contradiction.

Thus if $A(x_1)\cap A(x_2)\neq\emptyset$ whenever $x_1\neq x_2$ then the interior of the sets $A(x)$ is void for all elements x of the set D_0.

This work was supported in part by a grant from the George Washington University.

References

1. D. Henney, Set-valued quadratic functionals, Arkiv Mat., Sweden, 4 (1962) 4.
2. ———, The structure of set-valued additive functions, Tagesbericht, Math. Forschungsintitut, Oberwolfach, Germany, 1965 (Abstract).
3. H. Rådström, One-parameter semigroup, Arkiv Mat., 4 (1961).

A CHARACTERIZATION OF STAR-SHAPED SETS

C. R. SMITH, Oklahoma State University and Northeast Louisiana State College

The following characterization of a star-shaped set S in terms of the maximal convex subsets of S provides a simple answer to a question posed by Valentine [2, problem 9.3].

THEOREM. *In a linear space a set S is star-shaped if and only if the intersection of all the maximal convex subsets of S is nonempty.*

Proof. The collection of points with respect to which S is star-shaped is denoted by ck S and is called the convex kernel. As is known, ck S is convex, Brunn [1]. The prove the above it is sufficient to prove the following: If S is star-shaped, then ck $S=\bigcap C_\alpha$, $\alpha\in A$ where $\{C_\alpha, \alpha\in A\}$ is the collection of maximal convex subsets of S.

If $x\in\bigcap C_\alpha$, then $x\in C_\alpha$ for all $\alpha\in A$. Given $y\in A$, then by the lemma, $y\in C_\alpha$ for some $\alpha\in A$. Hence x and y belong to the same maximal convex subset of S, say C_α. Therefore the segment $xy\subset C_\alpha\subset S$, which implies $x\in$ck S.

If $x\in$ck S, suppose there exists a maximal convex subset of S, call it C, such that $x\notin C$. Then, for all $y\in C$, $\bigcup xy=\text{conv}[\{x\}\cup C]$ is a convex subset of S which properly contains C, a contradiction of the maximality of C. Therefore $x\in C_\alpha$, for all $\alpha\in A$, and ck $S\subset\bigcap C_\alpha$.

LEMMA. $\{C_\alpha, \alpha\in A\}$, *is nonempty and* $S=\bigcup C_\alpha$.

Proof. This follows from consideration of the totality of convex subsets of S which contain a point $x\in S$ and the Hausdorff Maximal Principle which implies that $x\in C_\alpha$, for some $\alpha\in A$.

The work on this paper has been supported by National Science Foundation Summer Fellowship No. 75098.

References

1. H. Brunn, Über Kerngebiete, Math. Annalen, 73 (1913) 436–440.
2. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

CORRECTION: Equation (1.1) in "Amicable Numbers with Opposite Parity," by A. A. Gioia and A. M. Vaidya (this MONTHLY, 74 (1967) 969), should read: $\sigma(n_1)=n_1+n_2=\sigma(n_2)$.

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the MONTHLY. *Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.*

NOTE ON REGULAR SEMIGROUPS

JIN BAI KIM, Michigan State University and West Virginia University

Let S be a regular semigroup with 0. Let $a\in S\backslash 0$. D_a, R_a, L_a mean respectively the D, R, L class containing a. D_a is said to be (h, k) regular if every R-class and every L-class in D_a respectively contain exactly h and k nonzero idempotents. Define $V(a)=\{x\in S\colon axa=a \text{ and } xax=x\}$. If $T\subset S$, $E(T)=\{e\in T\colon e=ee\}$. Every nonzero element a of S has n inverses, n being independent of a, while an inverse semigroup is a semigroup in which every element has a unique inverse [1]. As we shall show below, homogeneous n regular (if $|V(a)|=n$ for all $a\in S\backslash 0$ [5]) semigroups are not too far from inverse semigroups.

THEOREM 1. *S with 0 is homogeneous n regular iff there exists a set $\{(h_\xi, k_\xi)\colon \xi\in\Xi\}$ of pairs (h_ξ, k_ξ) of positive integers such that* (i) *$h_\xi k_\xi=n$ and* (ii) *D_ξ is (h_ξ, k_ξ) regular for every D_ξ in the set $D(S)=\{D_\xi\colon \xi\in\Xi\}$ of all D-classes of S with an index set Ξ.*

REMARK. There is a one-to-one correspondence between the set of all inverses a' of a and the set of all pairs (e, f) of idempotents with e in R_a and f in L_a [p. 60, 1].

Proof. (Necessity) Let $a\in S\backslash 0$. Let $b\in D_a$ for $b\in S\backslash 0$, and assume, by way of contradiction, that $|E(R_a)|=h$, $|E(L_a)|=k$, $|E(R_b)|=s$, $|E(L_b)|=t$, $hk=n=st$, and $h\neq s$. Choosing $c\in(R_a\cap L_b)\backslash 0$, by [1, Theorem 2.18], we see that $|V(c)|=|E(R_c)|\,|E(L_c)|=ht\neq n$. This contradiction shows that there exist h and k such that D_a is (h, k) regular with $hk=n$. The sufficiency follows from Remark stated above. Regarding Howie [2] and Kim [4], the natural question can be raised: What is a necessary and sufficient condition that $E(S)$ be a semigroup? For this we have

THEOREM 2. *Let S be a regular semigroup. $E(S)$ is a subsemigroup of S iff $V(e)\subset E(S)$ for every e in $E(S)$.*

For the proof see [3, Lemma 1.1].

Mail Address: Math. Dept., West Virginia University, Morgantown, W. Va. 26506.

References

1. A. H. Clifford and G. B. Preston, Amer. Math. Soc., Surveys no. 7, vol. 1, 1961.
2. J. M. Howie, J. London Math. Soc., 41 (1966) 707–716.
3. J. M. Howie and G. Lallement, Proc. Glasgow Math. Assoc., 7(1966) 145–159.
4. J. B. Kim, Singular matrices, to appear in Czechoslovak Math. Journal.
5. ———, Bull. Amer. Math. Soc., Vol. 71, No. 6 (1965) 867–971.

TWO OPERATOR FORMULAE AND THEIR APPLICATION

D. Ž. Djoković, University of Waterloo, Canada

Let $\mathbf{D}$, $\mathbf{S}$, $\mathbf{X}^a$ (a real) be operators defined by

$$\mathbf{D}f(x) = \frac{d}{dx}f(x), \qquad \mathbf{S}f(x) = f(x^2), \qquad \mathbf{X}^a f(x) = x^a f(x).$$

Theorem. *For any natural number n and any real a*

$$(\mathbf{X}^{-a}\mathbf{D}\mathbf{X}^{a+1}\mathbf{D})^n = \mathbf{X}^{-a}\mathbf{D}^n\mathbf{X}^{n+a}\mathbf{D}^n, \tag{1}$$

$$(\mathbf{D}^2 + (2a+1)\mathbf{X}^{-1}\mathbf{D})^n\mathbf{S} = 2^{2n}\mathbf{S}\mathbf{X}^{-a}\mathbf{D}^n\mathbf{X}^{n+a}\mathbf{D}^n. \tag{2}$$

The proof of (1) is by straightforward induction on n. The proof of (2) is based upon the easily established formula

$$(\mathbf{D}^2 + (2a+1)\mathbf{X}^{-1}\mathbf{D})\mathbf{S} = 4\mathbf{S}\mathbf{X}^{-a}\mathbf{D}\mathbf{X}^{a+1}\mathbf{D}.$$

One has only to apply this fomula n times and to use (1) afterwards.

We apply our theorem to get the following representation of the generalized Laguerre polynomials:

$$L_n^{\alpha}(x^2) = \frac{1}{n!}(-4)^{-n} e^{x^2}\left(\mathbf{D}^2 + \frac{2\alpha+1}{x}\mathbf{D}\right)^n e^{-x^2}. \tag{3}$$

This representation is due to Bragg [1]. We remind that ([3], p. 204)

$$L_n^{\alpha}(x) = \frac{1}{n!} e^x x^{-\alpha} \mathbf{D}^n (x^{n+\alpha} e^{-x}). \tag{4}$$

Formula (3) follows from (4) after applying (2) to $f(x) = e^{-x}$.

It is interesting to note that for $a=0$ formula (1) reduces to $(\mathbf{DXD})^n = \mathbf{D}^n\mathbf{X}^n\mathbf{D}^n$. An analogous formula, namely $(\mathbf{XDX})^n = \mathbf{X}^n\mathbf{D}^n\mathbf{X}^n$, is proved in [2], p. 80.

Mail Address: Dept. of Math., University of Waterloo, Ont., Canada.

References

1. L. R. Bragg, A Rodrigues' formula for the Laguerre polynomials, this Monthly, 73 (1966) 749–751.
2. D. S. Mitrinović and D. Ž. Djoković, Specijalne funkcije, Beograd, 1964.
3. E. D. Rainville, Special functions, Macmillan, New York, 1960.

ON QUASI-INJECTIVE IDEALS IN THE PRIME RINGS

Kwangil Koh, North Carolina State University

R. E. Johnson and E. T. Wong defined that a module M is *quasi-injective* if every homomorphism of a submodule of M into M may be realized by an endomorphism of M. Let R be a ring (not necessarily with 1). The purpose of this note is to establish the following statement:

THEOREM. *A necessary and sufficient condition that R is a simple artinian ring is that*

(1) *R is a prime ring with the descending chain condition on the annihilator right ideals;*

(2) *R contains a uniform right ideal and a nonzero right ideal which is quasi-injective.*

LEMMA A. *Let M be a right R-module such that the singular submodule of M is zero. If N is a quasi-injective submodule of M and T is a submodule of M such that no nonzero submodule of T is quasi-injective, then* $\mathrm{Hom}_R(N, T)=0$.

LEMMA B. *Let R be a semi-prime ring with a uniform right ideal U and zero (right) singular ideal. If U is injective then U is a minimal right ideal.*

Indication of a proof of Theorem. Since necessity is fairly clear, we indicate a proof for sufficiency. Let $\hat{R}$ be the injective hull of the right regular R-module R. The condition (2) and Lemma A enable us to choose a uniform right ideal U which is quasi-injective. Let $\hat{U}$ be the injective hull of U. Then $D=\mathrm{Hom}_R(U, U)$ $=\mathrm{Hom}_R(\hat{U}, \hat{U})=\mathrm{Hom}_R(\hat{U}, \hat{U})$ is a division ring and U and $\hat{U}$ are left vector spaces over D. By the condition (1) and Lemma B, we can show that U is a finite dimensional vector space over D and $R=\mathrm{Hom}_D(U, U)$.

Mail Address: Dept. of Math., North Carolina State University, Raleigh, N. C. 27607.

CORRECTION TO A NOTE ON ABELIAN GROUP AXIOMS

R. A. JACOBSON, Houghton College

A recent note [1] supposedly presented a set of absolutely independent axioms for an abelian group. However, C. I. Deisher pointed out that the example supporting axioms (i), (ii), (iii) was in error and the search for an absolutely independent set of axioms was once more alive.

In this note we modify axiom (ii) and investigate the resulting system. In particular, we include the restriction $xy\neq e$, so that the second axiom becomes: (ii) for all x, y, z; $x\neq e$, $x\neq y$, $x * y\neq e$; $(x * y) * z=y * (x * z)$.

Although examples verifying the absolute independence are readily constructed, the small change in (ii) presents a much more formidable challenge in proving that the new set of axioms define an abelian group. However, the struggle is not without reward: the axioms are indeed those of an abelian group and the quest is seemingly complete. A final temptation remains; that of finding a shorter, more elegant proof than that presented in the note.

Mail Address: Box 123, Houghton, New York.

Reference

1. R. A. Jacobson, Absolutely independent axioms for abelian groups, this MONTHLY, 72 (1965) 991–993.

ABSOLUTELY INDEPENDENT AXIOMS FOR A FINITE GROUP

E. L. WILSON, Wittenberg University

In a recent paper [1], Jacobson and Yocom have presented a set of absolutely independent axioms for a group. A question posed by the authors of that paper was whether their axioms constitute an absolutely independent set of axioms for a finite group of some order n. They do *not*, since there can be no finite model for the following triple of axioms from [1].

($\bar{\text{i}}$) For all x, y, if $x \neq y$, then $x * y \neq x$.

($\bar{\text{ii}}$) For all x, y, z, if $z \neq e$, then $x * (y * z) \neq (x * y) * z$.

(iii) $x * w = y$ has a unique solution w for all x, y, $x \neq y$.

The following axioms are absolutely independent for a finite group of order $n \geqq 4$.

(i) For all x, y, and for all z such that $w * z \neq w$ for some w, $(x * y) * z = x * (y * z)$.

(ii) For each x, y there is a unique w such that $x * w = y$.

(iii) For each x, y there is a unique v such that $v * x = y$.

Mail Address: Dept. of Math., University of the South, Sewanee, TN 37375.

Reference

1. R. A. Jacobson and K. L. Yocom, Absolutely independent group axioms, this MONTHLY, 72 (1965) 756–758.

SUBMODULES WHICH DETERMINE CERTAIN LINEAR MAPPINGS

J. C. PLEASANT, East Tennessee State University

Consider the following well-known result in topology:

THEOREM 1. *Two continuous mappings from a topological space X into a Hausdorff space which agree on a dense subset of X are equal.*

A. Weinstein noted that Hausdorff spaces are characterized by this property (problem 5232, (1965, 923)), this MONTHLY). But dense subsets are not characterized in Theorem 1, as is shown by the following

Example. Let $X = \{a, b\}$ be given the topology $T = \{\varnothing, \{a\}, X\}$. Then $\{b\}$ is not dense in X, but if $f: X \rightarrow Y$ (Hausdorff) is continuous with $f(b) = y$, then clearly $f(a) = y$.

An algebraic analogue of Theorem 1 will be given for a category of (unitary) left A-modules.

DEFINITION. *A submodule E' of a left A-module E is a vital submodule of E if for every x in E there is a regular element r (i.e. a nonzero-divisor) in A such that $rx \in E'$.*

We use Levy's [1] definition of torsion-free modules over arbitrary rings.

The proof of the following is straightforward.

THEOREM 2. *Two A-linear mappings from a left A-module E into a torsion-free left A-module which agree on a vital submodule of E are equal.*

If we restrict attention to rings A which satisfy (CM). Given a, r in A with r regular, there exist a', r' in A with r' regular such that $a'r=r'a$, then Theorem 2 yields a characterization of torsion-free-ness. For if F is a left A module (A satisfying (CM)) which is not torsion-free, there is a $0\neq t\in F$ with $rt=0$ for some regular r in A. Then Ar is a vital submodule of A_s (A considered as a left A-module in the natural fashion), and f, g: $A_s\to F$ agree on Ar where $f=0$ and $g(a)=at\forall a\in A$, but $f\neq g$. In fact, rings satisfying (CM) are characterized by this property.

The author thanks Dr. Edgar Enochs for suggesting this problem.

Mail Address: Math. Dept., East Tennessee State University, Johnson City, TN 37601.

Reference

1. L. Levy, Torsion-free and divisible modules over nonintegral domains, Canad. J. Math., 15(1963) 132–151.

THE SEPARATION AXIOMS FOR INVERTIBLE SPACES

S. K. HILDEBRAND AND R. L. POE, Texas Technological College

A topological space is said to be invertible iff for each nonempty open set U in S, there exists a homeomorphism h of S onto S such that $h(S-U)\subset U$. Several theorems have been proven (see references) which extend the local separation properties T_0, T_1, T_2, regularity, normality, T_3 (regularity and T_1), and T_4 (normality and T_1) to global properties of invertible spaces. It is the intent of this paper to establish that the remaining separation axioms as local properties may be extended to global properties of invertible spaces.

For the sake of brevity proofs of the following theorems are omitted. They may be obtained from the authors.

THEOREM 1. *If (S, τ) is an invertible space and S has a nonempty, open subset V, which as a subspace is completely regular, then (S, τ) is completely regular.*

THEOREM 2. *If the topological space (S, τ) is an invertible space and if S has a nonempty, open set V, which as a subspace is a Tychonoff space (completely regular T_1-space) then (S, τ) is a Tychonoff space.*

THEOREM 3. *If (S, τ) is an invertible space and S has a nonempty, open subset V, which as a subspace is completely normal, then (S, τ) is completely normal.*

THEOREM 4. *If (S, τ) is an invertible space and S has a nonempty, open set V, which as a subspace is a T_5-space (completely normal T_1-space), then (S, τ) is a T_5-space.*

THEOREM 5. *A topology for a set S is the uniform topology for some uniformity of S if and only if the topological space (S, τ) is completely regular.*

COROLLARY. *If the topological space (S, τ) is an invertible space and if S has a nonempty open subset V, which as a subspace is uniformizable, then (S, τ) is uniformizable.*

Mail Address: Dept. of Math., Texas Technological College, Lubbock, TX 79409.

References

1. P. H. Doyle and J. G. Hocking, Invertible spaces, this MONTHLY, 68 (1961) 959–965.
2. Norman Levine, Some remarks on invertible spaces, this MONTHLY, 70 (1963) 181–183.
3. W. J. Gray, On the metrizability of invertible spaces, this MONTHLY, 71 (1964) 533–534.
4. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.
5. William J. Pervin, Foundations of General Topology, Academic Press, New York, 1964.

COMMUTATIVITY IN RINGS OF ZERO DIVISORS

H. G. MOORE, S. J. PIERCE AND ADIL YAQUB, University of California, Santa Barbara

An *$\mathfrak{O}$-algebra* is an associative algebra A with identity over a field F such that every x in A which is not in F is a zero divisor. In this note we prove the following:

THEOREM. *Suppose A is an associative algebra with identity over a field F, and suppose that A has no units other than those in F. If, further, A satisfies the descending chain conditions for right ideals, then A is commutative. Moreover, either $A=F$ or A is a Boolean algebra. In particular, every $\mathfrak{O}$-algebra with* d.c.c. *for right ideals is commutative.*

Proof. If A satisfies d.c.c. for right ideals, then by the Wedderburn-Artin Theorem, A is the direct sum of a finite number m of complete matrix rings $A_i=M_{n_i}(\Delta_i)$ over division rings Δ_i. If for any $i=1, 2, \cdots, m$ n_i is larger than 1, then the element $u_i=I_{n_i}-E_{1n_i}$(E_{1n_i} is zero except for a "1" in the first row, last column) is a unit in A_i which is distinct from the unity of A_i. Then the element u of A whose jth component is I_{n_j} for $j\neq i$ and whose ith component is u_i is a unit in A which is not in F. This contradiction forces $n_i=1$, $i=1, \cdots, m$, so A is a finite direct sum of division rings Δ_i, each of which must contain F. Thus, if $u_i\in\Delta_i$, $i=1, \cdots, m$, are units, the element $(u_1, \cdots, u_m)$ is a unit in A and hence is an element of F. This can happen only if either $u_1=\cdots=u_m=1$ or $m=1$. Thus, if $A\neq F$, A is the direct sum of a finite number of copies of GF(2), and hence is Boolean. Therefore A is commutative, and the theorem is proved.

Mail Address: H. G. Moore, Dept. of Math., Brigham Young University, Provo, Utah 84601.

Reference

1. P. M. Cohn, Rings of zero-divisors, Proc. Amer. Math. Soc., 9 (1958) 909–914.

FUNCTIONS POSITIVE DEFINITE IN THE SPACE C

S. J. EINHORN, Keystone Computer Associates

A real, continuous function f is said to be positive definite in the metric space S if

$$\sum_{i,j=1}^{n} f(P_iP_j)x_ix_j \geqq 0$$

for $n=2, 3, \cdots$, arbitrary real x_i, and any n points P_i of S, where PQ denotes the distance from P to Q.

In particular, suppose f is positive definite in the space C whose points are the continuous functions on $[0, 1]$, with the usual metric. Then f must also be positive definite in every separable metric space, since it is known by a theorem of Banach and Mazur that every such space can be isometrically imbedded in C. Using suitably defined (non-euclidean) spaces, the following conjecture of I. J. Schoenberg is verified: *The only functions positive definite in C are the nonnegative constants.* In fact, if the assumption of continuity is dropped, then $f(x)=k\geqq 0$ if $x>0$, while $f(0)\geqq k$. Conversely, every such f is positive definite in C.

Mail Address: 409 North Easton Road, Willow Grove, PA 19090.

ON COMMUTING MAPPINGS BY SUCCESSIVE APPROXIMATIONS

JULIO CANO, Ball State University

In this note we give similar theorems (of which only one will be presented here) on the question of two continuous commuting mappings on a metric space having a common fixed point.

We take (1) f and g to be continuous commuting mappings from the nonempty metric space X into itself. $F(h)$ will denote the set of fixed points of h.

THEOREM. *Suppose*
(i) *f and g are as in* (1),
(ii) *$F(g)\neq\phi$ and compact,*
(iii) *for $x\in X$ and $x\notin F(f)$, $d(f^2(x), f(x)) < d(f(x), x))$.*
Then f and g have a common fixed point.

Proof. Take $x_0\in F(g)$ then $f^n(x_0)\in F(g)$ for every n. Let $\{f^{n_i}(x_0)\}_{i=1}^{\infty}$ of $\{f^n(x_0)\}_{n=1}^{\infty}$ converge to t. Suppose $f(t)\neq t$, otherwise the theorem follows. Let $Z=X-F(f)$. Consider $h\colon Z\to[0, 1)$ given by

$$h(x) = \frac{d(f^2(x), f(x))}{d(f(x), x)}\cdot$$

Take a neighborhood V of $t\in Z$ such that for $y\in V$, we have $1>p>h(y)\geqq 0$. Let $\epsilon=d(f(t), t)$, then there is N such that $d(f^{n_i+1}(x_0), f^{n_i}(x_0))>\epsilon$ for $i>N$ and where $f^{n_i}(x_0)\in V$. For this i, we have:

$$d(f^{n_i+2}(x_0), f^{n_i+1}(x_0)) < p\; d(f^{n_i+1}(x_0), f^{n_i}(x_0)).$$

Thus for $m>j>N$, we have

$$d(f^{n_m+1}(x_0), f^{n_m}(x_0)) < p^{m-j} d(f^{n_j+1}(x_0), f^{n_j}(x_0)) \to 0 \quad \text{as} \quad m \to \infty.$$

This contradiction establishes the theorem.

Mail Address: Dept. of Math., University of Toledo, Toledo, Ohio.

References

1. J. B. Diaz and F. T. Metcalf, On the structure of the set of subsequential limit points of successive approximations, Bull. Amer. Math. Soc., 73 (1967) 516–519.

2. M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc., 37 (1962) 74–79.

CLASSROOM NOTES

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT 06268.

INVARIANT DEFINITIONS FOR VECTOR CALCULUS

OSWALD WYLER, Carnegie-Mellon University

One of the most important aspects of vector operations is their invariance, i.e., their independence of a particular coordinate system. Unfortunately, almost all textbooks define the differential operators of vector calculus in terms of components, in a particular coordinate system, and their invariance is often not discussed at all. There is no need for this deficiency. It is the purpose of this note to show that coordinate-free, and thus invariant, definitions of the differential operators can easily be given, at the level of abstraction of an Advanced Calculus course.

We shall take the underlying vector algebra and its invariance for granted, as well as the invariance of limits and continuity for scalar and vector valued functions defined in a domain S of space. In the following, we shall keep S fixed. A continuous function defined on S will be called a *scalar field* if its values are scalars, and a *vector field* if its values are vectors.

We begin with directional derivatives. Let ϕ be a scalar or vector field, and $\mathbf{a}$ a fixed vector, or direction. The *directional derivative* $D_\mathbf{a}\phi$ is defined at a point P of S by the equation

$$(1) \qquad (D_\mathbf{a}\phi)(P) = \lim_{h\to 0} \frac{1}{h}\Big(\phi(P + h\mathbf{a}) - \phi(P)\Big),$$

provided of course that the limit at right exists. If $D_\mathbf{a}\phi$ is defined and continuous

at all points P of S, for every direction $\mathbf{a}$, then we call ϕ *continuously differentiable.*

We define a *gradient* of a continuously differentiable scalar field f as a vector field $\mathbf{V}$ which satisfies $\mathbf{a}\cdot\mathbf{V}(P)=(D_{\mathbf{a}}f)(P)$, for every direction $\mathbf{a}$ and all points P of S. We write $\mathbf{V}=\nabla f$ if $\mathbf{V}$ is a gradient of f.

LEMMA 1. *A scalar field f cannot have more than one gradient.*

Proof. If $\mathbf{V}$ and $\mathbf{V}'$ are gradients of f, then $\mathbf{a}\cdot\mathbf{V}(P)=\mathbf{a}\cdot\mathbf{V}'(P)$ for every vector $\mathbf{a}$, at any point P of S. Applying this to $\mathbf{a}=\mathbf{V}(P)-\mathbf{V}'(P)$, we obtain $\mathbf{V}(P)-\mathbf{V}'(P)=0$. Thus $\mathbf{V}=\mathbf{V}'$.

Our definitions are coordinate-free, and hence invariant. However, a coordinate system is quite useful for proving that a gradient, or another differential operator, exists. Thus let x, y, z be cartesian coordinates, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors in the coordinate directions. For $\mathbf{a}=\mathbf{i}, \mathbf{j}, \mathbf{k}$, the directional derivatives $D_{\mathbf{a}}f$ are the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

THEOREM 1. *The following statements for a scalar field f are equivalent:*
(i) *f has continuous partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$,*
(ii) *f is continuously differentiable,*
(iii) *f has a gradient.*

Proof. The implications (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) follow immediately from our definitions. If f satisfies (i), then the vector field with components $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ is a gradient of f, as defined in this note. For a proof of this, see almost any textbook. Thus (i)$\Rightarrow$(iii).

The proof of Theorem 1, together with Lemma 1, shows that our definition of a gradient is equivalent to the usual definition by components.

We now define a *divergence operator* as a correspondence which assigns to every continuously differentiable vector field $\mathbf{V}$ a scalar field denoted by $\nabla\cdot\mathbf{V}$, with the following properties.

A. $\nabla\cdot(\mathbf{V}+\mathbf{W})=(\nabla\cdot\mathbf{V})+(\nabla\cdot\mathbf{W})$, for any continuously differentiable vector fields $\mathbf{V}$ and $\mathbf{W}$.

B. $\nabla\cdot(f\mathbf{V})=(\nabla f)\cdot\mathbf{V}$ if $\mathbf{V}$ is a constant vector field, for any continuously differentiable scalar field f.

THEOREM 2. *There is exactly one divergence operator.*

Proof. We operate again with a coordinate system x, y, z. Every continuously differentiable vector field is of the form $\mathbf{V}=v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k}$, where v_x, v_y, v_z are continuously differentiable scalar fields, determined by $\mathbf{V}$. If we apply A and B to this, we obtain

$$\nabla\cdot\mathbf{V}=(\nabla v_x)\cdot\mathbf{i}+(\nabla v_y)\cdot\mathbf{j}+(\nabla v_z)\cdot\mathbf{k}. \tag{2}$$

Thus there is at most one operator which satisfies A and B.

In order to show that a divergence operator exists, we must show that the operator defined by (2) satisfies A and B. For A, this presents no difficulties, and

we omit this step. For B, we note that the components of $f\mathbf{V}$ are fv_x, fv_y, fv_z. If v_x is constant, then

$$(\nabla(fv_x))\cdot\mathbf{i} = (v_x(\nabla f))\cdot\mathbf{i} = (\nabla f)\cdot(v_x\mathbf{i}).$$

Adding this equation and the corresponding equations for the y- and z-components, we obtain at once B for the operator defined by (2). This completes the proof.

Working out (2), one obtains the usual formula for $\nabla\cdot\mathbf{V}$. This is, however, not our definition of $\nabla\cdot\mathbf{V}$, but a consequence of a coordinate-free, and hence invariant, definition.

For the *rotation* or *curl*, the situation is exactly similar. We require that the *rotation operator* assigns to every continuously differentiable vector field $\mathbf{V}$ a vector field $\nabla\times\mathbf{V}$, with the two properties:

A′. $\nabla\times(\mathbf{V}+\mathbf{W}) = (\nabla\times\mathbf{V})+(\nabla\times\mathbf{W})$;

B′. $\nabla\times(f\mathbf{V}) = (\nabla f)\times\mathbf{V}$ if $\mathbf{V}$ is constant.

THEOREM 3. *There is exactly one rotation operator.*

Proof. Using A′ and B′, we obtain at once

$$\nabla\times\mathbf{V} = (\nabla v_x)\times\mathbf{i} + (\nabla v_y)\times\mathbf{j} + (\nabla v_z)\times\mathbf{k}, \tag{3}$$

so that there is at most one rotation operator. One verifies, as in the proof of Theorem 2, that the operator defined by (3) satisfies A′ and B′, and thus this is a rotation operator. Working out (3), one obtains the usual formula for $\nabla\times\mathbf{V}$.

The method sketched in this note can be used to make all further definitions in vector differential calculus invariant, but we shall leave this to the reader.

A PROOF OF NEWTON'S POWER SUM FORMULAS

J. A. EIDSWICK, University of Nebraska

For a polynomial $P(z) = \alpha_0+\alpha_1 z+\cdots+\alpha_n z^n = \alpha_n(z-z_1)(z-z_2)\cdots(z-z_n)$, the power sums $S_m = \sum_{k=1}^n z_k^m$, $m = 1, 2, \cdots$, can be calculated from the formulas

$$\begin{aligned} m\alpha_{n-m} + \sum_{k=1}^{m} \alpha_{n-m+k} S_k &= 0 \quad \text{if } m \leqq n,\\ \sum_{k=m-n}^{m} \alpha_{n-m+k} S_k &= 0 \quad \text{if } m > n. \end{aligned} \tag{1}$$

For example, if $n = 3$,

$$S_1 = -\alpha_2\alpha_3^{-1}, \qquad S_2 = \alpha_2^2\alpha_3^{-2} - 2\alpha_1\alpha_3^{-1}, \qquad S_3 = -\alpha_2^3\alpha_3^{-3} + 3\alpha_1\alpha_2\alpha_3^{-2} - 3\alpha_0\alpha_3^{-1},$$

$$S_4 = \alpha_2^4\alpha_3^{-4} - 4\alpha_1\alpha_2^2\alpha_3^{-3} + 4\alpha_0\alpha_2\alpha_3^{-2} + 2\alpha_1^2\alpha_3^{-2}.$$

One quick, but rather vague, method of proving (1) is to differentiate $P(z)$ two different ways, equating like powers of z (see [1]). Another method is through the use of the theory of symmetric functions (see [2]). The student might find the following proof more satisfying: The logarithmic derivative of the polynomial $Q(z)=\alpha_n+\alpha_{n-1}z+\cdots+\alpha_0 z^n=\alpha_0(z-z_1^{-1})(z-z_2^{-1})\cdots(z-z_n^{-1})$ (assuming, without loss of generality, that $P(0)\neq 0$) is

$$F(z)=\frac{Q'(z)}{Q(z)}=\sum_{k=1}^{n}(z-z_k^{-1})^{-1}$$

which, when differentiated k times, gives $F^{(k)}(0)=-k!S_{k+1}$. Since

$$Q^{(m)}(z)=[F(z)Q(z)]^{(m-1)}=\sum_{k=0}^{m-1}\binom{m-1}{k}F^{(k)}(z)Q^{(m-1-k)}(z),$$

we have

$$\frac{Q^{(m)}(0)}{m!}=-\frac{1}{m}\sum_{k=0}^{m-1}\frac{Q^{(m-1-k)}(0)}{(m-1-k)!}S_{k+1}$$

or

$$-m\alpha_{n-m}=\sum_{k=0}^{m-1}\alpha_{n-m+k+1}S_{k+1}\qquad \text{if } m\leqq n,$$

$$0=\sum_{k=m-n-1}^{m-1}\alpha_{n-m+k+1}S_{k+1}\quad \text{if } m>n,$$

which is (1).

References

1. L. E. Dickson, Linear Groups, B. G. Teubner, Stuttgart, 1901, p. 53.
2. B. L. van der Waerden, Modern Algebra, Frederick Ungar, New York, 1953.

COUNTABLE AND NET CONVERGENCE

J. E. MARSDEN, Princeton University

It is well known that Lebesgue's dominated convergence theorem does not hold for nets; that is, having a countable sequence is essential (see [1], p. 95). On the other hand, for a real valued function on an interval, sequences do suffice; that is, $\lim_{x\to y}f(x)=a$ iff $\lim_{n\to\infty}f(x_n)=a$ for every sequence $x_n\to y$. The purpose of this note is to isolate the basic reasons for these phenomena.

DEFINITION. *A directed set A is called countably accessible iff there is a countable sequence a_n in A such that $a_n\to\infty$, that is, for any $b\in A$ there is an N such that $a_n\geqq b$ if $n\geqq N$.*

THEOREM. *Let X be a topological space and A a countably accessible directed set. Suppose $f: A \to X$ is a net and for every countable sequence $b_n \to \infty$ in A, $f(b_n)$ converges to $x \in X$. Then f converges to x.*

Proof. If f did not converge to x, there would be a neighborhood U of x such that for any $b \in A$ there is a $b' \geqq b$ with $f(b') \notin U$. However, if $a_n \to \infty$ then $f(a_n')$ does not converge to x even though $a_n' \to \infty$, a contradiction.

References

1. S. K. Berberian, Measure and Integration, Macmillan, New York, 1965.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1950.

AN ABSTRACTION

G. L. MUSSER, State University College at Buffalo

While instructing a class for prospective elementary school teachers, I noted the similarity of the usual proofs of the two theorems, "*for any natural numbers a, b, c, if $a|b$ and $b|c$, then $a|c$*" and "*for any natural numbers a, b, c, if $a \leqq b$ and $b \leqq c$, then $a \leqq c$.*" Generalizing and expanding the methods used in these proofs led to some general theorems which include among their special cases the two transitivity theorems above.

Let S be an arbitrary set, $*$ an associative and commutative binary operation defined on S, $\#$ a binary operation defined on S such that $\#$ is distributive over $*$, and R a binary relation defined by the statement "if a and b are elements of S, then aRb if and only if there exists an x in S such that $a * x = b$." For convenience we denote such a system by the ordered quadruple $(S, *, \#, R)$. Let a, b, c, d be elements of S. Then we have the following theorems, the proofs of which are easy exercises.

THEOREM 1. *If aRb and bRc, then aRc.*

THEOREM 2. *If aRb, then $(a * c)R(b * c)$.*

THEOREM 3. *If aRb and cRd, then $(a * c)R(b * d)$.*

THEOREM 4. *If aRb, then $(a \# c)R(b \# c)$.*

(Note that if S has an identity with respect to $*$, then we have cRc and Theorem 2 is a corollary to Theorem 3.)

Clearly, these four theorems hold for the following systems.

1. $(N, +, \cdot, \leqq)$ where N is the set of natural numbers.
2. $(N, \cdot, \exp, |)$ where '$a \exp b$' is a^b and $a|b$ means 'a divides b'.
3. $(2^N, \cup, \cap, \subseteq)$ where 2^N is the power set of N and if A, B are elements of 2^N, then $A \subseteq B$ if and only if $A \cup C = B$ for some C in 2^N.
4. $(Z, +, \cdot, \equiv)$ where Z is the set of integers and $a \equiv b$ if and only if $a + 5n = b$ for a, b, n in Z.

THE OPTICAL PROPERTY OF THE CONICS

HARLEY FLANDERS, Purdue University

The following proofs of the optical property of the conics provide for the calculus student a beautiful and convincing example of the power of vector methods.

Suppose $\mathbf{v}=\mathbf{v}(t)$ is a vector function of time t; then differentiating $|\mathbf{v}|^2=\mathbf{v}\cdot\mathbf{v}$, we have $d|\mathbf{v}|=(\mathbf{v}\cdot d\mathbf{v})/|\mathbf{v}|$.

Let $\mathbf{a}$ be the focus of a parabola and $\mathbf{u}$ a unit vector perpendicular to the directrix. The moving point $\mathbf{x}=\mathbf{x}(t)$ on the curve satisfies $\mathbf{x}\cdot\mathbf{u}=|\mathbf{x}-\mathbf{a}|$, hence

$$(d\mathbf{x})\cdot\mathbf{u}=\frac{(\mathbf{x}-\mathbf{a})\cdot d\mathbf{x}}{|\mathbf{x}-\mathbf{a}|},\qquad d\mathbf{x}\cdot\mathbf{u}=d\mathbf{x}\cdot\left(\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|}\right).$$

But $(\mathbf{x}-\mathbf{a})/|\mathbf{x}-\mathbf{a}|$ is a unit vector directed from the focus to $\mathbf{x}$; the equality means that this vector and $\mathbf{u}$ make the same angle with the tangent to the curve.

Let $\mathbf{p}$, $\mathbf{q}$ be the foci of an ellipse. Then

$$|\mathbf{x}-\mathbf{p}|+|\mathbf{x}-\mathbf{q}|=c,$$

where c is a constant. From this,

$$d\mathbf{x}\cdot\left(\frac{\mathbf{x}-\mathbf{p}}{|\mathbf{x}-\mathbf{p}|}\right)+d\mathbf{x}\cdot\left(\frac{\mathbf{x}-\mathbf{q}}{|\mathbf{x}-\mathbf{q}|}\right)=0,$$

from which one reads off the optical property. Of course, the hyperbola $|\mathbf{x}-\mathbf{p}|-|\mathbf{x}-\mathbf{q}|=\pm c$ is treated the same way.

The student should realize that, in each two-line proof, line one gives the defining geometrical property of the curve and line two the result of differentiating line one.

A REMARK ON A FIXED-POINT THEOREM FOR ITERATED MAPPINGS

V. W. BRYANT, University of Sheffield, England

The following result is proved in [**1**; p. 50], [**2**; p. 8] and [**3**; p. 150]:

"If f is a continuous mapping of a complete metric space into itself and if, for some positive integer k, the iterate f^k is a contraction, then f has a unique fixed point."

In each of the proofs listed above the assumption of continuity is used in the argument, but as we show below this condition is not necessary.

THEOREM. *If f is a mapping of a complete metric space into itself and if, for some positive integer k, f^k is a contraction, then f has a unique fixed point.*

It is well known that a contraction mapping of a complete metric space into itself has a unique fixed point; see e.g. [**1**; p. 43]. Denote by x_0 the unique fixed point of f^k. Since

$$f(x_0) = f(f^k(x_0)) = f^{k+1}(x_0) = f^k(f(x_0))$$

it follows that $f(x_0)$ is a fixed point of f^k and so $f(x_0) = x_0$. Thus f possesses a fixed point, and a fixed point of f is necessarily a fixed point of f^k and so is unique.

The theorem we have established constitutes a genuine extension of the result quoted above since the fact that f^k is a contraction of a complete metric space does not imply that f is continuous. For let f: $[0, 2] \to [0, 2]$ be defined by

$$f(x) = \begin{cases} 0 & x \in [0, 1] \\ 1 & x \in (1, 2]. \end{cases}$$

Then $f^2(x) = 0$ for all $x \in [0, 2]$ and so f^2 is a contraction of $[0, 2]$ although f is not continuous.

References

1. A. Kolmogorov and S. V. Fomin, Introduction to Functional Analysis, Volume 1, Graylock Press, New York, 1957.

2. F. F. Bonsall, Lectures on Some Fixed-point Theorems of Functional Analysis, Tata Institute of Fundamental Research, Bombay, 1962.

3. R. E. Edwards, Functional Analysis; Theory and Applications, Holt, Rinehart and Winston, New York, 1965.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

SOUTHERN METHODIST'S SUMMER PROGRAM IN MATHEMATICS FOR HIGH ABILITY SECONDARY SCHOOL STUDENTS

J. D. BROWN, Southern Methodist University

As a teacher, I am fascinated by a gifted student and feel particularly responsible for him. Sooner or later, most professionals face the problem of transmitting their own dedication and mastery to another generation. These natural concerns are transcended when one finds himself in the role of mentor to a group of gifted students. Such has been my privilege for the past seven summers.

The John von Neumann Mathematics Seminar has been held at S.M.U. for the past eleven summers. The 1957 and 1958 programs were supported by the Fund for the Advancement of Education. The program has been supported since then by the National Science Foundation.

The major objectives of the seminar are:

To challenge high ability students with some of the newer concepts of mathematics.

To give the students a view of the wide scope of the mathematics of the present by presenting significant mathematical topics and concepts which will be useful in connection with subsequent courses in high school and college.

To stimulate the participants to greater effort during their last year of secondary school.

To stimulate an interest in and show the opportunities of a career in science by having guest scientists and engineers speak on topics of their special interests.

To give these mentally gifted students with *superior* mathematical background an opportunity to compete and associate with one another.

Each summer, our group consists of 35 boys and girls selected from the 600 or more applications which are submitted. These applications are from 11th grade students in public, private or parochial schools of some 20 or more states. Their avowed interests range from engineering through the physical sciences to mathematics. Biological sciences are also represented, though to a lesser degree.

In the selection of participants we use a measure of their potential, inasmuch as it can be judged through scores on such standardized tests as the S.T.E.P. Math and Science tests and the *V-Q* ability test. The selection process always nets a group representing a wide spectrum of interest and abilities.

Each year, at the outset of the preparation of our proposal to the National Science Foundation, we are confronted with the dilemma as to what purpose should be served by a mathematics program for a collection of young individuals who have in common only eagerness, curiosity, an unbounded supply of vitality, and possibly an ultimate destiny in one of the sciences.

There are several choices open to us. We could decide upon a selection of some mathematical skills in the hope that they may be useful at some later date, or we could select some simple applications and treat jointly the mathematical tools and ideas of the related sciences. On the other hand, we could take advantage of the fact that with a proper selection of material one can remain within mathematics and yet exhibit a whole spectrum of ideas and thought processes which could confront any scientist in any field. This last approach is the one which we adopt and the one which the staff feels is particularly apropos today, when "theorizing" and deductive reasoning are so much a part of human activity. In our seminar, we have subordinated everything to the aim of providing our students with a rich experience in the thought processes of science, while still remaining within mathematics.

The topics chosen to carry us toward our objectives are number theory, probability and abstract structure theory.

Since the students have had a great deal of experience with arithmetic, the subject of number theory is accessible to them and one which is a favorite with them. Number theory provides an unusually fine opportunity for the development of the student's powers of observation, his intellectual curiosity, and his capacity for intellectual adventure. As an important by-product of all the courses, the student acquires facility in the effective, precise, and concise use of logic and deductive reasoning.

Because our world is becoming more and more statistically minded, it is felt that a study of the concepts of probability is necessary. The theory of probability, as the foundation upon which the methods of statistics are based, should command the attention of those who want to understand as well as apply (at some later date) statistical techniques. Less well known is the fact that probability concepts are finding increased use in the social sciences, business, psychology, economics and insurance.

Abstract structure theory arises out of basic physical experience. It is extremely instructive for the student to participate in the natural development of abstract structures, while being involved in the continual interplay of observation, experience, conjecture and counterexample or proof within the structure.

The principal part of the S.M.U. program consists of three series of lectures. These lectures are based to a large extent on text books and supplementary text materials which are given to the students. The materials are of a level of difficulty which these talented students can understand and at the same time challenges the best of them. The program demands that each participant do as high a quality of work as he is capable of doing.

Also, twice a week, we have guest scientists come and lecture to the students. These lectures range in scope from "Carcinogenic Hydrocarbons" to the "Physics of the Solar System." At least two field trips are taken to nearby industry to give the students an idea of how mathematics is used in industry.

Competition among the participants is very keen, even if friendly, and contributes much to the maintenance of high standards. For most of the participants, it is their first encounter with students who are of equal or better ability. They soon find out that competition at this level is far different than the hometown high school competition. It is interesting to note their reactions and adjustments to this situation. The students learn to use the library for supplementary reading and for obtaining information relating to topics of special interest to them. The availability of a large college library is a new experience for most of them and they withdraw books on every conceivable subject.

Those of us involved in the selection of participants for the program do not find it easy to single out those qualities of character and achievement which could be considered as an indication of promise. We try not to be too rigid in our selective process for fear of overlooking gifted nonconformists.

Remaining within the limits imposed by their immaturity, we try to bring these gifted young people into contact with the best scientific thinking of the Dallas area.

Working with these high-ability high school students is a very exacting task, but one is amply repaid for the effort when he observes the deep involvement and the keen pleasure shown by these young people as they meet and grasp new ideas and become aware of new horizons.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before August 31, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2075. *Proposed by Steven R. Conrad, Francis Lewis High School, Flushing, N.Y.*

Find all integral values of x, y, z for which

$$4^x + 4^y + 4^z$$

is a perfect square.

E 2076. *Proposed by Gregory Wulczyn, Bucknell University*

It has been proved that the only tetrahedral numbers $T_n = n(n+1)(n+2)/6$ which are perfect squares are T_1, T_2, T_{48}. Show that each of these tetrahedral numbers is the first term of an infinite series of tetrahedral numbers such that each partial sum is a square integer.

E 2077. *Proposed by Alvin Hausner, City College, New York*

Consider the curves given in polar coordinates by $r^n = a^n \sin n\theta$, where a is a positive constant and $n = 1, 2, \cdots$. Let L_n, A_n denote the length of, and the area enclosed by, all the n leaves of these curves. Find expressions for $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} A_n$.

E 2078. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

Let n be an integer $\geqq 210$. Prove or disprove: there are always at least 5 positive composite integers less than and relatively prime to n.

E 2079. *Proposed by D. S. Levine, Harvard University*

Let n be any integer $\geqq 2$. Prove that $\sum(1/pq) = 1/2$, where the summation is over all integers p, q which satisfy $0 < p < q \leqq n$, $p + q > n$, $(p, q) = 1$.

E 2080. *Proposed by Brian D. Wick, San Diego State College*

Prove: The product of the units of the ring $I/(n)$ is -1 if and only if the units form a cyclic group under multiplication.

E 2081. *Proposed by Leon Bankoff, Los Angeles, California*

With characteristic tenacity, Professor E. P. B. Umbugio has been struggling to solve the following paraphrased version of problem 9, page 201 of Hobson's *Plane and Advanced Trigonometry* (Dover reprint):

"If the orthocenter H, the incenter I, and the circumcenter O of a triangle ABC are the vertices of an equilateral triangle, show that $\cos A+\cos B+\cos C=3/2$."

Help terminate the professor's futile floundering by showing that (a) the triangle HIO can never be equilateral and (b) when $\cos A+\cos B+\cos C=3/2$, the triangle HIO is nonexistent.

E 2082. *Proposed by Ih-Ching Hsu, Fordham University*

F is a field of characteristic p ($p\neq 0$); T is a $p\times p$ nonsingular matrix over F. Prove that the only matrix of the form λT ($\lambda\in F$) similar to T is T itself.

E 2083. *Proposed by Erwin Just, Bronx Community College*

Find the total number of ways of arranging in a row the $2n$ integers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ with the restriction that for each i, a_i precede b_i, a_i precede a_{i+1}, and b_i precede b_{i+1}.

E 2084. *Proposed by J. J. Goode, Georgia Institute of Technology*

For all choices of integers k, n, $0\leqq k\leqq n$, prove the identity

$$\sum_{i=k}^{n}\binom{i}{k}\frac{1}{n+1-i}=\binom{n+1}{k}\sum_{i=k}^{n}\frac{1}{i+1}.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Another Triangle Inequality

E 1935 [1966, 1122]. *Proposed by W. J. Blundon, Memorial University of Newfoundland*

Prove that, for every triangle ABC, $s\leqq 2R+(3\sqrt{3}-4)r$, where s, R and r are the semiperimeter, circumradius and inradius, respectively. Equality holds only for the equilateral triangle.

Comment by Andrzej Makowski, Warsaw, Poland. The result is proved in the proposer's paper, *Inequalities associated with the triangle*, Canad. Math. Bull., 8(1965) 615-626, Formula 17. Here he shows that the given inequality is the strongest possible linear inequality in R, r and s.

Also solved by Ragnar Dybvik (Norway), M. G. Greening (Australia), and C. V. Subbarama Iyer (India).

A 60° Triangle Relation

E 1936 [1966, 1122]. *Proposed by W. J. Blundon, Memorial University of Newfoundland*

If in triangle ABC we have

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \sqrt{3},$$

prove that at least one angle of the triangle is 60°.

Solution by A. Krishnamurthi, Sri Ramakrishna Mission College, Vidyalaya, India. We use the following lemma which is easily proved: If $P+Q+R=0$, then

$$(*)\qquad \sin P + \sin Q + \sin R = -4 \sin \frac{P}{2} \sin \frac{Q}{2} \sin \frac{R}{2} \cdot$$

Since $\sqrt{3} = \sin 60°/\cos 60°$, the given relation may be put

$$\sum (\sin A \cos 60° - \cos A \sin 60°) = \sum \sin(A - 60°) = 0.$$

By (*) $\Pi \sin \frac{1}{2}(A - 60°) = 0$, and thus at least one angle must be 60°.

Also solved by Robert Abes, A. N. Aheart, Harvey Arnold, Jose Asseo, Anders Bager (Denmark), C. R. Berndtson, Arthur Bolder, A. K. Bose, Neil Cameron (New Zealand), L. Carlitz, J.P. Celenza, Tan Kok Chye (Malaysia), P. A. Clement, E. E. Cowan, Margaret Crawford, Ragnar Dybvik (Norway), R. B. Eggleton (Australia), Alexandra Forsythe, I. B. Goldberg, Michael Goldberg, D. Gootkind, M. G. Greening (Australia), J. W. Grossman, C. V. Subbarama Iyer (India), P. M. Kannen, P. S. Kornya, Lew Kowarski, J. C. Lanz, A. Makowski (Poland), D. C. B. Marsh, Norman Miller, Bohuslav Mišek (Czechoslovakia), P. L. Montgomery, C. B. A. Peck, J. M. Quoniam (France), Stanley Rabinowitz, Simeon Reich (Israel), Norman Schaumberger, David Shelupsky, H. Simpson (England), Sidney Spital, C. S. Venkataraman (India), Steven Weintraub, Charles Wexler, Qazi Zameeruddin (India), and the proposer.

Editorial Comment. Aheart also notes that the problem is equivalent to Corollary 1 of the proposer's paper, *A relation involving right triangles*, this MONTHLY, May 1967, pp. 566–568. The corollary states: At least one of the angles of a triangle is 60° if and only if $s = (R+r)\sqrt{3}$. Since $\sum \cos A = (R+r)/R$ and $\sum \sin A = s/R$, the present result follows immediately.

A Trigonometric Summation

E 1937 [1966, 1122]. *Proposed by J. M. Quoniam, Saint-Etienne, France*

Taking

$$S_\mu = \sum_{k=1}^{[n/2]} 4^\mu \cos^{2\mu} \frac{k\pi}{n+1},$$

prove that

$$S_\mu = (n+1)\binom{2\mu-1}{\mu-1} - 2^{2\mu-1},$$

where the usual symbols for greatest integer and binomial coefficient have been used.

Solution by M. G. Greening, The University of New South Wales, Australia. Put $z=\exp(\pi i/(n+1))$, so $z^{n+1}=-1$. We have

$$S_\mu = \sum_{k=1}^{[n/2]} (z^k + z^{-k})^{2\mu} = \sum_{k=1}^{[n/2]} \sum_{r=0}^{\mu-1} \binom{2\mu}{r} \left\{ z^{2(r-\mu)k} + z^{-2(r-\mu)k} \right\} + \sum_{k=1}^{[n/2]} \binom{2\mu}{\mu}$$

$$= \sum_{r=0}^{\mu-1} \binom{2\mu}{r} \sum_{k=1}^{[n/2]} (\alpha^k + \alpha^{-k}) + \left[\frac{1}{2}n\right] \binom{2\mu}{\mu} \quad \text{where} \quad \alpha = z^{2(\mu-r)}$$

$$= \sum_{r=0}^{\mu-1} \binom{2\mu}{r} \alpha^{-[n/2]} \sum_{h=0}^{2[n/2]} \alpha^h - \sum_{r=0}^{\mu-1} \binom{2\mu}{r} + \left[\frac{1}{2}n\right] \binom{2\mu}{\mu}.$$

We make use of

$$\sum_{h=0}^{2[n/2]} \alpha^h = \frac{\alpha^{2[n/2]+1}-1}{\alpha-1} = \begin{cases} 0, & n \text{ even} \\ -1/\alpha, & n \text{ odd,} \end{cases} \qquad (\alpha \neq 1)$$

and finally get, in case n is even,

$$S_\mu = -\frac{1}{2}\sum_{r=0}^{2\mu}\binom{2\mu}{r} + \frac{1}{2}\binom{2\mu}{\mu} + \left[\frac{1}{2}n\right]\binom{2\mu}{\mu} \qquad (n \text{ even})$$

$$= -2^{2\mu-1} + 2\left\{\left[\frac{1}{2}n\right] + \frac{1}{2}\right\}\binom{2\mu-1}{\mu-1} = (n+1)\binom{2\mu-1}{\mu-1} - 2^{2\mu-1}.$$

On the other hand, for n odd we have

$$\sum_{r=0}^{\mu-1}\binom{2\mu}{r}\alpha^{-[n/2]}\sum_{h=0}^{2[n/2]}\alpha^h = -\sum_{r=0}^{\mu-1}\binom{2\mu}{r}\alpha^{-[n/2]-1}$$

$$= -\sum_{r=0}^{\mu-1}\binom{2\mu}{r}(-1)^{\mu+r} = -(-1)^\mu\frac{1}{2}\sum_{r=0}^{2\mu}\binom{2\mu}{r}(-1)^r + \frac{1}{2}(-1)^{2\mu}\binom{2\mu}{\mu} = \frac{1}{2}\binom{2\mu}{\mu}.$$

It follows that

$$S_\mu = \left(\left[\frac{1}{2}n\right] + \frac{1}{2}\right)\binom{2\mu}{\mu} - \sum_{r=0}^{\mu-1}\binom{2\mu}{r}$$

$$= \left(\left[\frac{1}{2}n\right] + 1\right)\binom{2\mu}{\mu} - \frac{1}{2}\sum_{r=0}^{2\mu}\binom{2\mu}{r} = (n+1)\binom{2\mu-1}{\mu-1} - 2^{2\mu-1}.$$

It should be noted that in order to get the stated result we must assume $n+1>\mu>0$.

Also solved by L. Carlitz, D. Gootkind, Stephen Hoffman, Simeon Reich (Israel), Perry Scheinok, and the proposer.

A Ratio of Eccentricities

E 1938 [1966, 1123]. *Proposed by Stanimir Fempl, University of Belgrade, Yugoslavia*

A right cone and a right circular cylinder have a common base and altitude. If a plane meets these two bodies so that the intersections are ellipses, prove that the ratio of the eccentricities of the ellipses is equal to the ratio of the lengths of the generatrices of the bodies.

Solution by C. Stanley Ogilvy, Hamilton College, Clinton, N.Y. It is well known that if the plane cuts the axis of the cone at an angle θ and the generating angle of the cone is ϕ, then the eccentricity is $(\cos\theta)/(\cos\phi)$. (See, for example, Durell, *A Concise Geometrical Conics*, Macmillan, London, 1927, p. 76. See also Editorial Note to E 1353 [1959, 727].) In the cylinder, $\phi=0$, and the quotient of the two eccentricities reduces immediately to the required ratio.

Also solved by Anders Bager (Denmark), Leon Bankoff, Ragnar Dybvik (Norway), R. B. E. Eggleton (Australia), Michael Goldberg, M. G. Greening (Australia), Lew Kowarski, Charles McCracken, Norman Miller, Bohuslav Mišek (Czechoslovakia), D. A. Penner, J. M. Quoniam (France), H. Simpson (England), Sister Stephanie Sloyan, Jože Vrečko (Poland), Charles Wexler, Qazi Zameeruddin (India), and the proposer.

Divisibility of Triangular Numbers

E 1939 [1966, 1123]. *Proposed by Richard Stanley, California Institute of Technology*

How many of the first n triangular numbers $T_k=\frac{1}{2}k(k+1)$ are divisible by n?

Solution by D. C. B. Marsh, Colorado School of Mines. With $(k, k+1)=1$, p^a divides T_k if and only if $k\equiv 0$ or $-1 \pmod{p^a}$ when p is an odd prime, and 2^a divides T_k if and only if $k\equiv 0$ or $-1 \pmod{2^{a+1}}$. Thus, for $n=2^b\Pi p_i^{a_i}$ where there are $m(\geqq 0)$ distinct odd primes p_i dividing n, for $1\leqq k\leqq n$ the $m+1$ simultaneous congruences admit 2^m-e solutions k where $e=0$ or 1 as $b=0$ or $b>0$. This is the number of the first n triangular numbers which are divisible by n.

Also solved by Jack C. Abad, R. B. Eggleton (Australia), M. G. Greening (Australia), Donald Jeffords, L. J. Marx, P. L. Montgomery, D. H. Peterson, S. F. Robinson, A. M. Vaidya (India), and the proposer.

Editorial Comment. A related problem that appeared as no. 272 [1917, 427; 1934, 582] was to find how many from each of the following four sequences are relatively prime to n:

$$2\binom{k+1}{2}, \quad 6\binom{k+2}{3}, \quad \binom{k+1}{2}, \quad \binom{k+2}{3}.$$

A Simple Equivalence

E 1940 [1966, 1123]. *Proposed by J. V. Cornacchio and R. P. Soni, IBM, Endicott, N. Y.*

Given the finite sequence $\{p_k\}_{k=1}^{m}$ such that $0<p_k\leqq 1$ $(k=1, 2, \cdots, m)$, $\sum_{k=1}^{m} p_k=1$, and the $m\times m$ matrix $\|\alpha_{kl}\|$ each of whose elements satisfies the

condition $|\alpha_{kl}| \leqq 1$; prove or disprove

$$\sum_{k=1}^{m} \sum_{l=1}^{m} p_k p_l |\alpha_{kl}|^2 = 1 \quad \text{if and only if } |\alpha_{kl}| = 1, \qquad (k, l = 1, \cdots, m).$$

Solution by Mary R. Embry, University of North Carolina at Charlotte. Observe first that $\sum_{k=1}^{m} \sum_{l=1}^{m} p_k p_l = 1$. Thus

$$\sum_{k=1}^{m} \sum_{l=1}^{m} p_k p_l |\alpha_{kl}|^2 = 1 \Leftrightarrow \sum_{k=1}^{m} \sum_{l=1}^{m} p_k p_l (1 - |\alpha_{kl}|^2) = 0.$$

However, since $p_k p_l > 0$ and $1 - |\alpha_{kl}|^2 \geqq 0$ for $k, l = 1, \cdots, m$, then this last result holds true if and only if $1 - |\alpha_{kl}|^2 = 0$ for $k, l = 1, \cdots, m$.

Also solved by D. E. Crabtree, Philip Fung, D. Gootkind, M. G. Greening (Australia), P. Hantom, R. A. Jacobson, L. J. Marx, C. B. A. Peck, Mariano Rodrigues, G. S. Rogers, Perry Scheinok, David Shelupsky, Stephen Spindler, Sidney Spital, Jože Vrečko (Yugoslavia), and the proposer.

On Two Closure Problems

E 1941 [1966, 1123]. *Proposed by William Koenen, Highland Park High School, St. Paul, Minn.*

Let M be the set of natural numbers. For $A \subseteq M$, we let cA represent the complement of A in M.

(1) If kA is the smallest set containing A which is closed for addition, prove that there are no more than six sets of the form A, cA, kA, ckA, kcA, $ckcA$, $kckA$, $ckckA$, $kckcA$, etc., and display a set for which six are distinct.

(2) If hA is the smallest set containing A which is closed for multiplication, display a set A for which fourteen of the sets A, cA, hA, chA, hcA, $chcA$, $hchA$, $chchA$, $hchcA$, $chchcA$, etc., are distinct.

Solution by L. F. Meyers, Ohio State University. (1) If $1 \in A$, then $kA = M$; but also $1 \notin cA$, so that $1 \notin kcA$ and $1 \in ckcA$, thus yielding $kckcA = M$. Further application of k or ckc still yields M, so that the at most six sets are $ckcA$, A, M, and their complements kcA, cA, and $\emptyset$. If $1 \notin A$, then $1 \in cA$, and the six sets are ckA, cA, M, and their complements kA, A, and $\emptyset$.

If $A = \{2\}$, then kA is the set of all even positive integers and the six sets are distinct.

(2) Let $j = chc$. Then jA is the set of all elements of A which cannot be expressed as a product of positive integers not in A. If $A = \{2, 3, 14, 15, 25\}$, then the seven sets listed in the table below, and their complements in M, are distinct. A "+" or "−" in a box indicates that the number at the left does or does not, respectively, belong to the set at the top. The products in parentheses are examples or counterexamples; most of the other entries follow from the fact that $jB \subseteq B \subseteq hB$ for $B \subseteq M$.

	A	hA	jhA	$hjhA$	jA	hjA	$jhjA$
4	$-$	$+(2\cdot 2)$	$+$	$+$	$-$	$+(2\cdot 2)$	$+$
25	$+$	$+$	$-(5\cdot 5)$	$-$	$-(5\cdot 5)$	$-$	$-$
75	$-$	$+(3\cdot 25)$	$+$	$+$	$-$	$-$	$-$
210	$-$	$+(14\cdot 15)$	$-(10\cdot 21)$	$+(14\cdot 15)$	$-$	$+(14\cdot 15)$	$-(10\cdot 21)$

A form of the Kuratowski closure-complement problem (for partially ordered sets) shows that not more than fourteen distinct sets can be produced.

Also solved by J. Philip Smith, and the proposer. Partial solutions by Jerry Fischer and D. A. Marcus.

Editorial Comment. Other possible sets for part (2) are $A=\{3, 5, 10, 18, 21, 70\}$ given by Smith and $A=\{6, 7, 10, 11, 14, 15, 30, 33, 35, 42, 105, 110, 210, 231, 462, 1155\}$ given by the proposer. A natural question to ask here is: What is the size of the smallest set having the desired property?

The proposer notes that these problems are related to his paper, *The Kuratowski Closure Problem in the Topology of Convexity*, this MONTHLY, Aug.–Sept. 1966, pp. 704, 708. He also notes that (1) was solved by one of his high school juniors several years ago, and that (2) is related to P. C. Hammer, *Kuratowski's closure theorem*, Nieuw Archief voor Wiskunde (3) viii, 74–80, 1960. Hammer gives a proof that 14 is the maximum number of such sets for (2).

Another proof of this last fact is given in J. L. Kelley, *General Topology*, p. 57. Note also related problem 5349 [1966, 1132], *A Kuratowski Closure and Complement Problem.*

Square Root of a Matrix

E 1942 [1966, 1123]. *Proposed by C. F. McLaren, University of Michigan*

Let

$$A=\begin{pmatrix} -91 & 28 & 9 \\ 47 & -14 & -4 \\ -1113 & 341 & 108 \end{pmatrix}. \qquad \text{Find } \sqrt{A}.$$

Solution by E. J. F. Primrose, The University, Leicester, England. The characteristic equation of A is $\lambda^3-3\lambda^2-\lambda-1=0$, so

$$(1) \qquad A^3-3A^2-A-I=0.$$

This can be written as $A(I-A)^2=(I+A)^2$, so two solutions (out of eight) are given by $\sqrt{A}=\pm(I+A)(I-A)^{-1}$ which, by (1) may be expressed in the form $\sqrt{A}=\pm\frac{1}{2}(I+2A-A^2)$. This gives

$$\sqrt{A}=\pm\frac{1}{2}\begin{pmatrix} 239 & -73 & -23 \\ 577 & -175 & -55 \\ 668 & -208 & -66 \end{pmatrix}.$$

Also solved by Donald Batman, M. G. Greening (Australia), Louise S. Grinstein, Cornelius Groenewoud, Sidney Heller, J. E. Homer, Jr., L. N. Howard, and the proposer. Partial solutions by W. J. Blundon, and C. C. A. Sastri.

Editorial Comment. This simple solution above does not apply to the general matrix. A relatively easy method which will work for all 3×3 matrices is used by Batman who refers to R. Bell-

man, *Introduction to Matrix Analysis*, McGraw-Hill, N.Y., 1960, p. 101, Ex. 32. Assume $\sqrt{A} = rA^2+sA+tI$. Then by squaring and using the characteristic equation to eliminate the A^4 term, we can find equations for r, s, t by comparing the resulting cubic with the characteristic equation. This method will give the eight basic square roots (aside from similarity transformations, e.g., if X is a square root of I, so also is $P^{-1}XP$).

For a treatment of the problem of determining all mth roots of a matrix, see F. R. Gantmacher, *The Theory of Matrices*, I, Chelsea, N.Y., 1959, pp. 231–239.

On Sums of Triangular Numbers

E 1943 [1966, 1123]. *Proposed by J. M. Khatri, Baroda, India*

(1) Prove or disprove: There exists an infinite series of triangular numbers such that every partial sum is a perfect square number. (2) The same except that every partial sum is a triangular number.

Solution by Bernard Jacobson, Franklin and Marshall College. (1) If a is odd, $k\geqq 0$, and $x(k)=\frac{1}{4}(3^k a+1)^2$, then

$$x(k) + T_{2a\cdot 3^k} = x(k+1),$$

where T_n denotes the nth triangular number $n(n+1)/2$. This leads to the following permissible series:

$$\begin{aligned}
&\text{(a)} \quad T_1 + T_2 + T_6 + \cdots + T_{2\cdot 3^k} = \tfrac{1}{4}(3^{k+1}+1)^2,\\
&\text{(b)} \quad T_1 + T_5 + T_{14} + \cdots + T_{14\cdot 3^k} = \tfrac{1}{4}(7\cdot 3^{k+1}+1)^2,\\
&\text{(c)} \quad T_8 + T_{22} + T_{66} + \cdots + T_{22\cdot 3^k} = \tfrac{1}{4}(11\cdot 3^{k+1}+1)^2,\\
&\text{(d)} \quad T_1 + T_2 + T_9 + T_{26} + \cdots + T_{26\cdot 3^k} = \tfrac{1}{4}(13\cdot 3^{k+1}+1)^2,\\
&\text{(e)} \quad T_8 + T_9 + T_{34} + \cdots + T_{34\cdot 3^k} = \tfrac{1}{4}(17\cdot 3^{k+1}+1)^2.
\end{aligned}$$

It seems reasonable to conjecture that there exist infinitely many such series.

(2) Since

$$T_n + T_{T_n-1} = T_{T_n},$$

we can generate infinitely many sequences of the desired type by starting with arbitrary T_n for $n\geqq 2$.

Also solved by Anders Bager (Denmark), Dean Bandes, T. E. Elsner, Jerry Fischer, Michael Goldberg, Cornelius Groenewoud, J. A. H. Hunter, R. A. Jacobson, Donald Jeffords, Erwin Just, E. L. Magnuson, A. Makowski (Poland), D. A. Marcus, C. F. Marion, D. C. B. Marsh, Norman Miller, Steven Minsker, P. L. Montgomery, Robert Patenaude, Dale Peterson, Stanley Rabinowitz, Leo Schneider, D. R. Stark, and Gregory Wulczyn (two solutions).

A Minimum Number of Subsets

E 1944 [1966, 1123]. *Proposed by Philip Dwinger, University of Illinois at Chicago*

Let X be a set of n points, $n\geqq 1$. Let $A_1, A_2, \cdots, A_k$ be a family of subsets of X, such that every subset of X can be expressed in terms of the A_i, $1\leqq i\leqq k$, by

means of set-theoretic operations: union, intersection and complementation. Find the minimum value of k.

Solution by Daniel A. Marcus, Adelphi University. This is equivalent to the problem of expressing each singleton set in the prescribed manner. Define A_i as a carrier of $x \in X$ if and only if $x \in A_i$. To make the necessary distinction between singletons, each member of X must have a different set of carriers. This condition is also sufficient. With k sets there are 2^k possible sets of carriers. We must have $2^k \geqq n$. It follows that the minimum value of k is

$$-[-\log_2 n],$$

where the brackets denote the greatest integer function. To give an actual expression for $\{x\}$, we have

$$\{x\} = \bigcap_{x \in A_i} A_i \cap (\sim \bigcup_{x \notin A_j} A_j).$$

Also solved by Peter Ash, Dean Bandes, G. A. Heuer, Peter Kornya, D. C. B. Marsh, J. C. Morgan II, Robert Patenaude, R. C. Steinlage, Jože Vrečko (Yugoslavia), and the proposer.

An Impossible Diophantine Equation

E 1945 [1967, 75]. *Proposed by D. R. Rao, Secunderabad, India*

Show that there exists no solution of $x^{2n+1}=2^r \pm 1$ in positive integers r, n, x with $x>1$, $n>1$. (It has been proved that $x^r=2^n+1$ is impossible except for the special case $3^2=2^3+1$, and that $x^r=2^n-1$ is impossible when x is prime. See E 663 [1945, 519] and E 685 [1946, 159].)

Solution by E. S. Langford, U. S. Naval Postgraduate School. Suppose that $x^{2n+1}=2^r \pm 1$, with $x>1$, $n>1$. Then

$$x^{2n+1} \mp 1 = (x \mp 1)(x^{2n} \pm x^{2n-1} + \cdots \pm x + 1) = 2^r.$$

This is a contradiction since the second factor on the right is always odd and greater than 1.

Also solved by James Altman, Anders Bager (Denmark), Roxanne M. Byrne, Benedict Carlat, R. B. Eggleton (Australia), M. G. Greening (Australia), Donald Jeffords, Donald Kern, A. Makowski (Poland), and the propcser.

Makowski obtains the proof from a result in J. W. Cassels, *On the equation $a^x-b^y=1$*, Proc. Cambridge Philos. Soc., 56 (1960), 97–103. Eggleton, making use of E 663, establishes that $3^2=2^3+1$ is the only solution of $x^m=2^r\pm 1$ where x, m, r are integers >1.

An Identity

E 1946, [1967, 76]. *Proposed by G. E. Andrews, Pennsylvania State University*

Prove the identity

$$-1 = \sum_{n=1}^{p} (-1)^n \gamma^{n-1} q^{(n-1)^2} \prod_{j=0}^{p-n-1} \frac{(1-q^{n+j})(1+\gamma q^{n+j})}{(1-q^{j+1})},$$

where $|q| \neq 1$, p is a positive integer and we define $\prod_{j=0}^{-1}(\ )(\ )$ to be 1.

Solution by L. Carlitz, Duke University. Put

$$\begin{bmatrix} k \\ n \end{bmatrix} = \frac{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-n+1})}{(1-q)(1-q^2)\cdots(1-q^n)}$$

and recall that

$$\prod_{j=0}^{k-1}(1+q^jx) = \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix} q^{j(j-1)/2}x^j.$$

Then

$$\sum_{n=1}^{p}(-1)^n\gamma^{n-1}q^{(n-1)^2}\prod_{j=0}^{p-n-1}\frac{(1-q^{n+j})(1+\gamma q^{n+j})}{1-q^{j+1}}$$

$$= \sum_{n=1}^{p}(-1)^n\gamma^{n-1}q^{(n-1)^2}\begin{bmatrix} p-1 \\ n-1 \end{bmatrix}\prod_{j=0}^{p-n-1}(1+\gamma q^{n+j})$$

$$= \sum_{n=0}^{p-1}(-1)^{n+1}\gamma^n q^{n^2}\begin{bmatrix} p-1 \\ n \end{bmatrix}\prod_{j=0}^{p-n-2}(1+\gamma q^{n+j+1})$$

$$= \sum_{n=0}^{p-1}(-1)^{n-1}\gamma^n q^{n^2}\begin{bmatrix} p-1 \\ n \end{bmatrix}\sum_{s=0}^{p-n-1}\binom{p-n-1}{s}\gamma^s q^{(n+1)s}$$

$$= \sum_{k=0}^{p-1}\begin{bmatrix} p-1 \\ k \end{bmatrix}\gamma^k\sum_{n+s=k}(-1)^{n+1}\begin{bmatrix} k \\ s \end{bmatrix}q^{n^2+(n+1)s+s(s-1)/2}$$

$$= \sum_{k=0}^{p-1}(-1)^{k+1}\begin{bmatrix} p-1 \\ k \end{bmatrix}q^{k^2}\sum_{s=0}^{k}(-1)^s\begin{bmatrix} k \\ s \end{bmatrix}2^{s(s+1)/2-ks}$$

$$= \sum_{k=0}^{p-1}(-1)^{k+1}\begin{bmatrix} p-1 \\ k \end{bmatrix}q^{k^2}\prod_{j=1}^{k}(1-q^{-k+j}) = -1.$$

Also solved by Lois J. Reid, and the proposer.

A Functional Equation

E 1947 [1967, 76]. *Proposed by Underwood Dudley, Ohio State University*

Show that there exists a function f such that $\int_x^{x^2} f(t)dt = 1$ for all $x > 1$.

Solution by Christopher J. Henrich, Harvard University. Let F be any antiderivative of f; then $F(x^2) - F(x) = 1$. Set $x = e^y$ and $F(e^y) = G(y)$; then $G(2y) - G(y) = 1$. Set $y = e^z$ and $G(e^z) = H(z)$; then $H(z + \ln 2) - H(z) = 1$.

The general solution to the last equation is $H(z) = (z/\ln 2) + P(z/\ln 2)$ where P is a function with period 1. Since H is differentiable, P has a derivative which is also periodic with period 1, and $\int_0^1 p(z)dz = 0$. Retracing our substitutions, we find

$$F(x) = \{\ln(\ln x)\}/(\ln 2) + P\{\ln(\ln x)\}/(\ln 2).$$

Therefore

$$f(x) = (x \ln 2 \ln x)^{-1} + p\{(\ln(\ln x))/(\ln 2)\}/(x \ln 2 \ln x)$$

is the general solution.

Also solved by D. R. Anderson, Einar Andresen (Norway), Marcia Ascher, Günter Bach (Germany), Anders Bager (Denmark), A. P. Boblétt, Donald Batman, J. L. Brown, Jr., Roxanne M. Byrne, G. B. Chase, W. R. Derrick, Bob De Vore, R. B. Eggleton (Australia), T. E. Elsner, L. P. Epstein, J. A. Faucher, N. J. Fine, Mrs. A. C. Garstang, R. M. Gasper, D. L. George, Jan Gilbert, E. D. Gingerich, Daniel Goddard, Michael Goldberg, D. Gootkind, Rudolf Gorenflo, (Germany), Marjorie K. Gregg, Cornelius Groenewoud, J. B. Z. Gross, Harry Guess, J. E. Hafstrom, Alexia Henderson, G. A. Heuer, J. G. Hocking, Jim Howell, Ih-Ching Hsu, D. G. Huffman, R. A. Jacobson, Theodore Katsanis & Louis Goldman, V. H. Keiser, Jr., John Kieffer, D. Z. Kilhefner, Joe Kingston, B. G. Klein, A. F. Kleiner, Jr., E. S. Langford, R. D. Leitch (England), J. E. MacDonald, Jr., D. A. Marcus, W. D. Markel, D. C. B. Marsh, P. H. Mason, E. A. Memmott, Steven Minsker, S. S. Muchnick, J. S. Muldowney, J. B. Muskat, Lou Padulo, F. D. Parker, H. Penkuhn (Italy), David E. Penney, Walter Penney, W. J. Pervin, D. H. Peterson, L. J. Pratte, J. R. Purdy, Stanley Rabinowitz, L. A. Ringenberg, P. D. Ritger, L. E. Rogers, G. H. Ryder, David Ryeburn, L. J. Schneider, D. B. Shapiro, J. S. Shipman, Francis Siwiec, Sidney Spital, T. Teichmann, A. Thyagaraja (India), Julius Vogel, L. E. Ward, J. B. Wilker, R. Wong, and the proposer.

Editorial Comment. In the majority of contributions, the solution $f(t) = 1/t \ln t \ln 2$ was obtained simply by inspection.

The above solution can be easily adapted to solve $\int_x^{x^a} f(t)dt = g(x)$. The solution $f(t) = 1/t \ln t \ln a$ corresponding to $g(x) = 1$ was also obtained by inspection by a number of solvers. Ryeburn gives a constructive existence proof for f for the still more general equation $\int_x^{h(x)} f(t)dt = g(x)$. Katsanis and Goldman do the same for $g(x) = 1$.

N. J. Fine relates the problem to one of iterations of functions. If there is a function $u(x, t)$ such that

$$u(x, s + t) = u(u(x, s), t), \qquad -\infty < s, t < \infty, \tag{1}$$

subject to $u(x, 0) = x$, $u(x, 1) = h(x)$, then under suitable conditions,

$$\frac{\partial u}{\partial t} = \frac{1}{F(u)} = \frac{1}{F(x)} \frac{\partial u}{\partial x}, \tag{2}$$

where

$$\frac{1}{F(x)} = \left.\frac{\partial u}{\partial t}\right]_{t=0}. \tag{3}$$

Integrating (1) yields $t = \int_x^u F(\lambda)d\lambda$ and, in particular, with $t = 1$, $1 = \int_x^{h(x)} F(\lambda)d\lambda$. Thus, if one can find the general iterate of h (i.e., $u(x, t)$), then (3) provides a solution of $F(x)$.

For further material concerning the translation equation (1), see J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, N.Y., 1966, pp. 245–253. For a more general functional equation, see Chocewski and Kuczma, *On the "indeterminate case" in the theory of a linear functional equation*, Fund. Math., 58(1966) pp. 163–175. Here the functional equations treated are of the form $\phi(F(x)) = g(x)\phi(x) + F(x)$.

Correction: The name of P. N. Bajaj should be added to the list of solvers of Problem E 1914 [1968, 83].

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate signed sheets and should be mailed before October 31, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5580. *Proposed by G. F. Schumm, University of Chicago*

A collection of sets is said to be multiplicative (additive) if it contains as an element the intersection (union) of an arbitrary subcollection. If S is a set of n elements, find the number of (i) multiplicative, (ii) additive, and (iii) both multiplicative and additive, collections which can be formed of the subsets of S.

5581. *Proposed by J. E. Shirey, Purdue University*

What can be said about the points of discontinuity of a function $f: R\to R$, if every point is a local minimum?

5582. *Proposed by Olga Taussky, California Institute of Technology*

Let a_i be complex numbers. Let α be a root of

$$f(x) \equiv a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n = 0.$$

It can be shown easily that $g(x) \equiv f(x)/(x-\alpha) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$ where $b_r = \sum_{s\geq 0} a_{r+s+1}\alpha^s$, with $a_n = 1$, $a_{n+1} = a_{n+2} = \cdots = 0$ (see e.g. E. Artin, *Theory of Algebraic Numbers*, Göttingen 1959). Let β be a root of $g(x)$. Let C be the companion matrix of $f(x)$ and C' its transpose. Interpret $\sum_{i=0}^{n-1} b_i\beta^i = 0$ as an orthogonality condition on characteristic vectors of C and C'.

5583. *Proposed by S. W. Williams, Lehigh University*

Let D be a dense subset and S a discrete subset of a topological space X. If $|D| = m$ and $|S| > 2^m$ for some cardinal m, then X is not regular. ($|\cdot|$ denotes cardinality; regular does not imply T_1.)

5584. *Proposed by Erwin Just, Bronx Community College*

Let H_i $(i=1, 2, \cdots, m-1)$ be a set of $m-1$ subgroups of an abelian group in which $H_j \cap H_k = \{1\}$ for each $j \neq k$. Prove that there exists a subgroup $H_m \subset G$ such that $G/\prod_{i=1}^{m} H_i$ is a torsion group and $H_i \cap H_m = \{1\}$, $1 \leq i \leq m-1$.

5585. *Proposed by H. D. Keesing, University of Wisconsin*

Let K be the Cantor set on the real line R (the standard one in $[0, 1]$ plus all its integer translates). Suppose $f: R\to R$ is of class C^∞, and that the restriction of f to each component of $R-K$ is real analytic. Prove or disprove that f must be real analytic on R.

5586. *Proposed by H. D. Keesing, University of Wisconsin*

Let U be a dense open subset of E^2. Suppose $f: E^2 \to E^2$ is of class C^∞ and $f \mid U$ is complex analytic. Prove or disprove that f must be entire.

5587. *Proposed by G. F. Schumm, University of Chicago*

Let $\beta \neq 0$ be an ordinal number of the second kind. (β is of the second kind if it has no predecessor.) Then for $\gamma \leqq \omega^\alpha$ and $\alpha \neq 0$, prove that $(\omega^\alpha + \gamma)^\beta = \alpha\beta$ if and only if $\alpha\beta$ is an ϵ-number.

5588. *Proposed by C. B. Mehr, Ohio University, Athens, Ohio.*

Let X be a real linear space and f a seminorm on X. Let $x, y \in X$ and $a \in R$. Prove

$$\lim_{n\to\infty} f[(n+a)x+y] - f(nx+y) = af(x).$$

5589. *Proposed by Richard Bumby and Erik Ellentuck, Rutgers—The State University*

Let f be any real valued function defined on the natural numbers. Define $f^{(m)}(n) = (f(n)+f(n+1)+\cdots+f(n+m-1))/m$. Prove that $\lim\inf_m \mathrm{glb}_n f^{(m)}(n) = \mathrm{lub}_m \lim\inf_n f^{(m)}(n)$.

SOLUTIONS OF ADVANCED PROBLEMS

An Inverse Problem in Dynamics

4664 [1955, 734]. *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

Are there any other laws of attraction beside the inverse square law such that the time of descent (from rest) through any straight tunnel through a uniform spherical planet is independent of the path?

Editorial Note. The solution of this problem appears as a Scientific Laboratory of the Ford Motor Company publication, preprint of a paper by M. S. Klamkin and D. J. Newman, *On some inverse problems in dynamics*, to appear in the Quarterly of Applied Mathematics. The abstract of the paper follows.

It is known that the time of transverse of a freely falling body through a straight tunnel connecting any two points of the surface of a uniform spherical planet is isochronous. We show here that if the isochronous property is to hold for any planet with a spherical symmetric density, then the density must be constant. Also it is shown that if the isochronous property is to hold for any uniform spherical planet subject to a central force law, then the force law must be inverse square. However, the isochronous property can hold for one uniform spherical planet with a different force law of attraction.

A Topological Identity

5428 [1966, 897]. *Proposed by P. G. Jessup, Lehigh University*

For two vector topologies T, T' on a linear space, let $T \wedge T'$ be the largest

vector topology included in $T \cap T'$. Let $X^{\#}(X')$ be the set of all (continuous) linear functionals on X. For $S \subset X^{\#}$, let $\sigma(S)$ be the weak topology by S. For (X, T) a linear topological space, when is it true that $\sigma(X^{\#}) \wedge T = \sigma(X')$?

Solution by the proposer. We show this is always true.

Let (X, T) be a linear topological space. Clearly $\sigma(X^{\#}) \wedge T \supseteq \sigma(X')$. A typical neighborhood of 0 in $\sigma(X^{\#}) \wedge T$ is of the form $U+V$; V a T open neighborhood of 0 and $U = \bigcap_{i=1}^{n} (|f_i| < \epsilon_i), f_i \in X^{\#}$. Let $A = \bigcap_{i=1}^{n} f_i^{\perp}$; then the T closure of $A \equiv \overline{A}$ is a closed subspace of (X, T). Also, $X = b_1 \oplus b_2 \oplus \cdots \oplus b_m \oplus \overline{A}$; this means that any $x \in X$ can be written uniquely as

$$x = \sum_{i=1}^{m} \alpha_i b_i + a, \tag{1}$$

$a \in \overline{A}$, α_i scalars, [cf. A. Wilansky, *Functional Analysis* p. 39, Th. 2]. Define $g_i(x) = \alpha_i, i = 1, \cdots, m$. Then $g_i^{\perp} \supseteq \overline{A}$ for all i. Hence $g_i^{\perp}$ is equal to a closed subspace of (X, T) plus a finite dimensional one. By Cor. 5 on p. 192 of the above reference, $g_i^{\perp}$ is T closed, and from Th. 3, p. 186, it follows that g_i is continuous for each i.

Let w be a T open neighborhood of 0 such that $w+w \subseteq V$. Then $U+V \supseteq A + V \supseteq (A+w)+w \supseteq \overline{A}+w$ [p. 174 Fact iv].

Now choose N, a T open neighborhood of 0 such that $\sum_{i=1}^{m} N \subseteq w$, and choose $\epsilon > 0$ such that $|t| < \epsilon$ implies $tb_i \in N$ for all i. Then $\bigcap_{i=1}^{m} (|g_i| < \epsilon)$ is a $\sigma(X')$ neighborhood of 0 and is included in $w + \overline{A} \subseteq U+V$. For if $|g_i(x)| < \epsilon$ then $|\alpha_i| < \epsilon$ for all i where the α_i are as given in (1). So $x = \sum_{i=1}^{m} \alpha_i b_i + a \epsilon \sum_{i=1}^{m} N + \overline{A} \subseteq w + \overline{A}$. Thus $(\sigma(X^{\#}) \wedge T) \subseteq \sigma(X')$; hence they are equal.

Nonabelian Group with Cyclic Sylow Subgroups

5481 [1967, 446]. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Let G be a nonabelian group of squarefree order. Suppose that for any pair of subgroups H, K of G, $\{1\} \subsetneq H \subsetneq K$ implies that $N(H) \subseteq K$, where $N(H)$ denotes the normalizer of H in G. Prove that $|G| = p \cdot q$, where p and q are distinct primes.

Solution by K. R. Pierce, University of Wisconsin. Since every Sylow subgroup of G is cyclic, G is solvable (see Theorem 9.4.3 in M. Hall, *The Theory of Groups*). Hence G has a minimal nontrivial normal subgroup M of prime power order, and $|M| = p$ for some prime p. By hypothesis G/M cannot contain a proper subgroup. Hence $|G/M| = 1$ or $|G/M| = q$ for some prime q necessarily different from p. The first case is excluded since G is nonabelian, whence $|G| = pq$.

Also solved by Homer Bechtell, Aiden Bruen, E. R. Gentile (Argentina), M. G. Greening (Australia), Rudolf Kochendorffer (Germany), Brian Parshall, P. K. Subramanian, Hugo Sun, Z. Z. Uoiea, W. C. Waterhouse, Kenneth Yanosko, K. L. Yocom, and the proposer.

Generalizing Fixed Point Relationships?

5482 [1967, 446]. *Proposed by R. J. Weaver, University of Massachusetts*

Let I be a closed interval of real numbers. Let f and g be continuous mappings from $I^n = I\times I\times\cdots\times I$ into itself where f is surjective. Does there always exist a point P in I^n for which $f(P)=g(P)$?

Solution by Mason S. Osborne, student, University of Washington. Without loss of generality I is taken as $[0, 1]$. If $n=1$, the answer is yes as can be seen by studying the difference $f(x)-g(x)$.

If $n>1$, there need be no such P. It suffices to show this for $n=2$; the same situation holds for $n>2$. Let

$$f(x, y) = \left\{\begin{matrix} (2x, y), \quad 0 \leqq x \leqq \frac{1}{6} \quad \text{or} \quad \frac{1}{3} \leqq x \leqq \frac{1}{2} \\ (2x, y[\frac{11}{12} + |x-\frac{1}{4}|]), \quad \frac{1}{6} \leqq x \leqq \frac{1}{3} \\ (2-2x, y), \quad \frac{1}{2} \leqq x \leqq \frac{2}{3} \quad \text{or} \quad \frac{5}{6} \leqq x \leqq 1 \\ (2-2x, y[\frac{11}{12} + |x-\frac{1}{4}|] + \frac{1}{12} - |x-\frac{1}{4}|), \quad \frac{2}{3} \leqq x \leqq \frac{5}{6} \end{matrix}\right\} = (f^1(x, y), f^2(x, y));$$

$$g(x, y) = \left\{\begin{matrix} f(x+\frac{1}{2}, y), & 0 \leqq x \leqq \frac{1}{2} \\ f(x-\frac{1}{2}, y), & \frac{1}{2} \leqq x \leqq 1 \end{matrix}\right\} = (g^1(x, y), g^2(x, y)).$$

f and g are both continuous.

If for some (x, y), $f(x, y)=g(x, y)$, then either $f(x, y)=f(x+\frac{1}{2}, y)$ or $f(x, y)=f(x-\frac{1}{2}, y)$. We need consider only one of these equations. Suppose $f(x, y)=f(x+\frac{1}{2}, y)$; then $2-2(x+\frac{1}{2})=2x$ implies $x=\frac{1}{4}$, and $f(\frac{1}{4}, y)=f(\frac{3}{4}, y)$ implies $\frac{11}{12}y=\frac{11}{12}y+\frac{1}{12}$ which is impossible. Thus f and g never take the same value simultaneously. Moreover, both are surjective and at most 2 to 1.

Also solved by Robert Connelly, L. A. Gavin, J. R. Kuttler, W. R. Scott, and the proposer.

The Derivative of a Function in the Neighborhood of a Zero

5483 [1967, 446]. *Proposed by W. O. Egerland, Edgewood Arsenal, Md.*

Let $f(x)$ be continuous, $g(x)$ differentiable on I: $[a, b]$, $g(a)=0$, and $\lambda\neq 0$ a constant. If

$$|g(x)f(x)+\lambda g'(x)| \leqq |g(x)|$$

on I, then $g(x)\equiv 0$ on I.

I. *Solution by Michael Menn, Boston College.* f need only be bounded. If g is not identically zero then there is a subinterval $(\bar{a}, \bar{b})$ of (a, b) such that g is nonzero on $(\bar{a}, \bar{b})$ and $g(\bar{a})=0$. Then on $(\bar{a}, \bar{b})$ the given inequality implies

$$|f(x)+\lambda g'(x)/g(x)| \leqq 1.$$

Let $h(x)=\log|g(x)|$. Then $|f(x)+\lambda h'(x)|\leqq 1$. But $\lim_{x\to\bar{a}} h(x)=-\infty$. Since

$(\bar{a}, \bar{b})$ is a finite interval, it follows that $h'(x)$ is unbounded, which contradicts the boundedness of f.

II. *Solution by Robert Breusch, Amherst College.* Let $|f(x)| \leqq A$ on $[a, b]$. The given inequality implies that $|g'(x)| \leqq |g(x)| \cdot (1+A)/|\lambda|$, or, with $B = |\lambda|/(1+A)$,

$$|g'(x)| \leqq |g(x)|/B$$

for $x \in [a, b]$. Let $[c, d]$ be a subinterval of $[a, b]$ with $d-c \leqq B/2$, and $g(c)=0$. Then for every $x_0 \in [c, d]$,

$$|g(x_0) - g(c)| = |g(x_0)| = (x_0 - c)|g'(x_1)| \leqq (B/2)|g(x_1)|/B.$$

Thus there exists a decreasing sequence $\{x_0, x_1, \cdots, x_n, \cdots\}$, $x_i \geqq c$, such that

$$|g(x_0)| \leqq (1/2)|g(x_1)| \leqq \cdots \leqq (1/2^n)|g(x_n)| \leqq \cdots.$$

Therefore $g(x_0)=0$. The proof is now completed by partitioning $[a, b]$ into subintervals of lengths less than $B/2$.

Also solved by M. L. Berry, A. R. Brodsky, J. L. Brown, Jr., P. R. Chernoff, Robert Coen, Red Cougar, J. K. Cross, L. D. Crowson, D. Ž. Djoković (Yugoslavia), W. G. Dotson, Jr., R. J. Driscoll, M. A. Ettrick, N. J. Fine, W. B. Fulks, M. F. Friedell, L. A. Gavin, Rudolf Gorenflo (Germany), H. A. Guess, D. A. Hejhal, J. R. Kuttler, Sim Lasher, M. D. Mavinkurve (India), R. K. Meany, A. Meir & D. W. Boyd, E. A. Memmott, M. E. Muldoon, M. S. Osborne, L. J. Pratte, Simeon Reich (Israel), Charles Riley, A. B. Rochman, Steven Russ, F. E. Siwiec, F. W. Steutel (Netherlands), J. J. Swetik, J. H. van Lint (Netherlands), M. F. Walker, D. Weintraub, J. E. Wilkins, Jr., Chung-chun Yang, K. L. Yocom, and the proposer.

Editorial Note. The solutions above show that λ may be taken as a variable $\lambda(x)$ which is bounded away from zero. An immediate consequence of the problem is the familiar fact that $g(a)=0$ and $|g'(x)| \leqq |g(x)|$ implies $g(x) \equiv 0$ (known as Gronwall's lemma). See also Problem 16, p. 101 of Rudin, *Principles of Mathematical Analysis.*

In his solution, Fulks indicates that the boundedness condition on $f(x)$ may be replaced by an integrability condition when $\lambda(x)$ is equivalent to a function of constant sign with positive essential minimum.

Semigroups of Order pq

5484 [1967, 447]. *Proposed by W. A. McWorter, University of British Columbia*

Let p, q be distinct primes and consider the multiplication table of a semigroup S of $n=pq$ elements. If the element a of S occurs in those and only those positions on the off-diagonal (i.e., $s_i \cdot s_j = a$, s_i, $s_j \in S$, iff $j=n-i+1$), prove that S is the cyclic group of order n.

I. *Solution by D. P. Sumner, University of Massachusetts.* Since S is a finite semigroup, S contains an idempotent, s_k. Therefore $s_k s_k = s_k$. We see that

$$s_k a = s_k(s_k s_{n-k+1}) = (s_k s_k) s_{n-k+1} = s_k s_{n-k+1} = a,$$

$$a s_k = (s_k s_{n-k+1}) s_k = s_k(s_{n-k+1} s_k) = s_k a.$$

Hence we have $a s_k = s_k a = a$. Thus, if $s_i \in S$,

$$a = s_k a = s_k(s_i s_{n-i+1}) = (s_k s_i) s_{n-i+1},$$

$$a = a s_k = (s_{n-i+1} s_i) s_k = s_{n-i+1}(s_i s_k).$$

But s_i is the only element of S which when multiplied on either the left or right by s_{n-i+1} yields a. So it follows that $s_i s_k = s_i = s_k s_i$. So s_k is an identity for S. It is also clear from the above that any idempotent of S must be an identity. Thus from the uniqueness of the identity element we deduce that s_k is the only idempotent of S. Let $s_r \in S$ such that $s_r \neq s_k$. Then the finite semigroup consisting of all powers of s_r contains an idempotent. And so there exists a natural number $n > 1$ such that $s_r^n = s_k$. Therefore, $s_r s_r^{n-1} = s_r^{n-1} s_r = s_k$. So s_r has an inverse in S. It follows that S is a group.

Since $n = pq$ where p and q are distinct primes, S must contain an element s_i of order p and an element s_j of order q. Hence $s_i(s_i^{p-1}a) = a$, so $s_i^{p-1}a = s_{n-i+1}$. Similarly $s_j^{q-1}a = s_{n-j+1}$. Noting that for all $s_r \in S$, $as_r = (s_r s_{n-r+1}) s_r = s_r(s_{n-r+1} s_r) = s_r a$, we have $s_{n-i+1}^p = (s_i^{p-1}a)^p = (s_i^p)^{p-1}a^p = a^p$, and similarly $s_{n-j+1}^q = a^q$.

But a cannot be of both order p and order q, so either s_{n-i+1} is not of order p or s_{n-j+1} is not of order q. Suppose then, without loss of generality, that the order of s_{n-i+1} is not p. If the order of s_{n-i+1} were n, then s_{n-i+1} would generate S and we would be finished. So assume that s_{n-i+1} is of order q.

Therefore $s_{n-i+1}^q = s_k = (s_i^{p-1}a)^q = (s_i^q)^{p-1}a^q$, so $a^q = s_i^q \neq s_k$ since s_i is of order p. Thus a is not of order q. But a is not of order p since $a^p = s_{n-i+1}^p$ and s_{n-i+1} is assumed not of order p. Therefore a must be of order n, so S is cyclic.

II. *Solution by M. G. Greening, University of New South Wales.* Denote the element in the jth position in the bordering row and column as r_j so $a = r_j r_{n-j+1} = r_{n-j+1} r_j$ and call r_{n-j+1} the "a-inverse" of r_j. Every element possesses a unique a-inverse from the information given. $r_i r_j = r_i r_k = r_t$ implies $a = r_{n-t+1}(r_i r_j) = (r_{n-t+1} r_i) r_j$; similarly $a = r_{n-t+1}(r_i r_k) = (r_{n-t+1} r_i) r_k$. Hence both r_j and r_k are a-inverses of the same element, so $r_j = r_k$. If r is the a-inverse of a, then $r_j r_{n-j+1} = a = ra = (rr_j) r_{n-j+1}$, whence $rr_j = r_j$ by the a-inverse property; similarly $r_j r = r_j$.

As $r_i r_j = r_i r_k$ implies $r_j = r_k$ and there is a two-sided identity, the finiteness of S renders it a group. The center Z of S contains a, since $ar_j = r_j a$. If $Z \neq S$, S/Z is of prime order and so cyclic; but then S is Abelian. An Abelian group of order pq is necessarily cyclic.

Also solved by M. F. Friedell, M. D. Mavinkurve (India), Bob Prielipp, Simeon Reich (Israel), John Shafer, and the proposer.

Covering the Plane with Integral Squares

5485 [1967, 447]. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur*

Show that the plane may be covered by squares of sides 1, 2, 3, · · · so that no two squares have an interior point in common. The side of each square must be a positive integer i, and for each i there is to be one and only one square.

Higher Trigonometric Identities

5486 [1967, 447]. *Proposed by Ludwig Bruch*

For any positive integer n, prove

$$\sum_{k=0}^{n-1} \csc^2 \left\{\frac{\pi}{2n}(2k+1)\right\} = n^2.$$

I. *Solution by W. O. Egerland, U. S. Army Nuclear Defense Laboratory Edgewood, Md.* Let $z_k = \exp\{i(\pi/n)(2k+1)\}$, $k=0, 1, \cdots, n-1$. Then

$$\csc^2\left\{\frac{\pi}{2n}(2k+1)\right\} = \frac{4}{(1-z_k)(1-\bar{z}_k)}$$
$$= \sum_{j=0}^{n-1} z_k^j \sum_{j=0}^{n-1} \bar{z}_k^j = n + \sum_{j=1}^{n-1}(n-j)(z_k^j + \bar{z}_k^j).$$

Since $\sum_{k=0}^{n-1} z_k^j = 0$, $j=1, 2, \cdots, n-1$, summation over k yields the desired result.

II. *Solution by W. J. Blundon, Memorial University of Newfoundland.* Let $\tan(2k+1)\pi/4n = \sqrt{t_k}$, where $0 < t_k < 1$. Putting $z_k = (1+i\sqrt{t_k})(1-i\sqrt{t_k})$, we have $z_k = \text{cis}\,(2k+1)\pi/2n$, so that $z_k^{2n} = -1$, which reduces to $(i+\sqrt{t_k})^{2n} + (i-\sqrt{t_k})^{2n} = 1$. Thus the t_k are the n roots of the polynomial equation $t^n - n(2n-1)t^{n-1} + \cdots + (-1)^n = 0$. Thus $\sum_{k=0}^{n-1} t_k = n(2n-1)$. Since $\cot(\frac{1}{2}\pi - \theta) = \tan\theta$, we also have $\sum_{k=0}^{n-1} t_k^{-1} = n(2n-1)$. Finally, since $\csc\theta = \frac{1}{2}(\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta)$, the required sun is equal to

$$\frac{1}{4}\sum_{k=0}^{n-1}(t_k + t_k^{-1} + 2) = n^2.$$

III. *Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands.* We use the following formulas:

$$(1) \quad \cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \tfrac{1}{2}\sin 2n\theta \csc\theta,$$

$$\tfrac{1}{2}n + (n-1)\cos\theta + (n-2)\cos 2\theta + \cdots + \cos(n-1)\theta$$
$$(2) \qquad = \tfrac{1}{2}\sin^2 \tfrac{1}{2}n\theta \csc^2 \tfrac{1}{2}\theta.$$

These are easily obtained from the sums $x + x^3 + \cdots + x^{2n-1}$ and $(n-1)x + (n-2)x^2 + \cdots + x^{n-1}$ by substitution of $x = e^{i\theta}$. (cf. Pólya und Szegö, *Aufgaben und Lehrsätze aus der Analysis*, 1954, Bd. 2, pp. 77–78).

We have by (2) and (1):

$$\sum_{k=0}^{n-1}\csc^2\left\{\frac{\pi}{2n}(2k+1)\right\} = \sum_{k=0}^{n-1}\left\{n + 2\sum_{j=1}^{n-1}(n-j)\cos\frac{\pi j}{n}(2k+1)\right.$$
$$= n^2 + 2\sum_{j=1}^{n-1}(n-j)\sum_{k=0}^{n-1}\cos\frac{\pi j}{n}(2k+1) = n^2.$$

Also solved by M. G. Beumer (Netherlands), Robert Breusch, L. J. Burton, Leonard Carlitz, P. R. Chernoff, P. G. Comba, D. Ž. Djoković, Marco Ettrick, J. A. Faucher, H. E. Fettis, Bengt Fornberg (Sweden), M. G. Greening (Australia), Louise S. Grinstein, Eldon Hansen, John Kieffer, M. S. Osborne, Edwin A. Power (England), J. M. Quoniam (France), Stanley Rabinowitz, Simeon Reich (Israel), Henry Ricardo, Norman Schaumberger, N. T. Sheth, Franklin C. Smith, M. R. Spiegel, F. W. Steutel (Netherlands), W. F. Trench, C. S. Venkataraman (India), L. E. Ward, Sr., Louis Weisner, K. S. Williams, Qazi Zameeruddin (India), and the proposer.

Editorial Note. A variety of other solutions apply formulas which may be found in the mathematical literature. Hansen, Ricardo and Ward set $x=\pi/2n$ in the identity $\sum_{k=0}^{n-1}\csc^2(x+k\pi/n) = n^2\csc^2 nx$ which appears in Bromwich, *Infinite Series*, pp. 211–216. Weisner and Venkataraman use the similar identity $\sum_{k=0}^{n-1}\cot(\theta+k\pi/n)=n\cot n\theta$, from Siddons and Hughes, *Trigonometry*. Part IV, Cambridge (1953), p. 358, and in Weisner, *Introduction to the Theory of Equations*, p. 175.

Additional references include formulas 433, 441, 442, 720, 722, 1076 in Jolley, *Summation of Series*, and Problem 241 in M. G. Beumer, *Elemente der Mathematik*, II (1956), p. 90. Fettis offers the following formula which leads to the proposed identity:

$$\sum_{k=0}^{n-1}\frac{\sin\beta}{\cos\beta-\cos(2\theta+2k\pi/n)}=\frac{n\sin n\beta}{\cos n\beta-\cos 2n\theta}.$$

By rewriting the identity of the problem, Chernoff shows how to obtain the familiar equation $\pi^2/8=\sum_{j=0}^{\infty}(2j+1)^{-2}$. Williams offers another similar formula which appeared as a question in the Higher Certificate Mathematics, Oxford/Cambridge Schools Examination Board, Mathematics Group III (Paper 5)(1945), question 9:

$$\sum_{k=1}^{n}\cot^2\left(\frac{k\pi}{2n+1}\right)=\frac{n(2n-1)}{3}.$$

Finally Smith, with his solution, offers a time capsule for reviving the question at some future date by camouflaging the trigonometric functions in the present identity and posing for proof the identity

$$\sum_{k=0}^{n-1}\left[\Gamma\left(\frac{2k+1}{n}\right)\Gamma\left(\frac{2n-2k-1}{2n}\right)\right]^2=\pi^2n^2.$$

A Function Generator

5487 [1967, 447]. *Proposed by Roy O. Davies, The University, Leicester, England*

Suppose that $f_0(x)$ is positive and Lebesgue integrable over $(0, 1)$, that positive functions $f_1, f_2, \cdots$ are defined successively by the relations $f_{n+1}(x) = \{\int_0^x f_n(t)dt\}^{1/2}(n=0, 1, \cdots)$, and that $f_0(x)\geqq f_1(x)$ for $0<x<1$. Find $\lim_{n\to\infty}f_n(x)$. (Based on an unpublished lemma of the late E. R. Reisenberg.)

I. *Solution by H. A. Guess, Alexandria, Va.* The following is a proof that the desired limit is $x/2$. By induction, $0\leqq f_{n+1}(x)\leqq f_n(x)$ and hence the limit function $f(x)=\lim_{n\to\infty}f_n(x)$ exists for each x in $(0, 1)$. Since $0\leqq f_n(x)\leqq f_0(x)$ for all x it follows from the Lebesgue dominated convergence theorem that

$$(1)\qquad \lim_{n\to\infty}\int_0^x f_n(t)dt=\int_0^x f(t)dt$$

for all x such that $0\leqq x\leqq 1$. Hence

$$(2) \qquad f(x) = \left[\int_0^x f(t)dt \right]^{1/2}$$

for all x in (0, 1). Since the indefinite Lebesgue integral of a summable function is absolutely continuous, f_n is continuous for all $n \geqq 1$. Furthermore for $n \geqq 1$ and $k \geqq 1$,

$$(3) \qquad 0 \leqq [f_n(x)]^2 - [f_{n+k}(x)]^2 = \int_0^x [f_{n-1}(t) - f_{n+k-1}(t)]dt$$

$$\leqq \int_0^1 [f_{n-1}(t) - f_{n+k-1}(t)]dt.$$

Since (1) implies that $\{\int_0^1 f_n(t)dt\}_{n=1}^\infty$ is a Cauchy sequence of real numbers it follows from (3) that $\{f_n^2(x)\}_{n=1}^\infty$ is a uniform Cauchy sequence of functions on (0, 1) and hence converges uniformly on (0, 1). Since the functions f_n^2 are all continuous (for $n \geqq 1$), their uniform limit is continuous on (0, 1). Thus $[f(x)]^2$ and hence $f(x)$ are continuous on (0, 1). It will now be shown that $0 < f(x)$ for all x such that $0 < x < 1$.

Let $x_0 > 0$, then for $x_0 \leqq x \leqq 1$:

$$f_2(x) = \left[\int_0^x f_1(t)dt \right]^{1/2} \geqq \left[\int_{x_0}^x f_1(t)dt \right]^{1/2} \geqq [f_1(x_0)]^{1/2}(x - x_0)^{1/2}$$

$$\geqq [f_1(x_0)]^{1/2}(x - x_0).$$

Define $g_n(x)$ as follows:

$$g_2(x) = \begin{cases} 0 & \text{for } 0 \leqq x < x_0 \\ [f_1(x_0)]^{1/2}(x - x_0) & \text{for } x_0 \leqq x < 1, \end{cases}$$

$$g_{n+1}(x) = \left[\int_0^x g_n(t)dt \right]^{1/2} \qquad \text{for } n \geqq 2.$$

Then $f_2(x) \geqq g_2(x)$ for all x in (0, 1) and by induction $f_n(x) \geqq g_n(x)$ for all n and all x in (0, 1). Therefore $f(x) = \lim_{n\to\infty} f_n(x) \geqq \lim_{n\to\infty} g_n(x)$, but by direct computation

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \frac{1}{2}(x - x_0), & x_0 \leqq x < 1, \\ 0, & 0 \leqq x < x_0. \end{cases}$$

Hence $f(x) \geqq g(x) = \frac{1}{2}(x - x_0) > 0$ for $x > x_0$.

Since x_0 was an arbitrary point in (0, 1) it follows that $f(x) > 0$ on (0, 1). Since $f(x) > 0$ on (0, 1) and since f is continuous on (0, 1) it follows from (2) that f is differentiable, that $f'(x) = \frac{1}{2}$ and in fact that $f(x) = x/2$ on (0, 1).

II. *Solution by M. E. Muldoon, York University, Toronto.* The condition $f_0(x) \geqq f_1(x)$ may be omitted. Suppose $0 < \delta < 1$. The positivity and integrability of f_0 show that there are positive numbers m and M such that $f_1(x) \leqq M (0 \leqq x \leqq 1)$

and $f_1(x) \geqq m(\delta \leqq x \leqq 1)$. We can then show by induction that

$$m^{2^{-n}} a_n(x-\delta)^{1-2^{-n}} \leqq f_{n+1}(x) \leqq M^{2^{-n}} a_n x^{1-2^{-n}}$$

for $n=1, 2, \cdots$, the lower bound for $f_{n+1}(x)$ holding for $\delta \leqq x \leqq 1$ and the upper bound for $0 \leqq x \leqq 1$, where $a_1 = 1$, and

$$a_n = (2/3)^{2^{-n+1}}(4/7)^{2^{-n+2}} \cdots (2^{n-1}/[2^n - 1])^{1/2}, \qquad n = 2, 3, \cdots.$$

An analysis of a_n shows that $\lim_{n\to\infty} a_n = \frac{1}{2}$ whence it follows, since δ was chosen arbitrarily, that $\lim_{n\to\infty} f_n(x) = x/2$.

III. *Solution by A. C. Zaanen, University of Leiden, Netherlands.* We omit the condition $f_1(x) \leqq f_0(x)$ and suppose (a) $f_0(x)$ is nonnegative and Lebesgue summable on $[0, 1]$, (b) $f_0(x)$ is positive almost everywhere on some subinterval $[0, a]$ of $[0, 1]$.

(A) The case in which $f_1(x) \leqq f_0(x)$ has been treated in the above solutions and we can apply the result if $\psi_0(x) \equiv C \geqq 1$. If $\psi_{n+1}(x) = (\int_0^x \psi_n(t)dt)^{1/2}$, it follows that $\psi_1(x) \leqq \psi_0(x)$ and so $\lim \psi_n(x) = x/2$.

(B) Let $0 < x_0 < 1$, $0 < \alpha < \frac{1}{2}$, and define the sequence $\phi_n(x)$:

$$\phi_0(x) = \begin{cases} 0 & \text{for } 0 \leqq x \leqq x_0, \\ \alpha(x - x_0) & \text{for } x_0 \leqq x \leqq 1, \end{cases}$$

$$\phi_{n+1}(x) = \left(\int_0^x \phi_n(t)dt\right)^{1/2}, \qquad n = 0, 1, 2, \cdots.$$

Explicit computation shows that $\phi_n(x)$ converges (monotonely increasing) to

$$\phi(x) = \begin{cases} 0 & \text{for } 0 \leqq x \leqq x_0, \\ \frac{1}{2}(x - x_0) & \text{for } x_0 \leqq x \leqq 1. \end{cases}$$

(C) Returning to the general problem, we delete $f_0(x)$ if necessary, and reindex; we may then assume that $f_0(0) = 0$ and that f_0 is continuous and strictly increasing on $[0, a]$. Choose the constant $C \geqq 1$ such that $f_0(x) \leqq C$ on $[0, 1]$, and set $\psi_0(x) = C$. Hence $f_0(x) \leqq \psi_0(x)$ on $[0, 1]$.

Choose $0 < x_0 < 1$; let $f_0(x_0) = \epsilon$, so $\epsilon > 0$, and define $\phi_0(x)$ by

$$\phi_0(x) = \begin{cases} 0 & \text{for } 0 \leqq x \leqq x_0, \\ \alpha(x - x_0) & \text{for } x_0 \leqq x \leqq 1, \end{cases}$$

where $0 < \alpha < \frac{1}{2}$, and α is so small that $\phi_0(1) \leqq \epsilon$. Hence $\phi_0(x) \leqq f_0(x)$ on $[0, 1]$.

Applying the iteration procedure to ϕ_0, f_0 and ψ_0, we obtain $\phi_n \leqq f_n \leqq \psi_n$ for all n. We have $\lim \psi_n(x) = \frac{1}{2}x$ by (A) and $\lim \phi_n(x) = \phi(x)$ by (B). Letting $x_0 \downarrow 0$, it follows that $\lim f_n(x) = \frac{1}{2}x$.

REMARK. An analogous procedure may be used to prove problem 138 in Nieuw Archief voor Wiskunde, 14(1966), p. 271: *Let A and B be bounded and self-adjoint linear operators in the Hilbert space H such that* $0 \leqq (Ax, x) \leqq (x, x)$ *and*

$0 \leqq (Bx, x) \leqq (x, x)$ *for every* $x \in H$. *Let* A *and* B *commute. The operators* B_n $(n=0,1,2,\cdots)$ *are defined by* $B_0=B$, $B_{n+1}=B_n+\frac{1}{2}(A-B_n^2)$ *for* $n=0,1,2,\cdots$. *Then the sequence* $\{B_n\}$ *converges strongly (but not necessarily monotonely) to the positive square root* $A^{1/2}$ *of* A.

Also solved by D. W. Boyd & A. Meir, Robert Breusch, W. G. Dotson, Jr., Richard Gisselquist, G. A. Heuer, J. R. Kuttler & V. G. Sigillito, M. D. Mavinkurve, (India), R. K. Meany, Steven Russ, Rajinder Singh, R. C. Steinlage, J. H. van Lint (Netherlands), J. E. Wilkins, Jr., R. S. C. Wong, and the proposer.

Characteristic Values of a Continuant Matrix

5488 [1967, 447]. *Proposed by J. Z. Hearon, National Institute of Health, Bethesda, Md.*

For any $n \times n$ matrix A, write $A=H+D$, where $D=\text{diag}\ (a_{11}, a_{22}, \cdots, a_{nn})$ and define $B=H-D$. Show that, if A is a continuant and λ is any root of A then $-\lambda$ is a root of B. Thus prove that an nth order continuant with zero diagonal and distinct roots is singular if n is odd.

I. *Solution by J. R. Kuttler, The Johns Hopkins University Applied Physics Laboratory.* If λ is an eigenvalue with associated eigenvector $(x_1, x_2, \cdots, x_n)$ of the $n \times n$ continuant (tridiagonal) matrix $A=H+D$, we have

$$(a_{1,1}-\lambda)x_1 + a_{1,2}x_2 = 0,$$
$$a_{i,i-1} + (a_{i,i}-\lambda)x_i + a_{i,i+1}x_i = 0, \qquad i = 2, \cdots, n-1,$$
$$a_{n,n-1}x_{n-1} + (a_{n,n}-\lambda)x_n = 0,$$

and it is clear that $(-x_1, x_2, \cdots, (-1)^i x_i, \cdots, (-1)^n x_n)$ is an eigenvector of $B=H-D$ associated with the eigenvalue $-\lambda$. Thus, if A has zero diagonal and distinct roots, these roots must be symmetric about the origin and so, when n is odd, exactly one is zero and A is singular. When n is even, however, the number of nonzero roots being even implies the number of zero roots is even, hence zero since A has distinct roots, and in this case A is nonsingular.

II. *Solution by R. C. Thompson, University of California, Santa Barbara.* Let A_n be the given $n \times n$ continuant, that is, a tridiagonal matrix, and let the main diagonal be $(a_1, a_2, \cdots, a_n)$, the diagonal just above the main diagonal be $(b_1, b_2, \cdots, b_{n-1})$ and the diagonal just below be $(c_1, c_2, \cdots, c_{n-1})$. Set $d_n = \det A_n = d_n(a_1, \cdots, a_n)$, then expanding by minors we get $d_n = a_n d_{n-1} - b_{n-1}c_{n-1}d_{n-2}$. From this it follows by induction on n that: if n is even then $d_n(a_1, \cdots, a_n) = d_n(-a_1, \cdots, -a_n)$, and if n is odd, $d_n(a_1, \cdots, a_n) = -d_n(-a_1, \cdots, -a_n)$.

Now let A and B be as stated in the problem. Then $\lambda I - A$ and $-\lambda I - B$ are related as above, hence $\det(\lambda I - A) = \pm \det(-\lambda I - B)$. Hence λ is an eigenvalue of A with multiplicity m if and only if $-\lambda$ is an eigenvalue of B with multiplicity m. This proves the first assertion of the problem. In particular, if A has zero main diagonal it follows that the eigenvalues of A come in negative pairs. Thus

if n is odd and A has zero main diagonal then A must have zero as an eigenvalue, hence is singular.

Also solved by E. L. Allgower, D. Ž. Djoković (Yugoslavia), M. G. Greening (Australia), A. S. Householder, M. D. Mavinkurve (India), Jernej Polajnar (Yugoslavia), P. V. Subba Rao & B. Ramachandra Rao (India), Sidney Spital, J. F. Standish, and the proposer.

Spital observes that the formula $|B+\lambda I| = (-1)^n |A-\lambda I|$ is found in Muir, *Treatise on the Theory of Determinants* (Dover reprint), p. 522. Allgower observes that the distinctness of the roots is not required in the case n is odd. He also notes that if the continuant has the constant a in the main diagonal, then the characteristic values are symmetric about a.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

Topological Vector Spaces and Distributions, Vol. I. By John Horvath. Addison-Wesley, Reading, Mass., 1966. xii+449 pp. $13.50.

The goal the author has set himself for this book is to present the tools of L. Schwarz's theory of distributions in a book which assumes that the student begins with the knowledge only of some linear algebra and of elementary real function theory and carries the student to the very desirable properties possessed by the Fourier transform between spaces of distributions.

This book is written, of course, in the currently fashionable definition-theorem-proof-remark style. The chapters and sections have no introductions, but the many remarks do help the reader to understand the author's purpose. Another helpful feature is the use of the term "proposition" for most of the results proved throughout the book, reserving the name "theorem" for a few important results.

This book contains enough material for about a year of class work, if the student is expected to follow the details. There is a strong temptation to regard the book as an introduction, gathering together the tools of the subject but not yet using them on topics of external interest. From this point of view the book appears to be quite effective, but its true value can not be calculated until the second volume, which is promised to cover the part of the theory dependent on Lebesgue measure and integration and some applications, appears. At present, I would call it a good book, clearly written, following sensible unifying principles for selection and rejection of topics.

M. M. DAY, University of Illinois

Abstract Algebra. By Andrew O. Lindstrum, Jr. Holden-Day, San Francisco, 1967. xii+211 pp. $9.00.

The manner in which the author presents his material is outstanding. It shows much care and planning. All exercises appear in the main body of the text, and each exercise follows the appropriate theorem or definition it is designed to illustrate. Further, the number of such exercises is sufficient to be truly illuminating.

Wherever possible, the author introduces new concepts and theorems in terms of topics previously covered. This tends to tie together many topics which other authors present quite separately. However, the instructor who uses this book may find himself

forced to present some topics in greater detail than he might otherwise care to. For example, the concept of homomorphism for groups is not defined as such but is illustrated following the definition of homomorphism for semigroups, which appears at the middle of the twenty-page chapter on semigroups. There is little of this chapter (or of any chapter) that an instructor can omit if he intends to continue with the text.

The author says his book is intended for first-year graduate students and very mature undergraduates, and he has hit his mark very well. In the statements of all definitions, theorems, and proofs the author uses logical symbols. Thus the book is unsuitable for anybody who cannot learn the language quickly. For those who can master the symbols, the text is an excellent introductory survey of abstract algebra.

G. E. DIMITROFF, Knox College

Algebraic Topology: An Introduction. By William S. Massey. Harcourt, Brace and World, New York, 1967. xix+261 pp. $9.25. (Telegraphic Review, November 1967.)

During recent years, a rather dubious market has been created in the textbook industry by a large number of teachers who, dedicated to the principle of equality, are convinced that all students are capable of understanding rigorous mathematics courses providing they are taught correctly. Thus, most self respecting colleges can feel they are keeping up with the times by offering courses in algebraic topology ("undergraduate" algebraic topology, to be sure), differential geometry, measure theory, and so on, in spite of the fact that they may seldom graduate a student who has properly mastered the notion of a least upper bound. This trend has made it possible for a great number of poor textbooks to appear, and it is true more often than not that when a publisher approaches just anyone with a Ph.D. in mathematics to write a textbook, a disservice is being done the field. For this reason, when an outstanding mathematician takes the time to write an introductory book in his specialty, it is an event for which we should all be grateful. This is the case with Mr. Massey's book.

In the introduction, he offers the opinion that homology theory is not the proper place to jump in when teaching a student algebraic topology for the first time, and as one who has tried once (and only once) to teach homology theory to undergraduates, I can heartily agree. However, let there be no misunderstanding: this is not a book for young gentlemen liberal arts students, and anyone who is not capable of becoming a full-fledged mathematician is wasting his time here. There is nothing in the book which is not good mathematics from both professional and pedagogical points of view, and the student is never going to have to relearn any of the concepts he encounters here from a better standpoint. Indeed, the treatment is modern enough to put the author very close in a number of situations to defining a category. It is perhaps fortunate that he does not.

The first chapter contains a very clear exposition of the classification theorem for compact (and compact bordered) surfaces, unique in the fact that it enables one to be aware of precisely the points where rigour is lacking. This starting point paves the way for numerous examples in the rigorous chapters which are to follow. An elegant appendix on quotient and product spaces enables the reader to fill in many of the gaps on his own. This would have been perhaps even more possible with the addition of a couple of other cases in which the subspace topology agrees with the quotient space topology. Chapter II treats the fundamental group, with the Brouwer fixed point theorem in 2 dimensions as an application. Again, a perfectly clean exposition is made possible by the above mentioned appendix. Chapter III is purely algebraic, dealing with free abelian groups, direct sums, free products, and free groups, all from the viewpoint of universal mapping properties. The existence of free products is proved using a very neat argument of van der Waerden. Chapter IV is devoted to the theorem of Seifert and van Kampen on the fundamental group of an (arbitrary) union of open, arcwise connected subspaces. This is used to determine fundamental groups for compact surfaces, as well as to provide a tiny intro-

duction to knot theory with a discussion of the torus knots. Chapter V provides a thorough treatment of covering spaces, with many motivating examples, and an application to the Borsuk-Ulam theorem for the 2-sphere. A short chapter on graph theory follows, in which it is proved that the fundamental group of a connected graph is a free group. It is also proved, using the fact that a covering space for a graph is again a graph, that a subgroup of a free group is free. This is just one of a number of theorems of a purely algebraic nature which are proved by topological methods, and each serves to underline the most important feature of the work, namely the interaction and mutual reinforcement of the domains of algebra and topology. Chapter VI introduces the reader to *CW*-complexes, and applies them to the Kurosh subgroup theorem and (following John Stallings, but staying in the finite rank case) Grushko's theorem on free products. A short epilogue prepares the reader for homology theory and more advanced aspects of the subject.

At the end of each chapter there is a list of references, as well as very interesting historical notes, sometimes accompanied by sketches of proofs (for example, the proof that any finitely presented group can be the fundamental group of a compact manifold) which give the reader some idea of the recent work that has been done in the subject. The exercises are all workable, but there are none of the routine nature which delude the weak student into thinking he is understanding something.

The small number of misprints indicate that the book must have been fastidiously proofread, and as far as errors are concerned, I found none that are worth mentioning here, save possibly the following. On page 93, line 8, the sentence beginning "Because $\eta_i: F_i \to F$ is a monomorphism . . ." should be replaced by "Define $f_i = f\eta_i$." I might also add that I was not able to justify equation 4.2–5 in the proof of the Seifert, van Kampen theorem without making an additional assumption on the meshing of the unit square, but this may be my fault.

In any case this book is most highly recommended (I want to say "required") for anyone who is to take a regular graduate course in algebraic topology. The level of perfection it attains can only leave one hoping to see some day its sequel in homology theory.

BARRY MITCHELL, Bowdoin College

Computation: Finite and Infinite Machines. By Marvin L. Minsky. Prentice-Hall, Englewood Cliffs, N.J. 1967. xvii+317 pp. $12.50.

The theory of effective computability and the theory of finite state machines are two relatively young branches of mathematics that grew out of foundational studies and modern technology respectively. It is now known that each has relevance to the other. Indeed, viewed from certain points of view, they tend to merge. The importance of the two (?) theories makes it imperative that they take their place in undergraduate mathematics, science and engineering curricula. Suitable texts are rare.

The author has written a text primarily for "students specializing in computer-oriented studies," but he feels (and I agree) that the material should become part of "every scientific or engineering curriculum." There are no specialized prerequisites, and it can be used in any undergraduate year. I would not recommend it as a text for a mathematics course *per se* (as distinct from the aforementioned purpose), but I do recommend it as supplementary reading for such courses and as bed-time reading for experts.

The reasons for this are as follows. The book is a rather even mixture of expository material and mathematical-technical details, and gives the appearance of having been written at two levels: a rather sophisticated one in the expository parts; less so in the technical details. (The author presumably does not agree; he seems to have anticipated this criticism and gives his answer in the preface.) Consequently, the text is liberally

sprinkled with bold, provocative and intriguing philosophical comments which give it a rare charm. On the other hand, in some places the mathematical details are rather sketchy and tend to be misleading. The treatment of partial recursive functions is an example.

The text is divided into three parts. First, finite automata are treated, but rather briefly. All the algebraic theory and much of the logical theory is missing. However, quite adequate treatments are given of neural nets and regular sets. Next, computability is studied from several (equivalent) points of view: Turing machines, μ-recursiveness, McCarthy's formalism, and the author's program machines. Decision problems are discussed only briefly. Finally, the emphasis shifts to symbol-manipulation—principally Post's canonical systems. A few specialized variants of computability are given, and the correspondence problem is treated. There are a number of excellent exercises ranging from easy to very difficult

Since the foregoing contains a few mildly negative comments, let me conclude with the remarks that I consider this book to be a superior addition to the undergraduate texts of its genre, and that I personally enjoyed reading it.

D. A. CLARKE, University of Toronto

Analytic Functions of Several Complex Variables. By Robert C. Gunning and Hugo Rossi. Prentice-Hall, Englewood Cliffs, N. J., 1965. 317 pp. $13.50.

The study of analytic functions of several variables divides itself into two parts, local questions about the structure of the ring of germs of analytic functions at a point (i.e., the ring of convergent power series in several variables) and global questions about the structure of the ring of analytic functions on a domain—or, more generally, on an analytic space. These two aspects are interrelated by the cohomology theory of sheaves in a manner which is dramatically illustrated by the beautiful results of Oka and Cartan.

In the past in order to study this subject one started by reviewing those aspects of the theory of analytic functions of one variable that generalize easily to several variables (e.g., by selective reading in Bochner and Martin, *Several Complex Variables* or Kneser, *Funktionentheorie*). One then studied the ring of convergent power series in several variables from some convenient source (nowadays, Zariski and Samuel, *Commutative Algebra*, Vol. II) and found out that it is a local, Noetherian, unique factorization domain because of the Weierstrass preparation theorem. The next question that presented itself was the relation between ideals in this ring and the (analytic) varieties that they define. Alas, the Nullstellensatz does not follow easily from the polynomial case and here some genuine work was necessary. If one tried to avoid the problem by turning to the theory of sheaves (e.g., from Godement, *Théorie des Faisceaux*) one quickly discovered that the relevant notion was that of coherent sheaves. A little searching led to the beautiful paper by Serre, *Faisceaux algébriques cohérents*, Ann. Math., 61 (1955), pp. 197–278 which took care of the theory and left one face-to-face with the problem of proving that the sheaf of germs of holomorphic functions and the sheaf of ideals determined by an analytic variety are coherent. The only source was the 1952 Cartan Séminaire notes which were of little help unless one happened to discover that there is an understandable proof for the case of one variable in the now out of print Tata notes on *Sheaf Theory* by Dowker. Next one collided with Theorems A and B—the basic technical tools to prove the beautiful results that attracted one to the subject in the first place. At this point nothing was left but faith to carry one through—plus a detour through spectral sequences.

Happily, the horrors of the good old days represented by this do-it-yourself kit are no longer with us. In 1963, Tata published Hervé: *Several Complex Variables* which gives an adequate account of the local theory. Soon after, the 1960–61 Cartan Séminaire notes became available, which do much of the theory again with complete (in the French sense)

details. In 1964, Abhyankar *Local Analytic Geometry* appeared, which treats the local theory for an arbitrary algebraically closed complete valued field rather than just for the complex numbers. Finally with Gunning and Rossi's book we enter the golden age of the "American style" textbook. Take one good undergraduate with a solid background in Ahlfors, Kelley, and Birkhoff-MacLane, start him in at the beginning and watch a potential Grauert come out the other end. Moreover, those of us who never thought it feasible before can now scramble to be first in line in our universities to teach the soon-to-be-instituted graduate courses in several complex variables.

The book in fact will be useful in several ways. The first third of chapter I presents that part of the subject which extends trivially from the case of one variable. (See particularly what happens to the study of removable singularities.) It is of the same difficulty as Ahlfors and easily could (and should) be included in courses taught at this level. The entire first chapter taken by itself provides a non-technical introduction to the whole subject and the reader (or course) that only gets this far is already in a position to appreciate a good deal of currently interesting mathematics.

Chapter II and III nicely dispose of the Weierstrass preparation theorem and the analytic Nullstellensatz. Chapters IV, V, and VI are the heart of the book and are devoted respectively to sheaf theory, analytic spaces, and the cohomology theory of sheaves. Each chapter begins with a general treatment of the theory described by its title and then treats the applications to complex analysis in considerable detail. This second aspect of each chapter is handled very well. Thus, for example the required coherence proofs mentioned above, the important "Proper Mapping Theorem" and the local form of Theorems A and B, are all very understandably presented.

The theoretical parts of these chapters, however, apparently posed an unsolvable problem for the authors. In trying to be brief they have seriously distorted the relevant theories. Actually, perhaps the central problem in writing a graduate level textbook these days is to give an adequate and accurate account of the general theories whose intersection is the special topic with which one is concerned. The solution adopted here is not a model for emulation. Thus, for example, in chapter IV, sheaves are treated so crudely that the reader never discovers the main point of sheaf theory for this type of application; namely, that exactness means exactness at each point. Furthermore, the complicated notion of coherence is totally obscured by an opaque definition. The student is better off going back to Serre or Dowker here. In chapter V, direct and inverse images of sheaves by maps between ringed spaces are not defined. Since the authors (like everyone else) have to use them, the reader will have to fill this gap, providing he can recognize that this is what is missing. In chapter VI there are no occurences of the words "category," "functor," or "natural transformation" (perhaps a *tour de force!*). In any case, in this age of categories in kindergarten, there cannot be any graduate students left who will have the slightest difficulty in inserting them where they belong. Also, a technical point, the "visualization" of Leray's Theorem here is dubious at best, if not actually mischievious.

Once over these difficulties, the last third of the book is a rousing analytical finale. Chapter VII treats techniques leading up to the holomorphic imbedding theorem, chapter VIII proves Theorems A and B for Stein spaces and gives some of the fascinating consequences of these two theorems, and chapter IX begins the study of pseudoconvexity —essentially, the concept which provides the link between sheaf-oriented methods and classical complex analysis.

With this book let us hope that we can begin to see the end of the time when practically every sophomore know something about several real variables and practically every mathematician knows nothing about several complex variables.

J. W. GRAY, University of Illinois

Notes on Spectral Theory. By Sterling K. Berberian. Van Nostrand Mathematical Studies No. 5. Van Nostrand, Princeton, N. J., 1966. 121 pp. $2.50.

These notes contain a lucid exposition of the spectral theory of normal operators on complex Hilbert space. It is quite readable, offering complete proofs and presupposes only a minimal knowledge of Hilbert space. (Either a knowledge of the author's book *Introduction to Hilbert Space* or of one of the recent texts on functional analysis should suffice.)

The author approaches the spectral theorem using "elementary" measure-theoretic techniques so that positive operator measures are central. A certain amount of extra generality is indulged in but this does not seem to intrude.

We conclude with some very subjective criticism. Despite the splendid organization of these notes and the attractiveness of the style for the nonspecialist (in the theory of Hilbert space), the reviewer feels the author has erred either in choice of content or in choice of style. The spectral theorem for normal operators as an end is not of sufficient importance to justify all means of attaining it. In particular, the techniques acquired in the Banach algebras approach are perhaps of more importance than the spectral theorem itself. Thus the reviewer feels that this is a better approach for the student and nonspecialist.

On the other hand, while the measure-theoretic techniques are of interest to the specialist and do lead to current research the author fails to indicate this. Moreover, his diffuse style makes the reading of these notes somewhat tedious for the specialist.

R. G. Douglas, University of Michigan

Introduction to Numerical Analysis. By Carl-Erik Fröberg. Addison-Wesley, Reading, Mass. 1965. 340 pp. $9.50.

This is a translation of the Swedish edition published in 1962. The book is intended to be an introduction to the subject for readers of limited mathematical background. According to this purpose, chapters on Matrix Theory, Systems of Linear Equations, Linear Operators, etc., supply some of the needed theoretical material.

Although, as stated in the Preface, "quite a few older methods, which are never used nowadays, have been excluded," still the book touches such a large variety of problems and methods that almost every one of them receives a brief and superficial treatment. For example, less than nine pages are devoted to Simulation and Monte Carlo methods. Quadratic, Convex and Dynamic Programming share together little more than one page of text.

The quality of the exposition, which is generally acceptable, suffers somewhat from the natural consequences of this compression of topics. This is particularly obvious in the case of the chapters in "pure" mathematics.

The translation, although grammatically correct, often departs from the usual terminology and the reader is kept conscious all the time of the fact that he is reading a translation. Odd sounding words and expressions appear more frequently than they should.

A number of exercises follow each chapter; they are well selected and the solutions are collected at the end of the book. Appropriate bibliographic references are always given.

There are a number of minor errors and some misprints. For example, Chapters 9 and 12 are listed in the index with no identification of page.

A. G. Azpeitia, University of Massachusetts

Linear Operators in Hilbert Space. By Werner Schmeidler. Academic Press, New York, 1965. ix+122 pp. $6.00 (cloth), $2.95 (paper).

This book treats the study of linear operators in Hilbert Space with attention paid to applications. The treatment, although brief, is essentially complete. Part I considers the Hilbert space of sequences, abstract Hilbert space, and orthonormal systems. Part II, on linear operators, deals with bounded, completely continuous, and self-adjoint operators. Exercises give applications to finite-dimensional spaces and probability.

Part III treats the spectral theory of Hermitian, normal, and unbounded self-adjoint operators. There are applications to harmonic functions, crystal lattices, classical mechanics, and atomic theory.

The book is intended for students in science as well as in mathematics and gives a systematic development of the theory.

STEPHEN HOFFMAN, SUNY Cortland

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year). A boldface star (★) marks a notable book that might be overlooked.

Miscellaneous

Rules for Type-setting Mathematics. By Karel Wick. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1965. Mouton & Co., The Hague, Netherlands, 1965. 99 pp. 12 D.G. A carefully written informative manual of interest to all editors and authors. P, L.

Analysis

Calculus of Variations and Partial Differential Equations of the First Order. Vol. II. By C. Caratheodory. Holden-Day, San Francisco, 1967. xvi+224 pp. $9.75. The first volume was on partial differential equations (see telegraphic review this MONTHLY, April 1967). This second volume, translated from the second edition of 1966, edited by E. Holder, can be read independently of the first volume. It has a good bibliography with some historical comments on particular topics. T (16–17), S, P, L.

Fourier Series, A Modern Introduction. Vol. I. By R. E. Edwards (Australian National Univ.). Holt, Rinehart and Winston, New York, 1967. x+208 pp. $6.95. The subject is treated from a modern point of view with stress on global rather than pointwise aspects and on topics related to current interest and research in harmonic analysis on general groups, representation and conversion, and problems of an algebraic-topological nature in various function spaces. There are appendices on metric spaces and Baire's theorem, topological linear spaces, dual spaces and weak sequential completeness, and Runge's theorem. There is a good bibliography and a table of symbols. No clue is given to the contents of Volume II except the assertion that it is more difficult and less self-contained. Prerequisite is familiarity with such topics as Lebesgue integration, normed linear spaces, and point set topology. There is substantial historical and motivational material, references, and bibliography. The book bridges the gap between classical and abstract harmonic analysis. T (17–18), S, P, L.

Real Functions. By Casper Goffman (Purdue Univ.). Prindle, Weber and Schmidt, Boston, Mass., 1967. x+261 pp. $3.95 (paper). A paperback reprint with corrections of

the text originally published by Holt in 1953. Used for many years in beginning real variable courses, this book should be given serious consideration because of its very reasonable price. T (15–16), S.

A Hilbert Space Problem Book. By Paul R. Halmos (Univ. of Michigan). Van Nostrand, Princeton, N. J., 1967. xvii+384 pp. \$11.50. This is not a mere list of problems with solutions, but a collection of nontrivial problems with introductory definitions and explanations, hints and solutions. It is designed to teach by involving students who already know techniques and results of general topology, measure theory, real and complex analysis, as well as some Hilbert space theory. T (17–18), S.

Transcendental Numbers. By Joseph Lipman. Queen's Papers in Pure and Applied Mathematics, No. 7. Queen's University, Kingston, Ont., 1966. vii+83 pp. \$2.00. An introduction for advanced undergraduates with some training in modern algebra and analysis. Chapter headings are Approximation Methods, Schneider's Theorem, and Algebraic Independence. S, P.

Spherical Harmonics. By T. M. MacRobert. 3rd ed. revised with the assistance of I. N. Sneddon. International Series of Monographs in Pure and Applied Mathematics, Vol. 98. Pergamon, New York, 1967. xviii+349 pp. \$15.50. The foreword is a too brief biographical note on MacRobert, who died in 1962, followed by a list of publications. This edition incorporates changes inserted by Sneddon on the basis of discussion with the author. P, L.

Lectures on Calculus. Edited by Kenneth O. May (Univ. of Toronto). Holden-Day, San Francisco, 1967. vii+180 pp. \$6.50 (paper). This volume attempts to bring to the student of calculus at various levels material of the type usually provided by visiting lecturers of the Mathematical Association of America. Each paper is accompanied by biographical information and some indication of the content and level. The authors are A. H. Copeland, Sr., J. D. Mancill, D. E. Richmond, Hans Sagan, H. W. Guggenheimer, Albert Wilansky, M. E. Munroe, Oswald Wyler, and the late M. K. Fort, Jr. The price is too high for student purchase. S, L.

Approximation of Functions: Theory and Numerical Methods. By Günter Meinardus (Technische Hochschule Clausthal). Translated by Larry L. Schumaker. Springer-Verlag, New York, 1967. viii+198 pp. \$13.50. An expanded translation of the original German work published in 1964, this is vol. 13 of the Springer Tracts in Natural Philosophy. More than a third of the book is devoted to nonlinear approximation and the emphasis throughout is on constructive methods and their use in practice. S, P, L.

Maximum Principles in Differential Equations. By Murray H. Protter (Univ. of California, Berkeley) and Hans F. Weinberger (Univ. of Minnesota). Prentice-Hall, Englewood Cliffs, N.J., 1967. x+261 pp. \$8.00. Functions which satisfy a differential inequality in a domain and, because of it, achieve their maxima on the boundary, are said to possess a maximum principle. The concept is one of the most useful tools in dealing with partial differential equations. This is an elementary treatment stressing numerical approximations and applications. There are bibliographic and historical notes and a fifteen page bibliography. T (16–17), S, P, L.

Topics in the Theory of Elliptic Functions. By Peter Scherk (Univ. of Toronto). Queen's Papers on Pure and Applied Mathematics, No. 8. Queens University, Kingston, Ont., 1967. i+306 pp. \$4.00. A modern approach to "one of the most beautiful creations of 19th Century analysis." S, P.

Inequalities. Proceedings of a Symposium. Edited by Oved Shisha (Wright-Patterson Air Force Base). Academic Press, New York, 1967. xiv+360 pp. $18.00. Twenty-four papers covering a very wide range in geometry, algebra, differential equations, and statistics. S, P, L.

Applications

Analytical and Numerical Methods of Celestial Mechanics. By G. A. Chebotarev (USSR Academy of Sciences). Translated by Scripta Technica, edited by Ludwig Oster. American Elsevier, New York, 1967. xviii+331 pp. $17.50. The methods of Laplace-Newcomb, Hill (planetary and lunar), variation of arbitrary constants, periodic orbits, and Cowell. S, P.

Relativity Theory and Astrophysics. Edited by Jurgen Ehlers (Univ. of Texas). Vol. I. Relativity and Cosmology. Vol. II. Galactic Structure. Vol. III. Stellar Structure. Lectures in Applied Mathematics vols. 8, 9, 10. American Mathematical Society, 1967. Vol. I. xvi+292 pp. $9.40. Vol. II. viii+220 pp. $8.10. Vol. III. viii+136 pp. $6.70. This series reports the Proceedings of the Fourth Summer Seminar on Applied Mathematics, arranged by the AMS at Cornell University in 1965 for the purpose of acquainting graduate students and recent Ph.D's with the state of knowledge and current problems. P.

Handbook of Numerical Methods and Applications. Louis G. Kelly (Johns Hopkins Univ.). Addison-Wesley, Reading, Mass., 1967. xiv+354 pp. $14.50. Designed as a handbook and reference for engineers and scientists programming their own problems and for programmers and computing center personnel, it brings together materials collected over a period of ten years. There are few definitions or explanations but many references. The coverage is extensive and includes, for example, scaling of matrices, digital filtering, integral equations, vibration problems, Padé approximation, Gram-Schmidt orthogonalization and functional minimization. There are six appendices: four including supplementary mathematical theorems and methods, the fifth a table of Laplace transforms, and the last a supplementary bibliography of the most recent references. S, P.

Journal of Optimization Theory and Applications. Edited by Angelo Miele (Rice Univ). Plenum, New York, Vol. 1. No. 1. July 1967. Bi-Monthly. Subscription (1968) 6 issues: $18.00. The Journal plans to publish "carefully selected papers covering mathematical optimization techniques and their application to science and engineering." There is a board of 24 associate editors from the United States, Italy, Scotland, New Zealand, and the Soviet Union.

A Mathematical Theory of Systems Engineering: The Elements. By A. Wayne Wymore (Univ. of Arizona). Wiley, New York, 1967. xii+353 pp. $15.95. This book is addressed to engineers and to mathematicians "who may be at least a little bit applied" and whom the author hopes to interest in "general system theory *as mathematics*, for there are some very subtle and very deep mathematical problems involved . . .the mathematical public must accept general system theory *as mathematics* before these problems will be solved." (Italics in original). After the introduction, chapter headings are Systems Definitions, Modelling of Systems, Comparison of Systems, Coupling of Systems, Subsystems and Components, and Discrete Systems. P.

Computers

Digital Logic and Computer Operations. By Robert C. Baron (Honeywell, Inc.) and Albert T. Piccirilli (Honeywell Inc.). McGraw-Hill, New York, 1967. xii+330 pp. $13.50. A

general introduction based on courses taught by the authors to fellow engineers, scientists, technicians, secondary teachers and high school students. T, P.

Systems and Computer Science. Proceedings of a Conference held at the University of Western Ontario, September 10–11, 1965. Edited by John F. Hart (Univ. of Western Ontario) and Satoru Takasu (Univ. of Kyoto). Univ. of Toronto Press, Toronto, 1967. x+249 pp. $17.50. Revised and somewhat extended versions of ten papers given at the conference whose primary purposes were the promotion of research and teaching in the field of computer science in Canadian universities. Topics include automata, sequential machines, regular expressions, linguistics, heuristic problem solving, and systems theory. Authors are J. Hartmanis, J. A. Brzozowski, R. McNaughton, Michael A. Arbib, C. C. Elgot, A. Robinson, J. D. Rutledge, Saul Gorn, J. A. Robinson, S. Amarel, M.D. Mesarovic, T. G. Windeknecht. S, P, L.

Soroban, the Japanese Abacus, its use and practice. Prepared by the Japan Chamber of Commerce and Industry. Charles E. Tuttle, Rutland, Vermont, 1967. 96 pp. $1.50. This manual is preceded by a history of the abacus and an account of its present use in Japan. In spite of the widespread use of "a whole array of up-to-date electronic and electric calculating appliances . . . most calculations at these huge establishments, to say nothing of private stores and households, are done by the handy and simple abacus, which is still unrivaled as the most convenient and efficient instrument for everyday business calculation." P, L.

The Anatomy of a Compiler. By John A. N. Lee (Univ. of Massachusetts). Reinhold, New York, 1967. xi+275 pp. $13.75. "This text attempts to lift a portion of the veil of secrecy from compiler writing." (A compiler is a program for translating from user languages such as FORTRAN into machine language.) S, P.

Multichannel Time Series Analysis with Digital Computer Programs. By Enders A. Robinson, (Digital Consultants, Inc., and N.A.S.A.) Holden-Day, San Francisco, 1967. xxiii+298 pp. $11.75. First in a series in Time Series Analysis edited by G. M. Jenkins and E. Parzen, this book is designed for advanced undergraduates in computer science and as a supplement in statistics or in applied fields. There are a few references but no bibliography. FORTRAN is used. T, S, P.

FILMS

John von Neumann. A documentary on his life and works. Committee on Educational Media, Mathematical Association of America. Distributed by Modern Learning Aids. 63 min. Black and white. Rent or purchase.

This film is a biography of von Neumann combined with commentaries on various aspects of his work. The latter remarks are not too technical to be understood by college students or even by high school seniors, so that the film in fact addresses itself to a very wide audience.

Since mathematicians, unlike, shall we say, politicians, are seldom the subject of popular news stories during their lives, the collection of live movie material for a movie biography presents rather a problem. In fact there is only one live movie sequence of von Neumann in this film, taken from a television program. The rest of the biographical material is a sort of *collage* of photographs of von Neumann, of pages of his work, of places where he lived and worked, with narrations by some of his former colleagues. This leads to an unevenness of style which is somewhat damaging to the total impression of the film. Perhaps the only remedy for this defect would be to select the material with a view to overall effect rather than with the apparent aim of trying to include everything.

It is curious that the most vivid presentation of the more technical material is that of the most abstract, namely Halmos' remarks on the work of von Neumann in ergodic theory. Here the camera treatment is simple and natural and the speaker has a naturally vivid screen personality. Some of the other sequences show the mistake of trying to set up a dramatic situation with non-professional actors. And the trouble is sometimes compounded with a fussy use of camera movement and close-up.

A. H. WALLACE, University of Pennsylvania

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. J. Roth, Upsala College, represented the Association at the dedication of Puder Hall for the Sciences at Upsala College on December 2, 1967.

Grand Valley State College: Associate Professor D. A. Clock has been appointed Chairman of the School of General Studies; Assistant Professor D. W. Vander Jagt has been appointed Chairman of the Mathematics Department.

Lawrence Institute of Technology: Associate Professor Ellen M. Hartwell, Detroit Institute of Technology, has been appointed Associate Professor; Mrs. Sonia Henckel has been promoted to Assistant Professor; Mr. T. E. King has been promoted to Assistant Professor.

Ohio Northern University: Associate Professor J. A. Berton, Ripon College, has been appointed Professor and Chairman of the Mathematics Department; Assistant Professor D. L. Daly, Wesleyan University, has been appointed Assistant Professor.

Dr. T. A. Botts, Executive Director of the Committee on Support of Research in the Mathematical Sciences, has been appointed Executive Director of the Conference Board of the Mathematical Sciences.

Dr. R. L. Brabenec, Wheaton College, has been appointed Chairman of the Mathematics Department.

Dr. L. R. Carry, School Mathematics Study Group, Stanford University, has been appointed Assistant Professor at The University of Texas.

Professor Wade Ellis, Oberlin College, has been appointed Professor of Mathematics and Associate Dean in the Horace H. Rackham School of Graduate Studies, University of Michigan.

Dr. M. A. Hyman, Federal Systems Division, IBM Corporation, has been appointed Visiting Research Professor at the Mathematics Research Center, U. S. Army, University of Wisconsin.

Dr. M. D. Levin, University of Iowa, has been appointed Assistant Professor at Florida State University.

Mr. G. L. Thesing, Sacred Heart College, has been promoted to Assistant Professor and Chairman of the Mathematics Department.

Professor D. C. Dearborn, Catawba College, died on November 13, 1967. He was a member of the Association for thirty-four years.

Professor Constantine Kassimatis, Wayne State University, died in September, 1967. He was a member of the Association for ten years.

Professor Emeritus C. N. Moore, University of Cincinnati, died on December 12, 1967. He was a Charter Member of the Association.

Mr. M. F. Pollack, retired teacher, California, died in August, 1967. He was a member of the Association for sixteen years.

Dr. W. A. Shewhart, Bell Telephone Laboratories, died on March 11, 1967. He was a member of the Association for forty-two years.

UPSILON MU ALPHA

In the fall of 1966 a new type of information organization for mathematics was initiated at Assumption College in Worcester, Massachusetts. This organization, named *Upsilon Mu Alpha*, is in the process of expanding to the entire United States. Its purposes are: to make each mathematics club aware of other clubs in its area; to distribute information concerning possible club activities; and to distribute information of interest for individual mathematics majors.

There is no charge for membership and any Mathematics Club, local chapter of an honor society, or mathematics department may join. Those interested in joining may write to: Robert Daigler, Box 53, Assumption College, 500 Salisbury Street, Worcester, Massachusetts 01609.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FIFTY-FIRST ANNUAL MEETING OF THE ASSOCIATION

The Fifty-first Annual Meeting of the Mathematical Association of America was held at the San Francisco Hilton Hotel, San Francisco, California, from Thursday to Saturday, January 25 to 27, 1968, in conjunction with the Annual Meeting of the American Mathematical Society. There were registered 3936 persons including 1959 members of the Association.

Sessions of the Association were held on Thursday morning, on Friday morning, and on Saturday morning and afternoon in the Continental Ballroom. Presiding officers were Professor Samuel Eilenberg on Thursday morning, Professor R. L. Wilder for the first lecture on Friday morning, President E. E. Moise for the Retiring Presidential Address, Professor David Gale for the first lecture on Saturday morning, Professor A. H. Taub for the session on applications of mathematics in the undergraduate curriculum, Professor Victor Klee for the first lecture on Saturday afternoon, and Professor J. G. Herriot for the panel discussion on an international study of achievement in mathematics. The Program Committee for the meeting consisted of David Gale, Chairman; J. G. Herriot, L. H. Lange, S. K. Stein, and J. W. T. Youngs.

FIRST SESSION OF THE ASSOCIATION

What Is Automata Theory?

Programs and Machines, by Professor D. S. Scott, Stanford University.

A machine can be viewed as a device with an operational unit, a control unit, and a storage unit. The operational unit receives inputs and gives outputs and stores information under the direc-

tion of the control unit. The control can be simply identified with a program, given by a flow diagram in the usual way. Several examples of machines come to mind where for simplicity we take the input, output, and memory to be strings of symbols from finite alphabets: the Turing machine, the Post (Tag) machine, the multiple pushdown store machine, all of which are effectively equivalent. The finite automaton is similar but has only a finite (usually zero) memory. (Note: a program has itself a finite memory capability.) The automaton is weaker than the previously mentioned machines. Other intermediate examples are the single push-down store machine, the linear-bounded automaton, and the stack machine. These various machines can be used for a theoretical classification of languages (i.e. sets of strings). Sets can be *decidable*, *acceptable*, or *generable* by a machine. (These notions generalize the concepts of *recursive* and *recursively enumerable* sets of integers.) The machines can also be used for transformations (mappings) of strings and sets of strings. This direction of study has a strongly algebraic (semigroup) component which could not be touched on in the lecture. A final topic mentioned concerned iterative arrays of machines. These can be used, for example, to generate the primes (better: the characteristic function of the primes) and to get firing squads to fire. A short bibliography was provided.

Grammars and Languages, by Professor E. H. Spanier, University of California, Berkeley.

This paper contained a survey of the mathematical theory of languages, a language being a set of words in a free semi-group and generated by a finite scheme called a grammar. The paper began with definitions of those types of grammars introduced originally as models for the study of natural languages and currently of interest in the theory of programming languages. It continued with a description of the corresponding classes of languages in terms of acceptance by suitable automata and discussed their closure properties under various operations. The paper concluded with a presentation of the concept of an abstract family of languages as a unifying theme for the subject. (This paper will appear in an early issue of this MONTHLY.)

Theory of Computation, by Professor Marvin Minsky, Massachusetts Institute of Technology.

The Theory of Computation will develop its full power when mathematicians come to appreciate the challenge and importance of defining and establishing the properties of a number of fundamental "conjugacies" in finite computational procedures. Among these are: the exchange between computation time and available memory (in particular, for certain matching and sorting problems); the exchange between time and amount of parallel machinery (in particular for tree-search and for differential equations); and the relation between admitted primitive operations and minimal-length algorithms (for example, in matrix multiplication or inversion).

SECOND SESSION OF THE ASSOCIATION

What is Global Analysis?, by Professor Stephen Smale, University of California, Berkeley.

Global Analysis is defined as the study of differential equations, ordinary and partial, in the context of manifolds and vector space bundles. Some recent work was discussed to illustrate global analysis, with some background material being included.

Annual Business Meeting of the Association; the Association's Seventh Award for Distinguished Service to Mathematics, and the Award of the 1968 Chauvenet Prize.

Retiring Presidential Address: Evolution of the Concept of Rigor, by Professor R. L. Wilder, University of Michigan.

The speaker discussed when and how rigor first became of concern in mathematics; in particular, how it evolved in Babylonian and Greek mathematics. He showed how problems such as those raised by the discovery of the irrational rendered visual proof methods inadequate. He inquired into the evolution of deductive methods, with Euclid's *Elements* becoming the ideal of mathematical rigor. New type problems of the 17th–19th centuries necessitated new principles and methods, raising a new concern regarding rigor. The belief at the end of the 19th century that absolute rigor had been attained was promptly shattered and the problem of mathematical rigor became primary. He then surveyed the present-day situation.

THIRD SESSION OF THE ASSOCIATION

On Mathematical Reasoning in Science, by Professor M. M. Schiffer, Stanford University.

One aim of our teaching in undergraduate mathematics is to develop an understanding of the significance of mathematics in our scientific and technological civilization. This is achieved particularly by discussing and solving problems in engineering and physics as is ably shown in Noble's book on "Applications of Undergraduate Mathematics in Engineering"; but the speaker proposed to complement this approach by a discussion of the role of mathematics in the actual discovery of the laws of nature and by demonstrating the role of mathematics as a "logical microscope" to magnify and interpret a limited number of experimental facts into coherent theories. This is best done by a historic approach to classical physics where the facts were still simple and the mathematical tools within the reach of a modern undergraduate. Examples from mechanics and optics were presented to show how such a discussion can clarify the role of mathematical argument in science and can lead to significant and interesting applications of undergraduate mathematics.

Session on Applications of Mathematics in the Undergraduate Curriculum

Is it Possible or Desirable to Introduce Applications into the Undergraduate Curriculum?, by Professor Ben Noble, University of Wisconsin, Madison.

It is as difficult to define "applications" as it is to define "applied mathematics." We want topics that (a) give students an intuitive feeling for mathematical concepts (b) provide motivation for studying mathematics (c) illustrate the direct relevance of mathematics to other branches of knowledge. Even if we agree on topics, how much time should be spent on applications, and how can teachers be persuaded to include this type of material? Suitable textbooks, computer consoles, and audio-visual aids could revolutionize the situation. However desirable this might be, the speaker doubted if it is possible—at least now.

The Far-Flung Applications of Mathematics, by Dr. H. O. Pollak, Bell Telephone Laboratories.

Applications of mathematics are "far-flung" in many ways. Mathematical subject matter which has significant practical applications includes all of the material taught in the secondary schools, and just about everything taught on the undergraduate level. The number of "applied" fields of mathematics has thus increased enormously in recent years. Next, the fields to which mathematics is applied have also become much more numerous, and this has serious effects on the potential role of a mathematics department. Some examples of research problems were also given.

Applications of Mathematics and the Problems of the Small Department, by Professor G. S. Young, Tulane University.

Among the reasons for making material in the applications of mathematics available to all majors are the following: possible vocational interests of the student, giving a better picture of the structure of mathematics, and, most important, the very serious and pressing problems of our society that will need the help of mathematics for solution. The problems of providing such a background in the smaller college or university were discussed and some possible solutions indicated. In particular, a new course, now being developed, was proposed as a partial solution.

General Discussion by the Panel and the Audience

FOURTH SESSION OF THE ASSOCIATION

Some Combinatorial Problems in the Theory of Convex Sets, by Professor G. D. Chakerian, University of California, Davis.

Let K be a compact, convex subset of the plane. Is it true that whenever Q is a set with the property that every 2 of its points can be covered by a translate of K, then Q itself can be covered by 3 translates of K? The answer is affirmative for certain K, but not known in general. This problem is the prototype for a host of interesting combinatorial covering and intersection problems

which, because of their elementary formulation and the fact that they are often amenable to "experimental" attack, have (aside from their intrinsic interest) pedagogical value.

Panel Discussion of an International Study of Achievement in Mathematics, by Professor M. L. Hartung, University of Chicago, Professor R. P. Dilworth, California Institute of Technology, Professor B. W. Jones, University of Colorado, Professor E. G. Begle, Stanford University.

Professor Hartung described the *International Study of Achievement in Mathematics*, a comparison of the educational systems of twelve countries. Achievement was studied in relation to a wide range of independent variables including features of school organization, the curriculum, and social background of students. Data were collected by mathematics tests and several questionnaires responded to by students, teachers, and school officials. It was found that the educational system of Japan was producing the best all round results. The United States ranked low in terms of mean achievement. These and many other results were related to such factors as selectivity of the systems, the opportunity students had to learn the mathematics tested, etc.

Professor Dilworth surveyed the statistical methods used and the assumptions made in this *International Study*. He discussed the strong and weak points of the treatment and examined some possible misinterpretations.

Professor Jones considered various statements appearing in published reports about the findings of the *International Study*, with a view to comparing them with the statistics actually presented in the official two-volume report and the points of view expressed therein.

Professor Begle considered the study extremely valuable in that it has demonstrated that it is possible to obtain empirical evidence on achievements in mathematical education on a large scale. On the other hand, he felt that the Study did not measure some very important variables and that, as a result, the conclusions drawn could not always be supported. In particular, he listed the following variables as not having been measured: opportunity to learn, intellectual ability, and a variable indicating which students get into certain schools and how. He urged that serious attention be given to the question as to how it happened that the Study neglected such important variables and that in future studies of this kind consultation take place with people who fully understand what is being studied.

General Discussion by the Panel and the Audience.

SPECIAL SESSIONS OF THE ASSOCIATION

On Thursday, at 8:10 P.M., Professor C. B. Allendoerfer of the University of Washington gave the final report of the CEM Level I Project, including the presentation of some of the Level I films together with film strips not previously shown.

Other film showings were held in the Continental Ballroom as follows: Thursday, 7:30–8:03 P.M., MR. SIMPLEX SAVES THE ASPIDISTRA, with Frank Kocher, Leon Henkin, and Julius H. Hlavaty (A CEM Level I film in color).

Films of the College Geometry Project of the University of Minnesota (in color).

Thursday, 9:10–9:25 P.M., SYMMETRIES OF THE CUBE, by H. S. M. Coxeter and Wm. O. J. Moser; 9:26–9:52 P.M., ISOMETRIES, by Wm. O. J. Moser and S. Schuster.

CEM Individual Lectures Films (in b & w).

Friday, 7:30–8:17 P.M., MEASURES AND SET THEORY: A Lecture by Stanislaw Ulam; 8:30–9:30 P.M., WHO KILLED DETERMINANTS: A Lecture by Kenneth O. May.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Wednesday morning and afternoon in the Toyon Suite of the San Francisco Hilton Hotel with 42 members present.

The Board approved the appointment by President Moise of the following Nominating Committee for 1968: R. J. Walker, Chairman; R. C. James, and C. O. Oakley.

The Board elected Professor Raoul Hailpern Associate Secretary for an additional five-year term. It also reelected Professor G. B. Price for an additional four-year term as

a member of the Finance Committee and Professor H. M. Gehman to fill the unexpired part of the term, extending through 1969, of Professor E. A. Cameron as a member of the Finance Committee, which had become vacant due to Professor Cameron's election as Treasurer of the Association effective after the San Francisco meeting.

Professor R. A. Rosenbaum has requested to be relieved of the position of Editor of the MONTHLY at an early date because of additional responsibilities at Wesleyan University. Upon the recommendation of a Nominating Committee to select his successor, consisting of R. P. Boas, Chairman; R. C. Buck, R. A. Rosenbaum, and George Springer, the Board elected Professor Harley Flanders of Purdue University as Editor of the MONTHLY, beginning on or about June 1, 1968, for the unexpired part of the term of Professor Rosenbaum extending through 1971.

Upon the recommendation of a Nominating Committee, consisting of F. A. Ficken, Chairman; Roy Dubisch, H. W. Eves, Raoul Hailpern, and R. E. Horton, the Board elected Professor S. A. Jennings of the University of Victoria as Editor of the MATHEMATICS MAGAZINE for the period 1969–73.

The Board approved a motion calling attention to the Sections, where appropriate, of the revision in the By-Laws of the Pacific Northwest Section, approved at this meeting of the Board, which provides for the appointment of a Second Vice-Chairman whose duties "shall be to represent the two-year colleges on the Executive Committee, arrange the program for the two-year colleges, and preside at these sessions."

The Board approved a revision in the regulations governing the Chauvenet Prize (see page 448 of this MONTHLY).

The Board approved the following schedule of future meetings of the Association: University of Wisconsin, Madison, August 26–28, 1968; New Orleans, Louisiana, January 25–27, 1969; University of Oregon, August 25–27, 1969; Miami, Florida, January 24–26, 1970; University of Wyoming, Laramie, Wyoming, August, 1970; Atlantic City, New Jersey, January, 1971.

The Executive Director reported the membership of the Association as 17,859 individual members, an increase of 329 since the corresponding date last year, 3 corporate members, and 246 academic members.

The Board passed a resolution of thanks to The Boeing Company for its contribution of $1000 toward printing and distributing the brochure YOU'LL NEED MATH.

The Board approved a motion that henceforth the top five individuals in the William Lowell Putnam Mathematical Competition be known as Putnam Fellows. In cases of ties for fifth place, all those tying will be designated as Putnam Fellows.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The Annual Business Meeting was held on Friday, January 26, 1968, in the Continental Ballroom with President Moise presiding. The Association's Seventh Award for Distinguished Service to Mathematics was made to Professor A. W. Tucker of Princeton University. The citation (which appears on pages 1–3 of the January issue of this MONTHLY) was prepared and read by Professor H. W. Kuhn of Princeton University. An autographed copy of the January issue of the MONTHLY was presented to Professor Tucker.

Professor Tucker, in accepting the Award, expressed his deep gratitude to the Association for this high honor and expressed his belief that the service imparted to him was mainly a matter of teamwork. He added: "Over the years I have been most fortunate in my co-workers, colleagues, secretaries, and students. I hope that they will feel a share in this honor."

The 1968 Chauvenet Prize was awarded to Professor Mark Kac of the Rockefeller University for his paper "Can One Hear the Shape of a Drum?" published in this MONTHLY, 73(1966), Part II (Slaught Paper No. 11), 1–23. The Award was presented

by President Moise. (For further details on this Award, see the January issue of this MONTHLY, pages 3–4.) An autographed copy of the January issue of the MONTHLY was presented to Mrs. Kac.

Professor Kac, in accepting the award, stated that he was greatly honored and more than greatly flattered by the award. He added: "It would be unnatural if my pleasure was not heightened by the fact that I have received it for the second time." He then reviewed the history of the origin of the paper for which the award was given to him. He touched upon the tensions which have existed in the past between pure and applied mathematics, a separation which he viewed as a tragedy, and added: "I consider its prevention to be a major challenge in the years to come. The two great streams of mathematical creativity are a tribute to the universality of the human genius. Each carries its own dreams and its own passions. Together they create new dreams and new passions. Apart both may die—one in a kind of unembodied sterility of medieval scholasticism and the other as a part of military art."

The Secretary then announced the results of the balloting for officers in which 2504 votes were cast: Professor G. S. Young of Tulane University was elected President-Elect for 1968; Professor Victor Klee of the University of Washington was elected First Vice-President for 1968–69; Professor Leo Moser of the University of Alberta and Professor Walter Rudin of the University of Wisconsin, Madison, were elected Governors for three-year terms, 1968–70.

The Secretary announced with profound regret the death on November 11, 1967, of Professor Lester R. Ford. He made it known that it could now be revealed that the *Fund Established by an Anonymous Donor* was actually established by Professor Lester R. Ford. In accordance with his request, the Fund will be known hereafter as the *Lester R. Ford Fund*. Professor Ford's will carries a bequest of $15,000 to the Association which will be added to this Fund. The notice of Professor Ford's death, which appeared in the Charlottesville newspaper, stated that instead of sending flowers, contributions should be made to the Lester R. Ford Fund. The substantial contributions already received as a result of this notice have also been added to the Lester R. Ford Fund.

The Secretary then reported on some of the actions taken by the Board of Governors on Wednesday. He announced that the following two books will be published by the Association during the spring of 1968: MAA STUDIES No. 5, *Studies in Modern Topology*, edited by P. J. Hilton, and CARUS MONOGRAPH No. 15, *Non-commutative Rings*, by I. N. Herstein.

The Secretary expressed, on behalf of the Association, deep appreciation to the local Committee on Arrangements for its excellent planning for the meeting. He singled out for special commendation Professor D. W. Blakeslee, Chairman of the Committee, and Professor H. M. Bacon, who acted as Chairman during a brief period of illness of the Chairman of the Committee.

The Secretary then moved to amend Article VII, Section 6, of the By-Laws of the Association to read as follows: "Any ordinary member who because of age is no longer in active service, who is in good standing at the time of his retirement, and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from payment of dues, with the privilege of obtaining the official journal at a cost of half the dues of ordinary members." The Secretary noted that the Board of Governors has voted that the cost of the official journal to those members who became emeritus prior to January 1, 1968, remains at $2. The motion was approved without dissent.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Tuesday, January 23, to Friday, January 27. The forty-first Josiah Willard Gibbs Lecture was delivered by Professor E. P. Wigner of Princeton University on Tuesday evening at 8:00 P.M. in the Con-

tinental Ballroom on "Symmetry Principles in Old and New Physics." Dean A. A. Albert of the University of Chicago gave the Presidential Address on Tuesday at 1:30 P.M. in the Continental Ballroom, entitled "On Associative Division Algebras." Invited addresses were given by Professor Wolfgang Wasow of the University of Wisconsin, Madison, on Thursday at 1:30 P.M. on "Connection Problems for Asymptotic Series," and by Professor Louis Auslander of the City University of New York, on Friday at 1:30 P.M. on "A Survey of Solvable Lie Groups and Applications," both in the Continental Ballroom.

The first George David Birkhoff Prize in Applied Mathematics of the Society was awarded on Wednesday at 1:30 P.M. to Professor Jürgen K. Moser of the Courant Institute of Mathematical Sciences for his outstanding contributions to the theory of Hamiltonian dynamical systems.

The Society held an open meeting of its Committee to Monitor Problems in Communication in the Mathematical Sciences on Wednesday at 8:00 P.M. in the Continental Ballroom with Professor W. J. LeVeque, Chairman of the Committee, presiding. The Committee is concerned with new or better devices for communicating mathematics, through research publications, reviewing journals, expository writing, meetings, films, etc.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of D. W. Blakeslee, Chairman; H. L. Alder, H. M. Bacon, W. G. Bade, N. H. Fisher, Mrs. Dorothy Friedman, R. S. Lehman, R. S. Pierce, P. E. Thomas, G. L. Walker.

Registration headquarters were located in the East Lounge on the Ballroom Floor of the San Francisco Hilton Hotel. The Mathematical Sciences Employment Register was maintained in the Imperial Ballroom from 9:00 A.M. to 5:00 P.M. on Wednesday through Friday, and book and educational media exhibits were displayed in the North and West Lounges of the Ballroom floor from 9:00 A.M. to 5:00 P.M. on Tuesday through Friday.

A No-Host-Get-Together was held in the Hilton Plaza inside the San Francisco Hilton Hotel from 5:00 P.M. to 7:00 P.M. on Thursday.

HENRY L. ALDER, *Secretary*

OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1968

General Offices: SUNY at Buffalo, Buffalo, New York 14214
Executive Director: H. M. GEHMAN

OFFICERS

President, E. E. MOISE, Harvard University (1967–68)
President-Elect, G. S. YOUNG, Tulane University (1968)
First Vice-President, VICTOR KLEE, University of Washington (1968–69)
Second Vice-President, R. J. WALKER, Cornell University (1967–68)
Editor, R. A. ROSENBAUM, Wesleyan University (1967–71)
Secretary, H. L. ALDER, University of California, Davis (1965–69)
Treasurer, E. A. CAMERON, University of North Carolina (1968–72)
Associate Secretary, RAOUL HAILPERN, SUNY at Buffalo (1968–72)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

R. H. BING, University of Wisconsin, Madison (1965–70)
A. W. TUCKER, Princeton University (1963–68)
R. L. WILDER, University of Michigan (1967–72)

Elected Members of the Finance Committee

H. M. GEHMAN, SUNY at Buffalo (1968–69)
G. B. PRICE, University of Kansas (1968–71)

Governors at Large

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N. H. McCOY, Smith College (1967–69)
LEO MOSER, University of Alberta (1968–70)
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WALTER RUDIN, University of Wisconsin, Madison (1968–70)
M. F. SMILEY, SUNY at Albany (1966–68)

Sectional Governors (July 1, 1965–June 30, 1968)

Florida, D. B. GOODNER, Florida State University
Illinois, J. M. H. OLMSTED, Southern Illinois University
Iowa, D. W. WALL, State University of Iowa
Louisiana-Mississippi, P. K. REES, Louisiana State University
Maryland-D.C.-Virginia, DOROTHY L. BERNSTEIN, Goucher College
Michigan, K. W. FOLLEY, Wayne State University
Minnesota, E. J. CAMP, Macalester College
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Sectional Governors (July 1, 1966–June 30, 1969)

Allegheny Mountain, A. F. STREHLER, Carnegie-Mellon University
Indiana, G. N. WOLLAN, Purdue University
Kentucky, L. L. SCOTT, University of Louisville
Metropolitan New York, ABRAHAM SCHWARTZ, City University of New York
Nebraska, D. W. MILLER, University of Nebraska
Northern California, D. W. BLAKESLEE, San Francisco State College
Oklahoma-Arkansas, R. B. DEAL, University of Oklahoma Medical School
Rocky Mountain, F. M. STEIN, Colorado State University
Wisconsin, C. J. VANDERLIN, JR., University of Wisconsin, Madison

Sectional Governors (July 1, 1967–June 30, 1970)

Kansas, R. N. BRADT, University of Kansas
Missouri, P. B. BURCHAM, University of Missouri
New Jersey, JOSHUA BARLAZ, Rutgers—The State University
Northeastern, GRACE E. BATES, Mount Holyoke College
Ohio, H. M. MACNEILLE, Case Western Reserve University
Pacific Northwest, JOSEPH HASHISAKI, Western Washington State College
Southeastern, R. W. BALL, Auburn University
Southwestern, A. P. HILLMAN, University of New Mexico
Upper New York State, F. D. PARKER, St. Lawrence University

COMMITTEES OF THE ASSOCIATION

Terms of office of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms of office are listed, since they are automatically discharged at the expiration of the President's term of office, which is the Annual Meeting in January 1969.

EXECUTIVE COMMITTEE

E. E. MOISE, *Chairman* (1966–69); H. L. ALDER (1965–69), E. A. CAMERON (1968–72), VICTOR KLEE (1968–69), R. A. ROSENBAUM (1967–71), R. J. WALKER (1967–68), G. S. YOUNG (1968–71), all *ex officio*.

FINANCE COMMITTEE

E. E. MOISE, *Chairman* (1967–68), *ex officio;* H. M. GEHMAN (1968–69), G. B. PRICE (1968–71), H. L. ALDER, (1965–69), *ex officio,* E. A. CAMERON (1968–72), *ex officio.*

COMMITTEE ON ADVISEMENT AND PERSONNEL

E. A. DAVIS, *Chairman* (1967–69); K. J. ARNOLD (1966–68), K. W. FOLLEY (1968–70), Mrs. F. J. MACWILLIAMS (1967–69), C. R. PHELPS (1967–68), B. E. RHOADES (1967–69), W. H. SCHMIDT (1966–68).

COMMITTEE ON ASSISTANCE TO DEVELOPING COLLEGES

GEORGE SPRINGER, *Chairman;* R. D. ANDERSON, DAVID BLACKWELL, C. M. BRADEN, L. L. CLARKSON, C. O. OAKLEY, B. E. RHOADES, M. B. SMITH, JR., F. M. STEWART.

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

C. B. MORREY, *Chairman* (1966–68); N. J. FINE (1967–69), GIAN-CARLO ROTA (1968–70).

COMMITTEE ON EDUCATIONAL MEDIA

C. O. OAKLEY, *Chairman* (1966–69); A. A. BLANK (1967–68), E. A. CODDINGTON (1966–68), L. W. COHEN (1967–69), P. S. JONES (1967–69), P. E. MILES (1968–70), SEYMOUR SCHUSTER (1967–69), GEORGE SPRINGER (1966–68).

Advisory Committee for Individual Lecture Films: A. A. BLANK, *Chairman* (1967–69); R. G. LONG, *Director* (1967–69), R. C. BUCK (1968–70), L. W. COHEN (1968–70), MARK KAC (1967–69), N. D. KAZARINOFF (1967–70), P. D. LAX (1968–70), B. J. PETTIS (1967–69), H. O. POLLAK (1967–69), L. B. WILLIAMS (1968–70).

Advisory Committee for Programmed Learning: B. H. GERE, *Director* (1968–70); D. W. BLAKESLEE (1968–70), LEONARD GILLMAN (1968–70), ROBERT KALIN (1968–70), A. B. MEWBORN (1968–70), L. W. SMITH (1968–70).

Subcommittee on Television: P. S. JONES, *Chairman;* C. B. ALLENDOERFER, R. C. FISHER.

COMMITTEE ON HIGH SCHOOL CONTESTS

Terms of office of members of this committee expire on August 31 of the last year of service listed.

C. T. SALKIND, *Chairman* (1966–69); W. H. FAGERSTROM, *Director*, PAMELA AMES (1967–69), L. C. DALTON (1965–68), J. M. EARL (1965–68), HARRISON GIVENS, JR. (1966–69), N. S. MENDELSOHN (1965–68), HANS SAGAN (1967–70), E. E. STROCK (1967–70).

COMMITTEE ON INSTITUTES

D. L. THOMSEN, JR., *Chairman* (1967–69); D. K. HARRISON (1966–68), MARK KAC (1967–69), JOSEPH LANDIN (1966–68), V. O. MCBRIEN (1966–69), E. R. MULLINS (1966–68).

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Editorial Committee for a Volume of Collected Notes on Calculus: T. M. APOSTOL, *Chairman;* H. E. CHRESTENSON, C. S. OGILVY, D. E. RICHMOND, N. J. SCHOONMAKER.

Subcommittee on Carus Monographs: IVAN NIVEN, *Chairman* (1966–68); R. P. BOAS (1967–69), OLGA TAUSSKY (1966–68).

Subcommittee on Ford Awards: IVAN NIVEN, *Chairman* (1966–68), *ex officio;* C. W. CURTIS (1966–68), EDWIN HEWITT (1968–70).

Subcommittee on MAA Studies in Mathematics: DAVID GALE, *Chairman* (1967–69); P. R. HALMOS (1968–70), ALEX ROSENBERG (1968–70).

Subcommittee on Slaught Papers: I. I. HIRSCHMAN, JR., *Chairman* (1966–68); JOSHUA BARLAZ (1967–69), MARY P. DOLCIANI (1966–68), R. A. ROSENBAUM (1967–71), *ex officio.*

Subcommittee on Miscellaneous Publications: IVAN NIVEN, *Chairman* (1966–68); H. L. ALDER (1966–69), E. A. CAMERON (1968–72), all *ex officio.*

COMMITTEE ON SECONDARY SCHOOL LECTURERS

H. M. BACON, *Chairman* (1967–69); B. H. BISSINGER (1966–68), J. N. EASTHAM (1967–69), R. L. FINNEY (1968–70), JOSEPH HASHISAKI (1968–70), N. D. KAZARINOFF (1968–70), J. H. WAHAB (1967–69).

COMMITTEE ON SECTIONS

L. E. MEHLENBACHER, *Chairman* (1967–70); E. M. BEESLEY (1965–68), W. T. FISHBACK (1967–70), S. A. JENNINGS (1968–71), ARNOLD WENDT (1966–69), RAOUL HAILPERN (1968–72), *ex officio.*

COMMITTEE ON THE AWARD FOR DISTINGUISHED SERVICE TO MATHEMATICS

H. M. GEHMAN, *Chairman* (1966–68); W. L. DUREN, JR. (1967–69), A. W. TUCKER (1968–70).

COMMITTEE ON THE CHAUVENET PRIZE

SAMUEL EILENBERG, *Chairman* (1966–68); MARK KAC (1968–70), G. L. WEISS (1967–69).

COMMITTEE ON THE PREPARATION OF A 50-YEAR HISTORY OF THE ASSOCIATION

K. O. MAY, *Chairman*; C. B. BOYER, D. J. STRUIK.

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

N. D. KAZARINOFF, *Chairman* (1966–68); J. H. MCKAY, *Director* (1968–72), LEO MOSER (1967–69), ALBERT WILANSKY (1968–70).

COMMITTEE ON THE REVISION OF THE BY-LAWS OF THE ASSOCIATION

W. L. DUREN, *Chairman;* H. L. ALDER, R. H. BING, E. A. CAMERON, H. M. GEHMAN, ARNOLD WENDT.

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

R. P. BOAS, *Chairman* (1968–70); R. D. ANDERSON (1968–70), DOROTHY L. BERNSTEIN (1967–69), L. W. COHEN (1966–68), M. D. DONSKER (1967–69), D. T. FINKBEINER II (1967–69), D. B. GOODNER (1966–69), F. A. GRAYBILL (1968–70), H. J. GREENBERG (1967–69), I. N. HERSTEIN (1967–69), MEYER JERISON (1968–70), L. J. PAIGE (1965–68), ALEX ROSENBERG (1966–68), E. H. SPANIER (1968–70), R. M. THRALL (1965–68), A. L. YANDL (1968–70), G. S. YOUNG (1965–68), LEO ZIPPIN (1967–69), E. G. BEGLE, *ex officio,* E. E. MOISE (1967–68), *ex officio.*

Advisory Group on Applications of Mathematics: M. D. DONSKER, *Chairman* (1967–68).

Advisory Group on Communications: D. T. FINKBEINER II, *Chairman* (1967–69); J. D. BAUM (1967–69), C. R. DEPRIMA (1966–67), H. J. GREENBERG (1968–71), R. H. MCDOWELL (1967–69), R. M. THRALL (1966–67).

Committee on Qualifications for Two-Year College Faculty in Mathematics: J. W. JEWETT, *Chairman;* J. N. EASTHAM, L. R. FIBEL, I. N. HERSTEIN, R. D. LARSSON, B. E. MESERVE, L. J. PAIGE.

Panel on College Teacher Preparation: LEONARD GILLMAN, *Chairman* (1966–68); C. E. BURGESS (1966–68), D. W. BUSHAW (1966–68), D. E. CHRISTIE (1966–68), L. W. COHEN (1966–68), D. T. FINKBEINER II (1966–68), MEYER JERISON (1966–68), HERMAN MEYER (1966–68), ALEX ROSENBERG (1966–68), E. H. SPANIER (1966–68), J. H. WELLS (1966–68).

Panel on Computing: H. J. GREENBERG, *Chairman* (1967–69); DOROTHY L. BERNSTEIN (1967–69), L. K. DURST (1967–70), W. C. RHEINBOLDT (1967–70), PATRICK SUPPES (1967–69).

Panel on Mathematics for the Life Sciences: R. M. THRALL, *Chairman* (1967–69); WILLIAM BOSSERT (1968–71), W. C. HOFFMAN (1967–69), G. B. PRICE (1967–69), H. R. VAN DER VAART (1967–69), G. L. WEISS (1967–70).

Panel on Mathematics in Two-Year Colleges: D. B. GOODNER, *Chairman* (1966–69); JOSHUA BARLAZ (1966–69), L. J. DUNHAM (1966–68), J. N. EASTHAM (1966–67), M. GWENETH HUMPHREYS (1966–67), R. C. JAMES (1966–68), CAROL H. KIPPS (1966–68), RALPH MANSFIELD (1967–68), B. E. MESERVE (1966–68), J. W. METTLER (1966–67), R. Z. NORMAN (1966–68), B. E. RHOADES (1967–69), W. R. RICE (1966–69), K. C. SKEEN (1967–68).

Panel on Statistics: F. A. GRAYBILL, *Chairman*

Panel on Teacher Training: G. S. YOUNG, *Chairman* (1967–68); C. E. HARDGROVE (1966–68), SHIRLEY A. HILL (1968–70), P. J. HILTON (1968–70), E. R. KOLCHIN (1966–68), D. L. KREIDER (1968–70), R. H. MCDOWELL (1967–70), M. E. SHANKS (1968–70), GEORGE SPRINGER (1966–68), S. S. WILLOUGHBY (1966–68).

COMMITTEE ON VISITING LECTURERS

R. D. BOSWELL, JR., *Chairman* (1966–69); E. M. BEESLEY (1966–68), BRINDELL HORELICK (1968–70), W. K. MOORE (1966–68), M. W. POWNALL (1968–70), J. R. WESSON (1968–70), W. L. WILLIAMS (1966–68).

COMMITTEE TO STUDY THE REORGANIZATION OF ASSOCIATION JOURNALS

DOROTHY L. BERNSTEIN, *Chairman;* D. W. BLAKESLEE, D. E. CHRISTIE, C. R. PHELPS, G. S. YOUNG.

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES

Terms of office of members of this committee expire on February 28 of the last year of service listed.

M. L. HENRIKSEN *Chairman* (1966–70, AMS); G. S. JONES (1966–69, SIAM), R. J. THOMPSON (1968–72, MAA).

JOINT COMMITTEE ON PLACES OF MEETINGS

G. L. WALKER, *Chairman;* H. L. ALDER, H. M. GEHMAN, EVERETT PITCHER, all *ex officio*.

JOINT COMMITTEE TO FACILITATE COOPERATION BETWEEN THE AMS AND THE MAA

R. C. BUCK, *Chairman;* E. A. CAMERON, W. T. MARTIN, H. O. POLLAK, R. A. ROSENBAUM, JOHN WERMER, LEO ZIPPIN.

THE CHAUVENET PRIZE

The Board of Governors, at its meeting on January 24, 1968 at San Francisco, voted to adopt the following revised regulations governing the Association's Award of the Chauvenet Prize:

1. The Chauvenet Prize is to be awarded at the Annual Meeting in January of the Mathematical Association of America. The prize is to be \$500, together with a certificate, and is to be awarded for a noteworthy paper published in English, such as will come within the range of profitable reading for members of the Association. The purpose of the prize is to stimulate the writing of expository and survey articles.

2. In case of joint authorship the prize shall be divided equally and each of the authors shall receive a certificate.

3. An individual may receive the Chauvenet Prize more than once, for different papers.

4. Each year there shall be at most one award. No award shall be made if there is no suitable recipient.

5. The award is to be made for material published during the three calendar years beginning January 1 four years prior to the time of the award.

6. The paper for which the award is made shall be of an expository or survey nature

and should preferably have appeared in a journal, anthology, or other form of publication easily accessible to most members of the Association. In case the award is given for an exceptionally worthy paper not so accessible, the paper should, if possible, be republished in the MONTHLY.

7. The recipient of the award is to be recommended by a standing Committee on the Chauvenet Prize to be appointed by the President of the Association. This Committee is to consist of three members having staggered, three-year terms. The recommendation of the Committee shall be confirmed by the Board of Governors.

HENRY L. ALDER, *Secretary*

REPORT OF THE TREASURER FOR THE YEAR 1967

Following is a summary of the report of the Treasurer of MAA for the year 1967. The report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any MAA member who wishes a copy of the Treasurer's report may obtain one by writing to the Buffalo office of the Association.

I am glad to report that almost all of the funds of the Association increased during 1967. The only significant exception is the Dunkel Fund, where payment of printing expenses has caused a decrease in the balance in the fund.

	January 1, 1967	*December 31, 1967*
ASSETS OF THE ASSOCIATION		
M & T Trust Company, checking account	\$ 18,254	\$ 30,157
M & T Trust Company, special account	25,970	12,831
M & T Trust Company, third account	1,406	367
Securities at market values	189,158	195,861
Deposit accounts	–	1,925
	\$234,789	\$241,142
FUNDS OF THE ASSOCIATION		
Current Fund	\$ 4,437	\$ 9,520
MATHEMATICS MAGAZINE	−238	2,236
Carus Fund	63,548	77,887
Chace Fund	−2,108	164
Houck Fund	9,087	9,821
Dunkel Fund	26,296	20,849
L. R. Ford Fund	9,312	10,326
Awards Fund	1,992	1,753
General Fund	80,257	81,656
CEM-Subcommittee on TV	1,945	1,771
High School Contests	3,161	2,483
Institutes	1,370	1,476
Greenwood Fund	8,350	1,071
Committee on Advisement & Personnel	–	5,000
NSF Fund	25,970	14,756
Contributions Fund	1,406	367
	\$234,789	\$241,142

This is my twentieth and final report as Treasurer. In concluding my term of office, I want to express to all members of the Association and especially to the officers and governors my appreciation of the unfailing support which I have received during my term as Secretary-Treasurer and as Treasurer.

HARRY M. GEHMAN, *Treasurer*

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in San Francisco, California, on January 24, 1968, elected to membership the thirteenth set of applicants for academic membership (for election of the other twelve sets, see the April and November issues for 1962–67). Approval for election was given to the following 3 applicants for academic membership:

Spartanburg Jr. College, Spartanburg, South Carolina
State University College at Oswego, Oswego, New York
West Virginia University, Morgantown, West Virginia

HENRY L. ALDER, *Secretary*

NOVEMBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held on November 11, 1967 at Marian College in Indianapolis, Indiana. There were 112 persons in attendance, including 70 members of the Association.

The Right Reverend Monseigneur F. J. Reine, President of Marian College, welcomed the group. Professor K. J. Sidebottom, Chairman of the Section, presided.

At the business meeting, Professor Sidebottom reported some of the highlights of the Meeting of Section Officers at the University of Toronto, August 28, 1967. Professor B. E. RHOADES described the GUIDEBOOK and suggested some uses to which it might be put. Professor J. C. Polley announced that he was turning over to the Executive Committee of the Section a nearly complete set of the minutes of the Indiana Section; Professor P. D. Edwards suggested that the Committee might investigate the possibility of depositing the minutes in the Indiana State Library. Professor Earl McKinney reported on arrangements for the spring meeting of the Section, to be held in conjunction with the Indiana Council of Teachers of Mathematics at Ball State University on May 4, 1968.

The following program was presented:

1. *On the inadequacy of sequences*, by J. B. Conway, Indiana University.
2. *Counting n-dimensional trees*, by R. E. Pippert, Purdue University, Ft. Wayne.
3. *Some ovals I have known*, by H. Flanders, Purdue University.
4. *Non-standard models*, by A. Adler, Indiana University.
5. *Some seldom-mentioned non-measurable sets*, by R. P. Miller, Ball State University.
6. *Graphs and matrices*, by L. W. Beineke, Purdue University, Ft. Wayne.

M. J. MANSFIELD, *Secretary-Treasurer*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The thirteenth annual meeting of the Northeastern Section of the MAA was held at Phillips Academy, Andover, Massachusetts, on November 25, 1967. The registered attendance was 110, including 92 members of the Association. Chairman Robin Robinson of Dartmouth College presided at both the morning and afternoon sessions.

At the business meeting a nominating committee consisting of M. E. Munroe, Chairman, D. W. Blackett, and C. E. Richart proposed the following slate of officers for the coming year: Chairman, Guilford Spencer, II, Williams College; Vice-Chairman, W. S. H. Crawford, Mount Allison University; Secretary-Treasurer, G. W. Best, Phillips Academy. The slate was elected unanimously. A motion was also passed which

gives the Executive Committee authority to award one year memberships in the Association to individual students in institutions in the Northeastern Section ranking highest on the Putnam Mathematical Competition, the award being restricted to students not already receiving memberships from other sources.

The following talks were given:

1. *A new approach to elementary statistics*, by G. E. Noether, Boston University.

The main aim of an elementary statistics course should be to acquaint the student with the logic of the statistical way of thinking. The speaker feels that with students who have a relatively low mathematical preparation this purpose can be achieved much more successfully by discussing nonparametric rather than normal-theory procedures. Nonparametric procedures are much simpler both mathematically and conceptually than normal-theory methods.

2. *On a classical Diophantine equation*, by Emil Grosswald, University of New Hampshire.

The theory of lattice points on conic sections and, more generally, on unicursal curves, is classical. However, formulae that yield all, or at least infinitely many lattice points on a hyperbola have been published only recently and only for a few, rather special equations. The equation $x^2+bx+c=ky^2$ ($d=b^2-4c$ and $k>0$ not perfect squares, $d+4k$ a perfect square) was investigated and two theorems were proven. Theorem 1 gives formulae that yield an infinite sequence of lattice points; these are, in fact, all lattice points on the curve, under certain sufficient conditions stated in Theorem 2.

3. *Some implications of the African Mathematics Program*, by V. H. Haag, Franklin and Marshall College.

The speaker described the work of mathematicians and educators from ten English speaking tropical African countries in developing a K-12 program, under the sponsorship of Education Development center in Newton, Massachusetts. American consultants in the program are noticing interesting parallels between the African program and corresponding experimental programs in the United States. Some of these parallels were reported.

4. *Some questions concerning the Four-Color problem*, by Oystein Ore, Yale University.

The lecturer discussed the numerical methods applied in the Four-Color Problem and particularly the various possibilities for using computers. Among results mentioned was the fact that the Four-Color Theorem has now been proved for all graphs with fewer than 40 countries (Ore-Stempel).

G. W. BEST, *Secretary-Treasurer*

COMMITTEE ON ASSISTANCE TO DEVELOPING COLLEGES

The MAA has recently established a committee to study ways of assisting developing colleges. One critical factor, particularly at Negro developing institutions, is faculty recruitment. Each of these colleges could benefit greatly from a semester or year spent there by a competent mathematician. The committee has agreed to serve temporarily as a clearinghouse. We are compiling a list of vacancies sent to us by these colleges, and we will freely send this list to interested mathematicians. Please address correspondence to: Professor George Springer, Chairman CADC, Department of Mathematics, Indiana University, Bloomington, Indiana 47401.

CALENDAR OF FUTURE MEETINGS

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

Fifty-second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

FLORIDA

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA, Ball State University, Muncie, May 4, 1968.

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, College of St. Teresa, Winona, May 4, 1968.

MISSOURI

NEBRASKA

NEW JERSEY, Rider College, Trenton, May 4, 1968.

NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.

NORTHERN CALIFORNIA, University of Santa Clara, February 8, 1969.

OHIO

OKLAHOMA-ARKANSAS

PACIFIC NORTHWEST, Reed College, Portland, OREGON, June 14–15, 1968.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.

ROCKY MOUNTAIN, University of Denver, Colorado, May 10–11, 1968.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE, Hamilton College, Clinton, May 11, 1968.

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, University of Wisconsin, Madison, August 27–30, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, Chicago, Illinois, August 20–22, 1968.

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS, University of Wisconsin, Madison, August 27–28. 1968.

MU ALPHA THETA, Trinity University, San Antonio, Texas, August 11–14, 1968.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cedar Rapids, Iowa, August 22–24, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, St. Francis Hotel, San Francisco, May 1–3, 1968.

PI MU EPSILON, University of Wisconsin, Madison, August 27–28, 1968.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, King Edward Sheraton Hotel, Toronto, Canada, June 11–14, 1968. (Symposium on optimization.)

THE MATHEMATICS, STATISTICS AND COMPUTER SCIENCE SECTION of the NATIONAL REGISTER OF SCIENTIFIC AND TECHNICAL PERSONNEL

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American Mathematical Society
American Statistical Association
Association for Computing Machinery
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Biometric Society
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Industrial Mathematics Society
The Institute of Mathematical Statistics
Mathematical Association of America
Operations Research Society of America
Society for Industrial and Applied Mathematics
Society of Actuaries

The American Mathematical Society, at the request of the National Science Foundation, will mail the 1968 reporting forms to the Mathematical, Statistical and Computer Science Section of the National Register. The main objective of the National Register is to provide up-to-date information on the scientific manpower resources of the United States. It is also increasingly valuable to our profession as a source of statistical information.

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Harry M. Gehman, Executive Director
Mathematical Association of America

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CONTENTS

MAY 1968

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R. A. ROSENBAUM, *Editor*

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Articles should be typewritten on good quality paper, triple spaced with wide margins. Submit the original (and a duplicate if convenient) keeping a complete copy for your protection against loss. Please follow the format used in current issues of the MONTHLY.

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EDITORIAL

The editorship of the AMERICAN MATHEMATICAL MONTHLY is in the process of transfer to Harley Flanders, who is now prepared to receive manuscripts of articles. The individual with whom to correspond about Mathematical Notes, Classroom Notes, etc. is listed at the beginning of each of the various departments.

ROBERT A. ROSENBAUM, *Editor*

Statement of Policy

From its inception, the MONTHLY has been the journal of college mathematics, designed to compete neither with journals of school mathematics nor with journals of mathematical research. Its founders pledged themselves to promote and advance college mathematics. As have editors in the past so do we again endorse this pledge.

Most of the MONTHLY subscribers teach college mathematics. In our view we can best serve them by using the following criteria for selecting material.

MAIN ARTICLES. We seek expository and survey articles the subject matter of which is relevant to the current mathematical scene.

MATHEMATICAL NOTES. We seek brief papers which give new insights, new proofs of old theorems, and mathematical pearls. Notes should be one or two pages in length.

It is not the task of the MONTHLY to publish research which is of interest only to a small group of specialists. However, original research papers which are of wide interest, which can be read with profit by non-specialists, and which meet our standards of exposition will be published.

CLASSROOM NOTES. We seek short papers with mathematical content suitable for classroom presentation at the undergraduate and early graduate levels.

MATHEMATICAL EDUCATION. This is an area of experiment, rapid change, and considerable controversy. We shall encourage open discussion of all professional aspects of education pertinent to our work as teachers of mathematics.

PROBLEMS. From Vol. 1, No. 1, the problem section has been the backbone of the MONTHLY. We seek problems of merit, both elementary and advanced, particularly problems in modern mathematics.

REVIEWS. Textbooks, monographs, and films of interest to our readers will be reviewed. We shall encourage reviewers to be critical in addition to being informative.

HARLEY FLANDERS, *Editor-Elect*

WHAT TO DO TILL THE COMPUTER SCIENTIST COMES*

GEORGE E. FORSYTHE, Computer Science Department, Stanford University

Computer science departments. What is computer science anyway? This is a favorite topic in computer science department meetings. Just as with definitions of mathematics, there is less than total agreement and—moreover—you must know a good deal about the subject before any definition makes sense. Perhaps the tersest answer is given by Newell, Perlis, and Simon [8]: just as zoology is the study of animals, so computer science is the study of computers. They explain that it includes the hardware, the software, and the useful algorithms computers perform. I believe they would also include the study of computers that *might* be built, given sufficient demand and sufficient development in the technology. In an earlier paper [4], the author defines computer science as the art and science of representing and processing information. Some persons [10] extend the subject to include a study of the structure of information in nature (e.g., the genetic code).

Computer scientists work in three distinguishable areas: (1) design of hardware components and especially total systems; (2) design of basic languages and software broadly useful in applications, including monitors, compilers, time-sharing systems, etc.; (3) methodology of problem solving with computers. The accent here is on the principles of problem solving—those techniques that are common to solving broad classes of problems, as opposed to the preparation of individual programs to solve single problems. Because computers are used for such a diversity of problems (see below), the methods differ widely. Being new, the subject is not well understood, and considerable energy now goes into experimental solution of individual problems, in order to acquire experience from which principles are later distilled. But in the long run the solution of problems in field X on a computer should belong to field X, and computer science should concentrate on finding and explaining the principles of problem solving.

One example of methodological research in computer science is the design and operation of "interactive systems," in which a man and a computer are appropriately coupled by keyboards and console displays (perhaps within a time-sharing system) for the solution of scientific problems.

Because of our emphasis on methodology, Professor William Miller likens the algorithmic and heuristic aspects of problem solving in computer science to the methodology of problem solving in mathematics so ably discussed by Professor Pólya in several books [9]. In computer science there is great stress on the dynamic action of computation, rather than the static presentation of logical structure. It tends to attract men of action, rather than contemplative men. Our students want to *do* something from the first day.

Computer science is at once abstract and pragmatic. The focus on actual

* Expanded version of a presentation to a panel session before the Mathematical Association of America, Toronto, 30 August 1967. The author is grateful to Professors T. E. Hull, William Miller, and Allen Newell for various ideas used in the paper.

computers introduces the pragmatic component: our central questions are economic ones like the relations among speed, accuracy, and cost of a proposed computation, and the hardware and software organization required. The (often) better understood questions of existence and theoretical computability—however fundamental—remain in the background. On the other hand, the medium of computer science—information—is an abstract one. The meaning of symbols and numbers may change from application to application, either in mathematics or in computer science. Like mathematics, one goal of computer science is to create a basic structure in terms of inherently defined concepts that is independent of any particular application.

Computer science has hardly started on the creation of such a basic structure, and in our present developmental stage computer scientists are largely concerned with exploring what computers can and cannot economically do. Let me emphasize the *variety* of fields in which computing has become an important tool. One of these is applied mathematics, as Professor Lax emphasizes, but this is merely one. Others include experimental physics, business data processing, economic planning, library work, the design of almost anything (including computers), education, inventory management, police operations, medicine, air traffic control, national population inventories, space science, musical performance, content analyses of documents, and many others. I must emphasize that the amount of computing done for applied mathematics is an almost invisible fraction of the total amount of computing today.

There is frequent discussion of whether computer science is part of mathematics—i.e., applied mathematics or "mathematical science." In a purely intellectual sense such jurisdictional questions are sterile and a waste of time. On the other hand, they have great importance within the framework of institutionalized science—e.g., the organization of universities and of the granting arms of foundations and the Federal Government.

I am told that the preponderant opinion among administrators in Washington is that computer science is part of applied mathematics. I believe the majority of university computer scientists would say it is not; cf. [8]. I would have to ask you how mathematicians feel about the matter. COSRIMS (Committee on the Support of Research in the Mathematical Sciences, appointed by the National Academy of Science—National Research Council) has taken the position that computer science is a mathematical science, but many of the discussions emphasize differences between mathematics and computer science.

In spite of the infancy of our subject, there are approximately 40 computer science departments in the United States and Canada today. There is no longer any doubt that computer science will have a separate university organization for several coming decades. I believe that the creation of these separate departments is a correct university response to the computer revolution, for I do not think computers would be well studied in an environment dominated by either mathematicians or engineers. However, finding suitable faculty members is very difficult today.

What are these computer science departments doing? Answer: Roughly the same things that mathematics departments are doing: education, research, and service. We teach computer science to three types of students: to our majors at the B.S., M.S., and Ph.D. levels, to technical students who need computing as a tool, and to any students who wish to become acquainted with computing as an important ingredient of our civilization. We do research in our several specialties: e.g., numerical analysis, programming languages and systems, heuristic methods of problem solving, graphical data representation and processing, time-sharing systems, logical design, business data processing, etc. We perform an unusually large amount of community service in helping our colleagues with their computing problems, both individually and by advising or managing the university computation center.

At Stanford University our graduate students are distributed among roughly three major areas of computer science: numerical mathematics (about 10 percent), programming languages and systems (about 50 percent), and artificial intelligence (about 40 percent). I have to emphasize that my own research field —numerical mathematics—is drawing only about 10 percent of our students. This is because the other two areas have problems that seem more exciting, important, and solvable at this particular stage of computer science. Moreover, they require less prior education, permitting the student to start original research at a younger stage. Thus in the past fifteen years many numerical analysts have progressed from being queer people in mathematics departments to being queer people in computer science departments!

Computer science is rich in designs of programming systems and languages, full of techniques for meeting this and that difficulty, and heavily beset with colleagues who request help. We are poor in theorems and general theories; our deep intellectual questions are shared with logic, economics, applied physics, and mathematics. On the other hand, the totality of techniques and ideas built into many of our moderate-sized computing systems (say an Algol compiler or a large eigenvalue routine) is quite impressive, for a computer is extremely good at dealing with very complex situations.

Most of known computer science must be considered as *design technique*, not theory. This doesn't bother us, as we all know that a period of developing technique necessarily precedes periods of consolidating theory, whether the subject be physics, mathematics, biology or computer science. As long as computers continue changing drastically every three or four years, there is scarcely a chance to sit down and contemplate the creation of a theory. In this respect our subject is reminiscent of early engineering, and also of mathematical analysis in the time after Newton. I wish to emphasize my belief that this is a passing stage of computer science.

The most valuable acquisitions in a scientific or technical education are the general-purpose mental tools which remain serviceable for a lifetime. I rate natural language and mathematics as the most important of these tools, and computer science as a third. The mathematics you teach reaches its effective

application largely through digital computing, and hence you and your students need to know some computer science. The learning of mathematics and computer science together has pedagogical advantages, for the basic concepts of each reinforce the learning of the other (e.g., the concepts of *function* in mathematics and *procedure* in Algol 60).

I have emphasized certain differences between computer science and mathematics, particularly because I feel this audience may not be aware of them. However, in another sense computer science and mathematics are remarkably similar. The computer industry is overwhelmed by the pains of growing so large so fast. In 1967 there are over 40,000 computers in the United States. Many thousands of programmers are constantly at work, producing software and descriptions thereof. These people work under extreme pressure of time, and many have had little supervised practice in the twin arts of programming for computers and expounding for human beings. Many compromises are made in the hurried effort to make reasonably available to users programs that work reasonably well (if not perfectly).

Seen from this hurly-burly of production, we academic mathematicians and computer scientists look much alike. We both insist on high standards of rigor and exposition (in mathematicians' language), or performance and documentation (in computer science terminology), and place a higher premium on quality than on promptness. As the computer era matures, we may find ourselves more and more thrown together in defense of this intellectual attitude. For the typical industrial programmer has little sympathy for it. He knows that the computer is often powerful enough to overcome the slipshod way it is understood and used. As an academic type, I can hardly admit it, but I have seen enough computing to believe it. Despite some grave deficiencies in users' understanding of the operation of hardware and software, the fact is that most large programs yield results that are satisfactory to the user—results that satisfy him as well or better than the analyses he used to get from mathematicians!

We academic types must surely defend our premise that critical analyses and proofs are worthwhile in this age of wholesale number-crunching.

What can you do now? And now follow my answers to the question of the title.

First, you can get a little acquainted with computing. This involves two steps:

Step A: Learn to program some automatic digital computer in some language—e.g., Fortran Algol, PL/1—and actually use the computer enough to find out some of the fascination and frustrations of the computerman's world.

Step B: Read some books from the list at the end of this paper. Since computer science is not yet very deep and mathematicians are very smart people, this should not be onerous.

Second, you can study how computing intersects mathematics. Applied mathematics is no longer the same subject, now that you have a magnificent

experimental tool at hand. Moreover, there are several undergraduate courses that owe their large enrollments largely to their wide applications in technology and science: e.g., linear algebra, and ordinary differential equations. I think both of these courses should be substantially influenced by computers.

In a linear algebra course, along with concepts like rank, determinant, eigenvalues, linear systems, and so on, ought to go some constructive computational methods suitable for automatic computers. There is plenty of literature now, and I think some of it should be worked into courses in linear algebra. If not, then an instructor should loudly confess that he is ignoring these topics, and furnish some reading lists for his students.

The same goes for ordinary differential equations. Here the situation is slightly different, in that textbooks in this field usually do say something about numerical methods. The trouble is that it usually dates from before the days of computers. It should be expunged and replaced with at least an equivalent amount of orientation in today's useful numerical methods for computers. See [7] for Professor Hull's suggestions.

I think also that the calculus courses should be influenced by an awareness of computing, but I do not expect this to be a very large fraction of the courses. See [6] for some ideas.

The alternative to weaving computational material into various mathematics courses is to teach computational mathematics in separate courses, in either the department of mathematics or the computer science department. This alternative is the accepted method at present, but many have felt it should be only a temporary expedient. If computational mathematics is taught in the computer science department, what effective mechanism can there be to reunite the theoretical and the computational aspects of mathematics?

There is a good deal of interest nowadays in *computer-aided instruction.* I don't expect this to have a very large application to university mathematics teaching. However, I should like to call your attention to the usefulness of a computer-controlled cathode-ray-tube display and "light pen" in giving vivid graphical representations of sophisticated concepts. In one of these, developed by Professor William McKeeman and Mr. William Rousseau at Stanford University, the scope shows both the complex z plane and the plane of $f(z)$, for any simple elementary function f typed at the console. When the light pen traces any curve in the z-plane, a dot of light traces the curve $f(z)$. Many of the elementary theorems of analytic function theory receive an impressive illustration in this way. Professor Marvin Minsky has used similar displays in dealing with nonlinear ordinary differential equations.

At a more fundamental level, the emergence of computer science has added one more applier of mathematics. Along with operations research, economics, and other more recently mathematized subjects, computer science is relatively more interested in *discrete mathematics* (e.g., combinatorics, logic, graph and flow theory, automata theory, probability, number theory, etc.; see [1]), than

in *continuum* mathematics (e.g., calculus, differential equations, complex variables, etc.). Hence the mathematics department (in my view) should devote much thought to organizing its curriculum suitably from the standpoint of consumers of discrete mathematics. I feel that currently common curricula are inherited from the days when continuum mathematics was more in demand (from physics, mechanical engineering, etc.).

Third, you can help the computer scientist find his way to your campus, and make him feel welcome. Above all, please don't judge him as a mathematician, for he isn't one and isn't supposed to be one—his values are different. The difference in values between mathematics and numerical analysis is the subject of a provocative paper [5].

When the computer scientist does arrive on campus, be prepared for a rather large impact. He is tied to a rampant field of rapidly growing interest to students and scholars everywhere. He will need many colleagues and new buildings. He may take some of the heat off mathematics faculties by providing a partial substitute for mathematics as a research tool. This vast energy may have some undesirable side effects on your sense of importance and even your budget.

Fourth, if you are really enthusiastic, I recommend tackling some research problems of a mathematical nature that would help computer science (and your own publication list). There are serious and important mathematical questions at almost every turn, and most computer scientists aren't very good at mathematics. I will leave to Professor Lax the important area of experimental mathematics. One area of computer science with a probable payoff is the automation of algebra and analysis. So far, most actual computing consists of automated *arithmetic*. A Fortran program, for example, asks a computer to carry out addition, subtraction, multiplication and division of (simulated) real or complex numbers, in a sequence which is dynamically determined by the course of the computation. There is little else. It is clear that computers are capable of automated *algebra*, and there have been experimental systems for this since about 1961. They are still primitive. Some of the roadblocks to further development occur at surprising places. One is the question of *simplification* (e.g., of rational polynomial expressions in n variables). What do we mean by simplification? How shall we do it? See Brown [2] for one indication of the depth of the problem.

Proposed by Dr. R. W. Hamming, but still largely in the future, is the partial automation of *analysis*. Faced with an initial-value problem for an ordinary differential equation, for example, a computer should be able to put the problem into some sort of normal form (using automated algebra, of course). Then the computer should inspect the normal form to see whether it is a recognized standard equation. If it is, then a solution formula should be obtained from a table, and then transformed (by automated algebra) back into the variables originally presented. Of course, the user may want a table of values. The computer then must decide whether to use the solution formula (if one exists), or to

compute a numerical solution. In the latter case, a numerical integration formula must be automatically selected (or devised), and then used (by automatic arithmetic) to produce a table of answers and error bounds (more automated analysis). There are many unsolved problems in this program, and mathematicians are uniquely qualified to define the problems and start their solution.

Most computation to date has been *serial* in nature, with only one computation or decision being made at a time within the central processor. Soon to arrive will be *parallel* computers, in which from two to perhaps several hundred operations can be formed simultaneously. The general pattern of serial computation has been well understood since the work of Babbage, Aiken, von Neumann, and others. There are good research problems in analyzing parallel computation and identifying the important features. See [3] for a recent contribution.

There are good research problems in the theoretical aspects of the design of algorithms. Initiated by Post, Turing, and others, there is an important theory that tells us that some functions are computable on a "Turing machine," and some are not. (Turing machines differ in theoretical capability from existing computers only in having infinite storage capacity.) This theory has been extended to state that some problems can be solved on a Turing machine with a suitable algorithm, but for some problems no such algorithm can exist.

It is essential to know that a problem is solvable, but this is only the beginning. What is needed next is information about how much computer storage is required for the program and data, and how long the algorithm will run. In other words, we need theoretical information on the complexity of solvability. There are some results by Kolmogorov and others on the complexity of a computable function, but much more research is needed.

Other research problems lie in areas further removed from mathematics. One such area is computer graphics—the uses of computers for dealing directly with information in the form of structures. (Examples: representing graphs of mathematical trees, design of networks, recognition of three-dimensional block structures from photographs, automatic reading of bubble chamber pictures.) In this area there are problems of representing information, both visually and inside a computer store, and of processing the information. Most algorithms are being created by persons with only a modest knowledge of mathematics, and it seems likely that an interested mathematician could both help solve some computing problems and find worth-while mathematical problems.

In summary, here are my four answers to the question of the title:

(1) Learn a little about computer science.
(2) Consider how mathematics curricula should be affected by computer science.
(3) Help the computer scientist find his way, but expect a big blast after he gets there.
(4) Think of computer science as a possible source of mathematical research problems.

Some books to read

Here are some suggested book readings in computer science:

F. L. Alt (editor), *Advances in Computers*, annual serial volume, of which the eighth was issued in 1967, Academic Press. (These contain interesting survey articles on a wide variety of topics in computer science.)

Anonymous, *Information*, Freeman, 1966. (Originated as the September 1966 issue of the *Scientific American*.)

Jeremy Bernstein, *The Analytical Engine: Computers, Past, Present, and Future*, Random House, 1964. (A good book to start with; it originally appeared in the *New Yorker*.)

Edward A. Feigenbaum and Julian Feldman (editors), *Computers and Thought*, McGraw-Hill, 1963. (These articles are devoted to the topic of "artificial intelligence": to what extent can computers accomplish tasks heretofore performed by human minds?)

L. Fox (editor), *Advances in Programming and Non-Numerical Computation*, Pergamon, 1966. (Series of articles explaining programming and nonnumerical computation to the uninitiated mathematician. The main nonnumerical applications dealt with here are theorem-proving, game-playing, and information retrieval.)

T. E. Hull, *Introduction to Computing*, Prentice-Hall, 1966. (A first course in Fortran and its use in computing, both arithmetic and symbolic, by a mathematician and numerical analyst. It has a good annotated bibliography that can serve to expand the present list.)

Kenneth E. Iverson, *A Programming Language*, Wiley, 1962. (The author has created a notation useful for describing the logical design of automatic computers and for programming computers. In other works the author makes it clear that he would like his notation to replace mathematical notation, which he finds full of inconsistencies.)

Marvin Minsky, *Computation: Finite and Infinite Machines*, Prentice-Hall, 1967. (An advanced undergraduate textbook on automata, computability, and so on. Actual automatic computers are never far out of the author's mind.)

B. Randell and L. J. Russell, *Algol 60 Implementation*, Academic Press, 1960. (This book describes a program that translates a program written in Algol 60 into the machine-language program of an actual computer. Such programs are called "compilers," and are by far the most frequent programs run by computers.)

Saul Rosen (editor), *Programming Systems and Languages*, McGraw-Hill, 1967. (One of the most sophisticated of the emerging parts of computer science is the theory of programming languages. It extends from abstract theories of written linguistics over to the psychological questions of what languages human beings can most effectively use.)

Peter Wegner (editor), *Introduction to Systems Programming*, Academic Press, 1964. (By a *system* the author means any program that controls the course of programs through a computer, programs that translate from one language to another, etc. Such systems are the "intelligence" that turns a bare pile of electronic componentry into an effective "living" computing machine.)

J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, 1965. (This is devoted to computing the eigenvalues and eigenvectors of a finite square matrix, by a man who has personally tested and analyzed most known methods. You will be surprised at how little space is wasted in the 662 pages.)

References

1. Edwin F. Beckenbach (editor), Applied Combinatorial Analysis, Wiley, New York, 1964.

2. W. S. Brown, Rational exponential expressions and a conjecture concerning π and e, manuscript, Bell Telephone Laboratories, 1967.

3. A. B. Carroll and R. T. Wetherald, Application of parallel processing to numerical weather prediction, J. Assoc. Comput. Mach., 14 (1967) 591–614.

4. George E. Forsythe, A university's educational program in computer science, Comm. Assoc. Comput. Mach., 10 (1967) 3–11.

5. R. W. Hamming, Numerical analysis vs. mathematics, *Science*, 148 (23 April 1965) 473–475.

6. R. W. Hamming, Calculus and the Computer Revolution, CUPM, P.O. Box 1024, Berkeley, California 94701, 1966.

7. T. E. Hull, The Numerical Integration of Ordinary Differential Equations, CUPM, P.O. Box 1024, Berkeley, California 94701, 1966.

8. Allen Newell, Alan J. Perlis, and Herbert A. Simon, What is computer science? *Science*, 157 (22 Sept. 1967) 1373–4.

9. George Pólya, How To Solve It, 2nd ed. Anchor Book A93, Doubleday, New York. (Several other books.)

10. University of Chicago, *Graduate Programs in the Divisions, Announcements 1966–67*, pp. 175–177, describing their Committee on Information Sciences.

NULL SETS FOR A CLASS OF ANALYTIC FUNCTIONS

LAWRENCE ZALCMAN, Massachusetts Institute of Technology

1. Let D be a domain (i.e., open, connected set) in the compactified complex plane S. Let $K \subset D$ be a compact set that does not divide the plane. The purpose of this paper is to discuss the following question:

(*) *For what sets K can every function continuous on D and analytic on $D-K$ be continued across K to be analytic on all of D?*

A good answer to this question would consist of giving explicit necessary and sufficient conditions of a geometric nature for K to satisfy (*). However, as is so often the case with questions that concern the interplay between geometry and function theory, this appears to be quite difficult. We shall content ourselves, therefore, with exploring some of the ramifications of the problem and with recasting it in a somewhat different form. We shall also comment, in passing, on the interesting history of this problem.

One way of reformulating (*) is to ask

(**) *For what compact sets $K \subset S$ is the set of all functions continuous on S and analytic on $S-K$ nontrivial?*

Clearly, this is just (*) with $D=S$. That the answers to (*) and (**) are the same is shown in Theorem 1 below.

In Sections 2 and 3 K will always denote a compact set which does not divide the plane (i.e., such that $S-K$ is connected). We shall also assume that K lies in the *finite* plane $\mathbf{C}$; this entails no loss of generality, since (except in the trivial case $K=S$) we can effect a conformal transformation which will map K onto a bounded set without altering the function-theoretic nature of the problem.

2. We begin with some definitions.

DEFINITION 1. *Let K be given and let D be a domain containing K. We shall denote by $A(D; K)$ the set of all functions analytic on $D-K$ and continuous on $\overline{D}$. We write $A(D)=A(D; \varnothing)$.*

DEFINITION 2. *K is a c-null set if and only if $A(S; K)$ consists only of the constants.*

DEFINITION 3. *The c-capacity of K is $\gamma_c(K)=\sup|f'(\infty)|$, where the sup is taken over all f satisfying*

(i) *f is analytic on $S-K$*

(ii) *f is continuous on S*

(iii) $|f|\leqq 1$.

In calculating the c-capacity of a compact set we may restrict our attention to functions which satisfy (iv) $f(\infty)=0$. Indeed, if f satisfies (i)–(iii) above then

$$g(z) = \frac{f(z) - f(\infty)}{1-\overline{f(\infty)}f(z)}$$

satisfies (i)–(iv), and $|g'(\infty)|\geqq|f'(\infty)|$. We call a function satisfying (i)–(iv) an *admissible function.*

DEFINITION 4. *K is removable if, for every domain D containing K, $A(D;K)=A(D)$. Say K is D-removable if the above condition holds for (a particular) $D\supset K$.*

Note that if $D'\supset D$ and K is D-removable then it is D'-removable.

We can now state our basic equivalence.

THEOREM 1. *The following are equivalent:*

(i) *K is removable*

(ii) *K is D-removable for some $D\supset K$*

(iii) *K is S-removable*

(iv) $\gamma_c(K)=0$

(v) *K is a c-null set.*

Proof. (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are trivial. (iii)$\Rightarrow$(iv) follows from the fact that S supports no nonconstant analytic function. To get (iv)$\Rightarrow$(v), suppose K is *not* a c-null set. Let f be a nonconstant element of $A(S;K)$. Suppose f has the expansion $f(z)=a_0+a_n/z^n+a_{n+1}/z^{n+1}+\cdots$, $a_n\neq 0$, at $z=\infty$. Then

$$g(z) = Lz^{n-1}(f(z)-a_0) = L(a_n/z+a_{n+1}/z^2+\cdots)$$

is an admissible function if $L\neq 0$ is an appropriately chosen constant. Hence $\gamma_c(K)\geqq|g'(\infty)|=|La_n|\neq 0$. So $\gamma_c(K)\neq 0$. Finally, suppose (v) holds and let $D\supset K$. Since K does not divide the plane, we can find bounded simply connected domains D_j $(j=1,2)$, each of whose boundary is a simple closed analytic curve, such that $D\supset\overline{D}_1\supset D_1\supset\overline{D}_2\supset D_2\supset K$. Let $\Gamma_j=\partial D_j$. Suppose $f\in A(D_1;K)$ and set

$$f_1(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta - z}d\zeta \qquad z\in D_1$$

$$f_2(z) = -\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta)}{\zeta - z}d\zeta \quad z\in S-\overline{D}_2.$$

Each f_j is clearly analytic on its domain of definition; moreover, by the Cauchy integral formula we have $f(z)=f_1(z)+f_2(z)$ for $z\in D_1-\overline{D}_2$. Therefore, setting

$f_2(z) = f(z) - f_1(z)$ for $z \in D_1$, we obtain a continuation of f_2 to a function analytic on all of $S-K$ and continuous on S. By (v), f_2 is constant; and so, $f \in A(D_1)$. Thus K is D_1-removable and hence D-removable. Since D was arbitrary, we are done.

REMARK. There is an obvious alternative to Definition 4. Let $\mathfrak{A}(D; K)$ be the set of all functions analytic on $D-K$ and continuous on D, and let $\mathfrak{A}(D) = \mathfrak{A}(D; \varnothing)$. Say K is strongly removable if, for every domain $D \supset K$, $\mathfrak{A}(D; K) = \mathfrak{A}(D)$. Removability and strong removability are equivalent notions; for the proof, let $D \supset \overline{D'} \supset D' \supset K$ and note that

$$\mathfrak{A}(D'; K) \supset A(D'; K) \supset \mathfrak{A}(D; K) \supset A(D; K).$$

3. Obviously, Theorem 1 gives us very little concrete information as to what removable sets look like. The examples given in this section indicate that there is no simple characterization of removable sets in terms of such usual "geometric" notions as connectedness, category, interior, or measure.

Clearly, if K has interior it is not removable. The same reasoning shows that if K is of the second category it cannot be removable, since a closed set of the second category has nonempty interior. The first example of a set without interior that fails to be removable was given by Pompeiu in [**9**]. His set was a totally disconnected perfect set of positive planar measure. Later, in [**6**] Denjoy sketched the construction of a totally disconnected perfect set which is not removable, yet has zero (planar) measure. Both of these constructions are rather involved. In [**11**], Urysohn showed that one could construct such sets in a very simple manner; his basic construction is worth reproducing.

One constructs a planar Cantor set in the usual way: divide the unit square $0 \leqq x, y \leqq 1$ into nine equal squares and delete the union of the closed squares containing points (x, y) such that $\frac{1}{3} < x < \frac{2}{3}$ or $\frac{1}{3} < y < \frac{2}{3}$; then take the closure of the remaining points. This entire process is iterated in the usual manner. After the nth step one is left with 4^n squares of total area $(4/9)^n$; call the union of these squares K_n. Then $K = \bigcap_{n=1}^{\infty} K_n$ is a totally disconnected perfect set of plane measure zero. Denote the centers of the components of K_n by $z_{n,k}$ $(k = 1, 2, \cdots, 4^n)$ and let

$$f_n(z) = 4^{-n} \sum_{k=1}^{4n} (z - z_{n,k})^{-1}.$$

One can show that $\lim_{n\to\infty} f_n(z) \equiv f(z)$ exists for all z and that $f \in A(S; K)$. But f is not constant, since $\int_{|z|=2} f(z)dz = 2\pi i$ by the residue theorem.

It is now easy to construct a connected set K without interior and having zero measure, which fails to be removable. Indeed, let $[0, 1]_x$ and $[0, 1]_y$ denote the unit intervals on the x and y axes respectively and let K_x (resp. K_y) denote the projection of the planar Cantor set constructed above on $[0, 1]_x$ (resp. $[0, 1]_y$). (Of course, K_x is just the usual linear Cantor set.) Then $K = [0, 1]_x \cup ([0, 1]_y \times K_x)$ is a set of the required type. In particular, K is not removable,

since it contains the planar Cantor set $K_x \times K_y$ which, as we have remarked, is not removable.

A different construction, due to Denjoy [7], is also of interest. Let $\Gamma \subset \mathbf{C}$ be a Jordan arc (i.e., a homeomorphic image of $[0, 1]$) in the plane that has positive two-dimensional Lebesgue measure. Then

$$f(z) = \iint_\Gamma \frac{dxdy}{\zeta - z} \qquad (\zeta \text{ traverses } \Gamma)$$

is obviously analytic on $S - \Gamma$ and $f(\infty) = 0$. f is nonconstant since $\lim_{z\to\infty} zf(z) = -\iint_\Gamma dxdy \neq 0$. Moreover, f can be extended to be continuous on all of S. Thus Γ is not removable. Some consequences of the existence of such functions have been examined in [5] and [14]. On the other hand, if Γ is a rectifiable arc it is removable, by Morera's theorem. In fact, if K is a countable union of sets of finite linear measure then K is removable [4]. The countability requirement is essential, since $[0, 1]_y \times K_x$ is a union of sets of finite linear measure, yet is not removable.

Perhaps the best result along the lines sketched above is due to Arens [3], who has obtained an interesting sufficient condition for the existence of nontrivial functions in $A(S; K)$. Let μ be a nontrivial finite (regular) Borel measure on the plane. For $r > 0$ let

$$m_\mu(r) = \sup_\zeta \mu(\{ | z - \zeta | < r\});$$

$m_\mu(r)$ is a nondecreasing function of the real variable r on $(0, \infty)$. Suppose

$$I(\mu) = \int_0^\infty r^{-1} dm_\mu(r) < \infty.$$

Then one has

THEOREM 2. (Arens) *Let f be a bounded Borel-measurable function on the plane. Then*

$$h(z) = \int \frac{f(\zeta)}{\zeta - z} d\mu(\zeta)$$

is a continuous function on the plane which is analytic on each open set of μ-measure 0. If μ has bounded support, h is analytic at ∞ with $h(\infty) = 0$ and is bounded on the whole plane.

Thus if one can find a nonzero finite Borel measure μ on the plane whose support is contained in K and which satisfies $I(\mu) < \infty$, then $A(S; K)$ is nontrivial. This fact can be used to construct examples similar to those above. For instance, if f is the usual Cantor function ([8], pp. 193–194) the product measure $\mu = df \times df$ satisfies $I(\mu) < \infty$. This gives another proof of Urysohn's result above.

Because of the importance of Theorem 2 we shall offer a proof. Our proof is

modelled on Arens' original proof but is somewhat more detailed and is also (essentially) self-contained. We begin by proving some lemmas.

LEMMA 1. *Let m_1 and m_2 be nondecreasing functions on* $(0, \infty)$ *and suppose* $m_1(r) \geqq m_2(r)$ *for all* r. *Then*

$$\int_0^t r^{-1} dm_2(r) \leqq \int_0^t r^{-1} dm_1(r)$$

for all t in $(0, \infty)$.

Proof. If m is a nondecreasing function on $(0, \infty)$

$$\int_0^t r^{-1} dm(r) = \sup[M_m((r_1, r_2, \cdots, r_n))] \tag{1}$$

where the bracketed expression in (1) denotes

$$\frac{m(r_1)}{r_1} + \frac{m(r_2) - m(r_1)}{r_2} + \cdots + \frac{m(r_n) - m(r_{n-1})}{r_n} \tag{2}$$

and the sup is taken over all n-tuples $(r_1, r_2, \cdots, r_n)$ of arbitrary finite length which satisfy $r_1 < r_2 < \cdots < r_n \leqq t$. We claim that for any such n-tuple

$$M_{m_2}((r_1, r_2, \cdots, r_n)) \leqq M_{m_1}((r_1, r_2, \cdots, r_n)). \tag{3}$$

Indeed, simplifying (2) we obtain

$$M_m((r_1, r_2, \cdots, r_n)) = \sum_{k=1}^{n-1} m(r_k) \left[\prod_{j \neq k} r_j - \prod_{j \neq k+1} r_j \right] + m(r_n) \prod_{j=1}^{n-1} r_j,$$

which shows that the l.h.s. of (3) is dominated by the r.h.s. Taking sups, we obtain the desired result.

LEMMA 2. *For any* $z \in \mathbf{C}$, $\int |1/(\zeta - z)| \, d\mu(\zeta) \leqq I(\mu)$.

Proof. First of all, note that, for any z, $1/(\zeta - z)$ is a (Borel) measurable function of ζ, and hence so is $|1/(\zeta - z)|$. Now fix z and let $t > 0$. We claim

$$\int_{|\zeta - z| \leqq t} \frac{1}{|\zeta - z|} \, d\mu(\zeta) \leqq \int_0^t \frac{1}{r} \, dm_\mu(r). \tag{4}$$

Let $\tilde{\mu}(r) = \mu(\{|\zeta - z| < r\})$. Then $\tilde{\mu}$ is a nondecreasing function on $(0, \infty)$. Clearly,

$$\int_{|\zeta - z| \leqq t} \frac{d\mu(\zeta)}{|\zeta - z|} \leqq \int_0^t \frac{1}{r} \, d\tilde{\mu}(r). \tag{5}$$

Since $m_\mu(r) \geqq \tilde{\mu}(r)$, the l.h.s. of (5) is dominated by

$$\int_0^t \frac{1}{r} \, dm_\mu(r).$$

Thus (4) holds. Now

$$(6)\qquad \int \frac{1}{|\zeta - z|}\, d\mu(\zeta) = \int_{|\zeta - z| > t} + \int_{|\zeta - z| \leqq t} \leqq \frac{1}{t}\,\mu(\mathbf{C}) + \int_0^t \frac{1}{r}\, dm_\mu(r).$$

As $t \to \infty$ the r.h.s. of (6) tends to $I(\mu)$. The proof is complete.

LEMMA 3. *The set functions defined by*

$$F_z(E) = \int_E \frac{d\mu(\zeta)}{\zeta - z} \qquad (E \text{ a Borel set})$$

are uniformly absolutely continuous (see [8]).

Proof. Let $\epsilon > 0$. Choose t such that

$$(7)\qquad \int_0^t \frac{1}{r}\, dm_\mu(r) < \epsilon/2,$$

Let $\delta = \epsilon t/2$ and suppose $\mu(E) < \delta$. Clearly,

$$(8)\qquad |F_z(E)| \leqq \int_E \frac{d\mu(\zeta)}{|\zeta - z|}.$$

Let $E_1 = \{\zeta \in E \mid |\zeta - z| < t\}$, $E_2 = E - E_1$. Obviously,

$$(9)\qquad \int_{E_2} \frac{d\mu(\zeta)}{|\zeta - z|} < \delta/t = \epsilon/2.$$

By (4) and (7) we have

$$(10)\qquad \int_{E_1} \frac{d\mu(\zeta)}{|\zeta - z|} \leqq \int_0^t \frac{1}{r}\, dm_\mu(r) < \epsilon/2.$$

The conjunction of (8), (9), and (10) shows that $|F_z(E)| < \epsilon$ if $\mu(E) < \delta$, independently of z. This is just the assertion of the lemma.

LEMMA 4. *Let $z_n \to z$. Then*

$$\int \left| \frac{1}{\zeta - z_n} - \frac{1}{\zeta - z} \right| d\mu(\zeta) \to 0.$$

Proof. Let $g_n(\zeta) = 1/(\zeta - z_n)$, $g(\zeta) = 1/(\zeta - z)$. Since $I(\mu) < \infty$, μ contains no point masses, i.e., $\mu(\{z\}) = 0$ for any z. Clearly, then, $g_n(\zeta) \to g(\zeta)$ pointwise a.e. $[d\mu]$. Of course, $g, g_n \in L^1(d\mu)$, by Lemma 2. Let $\epsilon > 0$ be given. By Lemma 3, we can pick δ such that $\int_E |g_n|\, d\mu < \epsilon/2$, $\int_E |g|\, d\mu < \epsilon/2$ if $\mu(E) < \delta$. By Egoroff's theorem (see [8]), there exists a Borel set $E_{\epsilon,\delta}$ such that g_n converges uniformly to g on $E_{\epsilon,\delta}$ and $\mu(E_{\epsilon,\delta}) \geqq \mu(\mathbf{C}) - \delta$. Then

$$\int |g - g_n| \, d\mu \leqq \int_{E_{\epsilon,\delta}} |g - g_n| \, d\mu + \int_{C-E_{\epsilon,\delta}} |g| \, d\mu + \int_{C-E_{\epsilon,\delta}} |g_n| \, d\mu.$$

As $n\to\infty$, the first term on the r.h.s. tends to zero; the remaining terms have sum less than ϵ. Since ϵ was arbitrary, we are done.

We can now prove the theorem. To show that $h(z)$ is continuous merely note that if $z_n\to z$

$$|h(z_n) - h(z)| \leqq \|f\|_\infty \int \left| \frac{1}{\zeta - z_n} - \frac{1}{\zeta - z} \right| d\mu(\zeta) \to 0$$

by Lemma 4. It remains to show that h is analytic on each open set of μ-measure 0. Let z lie in such a set and let $z_n\to z$. Then

$$\lim_{z_n\to z} \frac{h(z_n) - h(z)}{z_n - z} = \lim_{z_n\to z} \int f(\zeta) \frac{1}{(\zeta - z_n)(\zeta - z)} \, d\mu(\zeta). \tag{11}$$

As $n\to\infty$, $f(\zeta)/(\zeta - z_n)(\zeta - z) \to f(\zeta)/(\zeta - z)^2$ pointwise a.e. $[d\mu]$. Also, for large n

$$\left| f(\zeta) \frac{1}{(\zeta - z_n)(\zeta - z)} \right| < 2 \left| f(\zeta) \frac{1}{(\zeta - z)^2} \right| \tag{12}$$

on a closed set containing the support of μ. Since z lies at a positive distance from the support of μ, the r.h.s. of (12) is bounded on the support of μ and hence is integrable (by the finiteness of μ). By Lebesgue's dominated convergence theorem, the limit (11) exists and is equal to $\int f(\zeta)/(\zeta - z)^2 d\mu(\zeta)$. Thus h is differentiable at z. The rest of Theorem 2 is now obvious.

Theorem 2 has the following interesting corollary.

COROLLARY. *If K has positive planar (Lebesgue) measure, then K is not removable.*

Proof. Let μ be planar Lebesgue measure restricted to K. It is enough to show that $I(\mu) < \infty$. Let $D = \{ |\zeta| \leqq R \}$, where R is so large that $K \subset D$, and let λ be planar Lebesgue measure restricted to D. Then $m_\lambda(r) \geqq m_\mu(r)$ so that, by Lemma 2, $I(\mu) \leqq I(\lambda) = \int_0^R 2\pi dr = 2\pi R < \infty$.

Before passing on to other matters, we should comment briefly on the relation of Section 3 to Section 2. Although the word "removable" occurs repeatedly in Section 3, it should be quite clear that the relevant notion is that of c-null set; in particular, to prove that a set is not removable we show that it is not a c-null set. Here, of course, we are making use of Theorem 1 (Section 2). If one desires to frame the discussion in Section 3 in terms of c-null sets instead of removable sets (in other words, if the reader reads "c-null" whenever he sees "removable"), the only reference to Section 2 necessary for understanding Section 3 is Definition 2.

4. A few remarks concerning the general notion of *null set* (no relation to the empty set) are in order; the interested reader should consult the paper of Ahlfors and Beurling [2] for further information. Let $K \subset \mathbf{C}$ be compact and let Ω be the component of $S-K$ that contains ∞. Let $\mathfrak{F}=\mathfrak{F}(\Omega)$ be some specified family of (analytic) functions defined on Ω and let

$$M_{\mathfrak{F}}(K) = \sup_{\mathfrak{F}} |f'(\infty)|.$$

We say K is an $\mathfrak{F}$-null set if $M_{\mathfrak{F}}(K)=0$. If $\mathfrak{F}=\mathfrak{F}(\Omega)$ is chosen as the set of analytic functions on Ω bounded by 1, $M_{\mathfrak{F}}(K)$ is the so-called analytic capacity of K. For a relatively detailed discussion of the properties of analytic capacity see [12].

Ahlfors [1] seems to have been the first to notice that the compact sets of analytic capacity zero are just the Painlevé null sets, i.e., sets whose complements in S support no nonconstant bounded analytic function. Of course, this statement is just a partial analogue of Theorem 1; and the proof is essentially the same. A Painlevé null set cannot divide the plane; in fact, such a set must be totally disconnected. To see this, note that if K is not totally disconnected it must contain an arc γ. Then the Riemann map of $S-\gamma$ onto the open unit disc is a nonconstant function bounded and analytic on $S-K$.

Obviously, Painlevé null sets are c-null sets (=removable sets). However, the class of c-null sets is much wider than that of Painlevé null sets, since c-null sets are not, in general, totally disconnected.

Rudin [10] has used Painlevé null sets in the proof of a theorem on isomorphisms of rings of bounded analytic functions. Vitushkin [13] has used analytic capacity in proving some interesting results in the theory of rational approximation.

5. It is natural to inquire how the situation changes if we no longer require that $S-K$ be connected. The obvious procedure would be to extend γ_c to all compact sets by the method sketched in Section 4. (Here $\mathfrak{F}=A(\Omega)$.) Of course, it is again true that $\gamma_c(K)=0$ implies that K is removable; however, the converse now fails. Indeed, let $\Gamma=\{|z|=1\}$. Clearly, Γ is removable; but $\gamma_c(\Gamma)=1$. It seems more profitable, therefore, to extend γ_c to arbitrary compacta simply by using Definition 3 (Section 2); according to this convention, $\gamma_c(\Gamma)=0$.

Since this paper was originally submitted, exciting developments in the qualitative theory of rational approximation of functions of one complex variable have occurred. These results, due to Vitushkin and Melnikov, are linked closely to some of the ideas in this paper; in particular, the c-capacity of certain sets is decisive in determining the possibility of rational approximation. For a detailed discussion of the theorems in question as well as an extensive bibliography see the monograph [15].

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References

1. L. Ahlfors, Bounded analytic functions, Duke J. Math., 14 (1947) 1–11.
2. L. Ahlfors and A. Beurling, Conformal invariants and function theoretic null sets, Acta Math., 83 (1950) 101–129.
3. R. Arens, The maximal ideals of certain function algebras, Pacific J. Math., 8 (1958) 641–648.
4. A. Besicovitch, On sufficient conditions for a function to be analytic and on behavior of analytic functions in the neighborhood of non-isolated singular points, Proc. London Math. Soc., (2) 32 (1931) 1–9.
5. A. Browder and J. Wermer, Some algebras of functions on an arc, J. Math. Mech., 12 (1963) 119–130.
6. A. Denjoy, Sur les fonctions analytiques uniformes à singularités discontinues, Comptes Rendus, 149 (1909) 258–260.
7. ———, Sur la continuité des fonctions analytiques singulières, Bull. Soc. Math. France, 60 (1932) 27–105.
8. M. Munroe, Introduction to Measure and Integration, Addison-Wesley, Reading, Mass., 1953.
9. D. Pompeiu, Sur la continuité des fonctions de variables complexes, Annales de Toulouse, (2) 7 (1905) 264–315.
10. W. Rudin, Some theorems on bounded analytic functions, Trans. Amer. Math. Soc., 78 (1955) 333–342.
11. P. Urysohn, Sur une fonction analytique partout continue, Fund. Math., 4 (1923) 144–150.
12. A. Vitushkin, The analytic capacity of a set and some of its properties, (in Russian) Doklady Akad. Nauk, 123 (1958) 778–781.
13. ———, Some theorems on the possibility of the uniform approximation of continuous functions by analytic functions, (in Russian) Doklady Akad. Nauk, 123 (1958) 959–962.
14. J. Wermer, Polynomial approximation on an arc in $\mathbf{C}^3$, Ann. of Math., 62 (1955) 269–270.
15. L. Zalcman, Analytic Capacity and Rational Approximation, Springer-Verlag, 1968.

SOME EXTENSIONS OF THE DOUGLAS-NEUMANN THEOREM FOR CONCENTRIC POLYGONS

YUNG-CHOW WONG, University of Hong Kong

1. Introduction. In this paper we study the polygons in the Euclidean plane which we regard as the complex plane by assuming that a rectangular coordinate system has been given in it. In general, we denote a point and the complex number it represents by the same letter.

By an n-gon, i.e. a *polygon* with n vertices, we understand a set of n points $(A_1, A_2, \cdots, A_n)$ which are taken in a definite cyclical order; thus, two n-gons $\mathbf{A}=(A_1, A_2, \cdots, A_n)$ and $\mathbf{A}'=(A_1', A_2', \cdots, A_n')$ are the same iff

$$A_j = A'_{j+k} \quad \text{for some } k: 0 \leqq k \leqq n-1, \quad \text{and} \quad j = 1, \cdots, n.$$

Here and in what follows, the lower indices used to distinguish the different vertices of an n-gon are taken modulo n. The *centroid* of the n-gon $\mathbf{A}=(A_1, A_2, \cdots, A_n)$ is the point $(\sum A_j)/n$ which is intrinsically connected with the n-gon $\mathbf{A}$. If $A_{m+j}=A_j$ $(j=1, 2, \cdots, n)$ for some factor $m(<n)$ of n, we say that the n-gon

$\mathbb{A}$ resolves into an m-gon described n/m times. An n-gon which does not resolve into an m-gon in this manner is called a *proper* n-gon. Two n-gons are said to be *similar* (resp. *indirectly similar*) if they have the same shape and the same orientation (resp. opposite orientation).

Let $\omega^0(=1), \omega^1, \omega^2, \cdots, \omega^{n-1}$, where $\omega = \exp(2\pi\sqrt{-1}/n)$, be the n-th roots of unity. An n-gon is called *regular of* ω^h*-type* $(0 \leqq h \leqq n-1)$ if it is similar to the n-gon $(1, \omega^h, \omega^{2h}, \cdots, \omega^{(n-1)h})$. The following facts can easily be verified. A regular n-gon of ω^h-type and one of ω^{n-h}-type are indirectly similar. A regular n-gon of ω^0-type collapses into an *n-tuple point*, i.e. a single point described n times. If $h \geqq 1$ and $(h, n) = m$ is the greatest common factor of h and n, then a regular n-gon of ω^h-type resolves into a proper (n/m)-gon described m times. If h is such that $1 \leqq h \leqq n/2$, then a regular n-gon of ω^h-type is convex or star-shaped according as h divides n or not. In particular, only regular n-gons of ω^1-type and ω^{n-1}-type are regular convex n-gons in the ordinary sense.

Recently, P. J. Kelly and D. Merriell [4] drew our attention to the following beautiful theorem on concentric polygons discovered independently by Jesse Douglas [2, 3] and B. H. Neumann [5, 6] about 1940 on the two sides of the Atlantic. (See also [1].)

Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ *be any n-gon. Using an* ω^k *(k fixed and* $1 \leqq k \leqq n-1$)' *we construct from* $\mathbb{A}$ *a new n-gon* $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ *whose vertices are the free vertices of the isosceles triangles* $(A_1, A_2, B_1), (A_2, A_3, B_2), \cdots, (A_n, A_1, B_n)$ *erected on the sides* $A_1A_2, A_2A_3, \cdots, A_nA_1$ *of* $\mathbb{A}$ *and all similar to the triangle* $(1, \omega^k, 0)$. *If this process is repeated with the n-gon* $\mathbb{B}$, *but with a different value of k, and so on until all the* $n-1$ *values of k have been used (in arbitrary order), then the final n-gon obtained collapses into the centroid of the original n-gon* $\mathbb{A}$. *Moreover, if* $\mathbb{A}$ *is a proper n-gon, and if the construction using a particular* ω^h $(1 \leqq h \leqq n-1)$ *is omitted, the final n-gon is a regular n-gon of* ω^h*-type.*

The purpose of this paper is to prove two generalizations and converses of this theorem. In Sections 2–3 we fix our notation and give the necessary preliminaries. Our main results are Theorems 4.1, 5.1 and 5.2 in Sections 4 and 5. Some consequences are given in Section 6 and the paper ends in Section 7 with a summary in the language of set theory. The author wishes to thank his young colleague Mr. H. F. Lai who read through a first draft of this paper and suggested a few improvements.

2. A preliminary result. Using a complex number $b(\neq 1)$, we can connect to any two points A_1, A_2 the point

$$A_{12} = \frac{-bA_1 + A_2}{-b + 1}. \tag{2.1}$$

Expressing b in terms of A_1, A_2 and A_{12}, we have

$$b = \frac{A_{12} - A_2}{A_{12} - A_1},$$

so that (2.1) is the condition for the triangle (A_1, A_2, A_{12}) to be similar to the triangle $(1, b, 0)$. In particular, if $b=\omega^k=\exp(2k\pi\sqrt{-1}/n)$, the triangle (A_1, A_2, A_{12}) is an isosceles triangle with base A_1A_2 and vertex angle equal to $2k\pi/n$; this triangle is positively or negatively oriented according as $2k\pi/n$ is $<\pi$ or $>\pi$.

From a given n-gon $\mathbf{A}=(A_1, A_2, \cdots, A_n)$ and using n complex numbers $a_1, a_2, \cdots, a_n$, none of which is equal to 1, let us construct an n-gon $\mathbf{B}=(B_1, B_2, \cdots, B_n)$ whose vertices are

$$B_j = \frac{-a_jA_j + A_{j+1}}{-a_j + 1}, \qquad j = 1, 2, \cdots, n.$$

Then,

$$\sum B_j = \sum A_j + \sum \frac{-A_j + A_{j+1}}{-a_j + 1}.$$

From this it follows at once that, for any $\mathbf{A}$ and fixed $a_1, a_2, \cdots, a_n$, the n-gon $\mathbf{B}$ is concentric with the n-gon $\mathbf{A}$ iff $a_1=a_2=\cdots=a_n$. Therefore, we have the following theorem of which the sufficiency part is known [3, 5]:

THEOREM 2.1. *Let $\mathbf{T}_j$ $(j=1, 2, \cdots, n)$ be n fixed triangles and $\mathbf{A}=(A_1, A_2, \cdots, A_n)$ any n-gon. If triangles (A_j, A_{j+1}, B_j) are constructed similar to $\mathbf{T}_j$, then for the n-gon $\mathbf{A}$ and the n-gon $\mathbf{B}=(B_1, B_2, \cdots, B_n)$ to be concentric for every choice of $\mathbf{A}$, it is necessary and sufficient that the fixed triangles $\mathbf{T}_j$ be all similar.*

3. Concentric n-gons. From now on we shall consider only concentric n-gons constructed from a given n-gon in the manner described in Theorem 2.1. Using a complex number $b_1(\neq 1)$, we construct from an n-gon $\mathbf{A}=(A_1, A_2, \cdots, A_n)$ a second n-gon A_{b_1} whose vertices are

$$A_{12} = \frac{-b_1A_1 + A_2}{-b_1 + 1}, \quad A_{23} = \frac{-b_1A_2 + A_3}{-b_1 + 1}, \cdots, A_{n1} = \frac{-b_1A_n + A_1}{-b_1 + 1},$$

so that the triangles (A_1, A_2, A_{12}), (A_2, A_3, A_{23}), $\cdots$, (A_n, A_1, A_{n1}) are all similar to the triangle $(1, b_1, 0)$, and the n-gons $\mathbf{A}$ and $\mathbf{A}_{b_1}$ are concentric. Using a second complex number $b_2(\neq 1)$, we construct in a similar manner from the n-gon $\mathbf{A}_{b_1}=(A_{12}, A_{23}, \cdots, A_{n1})$ a third n-gon $\mathbf{A}_{b_1b_2}$ whose vertices are

$$A_{123} = \frac{-b_2A_{12} + A_{23}}{-b_2 + 1}, \quad A_{234} = \frac{-b_2A_{23} + A_{34}}{-b_2 + 1}, \cdots, A_{n12} = \frac{-b_2A_{n1} + A_{12}}{-b_2 + 1}.$$

This construction can be continued and we obtain after p steps the following sequence of concentric n-gons:

$$\mathbf{A}, \ \mathbf{A}_{b_1}, \ \mathbf{A}_{b_1,b_2}, \cdots, \mathbf{A}_{b_1,b_2,\cdots,b_p},$$

where $b_1, b_2, \cdots, b_p$ are p complex numbers none of which is equal to 1.

It is easy to verify by induction that if

$$\mathbf{A}_{b_1,b_2,\cdots,b_p} \equiv (R_1, R_2, \cdots, R_n),$$

then

$$(3.1)\qquad \begin{cases} R_1 = d_0A_1 + d_1A_2 + \cdots + d_pA_{p+1}, \\ R_2 = d_0A_2 + d_1A_3 + \cdots + d_pA_{p+2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ R_n = d_0A_n + d_1A_{n+1} + \cdots + d_pA_{p+n}, \end{cases}$$

where the coefficients $d_0, d_1, \cdots, d_p$ are equal to those in the polynomial

$$(3.2)\qquad \phi(t) \equiv \prod_{i=1}^{p} \left(\frac{t-b_i}{1-b_i}\right) \equiv d_0 + d_1t + \cdots + d_pt^p.$$

Since $A_u = A_v$ if $u \equiv v(\text{mod } n)$, it is seen from equation (3.1) that *a cyclic permutation of the vertices of* $\mathbf{A}$ *induces a cyclic permutation of the vertices of* $\mathbf{A}_{b_1,b_2,\cdots,b_p}$; this is also obvious from the way $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ is constructed from $\mathbf{A}$.

A consequence of (3.1) and (3.2) is that

$$\mathbf{A}_{b_1,b_2,\ldots,b_p} = \mathbf{A}_{b_{\sigma(1)},b_{\sigma(2)},\ldots,b_{\sigma(p)}},$$

where σ is any permutation of $1, 2, \cdots, p$. Geometrically, this means that *the final n-gon* $\mathbf{A}_{b_1,b_2,\cdots,b_p}$, *which is constructed from* $\mathbf{A}$ *by using the complex numbers* $b_1, b_2, \cdots, b_p$, *is the same no matter in what order these complex numbers are used.*

Let $p+1 = rn+n'$, where r, n' are integers such that $r \geqq 0$ and $0 \leqq n' < n$. Then because $A_u = A_v$ if $u \equiv v(\text{mod } n)$, we can write formulas (3.1) as

$$(3.3)\qquad \begin{cases} R_1 = f_1A_1 + f_2A_2 + \cdots + f_nA_n, \\ R_2 = f_1A_2 + f_2A_3 + \cdots + f_nA_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ R_n = f_1A_n + f_2A_1 + \cdots + f_nA_{n-1}, \end{cases}$$

where

$$(3.4)\qquad \begin{aligned} f_\lambda &= \sum_{\nu=0}^{r} d_{\nu n+\lambda-1}, & \lambda &= 1, 2, \cdots, n'; \\ f_\mu &= \begin{Bmatrix} \sum_{\nu=0}^{r-1} d_{\nu n+\mu-1} & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{Bmatrix}, & \mu &= n'+1, \cdots, n. \end{aligned}$$

It is easy to see that the coefficients $f_1, f_2, \cdots, f_n$ in (3.3) can be obtained from the polynomial (3.2) in the following way: If we put $t = \tau$ in (3.2), where τ is any nth root of unity, the result is

$$\phi(\tau) = \prod_{i=1}^{p} \left(\frac{\tau - b_i}{1 - b_i}\right) = f_1 + f_2\tau + \cdots + f_n\tau^{n-1}. \tag{3.5}$$

In particular, since

$$\phi(1) = \prod_{i=1}^{p} \left(\frac{1 - b_i}{1 - b_i}\right) = 1,$$

we have

$$f_1 + f_2 + \cdots + f_n = 1. \tag{3.6}$$

J. Douglas [3] has proved that given any n complex numbers $f_1, f_2, \cdots, f_n$ satisfying condition (3.6), there exist $n-1$ complex numbers $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{n-1}$, none equal to 1 and unique except for their order, such that for any n-gon $\mathbf{A} = (A_1, A_2, \cdots, A_n)$, the n-gon $\mathbf{A}_{\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{n-1}}$ is precisely the n-gon $(R_1, R_2, \cdots, R_n)$ defined by (3.3). From this we easily deduce that if the p complex numbers $b_1, b_2, \cdots, b_p$ which we use in constructing the n-gon $\mathbf{A}_{b_1, b_2, \cdots, b_p}$ from any n-gon $\mathbf{A}$, are taken arbitrarily, the functions $f_1, f_2, \cdots, f_n$ of the b's as defined by (3.2) and (3.4) are not subject to any condition other than (3.6); moreover, given any $p(\geqq n)$ complex numbers $b_1, b_2, \cdots, b_p$, none equal to 1, then there exist $n-1$ complex numbers $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{n-1}$, none equal to 1 and unique except for their order, such that

$$\mathbf{A}_{b_1, b_2, \ldots, b_p} = \mathbf{A}_{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{n-1}}$$

for every choice of $\mathbf{A}$. These results will not be needed in this paper, but they help to give a better understanding of our results.

4. A converse and generalization of the Douglas-Neumann theorem. The Douglas-Neumann theorem quoted in Section 1 can be restated thus: If

$$\{b_1, b_2, \cdots, b_{n-1}\} = \{\omega, \omega^2, \cdots, \omega^{n-1}\},$$

then for every n-gon $\mathbf{A}$, the n-gon $\mathbf{A}_{b_1, b_2, \cdots, b_{n-1}}$ is an n-tuple point. If

$$\{b_1, b_2, \cdots, b_{n-2}\} = \{\omega, \cdots, \hat{\omega}^h, \cdots, \omega^{n-1}\},$$

(where the roof ^ indicates that the term under it should be omitted) then, for every proper n-gon $\mathbf{A}$, the n-gon $\mathbf{A}_{b_1, b_2, \cdots, b_{n-2}}$ is a regular n-gon of ω^h-type.

We observe that, in each case, the final n-gon is similar to some fixed n-gon. In this section, we shall prove the following converse theorem.

THEOREM 4.1. (i) *Let* $\mathbf{A}$ *be any n-gon and let* $b_1, b_2, \cdots, b_p$ *be some fixed complex numbers none of which is equal to* 1. *If there exists a fixed n-gon* $\mathbf{C}$ *such that the n-gon* $\mathbf{A}_{b_1, b_2, \cdots, b_p}$ *is similar to* $\mathbf{C}$ *for every choice of* A, *then* $\mathbf{C}$ *must be an n-tuple point or a regular n-gon of* ω^h*-type* (*for some* h: $1 \leqq h \leqq n-1$).

(ii) *The n-gon* $\mathbf{A}_{b_1, b_2, \cdots, b_p}$ *is an n-tuple point for every choice of* A, *iff* $\{b_1, b_2, \cdots, b_p\} \supset \{\omega, \omega^2, \cdots, \omega^{n-1}\}$.

(iii) *For a fixed integer h, $1 \leqq h \leqq n-1$, and for every proper n-gon A, the n-gon $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ is a regular n-gon of ω^h-type iff* $\{b_1, b_2, \cdots, b_p\} \supset \{\omega, \cdots, \hat{\omega}^h, \cdots, \omega^{n-1}\}$ *but* $\not\supset \{\omega^h\}$.

We note that (ii) implies that $p \geqq n-1$ and (iii) implies that $p \geqq n-2$.

Proof. Let $\mathbf{A} = (A_1, A_2, \cdots, A_n)$. Then $\mathbf{A}_{b_1,b_2,\cdots,b_p} = (R_1, R_2, \cdots, R_n)$ is determined by (3.2), (3.3) and (3.4). If $\mathbf{C} = (C_1, C_2, \cdots, C_n)$ is a fixed n-gon, then the condition for $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ to be similar to $\mathbf{C}$ is that

$$(4.1) \qquad \frac{R_1 - R_2}{R_n - R_1} = \frac{C_1 - C_2}{C_n - C_1}, \quad \frac{R_2 - R_3}{R_1 - R_2} = \frac{C_2 - C_3}{C_1 - C_2}, \cdots, \frac{R_n - R_1}{R_{n-1} - R_n} = \frac{C_n - C_1}{C_{n-1} - C_n}.$$

Using the values of the R_j ($j = 1, 2, \cdots, n$) from (3.3) in $(R_j - R_{j+1})/(R_{j-1} - R_j)$, we have by (4.1) that

$$\frac{(f_1 - f_n)A_j + (f_2 - f_1)A_{j+1} + \cdots + (f_n - f_{n-1})A_{j+n-1}}{(f_1 - f_n)A_{j-1} + (f_2 - f_1)A_j + \cdots + (f_n - f_{n-1})A_{j+n-2}} = \frac{C_j - C_{j+1}}{C_{j-1} - C_j}, \qquad j = 1, 2, \cdots, n.$$

For these relations to hold for every choice of $A_1, A_2, \cdots, A_n$, we must have

$$(4.2) \qquad \frac{f_1 - f_n}{f_2 - f_1} = \frac{f_2 - f_1}{f_3 - f_2} = \cdots = \frac{f_{n-1} - f_{n-2}}{f_n - f_{n-1}} = \frac{f_n - f_{n-1}}{f_1 - f_n} = \frac{C_j - C_{j+1}}{C_{j-1} - C_j}, \qquad j = 1, 2, \cdots, n.$$

These conditions for $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ to be similar to $\mathbf{C}$ for every $\mathbf{A}$ are equivalent to

$$(4.3) \qquad f_1 - f_n = \rho(f_2 - f_1), f_2 - f_1 = \rho(f_3 - f_2), \cdots, f_n - f_{n-1} = \rho(f_1 - f_n),$$

$$(4.4) \qquad \frac{C_1 - C_2}{C_n - C_1} = \frac{C_2 - C_3}{C_1 - C_2} = \cdots = \frac{C_n - C_1}{C_{n-1} - C_n} = \rho,$$

where ρ is some complex number.

From (4.3) it follows that

$$(4.5) \qquad (f_1 - f_n)(f_2 - f_1) \cdots (f_n - f_{n-1})(\rho^n - 1) = 0.$$

There are two possibilities: (a) at least one of the first n factors in (4.5) is zero, and (b) none of these factors is zero.

In the case (a), let us say that $f_1 - f_n = 0$. Then it follows step by step from (4.3) that all the first n factors in (4.5) are zero, so that

$$f_1 = f_2 = \cdots = f_n = f \quad \text{(say)},$$

and consequently, by (3.3),

$$R_1 = f(A_1 + A_2 + \cdots + A_n) = R_2 = R_3 = \cdots = R_n.$$

Therefore in this case $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ is an n-tuple point.

In the case (b), we have $\rho^n - 1 = 0$, so that ρ is one of the nth roots of unity.

But $\rho \neq 1$; for, if $\rho = 1$, we would have

$$f_1 - f_n = f_2 - f_1 = f_3 - f_2 = \cdots = f_n - f_{n-1} = b \quad \text{(say)},$$

and addition gives $b = 0$, which contradicts our assumption in this case. Now let $\rho = \omega^h$ (for some fixed h: $1 \leqq h \leqq n-1$). Comparison of (4.4) with

$$\frac{1 - \omega^h}{\omega^{(n-1)h} - 1} = \frac{\omega^h - \omega^{2h}}{1 - \omega^h} = \cdots = \frac{\omega^{(n-1)h} - 1}{\omega^{(n-2)h} - \omega^{(n-1)h}} = \omega^h$$

shows that $\mathbf{C} = (C_1, C_2, \cdots, C_n)$ is a regular n-gon of ω^h-type. This completes the proof of part (i) of Theorem 4.1.

To prove part (ii) of the theorem, we observe that by (3.3) the n-gon $\mathbf{A}_{b_1, b_2, \cdots, b_p}$ is an n-tuple point iff

$$f_1 = f_2 = \cdots = f_n = f \quad \text{(say)}, \tag{4.6}$$

where $f \neq 0$ because of (3.6). Assume that (4.6) holds. If $\tau (\neq 1)$ is any nth root of unity, then by (3.5),

$$\phi(\tau) = \prod_{i=1}^{p} \left(\frac{\tau - b_i}{1 - b_i} \right) = f(1 + \tau + \cdots + \tau^{n-1}) = 0.$$

From this it follows $\{b_1, b_2, \cdots, b_p\} \supset \{\omega, \omega^2, \cdots, \omega^{n-1}\}$. Conversely, let us assume that this is true. Then $\omega, \omega^2, \cdots, \omega^{n-1}$ are all roots of the equation

$$\phi(t) \equiv \prod_{i=1}^{p} \left(\frac{t - b_i}{1 - b_i} \right) = 0. \tag{4.7}$$

Therefore, by (3.5), if $\tau (\neq 1)$ is any nth root of unity, then

$$\phi(\tau) = f_1 + f_2 \tau + \cdots + f_n \tau^{n-1} = 0.$$

This requires that (4.6) must hold, and the proof of (ii) is complete.

To prove (iii), we assume that $\mathbf{C} = (C_1, C_2, \cdots, C_n)$ is a regular n-gon of ω^h-type (for some fixed h: $1 \leqq h \leqq n-1$), i.e. $(C_1, C_2, \cdots, C_n) = (1, \omega^h, \cdots, \omega^{(n-1)h})$. Then condition (4.4) reduces to $\rho = \omega^h$; and consequently, condition (4.3) is equivalent to

$$f_n - f_1 \omega^h = f_1 - f_2 \omega^h = \cdots = f_{n-1} - f_n \omega^h = a \quad \text{(say)}, \tag{4.8}$$

where a is some complex number which may be zero. On account of this and (3.5), if $\tau (\neq 1)$ is any nth root of unity, then

$$\begin{aligned} \phi(\tau)(\tau - \omega^h) &= (f_1 + f_2 \tau + \cdots + f_n \tau^{n-1})(\tau - \omega^h) \\ &= (f_n - f_1 \omega^h) + (f_1 - f_2 \omega^h)\tau + \cdots + (f_{n-1} - f_n \omega^h)\tau^{n-1} \\ &= a(1 + \tau + \cdots + \tau^{n-1}) = 0. \end{aligned}$$

From this it follows that $\omega, \omega^2, \cdots, \omega^{n-1}$, with the possible exception of ω^h, are all roots of equation (4.7). But if ω^h were also a root of this equation, then by

part (ii) of the theorem just proved, the n-gon $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ would be an n-tuple point and not a regular n-gon of ω^h-type. This proves that if $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ is a regular n-gon of ω^h-type, then $\{b_1, b_2, \cdots, b_p\} \supset \{\omega, \cdots, \hat{\omega}^h, \cdots, \omega^{n-1}\}$ but $\not\supset \{\omega^h\}$.

To prove the converse part of (iii), let us assume the last relation above. Then $\omega, \omega^2, \cdots, \omega^{n-1}$ are all roots of the equation $\phi(t)(t-\omega^h)=0$, and this implies that

$$\begin{aligned}\phi(\tau)(\tau - \omega^h) &= (f_1 + f_2\tau + \cdots + f_n\tau^{n-1})(\tau - \omega^h) \\ &= (f_n - f_1\omega^h) + (f_1 - f_2\omega^h)\tau + \cdots + (f_{n-1} - f_n\omega^h)\tau^{n-1}\end{aligned}$$

must be zero for each of the values $\omega, \omega^2, \cdots, \omega^{n-1}$ of τ. Consequently, (4.8) must hold. On account of this, if we put

$$(C_1, C_2, \cdots, C_n) = (1, \omega^h, \cdots, \omega^{(n-1)h}),$$

then condition (4.2) for $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ to be a regular n-gon of ω^h-type is satisfied. This completes the proof of our theorem.

5. Another converse and generalization of the Douglas-Neumann theorem. We observe that in the Douglas-Neumann theorem, the final n-gons obtained are regular n-gons of ω^h-type ($h=0, 1, 2, \cdots, n-1$), and some of these regular n-gons have the property that two or more of their vertices coincide. In this section we study the condition that the n-gon $\mathbf{A}_{b_1,b_2,\cdots,b_p}$, constructed from an arbitrary n-gon $\mathbf{A}$ by using some fixed complex numbers $b_1, b_2, \cdots, b_p$, should always have the property that some of their vertices coincide. Our results are the following two theorems; the first is preliminary in nature and the second is another converse and generalization of the Douglas-Neumann theorem.

THEOREM 5.1. *Let the n-gon $\mathbf{R}=(R_1, R_2, \cdots, R_n)$ be related to the n-gon $\mathbf{A}=(A_1, A_2, \cdots, A_n)$ by*

$$\begin{cases} R_1 = f_1A_1 + f_2A_2 + \cdots + f_nA_n, \\ R_2 = f_1A_2 + f_2A_3 + \cdots + f_nA_1, \\ \cdots\cdots\cdots\cdots\cdots \\ R_n = f_1A_n + f_2A_1 + \cdots + f_nA_{n-1}, \end{cases}$$

where $f_1, f_2, \cdots, f_n$ are fixed complex numbers. Then, the condition that

$$R_1 = R_{q+1} \quad (\textit{for some fixed } q\colon 1 \leqq q \leqq n-1)$$

is equivalent to the condition that $R_1=R_{m+1}$, where $m=(n, q)$, and this implies that

$$\begin{cases} R_1 = R_{m+1} = \cdots = R_{(r-1)m+1}, \\ R_2 = R_{m+2} = \cdots = R_{(r-1)m+2}, \\ \cdots\cdots\cdots\cdots\cdots \\ R_m = R_{2m} = \cdots = R_{rm}(=R_n), \quad \textit{where } r = n/m. \end{cases} \tag{5.1}$$

Thus, the condition $R_1 = R_{q+1}$ *(for some* q: $1 \leqq q \leqq n-1$) *is equivalent to the condition that the n-gon* $\mathbf{R}$ *resolves into an m-gon described* n/m *times, where* $m = (n, q)$ *is the greatest common factor of* n, q.

It should be noted here that, for some $\mathbf{A}$ and some f's, the m-gon $(R_1, R_2, \cdots, R_m)$ may not be a proper m-gon, but this is not a consequence of the condition $R_1 = R_{m+1}$.

It follows from Theorem 5.1 that in studying the condition $R_1 = R_{q+1}$, we need consider only the cases where $q = m(\leqq n-1)$ is a factor of n, so that $1 \leqq m \leqq n/2$.

THEOREM 5.2. *Let m be a factor of n such that* $1 \leqq m \leqq n/2$, *and let* $b_1, b_2, \cdots, b_p$ *be fixed complex numbers none of which is equal to* 1. *Then in order that for every proper n-gon* $\mathbf{A}$, *the n-gon* $\mathbf{A}_{b_1, b_2, \cdots, b_p}$ *shall resolve into an m-gon described* n/m *times, it is necessary and sufficient that the set of complex numbers* $\{b_1, b_2, \cdots, b_p\}$ *contains all those n-th roots of unity which are not at the same time m-th roots of unity.*

We note that for the special case $p = n-1$ and $m = 1$, the sufficiency of the condition in Theorem 5.2 is precisely the first part of the Douglas-Neumann theorem.

Proof of Theorem 5.1. Since

$$R_1 = f_1A_1 + f_2A_2 + \cdots + f_nA_n,$$
$$R_{q+1} = f_1A_{q+1} + f_2A_{q+2} + \cdots + f_nA_{q+n},$$

the condition for $R_1 = R_{q+1}$ (for arbitrary $\mathbf{A}$) is that $f_1 = f_{q+1}$, $f_2 = f_{q+2}, \cdots, f_n = f_{q+n} (= f_q)$. But this is precisely the condition for $R_k = R_{q+k}$ (for any k, $1 \leqq k < n$). Hence $R_1 = R_{q+1}$ implies $R_k = R_{q+k}$ ($k = 1, 2, \cdots, n-1$). This fact is also a direct consequence of the condition that $R_1 = R_{q+1}$ should be true for arbitrary $\mathbf{A}$; for we can take $(A_k, A_{k+1}, \cdots, A_{k-1})$ as $\mathbf{A}$ instead of $(A_1, A_2, \cdots, A_n)$.

Let $(n, q) = m$ be the greatest common factor of n and q, and $n = rm$, $q = sm$. Then

$$(r, s) = 1, \quad \text{and} \quad rq = rsm = sn.$$

We now prove that $R_{m+1} = R_1$. In fact, since $(r, s) = 1$, there exist two integers a and b such that

$$as - br = \pm 1, \quad 0 \leqq a < r, \quad 0 \leqq b < s.$$

Multiplication of the first equation by m gives

$$aq - bn = \pm m.$$

Therefore, we have

$$R_{m+1} = \begin{cases} R_{aq-bn+1} = R_{aq+1} = R_1, & \text{or} \\ R_{-aq+bn+1} = R_{(r-a)q+1} = R_1, \end{cases}$$

since $rq = sn$ and $0 \leqq a < r$. Thus, $R_{m+1} = R_1$ as was to be proved. From this it fol-

lows (as before for q) that $R_k = R_{m+k}$ $(k=1, 2, \cdots, n-1)$, and consequently, the n points $R_1, R_2, \cdots, R_n$ have the properties expressed in (5.1).

Proof of Theorem 5.2. Condition (5.1) is easily seen to be equivalent to the condition

$$\begin{cases} f_1 = f_{m+1} = \cdots = f_{(r-1)m+1}, \\ f_2 = f_{m+2} = \cdots = f_{(r-1)m+2}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_m = f_{2m} = \cdots = f_{rm}(=f_n). \end{cases} \tag{5.2}$$

Assume that (5.1) holds. Then (5.2) holds, and we have from (3.5) that if τ is any nth root of unity, then

$$\begin{aligned} \phi(\tau) &= f_1 + f_2\tau + \cdots + f_n\tau^{n-1} \\ &= (1 + \tau^m + \tau^{2m} + \cdots + \tau^{(r-1)m})(f_1 + f_2\tau + \cdots + f_m\tau^{m-1}). \end{aligned} \tag{5.3}$$

Since $1-t^n = 1-t^{rm} = (1-t^m)(1+t^m+t^{2m}+ \cdots +t^{(r-1)m})$, the roots of $1+t^m+t^{2m}+ \cdots +t^{(r-1)m}=0$ are those nth roots of unity which are not at the same time mth roots of unity. Let τ_k $(k=1, 2, \cdots, n-m)$ be any such root. Then

$$1 - \tau_k^n = 0 \quad \text{and} \quad 1 + \tau_k^m + \tau_k^{2m} + \cdots + \tau_k^{(r-1)m} = 0,$$

so that by (3.5) and (5.3)

$$\prod_{i=1}^{p}\left(\frac{\tau_k - b_i}{1 - b_i}\right) = \phi(\tau_k) = 0, \qquad k = 1, 2, \cdots, n - m.$$

Therefore, $\{b_1, b_2, \cdots, b_p\} \supset \{\tau_1, \tau_2, \cdots, \tau_{n-m}\}$, and the necessity of the condition in Theorem 5.2 is proved.

Conversely, assume that the last relation above holds. Let

$$\psi(t) \equiv f_1 + f_2 t + \cdots + f_n t^{n-1}.$$

Then, for each τ_k $(k=1, 2, \cdots, n-m)$,

$$\psi(\tau_k) = f_1 + f_2\tau_k + \cdots + f_n\tau_k^{n-1} = \phi(\tau_k) = \prod_{i=1}^{p}\left(\frac{\tau_k - b_i}{1 - b_i}\right) = 0.$$

Therefore, $\psi(t)$ must be of the form

$$\psi(t) = (1 + t^m + t^{2m} + \cdots + t^{(r-1)m})(a_1 + a_2 t + \cdots + a_m t^{m-1}).$$

From this it follows that $f_1, f_2, \cdots, f_n$ satisfy condition (5.2), and consequently the n-gon $\mathrm{A}_{b_1, b_2, \cdots, b_p}$ has the properties expressed in (5.1). Thus Theorem 5.2 is completely proved.

6. Some consequences of Theorem 5.2. For convenience, we shall denote by N, M, and M_a the sets of nth, mth and m_a-th roots of unity, respectively, and by $X \backslash Y$ the set of elements which belong to the set X but not to the set Y.

We first prove that if $\{b_1, b_2, \cdots, b_{n-1}\} = \{\omega, \omega^2, \cdots, \omega^{n-1}\}$ and b_{n-1} is a primitive nth root of unity, say, ω^h, then, for any proper n-gon $\mathbf{A}$, the n-gon $\mathbf{A}_{b_1,b_2,\cdots,b_{n-2}}$ is a proper n-gon.

In fact, if $\mathbf{A}_{b_1,b_2,\cdots,b_{n-2}}$ resolves into an m-gon described n/m times, then by Theorem 5.2

$$N\backslash\{1, \omega^h\} = \{b_1, b_2, \cdots, b_{n-2}\} \supset N\backslash M.$$

Therefore $M \supset \{\omega^h\}$. But this is not possible since ω^h, being a primitive nth root of unity, is not an mth root of unity for any $m<n$. Hence our assertion is proved.

The following is a less direct consequence of Theorem 5.2. Let n be a positive integer. A positive integer $m(1 \leqq m < n)$ is a factor of n if it divides n, so that m is always $\leqq n/2$. A factor m of n is called a *maximal factor* of n if it is not a factor of a factor of n, or equivalently, if n/m is a prime number.

For a given positive integer n, let the following sequence of positive integers

$$1, m_1, m_2, \cdots, m_r, n \tag{6.1}$$

be such that each number is a maximal factor of the next number, and let

$$1, \tau_1, \tau_2, \cdots, \tau_{m_1-1}, \tau_{m_1}, \tau_{m_1+1}, \cdots, \tau_{m_2-1}, \cdots, \tau_{m_r}, \tau_{m_r+1}, \cdots, \tau_{n-1} \tag{6.2}$$

be the nth roots of unity arranged in such a way that the first m_1 are the roots of $t^{m_1}-1=0$, the first m_2 are the roots of $t^{m_2}-1=0$, and so on. Let us construct from any proper n-gon $\mathbf{A}$ the sequence of n-gons

$$\mathbf{A}_{\tau_{n-1}}, \mathbf{A}_{\tau_{n-1},\tau_{n-2}}, \cdots, \mathbf{A}_{\tau_{n-1},\tau_{n-2},\cdots,\tau_1} \tag{6.3}$$

by using the $n-1$ nth roots of unity $\tau_{n-1}, \tau_{n-2}, \cdots, \tau_1$ in the reverse order to that given in (6.2). Then, we assert that

the first $n-m_r-1$ n-gons are proper n-gons,
the next m_r-m_{r-1} n-gons are proper m_r-gons described n/m_r times,
the next $m_{r-1}-m_{r-2}$ n-gons are proper m_{r-1}-gons described n/m_{r-1} times,
. . .
the m_1-1 n-gons just before the last one are proper m_1-gons described n/m_1 times, and
the last n-gon collapses into the centroid of $\mathbf{A}$.

To prove our assertion, we consider the n-gon

$$\mathbf{A}_{\tau_{n-1},\tau_{n-2},\cdots,\tau_{m_a+1}} (1 \leqq a \leqq r)$$

constructed from $\mathbf{A}$ by using the last $n-m_a-1$ nth roots of unity as given in (6.2). By Theorem 5.2, we know that this n-gon resolves into an m_{a+1}-gon $\mathbf{B}$ described n/m_{a+1} times but does not resolve into an m_a-gon described n/m_a times. Our assertion will be proved if we can show that $\mathbf{B}$ is a proper m_{a+1}-gon. Assume that this m_{a+1}-gon $\mathbf{B}$ resolves into an m-gon described m_{a+2}/m times, where $m(<m_{a+1})$ is a factor of m_{a+1}. Then by Theorem 5.2, we must have

$$\{\tau_{m_a+1}, \cdots, \tau_{n-1}\} \supset N \backslash M.$$

But

$$N\backslash M_a = \{\tau_{m_a}, \tau_{m_a+1}, \cdots, \tau_{n-1}\} \supset \{\tau_{m_a+1}, \cdots, \tau_{n-1}\}.$$

Therefore, $N\backslash M_a \supset N\backslash M$, i.e. $M_a \subset M$. This means that all the roots of $t^{m_a}-1=0$ are roots of $t^m-1=0$. Hence m_a is equal to m or is a factor of m. But m cannot be equal to m_a; otherwise, $\mathbf{A}_{\tau_{n-1},\tau_{n-2},\cdots,\tau_{m_a+1}}$ would resolve into an m_a-gon described n/m_a times (see remark above). Nor can m_a be a factor of m; otherwise, since m is itself a factor of m_{a+1}, we would have a contradiction to the hypothesis that m_a is a maximal factor of m_{a+1}. Therefore, the m_{a+1}-gon $\mathbf{B}$ cannot resolve into an m-gon described m_{a+1}/m times, and is consequently a proper m_{a+1}-gon described n/m_{a+1} times, as was to be proved.

As an example, we consider the case $n=15$. There are two sequences of the form (6.1):

$$1, 3, 15; \qquad 1, 5, 15.$$

If $\omega=\exp(2\pi\sqrt{-1}/15)$, the 15-th roots of unity arranged as in (6.2) are

$$\begin{gathered} 1, \omega^5, \omega^{10}, \omega^1, \omega^2, \cdots, \hat{\omega}^5, \cdots, \hat{\omega}^{10}, \cdots, \omega^{14}; \\ 1, \omega^3, \omega^6, \omega^9, \omega^{12}, \omega^1, \omega^2, \hat{\omega}^3, \cdots, \hat{\omega}^6, \cdots, \hat{\omega}^9, \cdots, \hat{\omega}^{12}, \cdots, \omega^{14}. \end{gathered} \tag{6.4}$$

Here a roof ^ indicates that the term under it should be omitted. Let us now construct from any proper n-gon $\mathbf{A}$ the sequence of n-gons (6.3) by using the 15-th roots $\omega, \omega^2, \cdots, \omega^{14}$ of unity in the order reverse to that given in (6.4). Then in the first case, the first eleven 15-gons are proper 15-gons, the next two 15-gons resolve into proper 3-gons described five times, and the last 15-gon collapses into the centroid of $\mathbf{A}$. In the second case, the first nine 15-gons are proper 15-gons, the next four 15-gons resolve into proper 5-gons described three times, and the last 15-gon collapses into the centroid of $\mathbf{A}$.

These results show that although the final n-gon $\mathbf{A}_{b_1,b_2,\cdots,b_p}$ is the same no matter in what order the complex numbers $b_1, b_2, \cdots, b_p$ are used, the nature of the sequence of n-gons $\mathbf{A}_{b_1}, \mathbf{A}_{b_1,b_2}, \cdots$ leading to it may be quite different. In some sense, the two sequences of 15-gons constructed in accordance with (6.4) are "minimal" sequences.

7. Remarks. The results in this paper can be conveniently summarized in the language of set theory. Let P be the set of all n-gons in the plane, R the equivalence relation in P defined by similarity of n-gons, and $P'=P/R$ the quotient set whose elements are classes of similar n-gons. We denote by $[\mathbf{A}]$ the element of P' one of whose representatives is the n-gon $\mathbf{A}$; in particular, we denote by $[\mathbf{W}_0]$ and $[\mathbf{W}_h]$ $(1 \leq h \leq n-1)$ the class of the n-tuple points and the class of regular n-gons of ω^h-type, respectively.

To any complex number $b(\neq 1)$, there corresponds the mapping g_b: $P \to P$ defined by

$$\mathbf{A} = (A_1, A_2, \cdots, A_n) \to \mathbf{A}_b$$
$$= \left(\frac{-bA_1 + A_2}{-b+1}, \frac{-bA_2 + A_3}{-b+1}, \cdots, \frac{-bA_n + A_1}{-b+1}\right).$$

This mapping induces a mapping g_b: $P' \to P'$ (again denoted by g_b) defined by $[\mathbf{A}] \to [\mathbf{A}_b]$. We denote by

$$g_{b_1,b_2,\cdots,b_p}: \qquad P' \to P'$$

the product of the p mappings $g_{b_1}, g_{b_2}, \cdots, g_{b_p}$.

Our results can now be summarized as follows:

(1) The mapping $g_{b_1,b_2,\cdots,b_p}$ is symmetric with respect to the b's.

(2) If $g_{b_1,b_2,\cdots,b_p}$ maps every proper $[\mathbf{A}]$ into the same $[\mathbf{C}]$, then $[\mathbf{C}]$ must be $[\mathbf{W}_0]$ or one of the $[\mathbf{W}_h]$'s $(1 \leqq h \leqq n-1)$.

(3) (a) $g_{b_1,b_2,\cdots,b_p}$ maps every $[\mathbf{A}]$ into $[\mathbf{W}_0]$ iff $\{b_1, b_2, \cdots, b_p\} \supset \{\tau: \tau^n = 1, \tau \neq 1\}$.

(b) $g_{b_1,b_2,\cdots,b_p}$ maps every proper $[\mathbf{A}]$ into $[\mathbf{W}_h]$ (for some h: $1 \leqq h \leqq n-1$) iff $\{b_1, b_2, \cdots, b_p\} \supset \{\tau: \tau^n = 1, \tau \neq 1, \omega^h\}$.

(4) If $m (1 \leqq m \leqq n/2)$ is a factor of n, $g_{b_1,b_2,\cdots,b_p}$ maps each proper $[\mathbf{A}]$ into a $[\mathbf{B}]$ (which depends on $[\mathbf{A}]$ as well as on the b's) whose representative $\mathbf{B}$ is an m-gon described n/m times iff $\{b_1, b_2, \cdots, b_p\} \supset \{\tau: \tau^n = 1, \tau^m \neq 1\}$.

References

1. H. F. Baker, A remark on polygons, J. London Math Soc., 17 (1942) 162–164.
2. Jesse Douglas, Geometry of polygons in the complex plane, J. Math. Phys., 19 (1940) 93–130.
3. ———, On linear polygon transformations, Bull. Amer. Math. Soc., 46 (1940) 551–560.
4. P. J. Kelly and D. Merriell, Concentric polygons, this Monthly, 71 (1964) 37–41.
5. B. H. Neumann, Some remarks on polygons, J. London Math. Soc., 16 (1941) 230–245.
6. ———, A remark on polygons, J. London Math. Soc., 17 (1942) 165–166.

A PROBLEM ABOUT LINES AND OVALS

RALPH ALEXANDER, University of Illinois

1. Introduction. The following problem was suggested by the paper of H. Davenport [5]:

Suppose n straight lines intersect a unit square I. How large a square S (with sides parallel to those of I) is one assured of finding in I so that no line enters the interior of S?

Professor Davenport's conjecture is that a square of side $1/(n+1)$ will always exist. By choosing the lines to be equispaced and parallel, it is quickly seen that a square of side $1/(n+1)$ is the largest one could hope to find in general. In the course of this article we will show that S is of side at least $.82/(n+1)$.

However, our main purpose is to consider the following convenient generalization of Davenport's problem:

Suppose n straight lines intersect an oval K. What is the largest number μ for which there generally exists a dilation with ratio μ and center O, contained in K, so that no line enters the interior of the image of K, denoted by μK, in this dilation? (For our purposes, an oval is a closed, bounded, convex, planar set with nonempty interior.)

The conjecture $\mu = 1/(n+1)$, independently of K, seems plausible. In fact, we will prove that this conjecture is true if and only if the unproven affine planking conjecture of T. Bang is true. Also we will show that there is an absolute positive constant c, independent of K, for which $\mu \geqq c/(n+1)$; and will give a method, based on the work of Bang, for estimating μ in specific cases. We conclude with a further discussion of problems related to Bang's conjecture.

2. The Tarski and Bang plank problems. Call the closed region between two parallel lines a strip. If K is a closed convex body in the plane, let $w(\alpha)$ be the width of the narrowest strip having the direction α which covers K. Let $w = \min w(\alpha)$. In his memorable papers [1], [2] T. Bang solved the Tarski plank-problem by proving that should K be covered by a collection of strips, the sum of the widths of the strips must be at least w. Elegant refinements of his method have been given by Fenchel [6] and, especially, Bognar [4]. Bang's method, as he notes in [3], actually gives a stronger result than is demanded by the Tarski problem. Let $v(\alpha)$ be the maximum of lengths of chords in K which have direction orthogonal to α. (It is well known that $w \leqq v(\alpha) \leqq w(\alpha)$ for each α.)

THEOREM 1. (Bang) *Suppose the planar convex body K is covered by n strips of widths $s_1, \cdots, s_n$ and directions $\alpha_1, \cdots, \alpha_n$. Then*

$$s_1/v(\alpha_1) + s_2/v(\alpha_2) + \cdots + s_n/v(\alpha_n) \geqq 1.$$

It is Bang's conjecture that the previous inequality remains true if $v(\alpha_i)$ is replaced by $w(\alpha_i)$ for each i. Later, Ohmann [9] made the same conjecture independently. Unfortunately, his proposed proof contains an error. The affirmation of the conjecture would be pleasing, since the ratios $s_i/w(\alpha_i)$, which we call relative widths, are affine invariants.

LEMMA 1. *Let O be a fixed point on the boundary of the oval K and let $0<\mu<1$. For each boundary point O' of K, consider the dilation with ratio μ and center O'. Then the images of O under these dilations form the boundary of the image of K under the dilation with ratio $1-\mu$ and center O.*

Proof. Consider the line segment OO'. Clearly the image of O under the dilation with ratio μ and center O' is the same point as the image of O' under the dilation with ratio $1-\mu$ and center O.

THEOREM 2. *Bang's conjecture is true for the convex body K if and only if $1/(n+1)$ is the correct answer for the generalized Davenport problem for K.*

Proof. (It is easily seen that Bang's conjecture is true if the convex body is a line segment.) Suppose the conjecture is true for ovals, and suppose n lines intersect the oval K. Let O be a point fixed on the boundary of K, and let K' be the image of K under the dilation with ratio $1/(n+1)$ and center O. Each point p in the plane will uniquely represent a translate of K' if we agree to translate O onto p. From Lemma 1, we deduce that any point representing a translate of K' contained in K must lie in the image K'' of K under the dilation with ratio $n/(n+1)$ and center O. Furthermore, for a given line L of direction α, the points representing translates of K' whose interiors have empty intersection with L must lie outside the interior of a strip S of direction α and width $[1/(n+1)]w(\alpha)$. (See figure 1.)

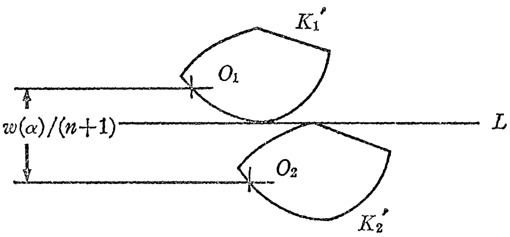

FIG. 1

The interiors of the n strips do not cover K''. For suppose they did. A simple compactness argument shows that for some $\rho>1$ the image of K'' under the dilation with ratio ρ and center O is covered by the strips. Since the relative width of each strip with respect to $\rho K''$ is

$$[w(\alpha)/(n+1)]\cdot[\rho n w(\alpha)/(n+1)]^{-1} = (\rho n)^{-1},$$

the sum of the relative widths is $n(\rho n)^{-1}=1/\rho<1$, contradicting Bang's conjecture. Thus we may choose a translate of K' associated with a point in K'' not in the interior of any strip. Clearly, none of the lines enters the interior of this translate.

As no new ideas are involved, we shall only sketch the proof of the "if" assertion. If the correct choice for μ in the generalized Davenport problem is $1/(n+1)$, the previous method shows that should K be covered by strips of equal relative width, then the sum of the relative widths must be at least one. To see this, let O be a point on the boundary of K and let K' be the image of K under the dilation with ratio $(n+1)/n$ and center O. The interior of each strip is the "forbidden area," as before, of a certain line which depends on the choice of O. If K is covered by m strips S_i of varying relative width, then we may cover any one of the S_i by strips of fixed relative width ϵ lying side by side so that at most one of these strips overlaps the boundary of the original strip. Hence the sum of the relative widths of the S_i is at least $1-m\epsilon$.

3. Estimates of μ. In case K is of constant width, we have $w=v(\alpha)=w(\alpha)$ for all α; and Theorems 1 and 2 insure that $\mu=1/(n+1)$. If $w(\alpha)$ varies rather mildly, Theorem 1 can be used to give a good estimate for μ. For instance, if K is a square, it is easy to show that $v(\alpha)/w(\alpha)\geqq 2(\sqrt{2}-1)=.82+$. Arguments paralleling those in Theorem 2 show that $\mu\geqq .82/(n+1)$ for a square. Since μ is an affine invariant, the above results are somewhat more general than they may at first appear. If K is an ellipse then $\mu=1/(n+1)$; if K is a parallelogram $\mu\geqq .82/(n+1)$.

THEOREM 3. *There is an absolute positive constant c for which $\mu\geqq c/(n+1)$ for any oval K.*

Proof. Of the rectangles contained in K, choose the rectangle A to be of maximum area. Consider the image A' of A under the dilation having ratio 7 and center at the center of A. If a point p in K lies outside of A', then K must contain a triangle of area greater than twice that of A, a contradiction. Thus A' contains K. By our previous estimate for parallelograms $\mu\geqq .8/(n+1)$ for A. Since A contains the image of K under the dilation with ratio $1/7$ and center at the center of A, it follows that μ for K is not less than $.1/(n+1)$. This estimate can doubtlessly be improved by elementary methods. In fact, Lemma 2 may be used to obtain $.5/(n+1)$, but nothing better. It would seem likely that a refinement of the previous argument could further improve this estimate for c.

COROLLARY. *Let the oval K be covered by strips of various widths. Then there is an absolute positive constant c, independent of K, such that the sum of the relative widths of the strips is at least c.*

The corollary may be obtained by the method of Theorem 2 or by application of Lemma 2 in the next section.

4. More on Bang's Problem. Bang [3] observed and Ohmann [9] proved, that in order to solve Bang's problem it suffices to consider convex sets in E^n which are covered by n mutually orthogonal strips of width 1. Treatments of Bang's conjecture in the case of two strips have been given by Bang [3] and Moser [8]. We give another proof which depends on the interesting

LEMMA 2. *Suppose the strip S has relative width $\rho\leqq 1$ with respect to the oval K of area A. The portion of K not covered by S must have area at least $(1-\rho)^2A$.*

Proof. Let O and O' be boundary points of K which lie on the lines of support in the direction of S. Consider the two images of K under the dilations with ratio $1-\rho$ and centers O and O'. Note that the portions of these two images not covered by S contain two disjoint pieces whose total area is precisely $(1-\rho)^2A$.

It is easy to see when the portion of K not covered by S has area exactly $(1-\rho)^2A$.

THEOREM 4. *Let the oval K be covered by the strips S_1 and S_2 with relative widths ρ_1 and ρ_2. Then $\rho_1+\rho_2\geqq 1$.*

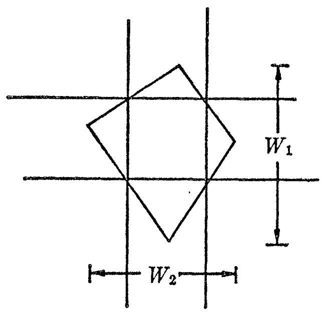

FIG. 2

Proof. (See figure 2.) Using the remarks at the beginning of this section, we assume that S_1 and S_2 are orthogonal and possess width one. Since K is contained in a convex quadrilateral circumscribing the unit square $S_1 \cap S_2$, we may assume K to be of this form. If W_1, W_2 are the widths of K in the directions of S_1, S_2, respectively, we must show that $1/W_1+1/W_2 \geqq 1$.

The portion of K covered by S_1 has area $1+\frac{1}{2}(W_2-1)$, while the total area of K is $\frac{1}{2}(W_1+W_2)$. Applying Lemma 2,

$$\tfrac{1}{2}[W_1 - 1] \geqq [1 - 1/W_1]^2 \cdot \tfrac{1}{2}[W_1 + W_2].$$

The desired result follows upon simplification.

Consideration of the situation when $\rho_1+\rho_2=1$ leads immediately to conjectures concerning "efficient" coverings of a convex body by strips. For instance, if the strips $S_1, \cdots, S_k$ have relative widths $\rho_1, \cdots, \rho_k$ with respect to an n-dimensional convex body K and if $\sum \rho_i \leqq 1$, we conjecture that the portion of K not covered by the strips possesses an n-volume at least $(1-\sum \rho_i)^n$ times the n-volume of K. The simple idea used in Lemma 2 shows this to be true in the case of one strip.

5. Special results for the square. The original Davenport problem led to certain results of interest. Suppose the sides of I are parallel to the axes. Professor Graham Higman showed that if "straight line" is replaced by "monotone curve," then a square S of side length $2/3(n+1)$ can always be found. Furthermore, he showed by simple example that this, in the asymptotic sense, is the best possible result.

The author has shown that if the additional condition that the monotone curves have derivatives not exceeding one in absolute value is imposed, then there is indeed a square S of side $1/(n+1)$.

Also, suppose one demands that S be a grid square, rather than arbitrary, in the original Davenport problem. Here the answer is certainly not $1/(n+1)$. We can not find a counterexample to $1/(n+2)$. If it could be shown that there is a fixed positive integer c so that a grid square S of side $1/(n+c)$ exists, then an affirmative answer to Davenport's problem would follow.

In conclusion, I wish to thank Professors Paul Bateman, George Orland and Oliver Aberth for helpful suggestions. Also, I wish to thank the referee for a number of comments contributing to the readability of the article.

References

1. Th. Bang, On coverings by parallel strips, Nordisk Mat. Tidskr. B, (1950) 49–53.
2. ———, A solution of the plank problem, Proc. Amer. Math. Soc., 2 (1951) 990–993.
3. ———, Some remarks on the union of convex bodies, Tolfte Skand. Kong., (1954) 5–11.
4. N. Bognar, On W. Fenchel's solution of the plank problem, Acta Math. Acad. Sci. Hungar., 12 (1961) 269–270.
5. H. Davenport, A note on Diophantine approximation, Studies in mathematical analysis and related topics, Stanford Univ. Press, (1962) 77–81.
6. W. Fenchel, On Th. Bang's solution of the plank problem, Nordisk Mat. Tidskr., B (1951) 49–51.
7. H. Moese, A contribution to Tarski's problem on the degree of equivalence of polygons, Odbitka Z Parametru, 2 (1932) 305–309.
8. W. Moser, On the relative widths of coverings by convex bodies, Canad. Math. Bull., 1 (1958) 154, 168, 174.
9. D. Ohmann, Eine Abschätzung für die Dicke bei Überdeckung durch konvexe Körper, J. Reine Angew. Math., 190 (1952) 125–128.
10. S. Straszewicz, Un théorème sur la largeur des ensembles convexes, Ann. Soc. Polon. Math., 21 (1948) 90–93.

INTERSECTIONS OF LINEARLY INDEPENDENT QUADRIC SURFACES

W. S. LOUD, University of Minnesota

In a study of certain complicated singular cases of the implicit function theorem the following question arose: Suppose we have in n-dimensional space a family of n linearly independent second degree hypersurfaces. What are the possibilities for the intersection of such a family of hypersurfaces? This is a complicated question since we are seeking real intersections. It appears likely that any intersection which has dimension $(n-1)$ is a linear manifold, but that intersections of dimension $(n-2)$ or less can be quite varied.

In this note we study what is probably the simplest nontrivial case of the above question. Namely we ask about the possible one-dimensional intersections of three linearly independent quadric surfaces in three-dimensional space. We find that such an intersection can indeed be a twisted curve. It is easy to give examples of three linearly independent quadric surfaces whose intersection is a straight line or a plane curve. We are able to find all twisted curves which lie on three linearly independent quadric surfaces. We normalize the problem to the extent of requiring that the curve in question be tangent to the x-axis at the origin and that its principal normal at the origin be the y-axis.

Our result is the following:

THEOREM. *For any choice of the constants* B_0, G_0, F_0, C_0, H_1, B_1, G_1, F_1, C_1, *and* K_1, *subject to the requirement* $B_0G_0H_1K_1\neq 0$ *the curve*:

$$(1)\qquad x = -\frac{K_1t(B_0 + 2F_0t + C_0t^2)}{\Delta}, \qquad y = \frac{2K_1G_0t^2}{\Delta}, \qquad z = \frac{2K_1G_0t^3}{\Delta}$$

$$(\Delta = H_1B_0 + (2H_1F_0 + G_1B_0 - G_0B_1)t + (H_1C_0 + 2G_1F_0 - 2G_0F_1)t^2 + (G_1C_0 - G_0C_1)t^3)$$

lies on all three of the linearly independent quadric surfaces

$$(2)\qquad F_0(x, y, z) \equiv B_0y^2 + 2G_0xz + 2F_0yz + C_0z^2 = 0,$$

$$(3)\qquad F_1(x, y, z) \equiv 2H_1xy + B_1y^2 + 2G_1xz + 2F_1yz + C_1z^2 + 2K_1z = 0,$$

$$(4)\qquad \begin{aligned} F_2(x, y, z) \equiv &- 4H_1G_0x^2 + (2G_1B_0 - 2B_1G_0 - 4H_1F_0)xy \\ &+ (2F_1B_0 - 2B_1F_0)y^2 - 2H_1C_0xz + (C_1B_0 - B_1C_0)yz \\ &+ 2K_1B_0y = 0. \end{aligned}$$

The curve (1) *is a twisted curve, tangent to the x-axis at the origin* ($t=0$) *and with principal normal at the origin the y-axis. Moreover all twisted curves passing through the origin with the above tangent and principal normal and which lie on three linearly independent quadric surfaces are included in the above set of curves.*

REMARK. The curve (1) also will lie on any quadric surface whose equation has the form

$$(5)\qquad \alpha_0F_0(x, y, z) + \alpha_1F_1(x, y, z) + \alpha_2F_2(x, y, z) = 0.$$

These surfaces are not linearly independent of (2), (3) and (4).

Proof. It can be verified by direct substitution of equations (1) into (2), (3), and (4) that the curve (1) lies on all three surfaces. The surfaces are clearly linearly independent because of the assumed nonvanishing of B_0, G_0, H_1, and K_1.

The main point of this discussion is to show that the set of curves given by (1) for all choices of the parameters does indeed include all cases, and to show how the result is discovered.

We shall be looking for three linearly independent quadric surfaces. As was remarked above, any quadric surface obtained by a linear combination (5) will also pass through the intersection of the original three surfaces. Hence what we really are seeking is the entire family of linear combinations, which will be described by a particular set of three linearly independent members of it, which form a basis for the family.

If a quadric surface is to contain a curve which is tangent to the x-axis at the origin, the surface must contain the origin and its normal at the origin must be perpendicular to the x-axis. This means that the equation of the surface will have its constant term and its linear term in x both zero. This property will be possessed by all three of the basis quadrics of the family and hence by every member of the family of all linear combinations.

Among the members of the family of linear combinations of three linearly independent quadrics tangent to the x-axis at the origin there can always be found two linearly independent quadrics which have the coefficient of x^2 equal to zero. We choose two such to be two of the three members of the basis of the family of linear combinations. Notice that these special quadrics which have the coefficient of x^2 equal to zero contain the x-axis, since their equations are satisfied when $y=z=0$. Also, given two linearly independent quadric surfaces, each having the coefficients of x^2 and x and the constant term equal to zero, a linear combination of them can be found for which the coefficient of z is zero. For one of the basic quadrics we choose a quadric having the constant term and the coefficients of x^2, x, and z all zero.

We therefore begin with two quadric surfaces:

$$2H_0xy + B_0y^2 + 2G_0xz + 2F_0yz + C_0z^2 + 2J_0y = 0 \tag{6}$$

$$2H_1xy + B_1y^2 + 2G_1xz + 2F_1yz + C_1z^2 + 2J_1y + 2K_1z = 0. \tag{7}$$

We determine the curve of intersection of (6) and (7). This will be the x-axis plus some twisted curve, in general. We then find further conditions on the coefficients in (6) and (7) such that the part of the intersection not the x-axis will be a twisted curve, tangent to the x-axis at the origin and having the y-axis as principal normal at the origin. After the curve of intersection is determined, we find all quadric surfaces which contain it.

If x is eliminated between equations (6) and (7), the resulting equation

$$\begin{aligned}(H_1y + G_1z)(B_0y^2 + 2F_0yz + C_0z^2 + 2J_0y)\\ = (H_0y + G_0z)(B_1y^2 + 2F_1yz + C_1z^2 + 2J_1y + 2K_1z)\end{aligned} \tag{8}$$

is the equation of the projection of the curve of intersection onto the y-z plane. If all four of H_0, G_0, H_1 and G_1 are zero, the elimination cannot be done. However, in this case (6) and (7) are cylinders parallel to the x-axis and any intersection is a straight line, which is not a twisted curve. Hence for a twisted curve we must have at least one of H_0, G_0, H_1, and G_1 nonzero.

The curve (8) can be parametrized by setting $z=ty$. The expressions for y and z are:

$$y = 2\,\frac{(H_0J_1 - H_1J_0) + (H_0K_1 + G_0J_1 - G_1J_0)t + G_0K_1t^2}{\begin{aligned}(H_1B_0 - H_0B_1) + (2H_1F_0 - 2H_0F_1 + G_1B_0 - G_0B_1)t\\ + (H_1C_0 - H_0C_1 + 2G_1F_0 - 2G_0F_1)t^2 + (G_1C_0 - G_0C_1)t^3\end{aligned}}$$

$$z = ty.$$

The expression for x is then found from whichever of

$$x = -\frac{B_0y^2 + 2F_0yz + C_0z^2 + 2J_0y}{2H_0y + 2G_0z},$$

$$x = -\frac{B_1y^2 + 2F_1yz + C_1z^2 + 2J_1y + 2K_1z}{2H_1y + 2G_1z}$$

has a nonvanishing denominator. The result is that x is also a rational function of t. Now at the origin, since the principal normal of the curve of intersection must be the y-axis, we must have $(z/y)\to 0$, which shows that the origin must correspond to $t=0$. Now as $t\to 0$, each of x, y, and z behaves asymptotically as an integral power of t, where the exponent may be positive, negative, or zero. But since the value $t=0$ corresponds to the origin, the coefficients in (6) and (7) will have to be chosen so as to make x, y, and z behave like positive powers of t. We have already seen that z/y must approach zero. Since the x-axis is the tangent to the curve, we must also have $y/x\to 0$. The formula for y as a function of t shows that $y=O(t^n)$ with n at most 2, $z=O(t^{n+1})$. Therefore we must have $x=O(t)$, $y=O(t^2)$ and $z=O(t^3)$ as $t\to 0$.

The requirement that $y=O(t^2)$ as $t\to 0$ implies that:

$$H_0J_1 - H_1J_0 = 0, \quad H_0K_1 + G_0J_1 - G_1J_0 = 0, \quad G_0K_1 \neq 0, \quad H_1B_0 - H_0B_1 \neq 0.$$

Let us first assume that $H_0\neq 0$. Then the requirement that $x=O(t)$ as $t\to 0$ gives $J_0=0$. But this implies in turn that

$$H_0J_1 = 0, \qquad H_0K_1 + G_0J_1 = 0$$

which together with $G_0K_1\neq 0$ imply that $H_0=J_1=0$. Thus the assumption that $H_0\neq 0$ leads to a contradiction. Therefore $H_0=0$ and we have, since $G_0\neq 0$, that $J_0=0$ in order that $x=O(t)$. We then have $G_0J_1=0$, whence $J_1=0$. Summarizing, we have found that in (6) and (7) we must have $H_0=J_0=J_1=0$, and our two basis quadrics take the forms

$$B_0y^2 + 2G_0xz + 2F_0yz + C_0z^2 = 0 \qquad B_0G_0 \neq 0$$

$$2H_1xy + B_1y^2 + 2G_1xz + 2F_1yz + C_1z^2 + 2K_1z = 0 \qquad H_1K_1 \neq 0$$

which agree with (2) and (3). Also the equations for x, y, and z become:

$$y = \frac{2G_0K_1t^2}{H_1B_0 + (2H_1F_0 + G_1B_0 - G_0B_1)t + (H_1C_0 + 2G_1F_0 - 2G_0F_1)t^2 + (G_1C_0 - G_0C_1)t^3}$$

$$= \frac{2G_0K_1t^2}{\Delta},$$

$$z = ty = \frac{2G_0K_1t^3}{\Delta},$$

$$x = -\frac{B_0y^2 + 2F_0yz + C_0z^2}{2G_0z} = -\frac{K_1t(B_0 + 2F_0t + C_0t^2)}{\Delta},$$

and these agree with (1). Note that we do have

$$x \sim -\frac{K_1}{H_1}t, \quad y \sim \frac{2G_0K_1}{H_1B_0}t^2, \qquad z \sim \frac{2G_0K_1}{H_1B_0}t^3 \text{ as } t \to 0.$$

So far we have shown that any twisted curve which is tangent to the x-axis at the origin and which has the y-axis as principal normal at the origin and which lies on two linearly independent quadric surfaces each of which contains the x-axis must of necessity have the form (1) and the family of linear combinations of the two quadrics includes two surfaces of the form (2) and (3). We now seek the most general quadric surface on which the curve (1) lies. Let us determine the coefficients of the quadric surface

$$Ax^2 + 2Hxy + By^2 + 2Gxz + 2Fyz + Cz^2 + 2Jy + 2Kz = 0 \tag{9}$$

so that the curve (1) lies on this surface. If we substitute the three equations (1) into (9), we obtain a rational function of t which must vanish identically. The numerator is found to be t^2 times a polynomial of degree four, and the identical vanishing of this polynomial gives five equations connecting the eight coefficients A, H, B, G, F, C, J, and K.

$$\begin{aligned}
&B_0^2K_1A + 4B_0G_0H_1J = 0;\\
&4B_0F_0K_1A - 4B_0G_0K_1H + 8F_0G_0H_1J + 4B_0G_0G_1J - 4G_0^2B_1J\\
&\quad + 4B_0G_0H_1K = 0;\\
&4F_0^2K_1A + 2B_0C_0K_1A - 8F_0G_0K_1H - 4B_0G_0K_1G + 4G_0^2K_1B + 4C_0G_0H_1J\\
&\quad + 8F_0G_0G_1J - 8G_0^2F_1J + 8F_0G_0H_1K + 4B_0G_0G_1K - 4G_0^2B_1K = 0;\\
&4C_0F_0K_1A - 4C_0G_0K_1H - 8F_0G_0K_1G + 8G_0^2K_1F + 4C_0G_0G_1J - 4G_0^2C_1J\\
&\quad + 4C_0G_0H_1K + 8F_0G_0G_1K - 8G_0^2F_1K = 0;\\
&C_0^2K_1A - 4C_0G_0K_1G + 4G_0^2K_1C + 4C_0G_0G_1K - 4G_0^2C_1K = 0.
\end{aligned} \tag{10}$$

Because the coefficients B_0, G_0, H_1, and K_1 are nonzero, the system (10) can be solved for A, H, B, F, and C in terms of G, J, and K. The results are:

$$\begin{aligned}
&A = -\frac{4G_0H_1}{B_0K_1}J; \; H = \left(-2\frac{F_0H_1}{B_0K_1} + \frac{G_1}{K_1} - \frac{G_0B_1}{B_0K_1}\right)J + \frac{H_1}{K_1}K;\\
&B = \left(-2\frac{F_0B_1}{B_0K_1} + \frac{C_0H_1}{G_0K_1} + 2\frac{F_1}{K_1}\right)J + \left(-\frac{B_0G_1}{G_0K_1} + \frac{B_1}{K_1}\right)K + \frac{B_0}{G_0}G;\\
&F = \left(-\frac{1}{2}\frac{C_0B_1}{B_0K_1} + \frac{C_0F_0H_1}{B_0G_0K_1} + \frac{1}{2}\frac{C_1}{K_1}\right)J + \left(-\frac{F_0G_1}{G_0K_1} + \frac{F_1}{K_1}\right)K\\
&\quad + \frac{F_0}{G_0}G;
\end{aligned} \tag{11}$$

$$C = \frac{C_0^2 H_1}{B_0 G_0 K_1} J + \left(- \frac{C_0 G_1}{G_0 K_1} + \frac{C_1}{K_1} \right) K + \frac{C_0}{G_0} G.$$

If we adjoin to the five equations (11) the two identities

$$G = \left(\frac{C_0 G_0 H_1}{B_0 G_0 K_1} - \frac{C_0 H_1}{B_0 K_1} \right) J + \left(- \frac{G_0 G_1}{G_0 K_1} + \frac{G_1}{K_1} \right) K + \frac{G_0}{G_0} G,$$

$$K = \frac{K_1}{K_1} K,$$

we can write

$$\begin{aligned}
&Ax^2 + 2Hxy + By^2 + 2Gxz + 2Fyz + Cz^2 + 2Jy + 2Kz \\
&= \left(\frac{C_0 H_1}{B_0 G_0 K_1} J - \frac{G_1}{G_0 K_1} K + \frac{1}{G_0} G \right) (B_0 y^2 + 2G_0 xz + 2F_0 yz + C_0 z^2) \\
&\quad + \frac{1}{K_1} K (2H_1 xy + B_1 y^2 + 2G_1 xz + 2F_1 yz + C_1 z^2 + 2K_1 z) \\
&\quad + \frac{1}{B_0 K_1} J(-4G_0 H_1 x^2 + (-4F_0 H_1 + 2B_0 G_1 - 2G_0 B_1) xy \qquad (12) \\
&\quad + (-2F_0 B_1 + 2F_1 B_0) y^2 - 2C_0 H_1 xz + (-C_0 B_1 + C_1 B_0) yz + 2B_0 K_1 y). \\
&= \left(\frac{C_0 H_1}{B_0 G_0 K_1} J - \frac{G_1}{G_0 K_1} K + \frac{1}{G_0} G \right) F_0(x, y, z) + \frac{1}{K_1} K F_1(x, y, z) \\
&\quad + \frac{1}{B_0 K_1} J F_2(x, y, z).
\end{aligned}$$

This shows that any quadric surface which contains the curve (1) must be a linear combination of the surfaces (2), (3), and (4). Thus the surfaces (2) (3) and (4) do form a basis for the linear combinations of three linearly independent quadric surfaces each of which contains the curve (1). From the way it was derived it is clear that only curves of the form (1) can lie on three linearly independent quadrics, so that all possibilities for space curves lying on three linearly independent quadrics (subject to the normalization of having the x-axis for tangent and the y-axis for principal normal at the origin) are included in (1) for the various choices of the ten parameters with $B_0 G_0 H_1 K_1 \neq 0$.

The simplest illustration of the result comes from choosing $B_0 = 1$, $G_0 = -\frac{1}{2}$, $H_1 = \frac{1}{2}$ and $K_1 = -\frac{1}{2}$, with the other parameters zero. The three surfaces are then:

$$y^2 - xz = 0; \qquad xy - z = 0; \qquad x^2 - y = 0$$

with the curve

$$x = t, \qquad y = t^2, \qquad z = t^3.$$

The problem of this paper would arise, for example, in the situation of pages 971–975 of the author's paper [1], in the case that there were three equations involving four unknowns.

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Reference

1. W. S. Loud, Some singular cases of the implicit function theorem, this MONTHLY, 68 (1961) 965–977.

ON STABLE TOPOLOGIES

N. A. FRIEDMAN AND R. C. METZLER, University of New Mexico

Introduction. Given a topology τ on a set X, the theory of uniform spaces suggests considering the topology $t(\tau)$ on X generated by the $\tau\times\tau$-neighborhoods of the diagonal in $X\times X$. In general $t(\tau)\subset\tau$ and we shall say τ is stable if $t(\tau)=\tau$. The stable topologies are precisely the R_0 topologies as defined by A. S. Davis [1]. In particular, τ is stable if and only if each open set contains the closures of each of its points. Thus every T_1 space is stable and, furthermore, Davis shows every stable space is topologically equivalent to a T_1 space. The relation between quasi-uniformities, as defined by W. J. Pervin [3], and stable spaces has been obtained by S. A. Naimpally [2].

In Section 1 we introduce notation and give a brief summary of the results of Davis that we shall need. It is shown that if X is finite then $t(\tau)$ is always stable for any τ. In Section 2 we consider the question of when a topology τ_2 satisfies $\tau_2=t(\tau_1)$ for some topology τ_1 and a necessary and sufficient condition is obtained. In section 3 we consider $t^n(\tau)=t(t^{n-1}(\tau))$, $n>1$, and construct a topology such that $t^n(\tau)\neq t^{n-1}(\tau)$ for each n. We conclude by defining $t_\omega(\tau)=\bigcap_{n=1}^{\infty} t^n(\tau)$ and construct a topology τ such that $t_\omega(\tau)$ is unstable.

1. Let $\Delta=\{(x,x): x\in X\}$ and let $A[x]=\{y: (x,y)\in A\}$ for $A\subset X\times X$. Given a topology τ on X, let $\mathcal{H}_\tau$ denote the $\tau\times\tau$-neighborhoods of Δ in $X\times X$.

DEFINITION 1. *Given a topology τ on X, let*

$$t(\tau)=\{G: x\in G \quad \text{implies} \quad H[x]\subset G \quad \text{for some} \quad H\in\mathcal{H}_\tau\}.$$

It is easily verified that $t(\tau)$ is a topology on X and $t(\tau)\subset\tau$.

DEFINITION 2. (Davis [1]) *τ is an R_0 topology if $G\supset\overline{\{x\}}$, $x\in G$, $G\in\tau$.*

We shall find it convenient to denote by F_x^i the closure of the singleton set $\{x\}$ in the τ_i topology. The superscript will be omitted when it is clear which topology is being considered.

THEOREM 1. *A subset G is open in* $t(\tau_1)$ *if and only if G is open in* τ_1 *and* $G\supset F_x^1$, $x\in G$.

Proof. Suppose $G\supset F_x^1$, $x\in G$, and G is open in τ_1. For any $x\in G$, let $H_x = (F_x^1\times G^c)^c$. Since H_x is the complement of a product of closed sets, it is an open set in $\tau_1\times\tau_1$. To show $H_x\supset\Delta$, let $a\in X$ and first suppose $a\notin F_x^1$. Therefore $(a, a) \notin F_x^1\times G^c$, hence $(a, a)\in H_x$. If $a\in F_x^1$ then $a\in G$, hence $(a, a)\notin F_x\times G^c$, hence $(a, a)\in H_x$. Thus $H_x\in\mathcal{H}_{\tau_1}$ and it is easily seen that $H_x[x]=G$. Since x was arbitrary, G is open in $t(\tau_1)$.

Conversely, if G is open in $t(\tau_1)$ then G is open in τ_1. By definition of $t(\tau_1)$, for each $x\in G$ there exists $H\in\mathcal{H}_{\tau_1}$ such that $H[x]\subset G$. Now we also have $H_1 = H\cap H^{-1}\in\mathcal{H}_{\tau_1}$ where H_1 is symmetric. If $y\in F_x^1$ then $x\in H_1[y]$ since $H_1[y]$ is open in τ_1. Thus by symmetry $y\in H_1[x]\subset H(x)\subset G$. Hence we conclude $F_x^1\subset G$.

COROLLARY 1. (Davis [1]) τ *is stable if and only if* τ *is an* R_0 *space.*

COROLLARY 2. *If G is open and closed in* τ *then G is open and closed in* $t(\tau)$.

COROLLARY 3. *If X is finite then* $t(\tau)$ *is stable for any topology* τ.

Proof. Suppose $G\in t(\tau)$, hence $G=\bigcup_{x\in G} F_x$. Therefore G is open and closed in τ because G is a finite union of closed sets. Therefore G is open and closed in $t(\tau)$, hence Corollary 2 implies G is open and closed in $t(t(\tau))$. Thus $t(t(\tau))\supset t(\tau)$ and hence $t(\tau)$ is stable.

Let us consider the relation R on the topological space (X, τ) defined by xRy if $F_x=F_y$. It is clear that R is an equivalence relation and we shall denote the equivalence classes by M_x, $x\in X$. It is shown in [1] that $X/R=\{M_x: x\in X\}$ with the quotient topology is always a T_0 space which is topologically equivalent to (X, τ). Furthermore, X/R is a T_1 space if and only if τ is stable in which case $M_x=F_x$, $x\in X$.

2. We shall now consider a topology τ_2 and ask if there exists a topology τ_1 such that $\tau_2=t(\tau_1)$. Of course, if τ_2 is stable, then we may choose $\tau_1=\tau_2$. In particular, Corollary 3 implies that in a finite space the only τ_2 topologies are the stable topologies.

LEMMA 1. *The sets* M_x, $x\in X$, *in a space* (X, τ) *have the following properties:*

(P1) $\overline{M}_x=F_y$, $y\in M_x$.

(P2) *If F is closed, then* $F=\bigcup_{x\in F} M_x$.

(P3) *If G is open, then* $G=\bigcup_{x\in G} M_x$.

(P4) *If* $y\in\overline{M}_x$, *then* $y\in M_x$ *or* $F_y\cap M_x=\varnothing$.

Proof. For (P1) we note that if $y\in M_x$ then $F_y\subset\overline{M}_x$. Also since $F_x=F_y$ for $y\in M_x$, we have $F_x\supset M_x$, hence $F_x\supset\overline{M}_x$. Thus $F_y\supset\overline{M}_x$ and (P1) follows. For (P2) suppose $x\in F$, hence $F\supset F_x\supset M_x$. (P3) follows by complements since we have an equivalence relation. For (P4) we consider $y\in\overline{M}_x-M_x$. If $z\in F_y\cap M_x$ then (P1) $\overline{M}_x=F_z$. Now $y\in F_z$ since $y\in\overline{M}_x$, hence $F_y\subset F_z$. But $z\in F_y$ implies $F_z\subset F_y$, hence $F_y=F_z$. Therefore $y\in M_x$ since $z\in M_x$ which is a contradiction.

DEFINITION 3. *We say b dominates a if $F_a \subset F_b$ but $F_a \neq F_b$. Thus τ is stable if and only if there exist no dominating points.*

LEMMA 2. *If $y \in F_x^1$, then $x \in F_y^2$.*

Proof. If $x \notin F_y^2$ then $x \in F_y^{2c}$ and F_y^{2c} is open in $\tau_2 = t(\tau_1)$. Therefore $F_x^1 \subset F_y^{2c}$, hence $y \notin F_x^1$ which is a contradiction.

We let M_x^i correspond to a topology τ_i.

THEOREM 2. *If there exists a dominating point b such that M_b^2 is finite, then $\tau_2 \neq t(\tau_1)$ for any τ_1.*

Proof. Let $M_b^2 = \{c: F_c^2 = F_b^2\}$ and assume $\tau_2 = t(\tau_1)$. We first show M_b^2 is closed in τ_1. Let $c \in M_b^2$ and let $a \in F_c^1$, hence $c \in F_a^2$ by Lemma 2. Therefore $F_c^2 \subset F_a^2$. Also $a \in F_c^1$ implies $a \in F_c^2$, hence $F_a^2 \subset F_c^2$ and thus $F_a^2 = F_c^2 = F_b^2$. Hence $a \in M_b^2$ and we have

$$M_b^2 = \bigcup_{c \in M_b^2} F_c^1.$$

Since M_b^2 is finite we conclude M_b^2 is closed in τ_1. Therefore M_b^{2c} is open in τ_1 and we now show M_b^{2c} is also open in τ_2. Let $a \in M_b^{2c}$ and first assume $a \in F_b^2$. Then (P4) implies $F_a^2 \subset M_b^{2c}$, hence $F_a^1 \subset F_a^2 \subset M_b^{2c}$. Next assume $a \notin F_b^2$, hence $a \in F_b^{2c}$ which is open in τ_1. Therefore $F_a^1 \subset F_b^{2c}$. Now $M_b^2 \subset F_b^2$ implies $M_b^{2c} \supset F_a^1$. Thus $M_b^{2c} \supset F_a^1$, $a \in M_b^{2c}$, hence M_b^{2c} is open in τ_2. Since M_b^{2c} is τ_1-open and τ_2-open, we conclude M_b^2 is τ_2-closed. Therefore $b \in M_b^2$ implies $F_b^2 \subset M_b^2$, hence $F_b^2 = M_b^2$. This contradicts b being a dominating point.

We shall now give an example of a topology τ_1 such that $t(\tau_1)$ is not stable. Let X be the positive integers. We define the basic closed sets of τ_1 to be $\emptyset$, X, $\{1\}$, $\{2\}$, and $\{2, n\}$, $n > 2$. An intersection of basic closed sets is a basic closed set and thus we may let the closed sets of τ_1 be finite unions of basic closed sets. It follows that $F_1^1 = \{1\}$, $F_2^1 = \{2\}$, and $F_n^1 = \{2, n\}$, $n > 2$. If $G_* = \{n: n \geqq 2\}$ then $G_* \in \tau_1$ and since $G_* \supset F_n^1$, $n \in G_*$, we have $G_* \in \tau_2 = t(\tau_1)$. However, if $G \in \tau_1$ and $G \neq G_*$ then $2 \notin G$ and hence $G \notin \tau_2$. Therefore $\tau_2 = \{\emptyset, X, G_*\}$ and $F_1^2 = \{1\}$, $F_n^2 = X$, $n > 1$. Thus $G_* \notin t(\tau_2)$ where $t(\tau_2) = \{\emptyset, X\}$, i.e., $t(\tau)$ is not stable.

We shall now generalize the basic idea in the previous example in order to obtain a converse to Theorem 2 as stated in Theorem 3 below. Let M be an infinite set and let us fix $x_M \in M$. Let $M - \{x_M\}$ be partitioned into a countable number of disjoint infinite sets M_n, $n = 1, 2, \cdots$. Thus $M = \{x_M\} \cup \bigcup_{n=1}^{\infty} M_n$. We say C is a *chosen* subset of M if

$$C = \{x_M\} \cup \bigcup_{j=1}^{r} M_{n_j} \quad \text{or} \quad C = \{x_M\}.$$

THEOREM 3. *Let (X, τ) be a topological space such that if b is a dominating point then M_b is infinite. Then for any $n = 1, 2, \cdots$, there exists a topology τ_n on X such that $\tau = t^n(\tau_n)$.*

Proof. We first construct τ_1 such that $\tau = t(\tau_1)$. For each $M = M_b$ corresponding to a dominating point b, choose $x_M \in M$ and apply the above decomposition to obtain the chosen subsets of M. The basic closed sets of τ_1 are finite unions of τ-closed sets and chosen sets. It is easily seen that if N is a basic closed set then $N = F \cup H$ where F is closed in τ and H is a finite union of chosen sets, each from distinct M sets. The τ_1-closed sets are arbitrary intersections of the τ_1-basic closed sets.

In τ_1 each set M_x is a union of τ_1-closed sets. This follows since if x is not a dominating point then $F_x = M_x$, hence M_x is closed in τ. If b is a dominating point then $M_b = M = \bigcup_{n=1}^{\infty}(M_n \cup \{x_M\})$ and each set $M_n \cup \{x_M\}$ is a chosen set and closed in τ_1. Since (P3) implies $G \in \tau$ is a union of M_x sets, G is therefore a union of τ_1-closed sets. Hence $G \supset F_x^1$, $x \in G$, and therefore $G \in t(\tau_1)$. Thus we have $t(\tau_1) \supset \tau$.

If N is closed in τ_1 then $N = \bigcap_{i \in I}(F_i \cup H_i)$ where F_i is closed in τ and $H_i = \bigcup_{l=1}^{l_i} C_{i,l}$ where $C_{i,l}$ is a chosen subset of distinct $M_{b_{i,l}}$, $1 \leqq l \leqq l_i$. We first show that if $z \in M = M_b$ for a dominating point b then if $z \neq x_M$ we have $F_z^1 = M_{n_z} \cup \{x_M\}$ where $z \in M_{n_z} \subset M$ for some positive integer n_z. Since $M_{n_z} \cup \{x_M\}$ is a chosen set in τ_1 we have $F_z^1 \subset M_{n_z} \cup \{x_M\}$. Now suppose N is closed in τ_1 and $z \in N$. Therefore $z \in F_i \cup H_i$, $i \in I$, where N is as above. If $z \in F_i$ then (P2) implies $F_i \supset M_z = M_b = M$. If $z \notin F_i$ then $z \in H_i$ where H_i is as above. Therefore $z \in C_{i,l_z}$ where C_{i,l_z} is a chosen subset of M. But then $C_{i,l_z} \supset M_{n_z} \cup \{x_M\}$. Thus we conclude that $N \supset M_{n_z} \cup \{x_M\}$, hence $F_z^1 = M_{n_z} \cup \{x_M\}$.

Now suppose N is closed in τ_1 but N is not closed in τ and let N be as above. Let $y \in N - \bigcap_{i \in I} F_i$ which is nonempty because N is not closed in τ. Therefore there exists $i \in I$ such that $y \notin F_i$, hence $y \in H_i$. Let $y \in M$, hence there is a chosen set C_{i,l_y} such that $C_{i,l_y} \subset H_i$ and $y \in C_{i,l_y}$. Now

$$M \cap N = \bigcup_{k=1}^{r} M_{n_k} \cup \{x_M\} \subset C_{i,l_y}.$$

Choose $z \in M_n \subset M$ where $n \neq n_k$, $1 \leqq k \leqq r$. Now $z \in N^c$ and $F_z^1 = M_n \cup \{x_M\}$. Since $x_M \in F_y^1 \subset N$, we conclude $N^c \not\supset F_z^1$, hence $N^c \notin t(\tau_1)$. Therefore $t(\tau_1) \subset \tau$ and thus $t(\tau_1) = \tau$.

Since the topology τ_1 which we have obtained also satisfies the hypothesis of the theorem, it is clear there exists a topology τ_2 which satisfies the hypothesis such that $t^2(\tau_2) = \tau$. Proceeding by induction, we obtain τ_n such that $t^n(\tau_n) = \tau$.

3. We shall now construct an example of a space (X, τ) such that $t^n(\tau) \neq t^{n-1}(\tau)$ for all n. Let X_n, $n = 1, 2, \cdots$, be disjoint countable sets and apply Theorem 3 to obtain a topology τ_n on X_n such that $t^k(\tau_n) \neq t^{k-1}(\tau_n)$, $k \leqq n$. Let $X = \bigcup_{n=1}^{\infty} X_n$ and let $\tau = \{G \colon G \cap X_n \in \tau_n, \text{ all } n\}$. It is easy to see that τ has the desired property. However, $t_\omega(\tau) = \bigcap_{n=1}^{\infty} t^n(\tau)$ can be shown to be stable.

We shall now construct a space (A, τ) such that $t_\omega(\tau)$ is unstable. Let B be an infinite set and suppose $B = \bigcup_{n=1}^{\infty} B_n$, where each B_n is infinite and $B_n \cap B_m = \{b\}$, $n \neq m$. Let $A = B \cup \{a\}$ where $a \notin B$.

DEFINITION 4. *Let $C \subset A$. We say C is a Q-set if $C \cap B_n = \varnothing$ or $\{b\}$ for all but a finite number of B_n's.*

Now given any sequence of topologies σ_n on B_n, $n=1, 2, \cdots$, we can consider the class of all Q-sets C such that $C \cap B_n$ is closed in σ_n for all $n=1, 2, \cdots$. Since this class is closed under finite unions and arbitrary intersections, the addition of the whole set A gives the closed sets of a topology on A which we denote by $Q_\sigma = Q\{\sigma_n : n=1, 2, \cdots\}$. Note that $\{a\}$ is closed in Q_σ but B is not closed in Q_σ since it is not a Q-set. We denote $Q_{t(\sigma)} = Q\{t(\sigma_n): n=1, 2, \cdots\}$.

LEMMA 3. *If $\{b\}$ is closed in σ_n for all but finitely many n then $t(Q_\sigma) = Q_{t(\sigma)}$.*

Proof. If G $(\neq \varnothing)$ is open in $t(Q_\sigma)$ then G is open in Q_σ and $G \supset F_x^{Q_\sigma}$ for all $x \in G$. Hence $G \cap B_n \supset F_x^{Q_\sigma} \cap B_n$ for all $x \in G \cap B_n$. This implies that $G \cap B_n$ is open in $t(\sigma_n)$ for all n. Since G^c is a Q-set, G is open in $Q_{t(\sigma)}$.

Conversely, if G $(\neq \varnothing)$ is open in $Q_{t(\sigma)}$, then G^c is a Q-set and G is open in Q_σ. We wish to show $G \supset F_x^{Q_\sigma}$, $x \in G$. Note that if $a \in G$ then $F_a^{Q_\sigma} = \{a\} \subset G$. Now consider $x \neq a$ and first assume $b \notin G$. Since $x \neq a$, there is some n such that $x \in G \cap B_n$ where $G \cap B_n$ is open in $t(\sigma_n)$. Therefore $F_x^n \subset G \cap B_n$ and $F_x^n \cap B_m = \varnothing$ for $n \neq m$ since $b \notin G$. Therefore $G \supset F_x^n$ which is closed in Q_σ. Thus $G \supset F_x^{Q_\sigma}$, hence $G \in t(Q_\sigma)$. Secondly, assume $b \in G$. Let $H_x = F_x^n \cup (\bigcup_{m=1}^\infty F_b^m)$. Now $H_x \subset G$ since $G \cap B_m$ is open in $t(\sigma_m)$ for all $m=1, 2, \cdots$. By hypothesis, $F_b^m = \{b\}$ for all but finitely many m, hence H_x is a Q-set. Clearly $H_x \cap B_m$ is closed in σ_m for all $m=1, 2, \cdots$. Hence $G \in t(Q_\sigma)$ in this case.

Now consider the topology $\tau_n = \{B_n, \varnothing, B_n - \{b\}\}$ on B_n. By Theorem 3 there exists a topology τ_n^* on B_n such that $t^n(\tau_n^*) = \tau_n$.

LEMMA 4. $t^k(Q_{\tau^*}) = Q_{t^k(\tau^*)}$, $k=1, 2, \cdots$.

Proof. We first note that for any $k=1, 2, \cdots$, $\{b\}$ is closed in $t^k(\tau_n^*)$ for all $n \geqq k$. But this implies that for every $k=1, 2, \cdots$, the sequence of topologies $\{t^k(\tau_n^*): n=1, 2, \cdots\}$ satisfies the hypothesis of Lemma 3. Therefore the result follows by induction.

LEMMA 5. *B is the only nontrivial set which is open in $t^k(Q_{\tau^*})$ for all k.*

Proof. To show that B is open in $t^k(Q_{\tau^*})$, Lemma 4 implies it suffices to show B is open in $Q_{t^k(\tau^*)}$. We first show B is a union of sets which are closed in $Q_{t^k(\tau^*)}$. Let $D_{k,m} = \bigcup_{l=1}^{k+m} B_l$. Now $D_{k,m}$ is a Q-set and for $n \leqq k+m$ we have $D_{k,m} \cap B_n = B_n$ which is closed in $t^k(\tau_n^*)$. For $n > k+m$ we have $D_{k,m} \cap B_n = \{b\}$ which is closed in $t^k(\tau_n^*)$. Therefore $B = \bigcup_{m=1}^\infty D_{k,m}$ is a union of sets which are closed in $Q_{t^k(\tau_n^*)}$. Since B is open in Q_{τ^*}, it follows by induction that B is open in $Q_{t^k(\tau^*)}$ for all $k=1, 2, \cdots$.

Now suppose F is not $\{a\}$ or $\varnothing$ and F is closed in $t^k(Q_{\tau^*})$ for all $k=1, 2, \cdots$. Then $F \cap B_n \neq \varnothing$ for some n and since F is closed in $t^{n+1}(Q_{\tau^*})$ we have $F \cap B_n = B_n$. Therefore $b \in F$ and hence $F \cap B_l \neq \varnothing$ for all $l=1, 2, \cdots$. Thus, as above, $F \supset B_l$, $l=1, 2, \cdots$, hence $F \supset B$. Since B is not closed in Q_{τ^*}, we must have $F=A$.

THEOREM 4. *There exists a topological space (A, τ) such that $t_\omega(\tau)$ is unstable.*

Proof. Let A be as above and let $\tau = Q_{\tau^*}$. Lemma 5 implies $t_\omega(\tau) = \{A, \emptyset, B\}$ which is clearly unstable.

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References

1. A. S. Davis, Indexed systems of neighborhoods for general topological spaces, this MONTHLY, 68 (1961) 886–893.
2. S. A. Naimpally, Separation axioms in quasi-uniform spaces, this MONTHLY, 74 (1967) 283–284.
3. W. J. Pervin, Quasi-uniformization of topological spaces, Math. Annalen, 147 (1962) 316–317.

MATHEMATICAL NOTES

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

SOME REMARKS ON A PAPER OF PAUL LÉVY ON HADAMARD MATRICES

HENRYK MINC, University of California, Santa Barbara

In a recent paper [1] Lévy defined a function ϕ on positive integers as follows. For a given positive integer n the value $\phi(n)$ is the largest integer for which there exists a $\phi(n) \times n$ $(1, -1)$-matrix A (i.e., a matrix all of whose entries are 1 or -1) satisfying

$$AA^T = nI_{\phi(n)}, \tag{1}$$

where $I_{\phi(n)}$ denotes a $\phi(n)$-square identity matrix. Recall that an n-square $(1, -1)$-matrix A is an Hadamard matrix if $AA^T = nI_n$. Thus, there exists an n-square Hadamard matrix if and only if $\phi(n) = n$ and the well-known conjecture on Hadamard matrices [4, p. 106] can be stated:

$$\phi(4m) = 4m \tag{2}$$

for all positive integers m.

Lévy proved in [1] that $\phi(2n) \geqq 2\phi(n)$. In the present note I show that $\phi(mn) \geqq \phi(m)\phi(n)$. The method of proof is similar to that used in the proof of Theorem 2.4 in [4; p. 106].

THEOREM. (a) *For any positive integers m and n*

$$\phi(mn) \geqq \phi(m)\phi(n). \tag{3}$$

(b) *If p is an odd positive integer then*

$$\phi(p) = 1, \tag{4}$$ $$\phi(2p) = 2. \tag{5}$$

Part (b) of the Theorem has somewhat unfortunate consequences. If conjecture (2) is true, as is generally believed, then the function ϕ can be of little importance: $\phi(n)=1, 2$ or n, according as n is odd, $n\equiv 2$ or $n\equiv 0 \bmod 4$. However, conjecture (2) is still unresolved at present and it may be of interest to find a significant lower bound for any n for which the value $\phi(n)$ is not presently known. For example, the value of $\phi(260)$ has not yet been determined, but it is known that $\phi(52)=52$ and thus that $\phi(260)\geqq 52$. Is $\phi(260)$ strictly greater than 52? The answer to this question may be of the same order of difficulty as proving the existence of a 260-square Hadamard matrix.

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References

1. Paul Lévy, Remarques sur le théorème d'Hadamard sur les déterminants, C. R. Acad. Sci. Paris Ser. A et B, 262 (1966) A22–A24.
2. Marvin Marcus and Henryk Minc, A survey of matrix theory and matrix inequalities, Allyn and Bacon, Boston, 1964.
3. R. E. A. C. Paley, On orthogonal matrices, J. Math. and Physics, 12 (1933) 311–320.
4. Herbert John Ryser, Combinatorial mathematics, MAA, Carus Math. Mon. no. 14, 1963.

CONTINUITY AND COMPOSITION OF FUNCTIONS

R. C. MOORE, Florida State University and Pennsylvania State University

1. Introduction. Throughout this paper, let R denote the reals with their usual topology. Define $t_n\colon R\to R$ by

$$t_n(x) = \max\{-n, \min\{x, n\}\} \qquad \text{for all } x \in R \quad (n = 1, 2, \cdots).$$

It can be proved that if $f\colon S\to R$, where S is any topological space, then f is continuous if and only if $t_n \circ f$ is continuous for all $n=1, 2, \cdots$. This is just a special case of Theorem 2.1 below. This example suggests the following definition.

DEFINITION 1.1. *Let X be any topological space and let Φ be a nonempty family of continuous functions from X into X (i.e. $\Phi\subseteq C(X, X)$). We call Φ an admissible family for X if and only if for every topological space S and every $f\colon S\to X$, f is continuous if and only if $\phi\circ f$ is continuous for each $\phi\in\Phi$.*

Given X, we wish to determine the nature of families of admissible functions for X. It will be convenient to have the following definition.

DEFINITION 1.2. *We say that $\Phi\subseteq C(X, X)$ determines the topology of X if and only if $\{\phi^{-1}[O]\colon O$ is open in X and $\phi\in\Phi\}$ is a subbase for the topology of X.*

The sufficiency in the following theorem is proved in Gillman and Jerison [1], and it is easy to prove the necessity.

THEOREM 1.3. *Φ is an admissible family for X if and only if Φ determines the topology of X.*

2. Sufficient conditions that $\Phi \subseteq C(X, X)$ be an admissible family for X. We shall use the following notation: For any $\phi \in \Phi$, let $\operatorname{int} \phi[X]$ denote the interior of $\phi[X]$ and let $\phi[X]'$ denote the complement of $\phi[X]$.

THEOREM 2.1. *$\Phi \subseteq C(X, X)$ is an admissible family for X if the following three conditions hold:*

(i) *for all $\phi \in \Phi$, $\phi(x) = x$ for every $x \in \operatorname{int} \phi[X]$,*

(ii) *for every $x \in X$, there is $\phi \in \Phi$ such that $x \in \operatorname{int} \phi[X]$, and*

(iii) *each $\phi \in \Phi$ carries $\phi[X]'$ into the boundary of $\phi[X]$.*

Before proving this theorem, we make the following remark which is easy to verify.

REMARK 2.2. Since each $\phi \in \Phi$ is continuous, (iii) is equivalent to the following:

(iii)$'$ each $\phi \in \Phi$ carries $(\operatorname{int} \phi[X])'$ into the boundary of $\phi[X]$.

Proof of Theorem 2.1. Let S and $f: S \to X$ be given. If f is continuous, clearly $\phi \circ f$ is continuous for each $\phi \in \Phi$. Now suppose $\phi \circ f$ is continuous for each $\phi \in \Phi$. It needs to be shown that f is continuous.

Let $s_o \in S$ be given and let H be any open set in X containing $f(s_o)$. We must show that there exists an open set H^* in S with H^* containing s_o such that $f[H^*] \subseteq H$.

By (ii), there is $\phi_o \in \Phi$ such that $f(s_o) \in \operatorname{int} \phi_o[X]$. Let $G = \operatorname{int} \phi_o[X]$. By (i), $\phi_o \circ f(s_o) = f(s_o)$ and thus $\phi_o \circ f(s_o) \in G \cap H$.

Let $U = G \cap H$. Thus U is an open set in X containing $f(s_o) = \phi_o \circ f(s_o)$. Now let $H^* = (\phi_o \circ f)^{-1}[U]$. Clearly H^* is open in S and $s_o \in H^*$. That $f[H^*] \subseteq H$ can be shown as follows: let $s \in H^*$ be given. Then $(\phi_o \circ f)(s) \in U$ and so $f(s) \in \phi_o^{-1}[U] \subseteq \phi_o^{-1}[\operatorname{int} \phi_o[X]] \subseteq \operatorname{int} \phi_o[X]$. This last inclusion follows from (iii)$'$. Thus by (i), $\phi_o(f(s)) = f(s)$. But since $\phi_o(f(s)) \in U \subseteq H$ and $\phi_o(f(s)) = f(s)$, we see that $f(s) \in H$. Thus $f[H^*] \subseteq H$.

3. Application of Theorem 2.1. One application of Theorem 2.1 is to the case where X is Euclidean m-space and Φ is the collection of functions $\phi_n (n = 1, 2, \cdots)$ defined by:

$$\phi_n((x_1, x_2, \cdots, x_m)) = \begin{cases} (x_1, x_2, \cdots, x_m) & \text{if } p \leqq n \\ \left(\dfrac{nx_1}{p}, \dfrac{nx_2}{p}, \cdots, \dfrac{nx_m}{p}\right) & \text{if } p > n \end{cases}$$

where

$$p = \sqrt{(x_1^2 + x_2^2 + \cdots + x_m^2)}.$$

The reader should observe that the case $m = 1$ is the example of the opening paragraph.

4. Discussion of Condition (iii)$'$. It is easy to construct examples satisfying (i) and (ii) for which Φ is an admissible family of functions for x but (iii)$'$ is not

satisfied; and thus *the converse of Theorem* 2.1 *does not hold even if we restrict our attention to those subsets of* $C(X, X)$ *which satisfy* (i) *and* (ii). For example, let $X=R$ and let $\Phi_1=\{\phi_n: n=1, 2, \cdots\}$ where (for $n=1, 2, \cdots$; and $x\in R$):

$$\phi_n(x) = \begin{cases} x & \text{if } x \leqq n \\ 2n - x & \text{if } n < x < n+1 \\ n-1 & \text{if } x \geqq n+1 \end{cases}$$

We next give an example to show that *conditions* (i) *and* (ii) *are not sufficient to guarantee that* Φ *be an admissible family for* X. To this end, let $X=R$ and let $\Phi_2=\{\phi_n: n=1, 2, \cdots\}$ where (for $n=1, 2, \cdots$; and for $x\in R$):

$$\phi_n(x) = \begin{cases} x & \text{if } |x| \leqq n \\ 2n - x & \text{if } n < x < 2n \\ -2n - x & \text{if } -2n < x < -n \\ 0 & \text{if } |x| \geqq 2n \end{cases}$$

The reader can easily verify that Φ_2 satisfies (i) and (ii). To see that Φ_2 is not an admissible family for R, consider $f: R\to R$ defined by:

$$f(x) = \begin{cases} |1/x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This function f is not continuous, but $\phi_n \circ f$ is continuous for each $\phi_n\in\Phi_2$.

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Reference

1. Leonard Gillman and Meyer Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, N. J., 1960.

A CHARACTERIZATION OF HEREDITARILY INDECOMPOSABLE CONTINUA

E. J. VOUGHT, California State Polytechnic College, Pomona

Let M be a compact, Hausdorff continuum. In [1] Zame has shown that M is hereditarily indecomposable if and only if for every pair of subcontinua H and K of M, $H-K$ is connected. The proof of sufficiency is the more significant and this can be accomplished with the additional restriction that K is a subcontinuum of H; i.e., no subcontinuum of M separates M and this property is hereditary. Thus it is possible to establish a somewhat stronger characterization of an hereditarily indecomposable continuum than the preceding one and at the same time use simple techniques.

THEOREM. *The continuum M is hereditarily indecomposable if and only if every subcontinuum N of M has the property that no subcontinuum of N separates N.*

Proof. Suppose that the subcontinuum N of M is separated by one of its subcontinua H; i.e., $N-H=A+B$ where A and B are mutually separated subsets of N. Then $A+H$ and $B+H$ are subcontinua of N whose sum is N. Thus N is decomposable and this establishes the necessity.

To prove the sufficiency suppose that the subcontinuum N of M is decomposable; i.e., $N=H+K$ where H and K are proper subcontinua of N. Now $H\cdot K$ separates N; so by hypothesis $H\cdot K$ is the sum of two disjoint closed sets P and Q. Let U and V be open sets relative to H such that $P\subset U$, $Q\subset V$ and $\overline{U}\cdot\overline{V}=\emptyset$. Denote by C_u a component of U containing a point of P and similarly let C_v be a component of V intersecting Q. Now $\overline{C}_u$ and $\overline{C}_v$ contain points of the boundaries (rel. H) of U and V respectively; so $\overline{C}_u-K\neq\emptyset\neq\overline{C}_v-K$. In addition, $\overline{C}_u\cdot\overline{C}_v=\emptyset$ and hence the continuum $\overline{C}_u+\overline{C}_v+K$ is separated by K into the sets $\overline{C}_u-K$ and $\overline{C}_v-K$. Such a separation is impossible by hypothesis and the proof is complete.

Reference

1. W. R. Zame, A characterization of hereditarily indecomposable continua, Proc. Amer. Math. Soc., 17 (1966) 709–710.

STONE'S REPRESENTATION THEOREM FOR BOOLEAN ALGEBRA

N. M. Rice, Queen's University, Canada

In 1949 Rado [4] proved Theorem 1 below concerning the existence of certain choice functions. Subsequently Gottschalk [1] gave a simple proof using Tychonoff's theorem on the product of compact spaces, and later Luxemburg [2] gave another interesting proof using the method of ultrapowers. The purpose of this note is to show how Rado's theorem can be used to give a simple proof of Stone's theorem (Theorem 2, below).

Theorem 1 (R. Rado [1], [2], [4]). *Let $\{A_\mu: \mu\in M\}$ be a nonempty family of nonempty finite sets. Assume that for every finite subset N of M a choice function a_N is given (i.e., $a_N(\mu)\in A_\mu$ for all $\mu\in N$). Then there exists a choice function a of the family $\{A_\mu: \mu\in M\}$ (i.e., a mapping a of M into $A=\cup(A_\mu: \mu\in M)$ such that $a(\mu)\in A_\mu$ for all $\mu\in M$) which has the following property: for every finite subset N of M there exists another finite subset N' of M such that $N\subset N'$ and $a(\mu)=a_{N'}(\mu)$ for all $\mu\in N$.*

Theorem 2. (M. H. Stone [5], [6]). *Every Boolean algebra R is isomorphic to a subring of the algebra of all subsets of some set X.*

Proof. Let X be the set of all homomorphisms of R into the Boolean algebra of two elements $\{0, 1\}$. If, for every $e\in R$, $T(e)=\{x: x\in X, x(e)=1\}$, then T is a homomorphism from R into the algebra of all subsets of X.

The crucial step now (cf. [5]) is to show that T is an isomorphism, by showing that if $e\in R$ and $e\neq 0$ then there exists an $x\in X$ for which $x(e)=1$. But this

follows easily from Rado's theorem. Indeed, in Theorem 1 let $M=R$, and for each $\mu\in M=R$ let $A_\mu=\{0, 1\}$. It is easy to see that for each finite subset N of R there is a homomorphism h_N defined on the finite Boolean algebra generated by $N\cup\{e\}$ such that $h_N(e)=1$. Then the function from R to $\{0, 1\}$ which exists by Rado's theorem is easily seen to be a homomorphism mapping e to 1.

N. H. McCoy and D. Montgomery have proved [3] a generalization of Stone's theorem, namely: *every p-ring is isomorphic to a subring of a direct sum of fields F_p.* (A p-ring is a commutative ring for which there is a prime p such that $a^p=a$ and $pa=0$ for every element a in the ring, and F_p is the field of integers mod p. A Boolean algebra is a 2-ring with unit if the operations of $+$ and $\cdot$ are taken to be $\triangle$ (symmetric difference) and $\wedge$ (cf. [6])).

As McCoy and Montgomery point out, the essential step in the proof of the theorem is to show, given $a\neq 0$ in the p-ring, that there exists a homomorphism from the p-ring to F_p which doesn't annihilate a. This can also be proved using Rado's theorem, by taking account of the fact (cf., Theorems 3 and 1 in [3]) that every finite p-ring does have such homomorphisms.

References

1. W. H. Gottschalk, Choice functions and Tychonoff's theorem, Proc. Amer. Math. Soc., 2 (1951) 172.

2. W. A. J. Luxemburg, A remark on a paper by N. G. deBruijn and P. Erdös, Nederl. Akad. Wetensch., Proc. Ser. A, 65 (1962) 343–345.

3. N. H. McCoy and D. Montgomery, A representation of generalized Boolean rings, Duke Math. J., 3 (1937) 455–459.

4. R. Rado, Axiomatic treatment of rank in infinite sets, Canad. J. Math., 1 (1949) 337–343.

5. E. R. Stabler, Boolean representation theory, this MONTHLY, 51 (1944) 129–132.

6. M. H. Stone, The theory of representations for Boolean algebras, Trans Amer. Math. Soc., 40 (1936) 37–111.

A COMMENT ON COMPACTNESS

B. J. PETTIS, University of North Carolina

The following occurred to the author in the early 1950's; perhaps it may, even at this late date, be of some use to others.

Let a paratopology be a topology without the requirement that the intersection of two open sets be open. The proofs that a topology $\mathfrak{I}$ for a set X is compact iff every universal net in X is $\mathfrak{I}$-convergent can be carried over, without change, for a paratopology. Two immediate consequences are the observation that the Tychonoff theorem holds for paratopologies and the following simple proof of Alexander's theorem. In X let $\mathfrak{U}$ be a collection of subsets covering X, $\mathfrak{V}$ be all intersections of finite subcollections of $\mathfrak{U}$, $\mathfrak{P}$ all unions of subcollections of $\mathfrak{U}$, and $\mathfrak{I}$ all unions of subcollections of $\mathfrak{V}$; then $\mathfrak{P}$ is a paratopology, $\mathfrak{I}$ is a topology, and $\mathfrak{U}$ and $\mathfrak{V}$—and therefore $\mathfrak{P}$ and $\mathfrak{I}$ as well—have precisely the same class of convergent nets. Since this class either does or does not include the universal nets, it is obvious that $\mathfrak{P}$ and $\mathfrak{I}$ are both compact or both not compact, which is Alexander's theorem.

A CHARACTERIZATION OF LOWER SEMI-CONTINUITY

R. E. SMITHSON, University of Florida and University of Wyoming

A multifunction is a point to set correspondence from a set X to a set Y. A multifunction $F: X \to Y$ is called upper semi-continuous in case for every closed set $A \subset Y$, the set $F^{-1}(A) = \{x \in X \mid F(x) \cap A \neq \square\}$ is closed. The multifunction F is called lower semi-continuous in case for every open set $U \subset Y$, the set $F^{-1}(U)$ is open. The purpose of this note is to give a characterization of lower semi-continuity in terms of directed families which is analogous to the result G. T. Whyburn [1] obtained for upper semi-continuous multifunctions.

DEFINITIONS. (1) *A family $\mathfrak{A}$ of subsets of a set X is directed if and only if for $A_1, A_2 \in \mathfrak{A}$, there is an $A_3 \in \mathfrak{A}$ such that $A_3 \subset A_1 \cap A_2$.*

(2) *A point x is a cluster point of a directed family $\mathfrak{A}$ if and only if whenever U is an open set containing x, we have $U \cap A \neq \square$ for all $A \in \mathfrak{A}$.*

(3) *A directed family $\mathfrak{A}$ converges to a point x if and only if for each open set U containing x, there is a member $A \in \mathfrak{A}$ such that $A \subset U$.*

If $A \subset X$, then $F(A) = \cup \{F(x) \mid x \in A\}$. If $\mathfrak{A}$ is a directed family then the family $F(\mathfrak{A}) = \{F(A) \mid A \in \mathfrak{A}\}$ is also a directed family.

In the following theorem we shall assume that each member of the family $\mathfrak{A}$ is nonempty.

THEOREM. *A multifunction $F: X \to Y$ is lower semi-continuous if and only if whenever $x_0 \in X$ and whenever $\mathfrak{A}$ is a directed family converging to x_0, each $y \in F(x_0)$ is a cluster point of the family $F(\mathfrak{A})$.*

Proof. Suppose that F is lower semi-continuous, and that $x_0 \in X$, and that $\mathfrak{A}$ is a directed family which converges to x_0. Let $y \in F(x_0)$ and let V be an open set containing y. Let U be an open set containing x_0 such that $F(x) \cap V \neq \square$ for all $x \in U$. Let $F(A) \in F(\mathfrak{A})$. Then there is an A_1 such that $A_1 \subset U$ and an A_2 such that $A_2 \subset A_1 \cap A$. If $x \in A_2$, then $F(x) \cap V \neq \square$, since $A_2 \subset A_1 \subset U$; since $x \in A$, $F(x) \subset F(A)$ and so $F(A) \cap V \neq \square$. Thus, y is a cluster point of $F(\mathfrak{A})$.

On the other hand, let $x_0 \in X$, and let $\mathfrak{A}$ be any directed family which converges to x_0, and suppose that every $y \in F(x_0)$ is a cluster point of $F(\mathfrak{A})$. Let $F(x_0) \cap V \neq \square$ where V is an open subset of Y, and let $y \in F(x_0) \cap V$. We let $\mathfrak{A}$ be the collection of open sets containing x_0. If F is not lower semi-continuous, then for each $U \in \mathfrak{A}$ there is a point $x \in U$ such that $F(x) \cap V = \square$. For each $U \in \mathfrak{A}$, set $\tilde{U} = \{x \in U \mid F(x) \cap V = \square\}$, and set $\tilde{\mathfrak{A}} = \{\tilde{U} \mid U \in \mathfrak{A}\}$. The family $\tilde{\mathfrak{A}}$ converges to x_0, since $\tilde{U} \subset U$, and thus, y is a cluster point of $F(\tilde{\mathfrak{A}})$. But $V \cap F(\tilde{U}) = \square$, which is a contradiction. Therefore F is lower semi-continuous.

Reference

1. G. T. Whyburn, Continuity of multifunctions, Proc. Nat. Acad. of Sciences, U.S.A., 54 (1965) 1494–1501.

THE CHARACTERISTIC POLYNOMIAL OF A SINGULAR MATRIX

D. E. VARBERG, Hamline University

Let $A = [a_{ij}]$ be an $n\times n$ matrix of rank $r<n$. Then $\lambda=0$ is a characteristic value of A of multiplicity at least $n-r$. Thus, if $\phi(\lambda)=|\lambda I-A|$ is the characteristic polynomial of A, $\phi(\lambda)=\lambda^{n-r}\psi(\lambda)$ where $\psi(\lambda)$ is a polynomial of degree r. We ask the question: *Is there a way to find $\psi(\lambda)$ without first finding $\phi(\lambda)$?* The purpose of this note is to give an affirmative answer and to show that $\psi(\lambda)$ is the characteristic polynomial of an $r\times r$ matrix D which is related to A in a simple way.

Since A is of rank $r<n$, it may be represented (actually in many ways) as a product BC of two $n\times n$ matrices where the last $n-r$ columns of B and the last $n-r$ rows of C consist of zeros. For example, the first r rows of C could be any r independent rows of A; B would then contain the proper multipliers to generate A.

Now CB will be of the form diag$\{D, 0\}$ where D is an $r\times r$ matrix. By a theorem originally stated by Sylvester [2] and proved in many modern textbooks (e.g. [1, p. 23]), BC and CB have the same characteristic polynomial. Thus

$$\phi(\lambda) = |\lambda I - A| = |\lambda I - CB| = \lambda^{n-r}|\lambda I - D| = \lambda^{n-r}\psi(\lambda).$$

If D is of rank less than r, the procedure could be repeated.

The result would seem to be particularly useful when r is small in comparison to n. We illustrate for $r=1$ and $r=2$, noting that the representation $A=BC$ above is equivalent to $a_{ij}=\sum_{k=1}^{r} b_{ik}c_{kj}$, $1\leqq i, j\leqq n$, and that the i, jth element of D is then given by $d_{ij}=\sum_{k=1}^{n} c_{ik}b_{kj}$, $1\leqq i, j\leqq r$.

Example 1: $(r=1)$. If $a_{ij}=b_ic_j$, then

$$|\lambda I - A| = \lambda^{n-1}\left(\lambda - \sum_{i=1}^{n} b_ic_i\right).$$

Example 2: $(r=2)$. If $a_{ij}=b_ic_j+d_ie_j$, then

$$|\lambda I - A| = \lambda^{n-2}\begin{vmatrix} \lambda - \sum_{i=1}^{n} b_ic_i & -\sum_{i=1}^{n} c_id_i \\ -\sum_{i=1}^{n} b_ie_i & \lambda - \sum_{i=1}^{n} d_ie_i \end{vmatrix}.$$

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References

1. C. C. MacDuffee, The theory of matrices, Chelsea, New York, 1956.
2. J. J. Sylvester, Philos. Mag. V, 16 (1883) 267–269.

TWO ASPECTS OF ANTIDIFFERENTIATION BY SUBSTITUTION

Z. YEH, University of Hawaii

The familiar method of substitution for antidifferentiation has in practice two distinct aspects involving different assumptions. We use a diagram to formulate these aspects into two different propositions.

Let F and f be functions on the interval (a, b) such that $F'=f$. Let ϕ be a differentiable mapping of the interval (c, d) onto (a, b), and let G and g be functions on the interval (c, d) such that $G=\phi_*(F)=F\circ\phi$ and $g=\phi_*(f)=f\circ\phi$ (see the diagram below), then the chain rule of differentiation gives $G'=g\cdot\phi'$.

$$\begin{array}{ccc} F & \xrightarrow{\phi_*} & G \\ \Big\downarrow & & \Big\downarrow \\ & & g\cdot\phi' \\ f & \xrightarrow{\phi_*} & g \\ (a,b) & \underset{\phi}{\longleftarrow} & (c,d) \end{array}$$

Given a function h to antidifferentiate, we can identify h with either f or $g\cdot\phi'$ in the above diagram. Accordingly, we have two cases.

CASE I. If we identify h with f, then we must find F. We can do this by selecting a suitable ϕ so that $g\cdot\phi'$ or $(f\circ\phi)\cdot\phi'$ is readily antidifferentiable to give G. From G we find F by $F=\phi_*^{-1}(G)=G\circ\phi^{-1}$. Clearly ϕ should have been one-to-one in order for ϕ^{-1} to exist. We summarize this, using customary notations, as follows.

PROPOSITION 1. *Given $f(x)$ on an interval (a, b), let $x=\phi(u)$ be a one-to-one differentiable mapping of some interval (c, d) onto (a, b) such that $f(\phi(u))\phi'(u)$ is antidifferentiable, then the antiderivative of $f(x)$ can be found by*

$$\int f(x)dx = \left(\int f(\phi(u))\phi'(u)du\right)_{u=\phi^{-1}(x)}$$

CASE II. If we identify h with $g\cdot\phi'$, then we must find G. In practice, we make this identification when h is in the form $g\cdot\phi'=(f\circ\phi)\cdot\phi'$. If f is readily antidifferentiable to give F, we find G by $G=\phi_*(F)=F\circ\phi$.

PROPOSITION 2. *If $h(x)=f(\phi(x))\phi'(x)$ with f antidifferentiable and ϕ differentiable (but not necessarily one-to-one), then the antiderivative of $h(x)$ can be found by*

$$\int f(\phi(x))\phi'(x)dx = \left(\int f(u)du\right)_{u=\phi(x)}.$$

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A PROOF OF THE SCHROEDER-BERNSTEIN THEOREM

R. H. Cox, University of Kentucky

The purpose of this short note is to indicate a proof of the Schroeder-Bernstein Theorem which, at least in the opinion of the author, is more easily followed and smoother than any of the usual arguments. The proof rests on the following lemma:

LEMMA. *If A is a set, B is a subset of A, and f is a* 1-1 *function from A into B; then there is a* 1-1 *function h from A onto B.*

Proof. If A is B, then the identity function on A is such an h. Thus, suppose that B is a proper subset of A, and let C denote the set of all elements y in A for which there is a nonnegative integer n and an element x in $A-B$ (the complement of B in A) such that $y=f^n(x)$; where f^0 is the identity function and, for each positive integer $k, f^k=f^{k-1}[f]$. For each z in A, define $h(z)$ as follows:

$$h(z) = \begin{cases} f(z) & \text{if } z \text{ is in } C \\ z & \text{if } z \text{ is not in } C. \end{cases}$$

Observe that $A-B$ is a subset of C; for each y in C, there is only one nonnegative integer n and only one element x in $A-B$ such that $y=f^n(x)$; and, finally, that $f(C)$ is a subset of C. From these observations, and the fact that f is 1-1, it follows easily that h is 1-1 from A onto B.

THEOREM. *If each of A and B is a set, f is a* 1-1 *function from A into B, and g is a* 1-1 *function from B into A; then there is a* 1-1 *function from A onto B.*

Proof. Since $g[f]$ is a 1-1 function from A into $g(B)$, there is, by the lemma, a 1-1 function h from A onto $g(B)$. Thus, $g^{-1}[h]$ is a 1-1 function from A onto B, and the proof is complete.

A NOTE ON CONTINUOUSLY NEAR-HOMOGENEOUS SPACES

H. V. KRONK, SUNY at Binghampton

In [1], Doyle and Hocking used the concept of a continuously invertible space to obtain an extrinsic characterization of the n-sphere. In this note we obtain a corresponding characterization using the analogous but weaker concept of continuous near-homogeneity.

DEFINITION. *A topological space S is said to be continuously near-homogeneous (continuously invertible) if, for each point $p\in S$ (closed proper subset C) and each nonempty open set U in S, there is an isotopy $\{h_t\}$ of S onto itself such that $h_1(p)\in U$ ($h_1(C)\subset U$).*

We shall need the following known result [3, page 350].

THEOREM 1. *Let X be a continuum in E^{n+1} separating the points p and q and let H: $X\times[0, 1]\to E^{n+1}$ be a continuous map such that* (1) $H|X\times\{t\}$ *is a homeomorphism of $X\times\{t\}$ into E^{n+1} for all $t\in[0, 1]$ and* (2) $H(x, 0)=x$ *for all $x\in X$. If $H(X\times[0, 1])$ doesn't contain either p or q, then p and q are separated by $H(X\times\{t\})$, for all $t\in[0, 1]$.*

THEOREM 2. *Let M be a compact continuously near-homogeneous subspace of E^{n+1}. If M contains an n-sphere S, then $M=S$.*

Proof. Assume that $M-S$ is not empty. Without any loss of generality, we may assume that there exists a point p of M in the bounded component A of $E^{n+1}-S=A\cup B$. Let q be any point of S and $U\subset A$ an open neighborhood of p such that $S\cap U$ is empty. Since M is continuously near-homogeneous there is an isotopy $\{h_t\}$ of M onto itself such that $h_1(q)\in U$. Now consider the intersection V of A and the unbounded component of $E^{n+1}-h_1(S)$. Clearly V is not empty. Either V lies entirely in the isotopy path of S or there is a point x in V not covered by $H(S\times[0, 1])$. In the latter case, the point x and a point y in $B\cap(E^{n+1}-M)$ are separated by S but they are not separated by $h_1(S)$. This contradicts Theorem 1 since the isotopy path of S doesn't contain either x or y. On the other hand, if V lies entirely in the isotopy path of S, then V lies in M and hence M contains an open $(n+1)$-cell. Since M is continuously near-homogeneous it easily follows that M is an $(n+1)$-manifold. But a compact $(n+1)$-manifold can't be embedded in E^{n+1}. Having been led to a contradiction in either case, it follows that the point p is nonexistent and hence $M=S$.

Since every continuously near-homogeneous Peano continuum contains a simple closed curve [2, page 830] we have as a corollary Theorem 10 of [2].

COROLLARY. *The only continuously near-homogeneous plane Peano continua are the simple closed curves.*

References

1. P. H. Doyle and J. G. Hocking, Continuous invertibility, Pacific J. Math., 12 (1962) 499–503.

2. M. D. Guay, J. G. Hocking, and H. V. Kronk, Local near-homogeneity, this MONTHLY, 70 (1963) 827–833.

3. K. Kuratowski, Topologie, Vol. 2, 2nd. ed. Warsaw, 1948.

A PROPERTY OF ARITHMETIC FUNCTIONS

AL SOMAYAJULU, Canisius College, Buffalo

Introduction. The function $\tau(x)$ represents the number of positive integral divisors of the positive integer x. N. C. Scholomiti [1] proved a property of $\tau(x)$, given by M. Lerch in 1887, by mathematical induction.

The property (Lerch's formula) is

$$\tau(n) = n - \sum_{k=1}^{n-1} t(n-k, k) \tag{1}$$

for each positive integer n, where $t(n-k,\ k)$ represents the number of divisors of $(n-k)$, each of which is greater than k.

In this note we shall generalize this result with an elementary proof without using mathematical induction.

Let $g(n)$ be a real or complex-valued arithmetic function.

Let

$$\alpha(n-k,\ k) = \sum_{\substack{d \mid (n-k) \\ d>k}} g(d),$$

n a positive integer. We shall prove the

THEOREM.

$$\sum_{k=0}^{n-1} \alpha(n-k,\ k) = \sum_{i=1}^{n} g(i). \tag{2}$$

Proof. Let $\lambda(x)$ be the characteristic function of $[1,\ \infty)$. Then

$$\alpha(n-k,\ k) = \sum_{d \mid (n-k)} \lambda\left(\frac{d}{k+1}\right) g(d).$$

Therefore

$$\begin{aligned}\sum_{k=0}^{n-1} \alpha(n-k,\ k) &= \sum_{k=0}^{n-1} \sum_{d \mid (n-k)} \lambda\left(\frac{d}{k+1}\right) g(d) \\ &= \sum_{i=1}^{n} \sum_{d \mid i} \lambda\left(\frac{d}{n-i+1}\right) g(d) \\ &= \sum_{d=1}^{n} g(d) \left\{ \sum_{r=1}^{[n/d]} \lambda\left(\frac{d}{n-rd+1}\right) \right\}.\end{aligned}$$

($[x]$ = greatest integer in x), $\lambda(d/(n-rd+1))=0$ for all $1 \leqq r < [n/d]$ and

$$\lambda\left(\frac{d}{n-[n/d]d+1}\right) = 1.$$

Hence,

$$\sum_{k=0}^{n-1} \alpha(n-k,\ k) = \sum_{d=1}^{n} g(d), \text{ which was to be proved.}$$

We obtain (1) as a corollary of (2) by taking $g(n) \equiv 1$ and observing that $t(n,\ 0) = \alpha(n,\ 0) = \tau(n)$.

If $g(n) = n^r$ for any positive integer r, arbitrary but fixed, (2) gives

$$\sigma_r(n) = \sum_{d \mid n} d^r = \sum_{i=1}^{n} i^r - \sum_{k=1}^{n-1} t_r(n-k,\ k),$$

where

$$t_r(n-k,k) = \sum_{\substack{d \mid (n-k) \\ d>k}} d^r.$$

Reference

1. N. C. Scholomiti, A property of the τ-function, this MONTHLY, 72 (1965) 745–747.

REMARK ON AN INEQUALITY OF SHAMPINE

D. L. PHILLIPS, Argonne National Laboratory

The inequality of Shampine derived in [1] is in error. The correct inequality is

$$(1) \qquad -\left[\frac{n}{2}\right]\int_0^\infty P_n(x)e^{-x}dx \leqq \int_0^\infty P_n'(x)e^{-x}dx \leqq \int_0^\infty P_n(x)e^{-x}dx,$$

where $P_n(x)$ is a polynomial of degree n with the property that $P_n(x) \geqq 0$ for $x \geqq 0$. Equality is obtained on the left if and only if P_n is a nonnegative multiple of

$$\left(\sum_{j=0}^{[n/2]} L_j(x)\right)^2,$$

where $L_j(x)$ is the Laguerre polynomial of degree j. Equality on the right is obtained if and only if $P_n(0)=0$.

The inequality has a simpler proof than the one given in [1]. Integrating $\int_0^\infty P_n'(x)e^{-x}dx$ by parts we get

$$(2) \qquad \int_0^\infty P_n'(x)e^{-x}dx = \int_0^\infty P_n(x)e^{-x}dx - P_n(0).$$

As stated in equation (1) of [1], $P_n(x)$ with $P_n(x) \geqq 0$ for $x \geqq 0$ can be expressed as

$$(3) \qquad P_n(x) = \left|\sum_{j=0}^{[n/2]} u_jL_j(x)\right|^2 + x\left|\sum_{j=0}^{[(n-1)/2]} v_jr_j(x)\right|^2,$$

where the u_j and v_j are certain, possibly complex, numbers. The L_j (Laguerre polynomials) and r_j are orthonormal with respect to e^{-x} and xe^{-x}, respectively. Since $L_j(0)=1$, it follows from (3) and the Cauchy inequality that

$$(4) \qquad P_n(0) = \left|\sum_{j=0}^{[n/2]} u_j\right|^2 \leqq \left(\left[\frac{n}{2}\right]+1\right)\sum_{j=0}^{[n/2]}\left|u_j\right|^2,$$

equality holding if and only if all u_j are equal. It also follows from (3) that

$$\int_0^\infty P_n(x)e^{-x}dx \geqq \int_0^\infty \left| \sum_{j=0}^{[n/2]} u_j L_j \right|^2 e^{-x}dx = \sum_{j=0}^{[n/2]} \left| u_j \right|^2, \tag{5}$$

equality holding if and only if the $v_j = 0$. Combining (4) and (5) we have

$$0 \leqq P_n(0) \leqq \left(\left[\frac{n}{2} \right] + 1 \right) \int_0^\infty P_n(x)e^{-x}dx,$$

which with (2) gives the desired inequality (1) as well as the appropriate conditions for when the equalities hold.

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Reference

1. L. F. Shampine, An inequality for the derivatives of nonnegative polynomials, this MONTHLY 73 (1966) 1090–1092.

ORDERS FOR FINITE NONCOMMUTATIVE RINGS WITH UNITY

K. E ELDRIDGE, Ohio University

Erickson [2] has shown that the order of a finite, noncommutative ring must have a square factor. His constructive proof for the existence of such rings gives only rings without unity.

We extend his result by proving the following theorem.

THEOREM. *Let R be a finite ring of order m with a unity. If m has a cube free factorization, then R is a commutative ring.*

For the proof we make use of a simple lemma.

LEMMA. *Let R be a finite ring of order p^n with unity e, where p is a prime. If $n<3$, then R is commutative.*

Proof. For $n=1$, the proof is obvious. For $n=2$, there are two cases. Case 1: the characteristic of R is p^2. But then the additive group of R is cyclic and hence R is commutative. Case 2: the characteristic of R is p. Here the additive group of R is generated by two elements. Since we may choose e as one of the additive generators, it should be clear that R is also commutative in this case.

Proof of the Theorem. Let $m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be the canonical decomposition of m. Then we know that

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_k \quad \text{(ideal direct sum)},$$

where each ideal R_i is of order $p_i^{n_i}$ and has a unity. Since m is cube free, each $n_i < 3$, and so each R_i is commutative by the Lemma. From the direct sum of ideals, it now follows that R is a commutative ring.

REMARK. In view of the lemma, we have an answer to the question raised at the end of the solution to problem E1529 (this MONTHLY, 70 (1963) 441): the order of the smallest noncommutative ring with unity is 8. An example of such a ring is the ring of 2×2 upper-triangular matrices with entries in GF(2).

It is interesting to note that this example generalizes in the following way:

PROPOSITION. *Let R be a finite noncommutative ring with unity. If the order of R is p^3, p a prime, then R is isomorphic to the ring of 2×2 upper-triangular matrices with entries from* $\mathrm{GF}(p)$.

Proof. Let e be the unity of R. Clearly, the characteristic of R is either p or p^2. Suppose it is the latter. Then the additive group of R is of type (p^2, p). With e as one of the two generators of the additive group, we find R to be commutative. Hence the characteristic of R must be p and the additive group must be of type (p, p, p), that is, R is a 3-dimensional algebra with unity over $\mathrm{GF}(p)$. Note that R must have a nonzero radical J of order p. For if $J=(0)$, R is isomorphic to a direct sum of total matrix rings by the Wedderburn-Artin Theorem [4, p. 106], and the only total matrix rings of order p^n with $n\leqq 3$ are finite fields. Hence, in this case, R would be commutative, contrary to the hypothesis.

On the other hand, if J is of order p^2, J would be commutative since it is a 2-dimensional nilpotent algebra over $\mathrm{GF}(p)$. Now in this case, $R/J\cong\mathrm{GF}(p)$. Hence, every element of R belongs to some coset $ke+J$ with $k\in\mathrm{GF}(p)$. However, since e is in the center of R and the elements of J commute with each other, we again have that R is commutative, contrary to the hypothesis. Therefore the order of J is p, and we can write

$$J = \{ku \mid k \in \mathrm{GF}(p)\}$$

for some $u\neq 0$ in J. Observe that $J^2=(0)$.

We now examine R/J. Since its order is p^2 and it has a unity, we know that R/J is either a field of p^2 elements or the direct sum of two fields with p elements each. Suppose it is a single field. Then J is a maximal ideal in R. From statement (2.1) of [3] we know that the natural homomorphism θ: $R\rightarrow R/J$ maps units onto units and nonunits onto nonunits. Hence, let g be the unit in R whose image $\theta(g)$ generates the cyclic group of units of R/J. Now for $u\neq 0\in J$ we have $gu=ku$ for some $k\neq 0\in\mathrm{GF}(p)$. Since $k^{p-1}=1$ in $\mathrm{GF}(p)$, $g^{p-1}u=u$, which implies that $g^{p-1}-e\in J$, since J is maximal. But then $[\theta(g)]^{p-1}=\theta(e)$, contrary to our choice of g. Therefore $R/J\cong\mathrm{GF}(p)\oplus\mathrm{GF}(p)$. Using Theorem 9.3C of [1] and the fact that R has a unity e, we can find an idempotent $f\in R$ such that $R=fR+(e-f)R$. Knowing that $\dim R=3$ over $\mathrm{GF}(p)$, we either have $\dim fR=2$ and $\dim(e-f)R=1$ or $\dim fR=1$ and $\dim(e-f)R=2$. Suppose $\dim fR=2$. Then

$$(e-f)R = \{k(e-f) \mid k \in \mathrm{GF}(p)\},$$

and since $(e-f)R$ is not nilpotent, $J\subset fR$. Clearly, f is a left unity for fR. This together with the noncommutativity of R implies $fu=u$ and $uf=0$ for all $u\in J$.

Therefore, the equations $f^2=f$, $uf=0$, $u^2=0$, and $er=re=r$ for all $r\in R$ and for all $u\in J$, completely determine the multiplication table for R. Then, if we set up the correspondences

$$e\to\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad f\to\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad u\to\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

the isomorphism between R and the ring of 2×2 upper-triangular matrices, with entries from $\mathrm{GF}(p)$, is obvious.

Similar arguments apply when $\dim fR=1$ over $\mathrm{GF}(p)$.

References

1. Artin, Nesbitt, and Thrall, Rings with Minimum Condition, University of Michigan Press, Ann Arbor, 1944.
2. D. B. Erickson, Orders for finite noncommutative rings, this MONTHLY, 73 (1966) 376–377.
3. H. K. Farahat, The multiplicative groups of a ring, Math. Zeitsch., 87 (1965) 378–384.
4. Neal H. McCoy, The Theory of Rings, Macmillan, New York, 1964.

ON EMBEDDING IN QUASI-CUBES

R. NIELSEN AND C. SLOYER, University of Delaware

It is well known that a topological space X is a Tychonoff space if and only if it is homeomorphic to a subspace of a cube [2, 118]. The purpose of this note is to prove an analogous theorem for any topological space satisfying the T_0 separation axiom. The construction is carried out in such manner as to yield the compactification and extension theorems usually associated only with Tychonoff spaces.

DEFINITION 1. *Let Q denote the unit interval equipped with the upper topology (a base for this topology is the family of all half-open intervals $[0, t)$, $0<t\leqq 1$). A quasi-cube is defined to be a Cartesian product of the spaces Q described above.*

We note that each such Q is T_0, compact and is not T_1. It is also observed that a function f from a topological space X to the closed unit interval with the usual topology is upper semi-continuous if and only if f is continuous as a function from X to Q [2, 101]. Denote by $F(X)$, the family of all upper semi-continuous functions from X to $[0, 1]$ and the evaluation map of X into $\times\{Q_f: f\in F\}$ by e. Finally, we observe that if $A\subset X$ is a closed set, then the characteristic function of A, ξ_A, is in $F(X)$ [1, 89].

THEOREM 2. *Any topological space satisfying the T_0 separation axiom can be embedded in a quasi-cube.*

Proof. It must be proved that the function e is a homeomorphism from X to its range $e(X)$. Thus, by the usual Embedding Lemma [2, 116], we must show that $F(X)$ is a family of continuous functions which distinguishes points and distinguishes closed sets and points. That $F(X)$ is a family of continuous

functions from X to Q has already been noted. If $x, y \in X$ and $x \neq y$, we suppose U is a neighborhood of x which does not contain y. The characteristic function $\xi_{U'}$ of the complement of U is in $F(X)$ and assumes different values at x and y. Finally, let A be a closed subset of X and $x_0 \notin A$. The characteristic function of A, ξ_A, is in $F(X)$ and $0 = \xi_A(x_0) \notin \overline{\xi_A(A)} = \overline{\{1\}}$, where the closure is taken in Q. Thus, $F(X)$ separates points and closed sets.

Note that the converse of the theorem holds since the T_0 property is productive and hereditary.

Since quasi-cubes are compact by the Tychonoff theorem, the closure of $e(X)$ in $\times\{Q_f : f \in F\}$ is compact and we refer to this as the upper Stone-Čech compactification of X. Thus, the following corollary is immediate:

COROLLARY 3. *Any T_0 space has an upper Stone-Čech compactification.*

With the usual methods (e.g. [2, 152]) one obtains the following extension theorem.

THEOREM 4. *If X is a T_0 space and f is a continuous function from X to a compact Hausdorff space Y, then there is a continuous extension of f which carries the upper Stone-Čech compactification of X into Y.*

The first author's research was supported in part by N. S. F. Grant No. GP6861.

References

1. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, New York, 1965.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N.J., 1955.

A SMOOTHING PROPERTY FOR CONDITIONAL EXPECTATIONS GIVEN σ-LATTICES

TIM ROBERTSON, University of Iowa

Some of the properties of conditional expectations given σ-fields are also valid for the more general concept of conditional expectation given a σ-lattice. It is well known (see Loeve [3]) that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are σ-fields of measurable subsets of a totally finite measure space and if $\mathcal{L}_1 \subset \mathcal{L}_2$ then for any integrable random variable X we have $E[E(X|\mathcal{L}_2)|\mathcal{L}_1] = E(X|\mathcal{L}_1)$. Loeve [3] calls this a smoothing property. In this note we present an example which shows that the above relation does not, in general, hold when $\mathcal{L}_1$ and $\mathcal{L}_2$ are σ-lattices and prove that it is valid if just one of these classes is a σ-field.

Suppose $(\Omega, \mathcal{A}, \mu)$ is a totally finite measure space. A random variable is an equivalence class of $\mathcal{A}$-measurable functions. If $\mathcal{L}$ is a σ-lattice of measurable subsets of Ω then $R(\mathcal{L})$ denotes the collection of all random variables X with property that $[X > a] \in \mathcal{L}$ for each real number a. Such random variables are said to be $\mathcal{L}$-measurable. Let L_1 be the class of integrable random variables and $L_1(\mathcal{L}) = L_1 \cap R(\mathcal{L})$. Let β denote the class of Borel subsets of reals. We adopt the

following definition for the conditional expectation, $E(X|\mathcal{L})$, of an integrable random variable X given the σ-lattice $\mathcal{L}$ (see Brunk [1]).

DEFINITION. *If $X \in L_1$ then $E(X|\mathcal{L})$ is that unique random variable Y in $L_1(\mathcal{L})$ such that*:

$$(1) \qquad \int (X - Y)Z d\mu \leqq 0 \quad \textit{for all bounded } Z \in R(\mathcal{L})$$

and

$$(2) \qquad \int_B (X - Y) d\mu = 0 \quad \textit{for all } B \in Y^{-1}(\beta).$$

If $\mathcal{L}$ is actually a σ-field then Properties (1) and (2) can be replaced by the single property:

$$(3) \qquad \int_B (X - Y) d\mu = 0 \quad \textit{for all } B \in \mathcal{L}$$

(see Loeve [3]).

THEOREM. *If $X \in L_1$ and $\mathcal{L}_1$ and $\mathcal{L}_2$ are σ-lattices of measurable subsets of Ω such that $\mathcal{L}_1 \subset \mathcal{L}_2$ and one is a σ-field then*

$$E[E(X|\mathcal{L}_2)|\mathcal{L}_1] = E(X|\mathcal{L}_1).$$

Let $X_1 = E(X|\mathcal{L}_1)$, $X_2 = E(X|\mathcal{L}_2)$ and $Y = E(X_2|\mathcal{L}_1)$. The following example illustrates that the theorem is not valid when we delete the hypothesis that either $\mathcal{L}_1$ or $\mathcal{L}_2$ is a σ-field. Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{A}$ be the collection of all subsets of Ω and μ be defined on $\mathcal{A}$ by $\mu(\{i\}) = 1$, $i = 1, 2, 3, 4$. Let the σ-lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ be defined by:

$$\mathcal{L}_1 = \{\phi, \{3, 4\}, \Omega\}$$

and

$$\mathcal{L}_2 = \{\phi, \{4\}, \{3, 4\}, \{2, 3, 4\}, \Omega\}.$$

The following table gives the values for a random variable X and the associated random variables X_1, X_2, and Y. Clearly $Y \neq X_1$.

w	$X(w)$	$X_1(w)$	$X_2(w)$	$Y(w)$
1	2	4	2	15/4
2	6	4	11/2	15/4
3	5	8	11/2	33/4
4	11	8	11	33/4

Proof of Theorem. We consider two cases. In each case it is clear that $Y \in L_1(\mathcal{L}_1)$. First suppose $\mathcal{L}_1$ is a σ-field. We must demonstrate that Y has

Property (3). First note that

$$\int (X - Y)d\mu = \int (X - X_2)d\mu + \int (X_2 - Y)d\mu = 0$$

by (2). Similarly for any B in $\mathcal{L}_1$

$$\int_B (X - Y)d\mu = \int_B (X - X_2)d\mu + \int_B (X_2 - Y)d\mu \leqq 0$$

since the second term on the right is zero by (3) and the first term is nonpositive by (1) because $R(\mathcal{L}_1) \subset R(\mathcal{L}_2)$. Since $\mathcal{L}_1$ is a σ-field the complement of B is also a member of $\mathcal{L}_1$ and this together with the above observations implies the desired conclusion.

On the other hand suppose $\mathcal{L}_2$ is a σ-field. We must show that Y has Properties (1) and (2). If Z is any bounded member of $R(\mathcal{L}_1)$ then

$$\int (X - Y)Zd\mu = \int (X - X_2)Zd\mu + \int (X_2 - Y)Zd\mu \leqq 0$$

since both terms on the right are nonpositive by (1). Finally we can show in a similar fashion that

$$\int_B (X - Y)d\mu = 0 \quad \text{for} \quad B \in Y^{-1}(\beta)$$

and the proof is completed.

Solutions for several extremum problems can be represented as conditional expectations given σ-lattices (cf. [2] and [4]). In some of these problems the solution is of the form $Y = E(X \mid \mathcal{L}_1 \cap \mathcal{L}_2)$ where $\mathcal{L}_1$ is a σ-field, $\mathcal{L}_2$ is a σ-lattice and X is the solution to the extremum problem which is measurable with respect to $\mathcal{L}_1$. In certain special cases Y is also equal to $E(X \mid \mathcal{L}_2)$. It is not true, in general, that $E(X \mid \mathcal{L}_1 \cap \mathcal{L}_2) = E(X \mid \mathcal{L}_2)$ when X is $\mathcal{L}_1$-measurable as the following example illustrates. It would be of interest to have conditions on $\mathcal{L}_1$ and $\mathcal{L}_2$ which are sufficient to insure that this is true.

Suppose $\Omega = \{1, 2, 3\}$ and μ is defined on the collection of all subsets of Ω by $\mu(\{i\}) = \frac{1}{3}$, for $i = 1, 2, 3$. If we let $\mathcal{L}_1 = \{\phi, \{1\}, \{2, 3\}, \Omega\}$ and $\mathcal{L}_2 = \{\phi\ \{1, 2\}, \{3\}, \Omega\}$ then $\mathcal{L}_1 \cap \mathcal{L}_2 = \{\phi, \Omega\}$. Further if X is defined as follows then $E(X \mid \mathcal{L}_1 \cap \mathcal{L}_2) \neq E(X \mid \mathcal{L}_2)$.

w	$X(w)$	$E(X \mid \mathcal{L}_1 \cap \mathcal{L}_2)(w)$	$E(X \mid \mathcal{L}_2)(w)$
1	1	5/3	3/2
2	2	5/3	3/2
3	2	5/3	2

References

1. H. D. Brunk, On an extension of the concept conditional expectation, Proc. Amer. Math. Soc., 14 (1963) 298–304.

2. H. D. Brunk, Conditional expectation given a σ-lattice and applications, Ann. Math. Statist., 36 (1965) 1339–1350.

3. Michel Loeve, Probability Theory, 3rd ed. Van Nostrand, Princeton, N. J. 1963.

4. Tim Robertson, On estimating a density which is measurable with respect to a σ-lattice, Ann. Math. Statist., 38(1967)482–493.

CORRECTION TO "A DIRECT PROOF OF STIRLING'S FORMULA"

WILLIAM FELLER, Princeton University

In Vol. 74, Number 10, December 1967, pages 1223–1225, of this MONTHLY, an unfortunate notational confusion occurred in the concluding argument. The functional equation (3.5) is satisfied by the *derivative* f', which is a continuous periodic function with $f'(0)=0$. (The function f itself satisfies $f(2x)=f(x)+f(x+\frac{1}{2})+\text{const.}$, from which (3.5) follows at once.) I am deeply apologetic for this error.

In (2.4) the signs of the integrals should be interchanged, and the sin in (2.10) should be replaced by log sin.

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the Monthly. Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.

A GENERALIZATION OF HILBERT'S THEOREM 90

M. G. MONZINGO, Southern Methodist University

Let E be a field extension of a field F and G the group of E/F.

DEFINITION. *If H is a subgroup of G, $H=\{I, \sigma_1, \cdots, \sigma_k\}$, then $N_H(x)=x\sigma_1(x)\cdots\sigma_k(x)$, for $x\in E$.*

The norm with respect to H, N_H, is shown to have some of the properties of N, the norm with respect to G.

THEOREM. *Let G be the group of E/F. If $a\in E$, then $N(a)=1$ if and only if there exists a subgroup H of G such that $N_H(a)=1$. Also, there exist $b\in E$ and $\sigma\in G$ such that $a=b^{I-\sigma}$ if and only if there exists a cyclic subgroup H of G, generated by σ, such that $N_H(a)=1$.*

Although the above theorem characterizes the elements of E which have norm 1, these characterizations may only be in terms of subgroups of G. There appears to be a possibility that a characterization of the elements having norm 1

in terms of elements of E has been achieved. That is, if for each $a \in E$ having norm 1 there is a cyclic subgroup H of G, generated by σ, such that $N_H(a)=1$, then all the elements of E having norm 1 would be of the form $b^{I-\sigma}$, for some $b \in E$ and $\sigma \in G$. The following example shows that this is not the case.

Let F be the field of rational numbers and $E=F(\sqrt{2}, \sqrt{3})$. The element $1+\sqrt{2}$ of E has norm 1, but for no cyclic subgroup H of G is $N_H(1+\sqrt{2})=1$

Mail Address: M. G. Monzingo, Dept. of Math., Southern Methodist University, Dallas, TX 75222.

SUMS OF POWERS OF NUMBERS

A. A. AUCOIN, University of Houston

We give in this paper formulas involving sums of powers of numbers. The main results are Theorems 4 and 8. We define the following numbers: the sequence $\{k_i\}$, for m a fixed positive integer $t_m, t_{m-1}, \cdots$, the arrays (a_{ij}), (b_{ij}), $(i, j=1, 2, \cdots)$ which satisfy the relations

$$a_{n1} = k_1^n \quad (n = 1, 2, \cdots), \qquad a_{1r} = k_r - k_{r-1} \quad (r = 2, 3, \cdots),$$
$$a_{n+1r} = k_r a_{nr} + a_{1r}(a_{nr-1} + \cdots + a_{n1}) \quad (r = 2, 3, \cdots),$$
$$b_{n1} = t_m^n \quad (n = 1, 2, \cdots), \qquad b_{1r} = t_{m-r+1} - t_{m-r+2} \quad (r = 2, 3, \cdots),$$
$$b_{n+1r} = t_{m-r+1} b_{nr} + b_{1r}(b_{nr-1} + \cdots + b_{n1}) \quad (r = 2, 3, \cdots).$$

THEOREM 1.

$$k_q + \sum_{j=q+1}^{s} a_{1j} = k_s.$$

THEOREM 2.

$$\sum_{r=1}^{s} a_{nr} = k_s^n.$$

THEOREM 3.

$$a_{ns} = k_s^n - k_{s-1}^n, \qquad s > 1.$$

THEOREM 4.

$$\sum_{i=1}^{m} k_i^p = \sum_{r=1}^{m} a_{jr} \sum_{i=r}^{m} k_i^{p-j}.$$

THEOREM 5.

$$t_{m-q} + \sum_{j=q+2}^{s} b_{1j} = t_{m-s+1}.$$

THEOREM 6.

$$\sum_{r=1}^{s} b_{nr} = \overset{n}{t}_{m-s+1}.$$

THEOREM 7.

$$b_{ns} = \overset{n}{t}_{m-s+1} - \overset{n}{t}_{m-s+2}, \qquad s > 1.$$

THEOREM 8.

$$\sum_{i=1}^{m} \overset{p}{t_i} = \sum_{r=1}^{m} b_{jr} \sum_{i=1}^{m-r+1} \overset{p-j}{t_i}.$$

Mail Address: Department of Mathematics, University of Houston, Houston, Texas 77004.

ON THE UNIFORM CONVERGENCES OF THE DISTRIBUTIONS OF NORMED SAMPLE QUANTILES

LAI KOW CHAN, University of Western Ontario

THEOREM. *Let $X_{1n} < X_{2n} < \cdots < X_{nn}$ be an ordered random sample of size n from an absolutely continuous distribution (with distribution function F and probability density function f) which satisfies the following condition: there exists a fixed closed interval $I = [p', p'']$, $0 < p' < p'' < 1$ such that $f(x)$ is continuous and positive at each point of $J = [F^{-1}(p'), F^{-1}(p'')]$ (end points included). Also let $g_{p,n}(y)$ be the probability density function of the normed sample quantiles*

$$Y_{n_p,n} = n^{1/2}(f(a_p)/\sqrt{p(1-p)})(X_{n_p,n} - a_p),$$

where $p \in I$, $n_p = 1 + $(the greatest integer $\leqq np$), $a_p = F^{-1}(p)$. Then

$$\lim_{n\to\infty} \sup_{p\in I} \left| g_{p,n}(y) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \right| = 0$$

uniformly on every bounded closed interval K.

Proof. The proof mainly consists of writing

$$g_{p,n}(y) = A_{1n} A_{2n}(y) A_{3n}(y)$$

and then proving that as $n \to \infty$

$$A_{1n} = [p(1-p)]^{1/2} \left[\frac{n - n_p + 1}{n(1-p)}\right] \left[\sqrt{n} \binom{n}{n_p - 1} p^{n_p-1}(1-p)^{n-n_p+1}\right]$$
$$= \frac{1}{\sqrt{2\pi}} + o\left(\frac{1}{n}\right)$$

$$A_{2n} = f(x)/f(a_p) = 1 + o(1), \quad \text{where} \quad x = a_p + (y/\sqrt{n})(\sqrt{p(1-p)})/f(a_p)$$
$$A_{3n} = [F(x)/p]^{n_p-1}[(1-F(x))/(1-p)]^{n-n_p} = \exp(-y^2/2) + o(1).$$

uniformly on I and K.

This theorem is a generalization to uniform convergence on I of the well-known result on convergence of normed sample quantiles appearing in the standard texts by H. Cramér, M. Fisz and S. S. Wilks.

COROLLARY.

$$\lim_{n\to\infty} \sup_{p\in I} \int_R \left| g_{p,n}(y) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)\right| dy = 0, \qquad R = (-\infty, \infty).$$

The proof of the corollary follows from the following lemma which is a uniformized version of Scheffe's useful convergence theorem (Ann. Math. Statist., 1947, pp. 434–438).

LEMMA. *Let μ be a σ-finite measure defined on the class of Borel sets in R and $f_n(x|\theta)\geqq 0$, $n=1, 2, \cdots$, $x\in R$, $\theta\in\Theta$ be such that* $\lim_{n\to\infty}\sup_\theta |f_n(x|\theta)-\Phi(x|\theta)| = 0$ *for some $\Phi(x|\theta)$ with* $\sup_\theta \Phi(x|\theta) \leqq B(x)$ *μ-integrable and* $\lim_{n\to\infty}\sup_\theta |\int_R (f_n(x|\theta)-\Phi(x|\theta))d\mu(x)| = 0$, *then* $\lim_{n\to\infty}\sup_\theta \int_R |f_n(x|\theta)-\Phi(x|\theta)|\, d\mu(x) = 0$.

Mail Address: Department of Mathematics, University of Western Ontario, London, Ontario, Canada.

THE NUMBER OF UNRESTRICTED kth POWER RESIDUES

J. H. JORDAN, Washington State University

An integer, A, is called an unrestricted kth power residue modulo m if $X^k\equiv A \pmod m$ is solvable.

Let $w_k(m)$ be the number of unrestricted kth power residues modulo m.

R. C. Buck (Am. Math J., vol. 68, 1946) exhibited the properties of $w_2(m)$. The general result is

THEOREM. *The functions w_k have the following properties which suffice to define them completely.*

A. *When $n=mk+s$, $0<s<k$;*

(i) *If $(a, b)=1$ then $w_k(ab)=w_k(a)w_k(b)$.*

(ii) *For p an odd prime* $w_k(p^n)=1+\sum_{i=0}^m \phi(p^{ki+s})/(k, \phi(p^{ki+s}))$.

(iii) *For k odd* $w_k(2^n)=1+\sum_{i=0}^m \phi(2^{ki+s})$.

(iv) *For k even* $w_k(2^n)=1+\sum_{i=0}^m \lambda(2^{ki+s})/(k, \lambda(2^{ki+s}))$, *where ϕ and λ are the Euler and Lucas function respectively.*

B. *When $n=mk$ the 1's are eliminated from the above formulas.*

Part (i) of the theorem can be proved by a simple counting argument. Parts (ii), (iii) and (iv) can be proved by induction on m, utilizing the known structure of the multiplicative group of the coprime residue classes modulo a prime power.

Mail Address: Mathematics Department, Washington State University, Pullman, WA 99163.

A DECOMPOSITION OF THE PLANE INTO BOUNDED CLOSED LINE SEGMENTS

H. J. CHARLTON, North Carolina State University

Let $\mathcal{A}$ be the collection of disjoint closed line segments $\mathcal{A} = \{[0,1]\times\{0\}, [2,3]\times\{0\}, [0,3]\times\{y\}$ and $\{x\}\times[0,3]$ where $y\in(0,3)$ and $x\in(3,4)\}$. Let G be the multiplicative group generated by the homeomorphisms L: $(x, y)\to(x+4, y)$ and U: $(x, y)\to(x+2, y+3)$ of $E^2=E\times E$ onto E^2.

LEMMA 1. *Let a be a positive integer. Then given a real number b there exist a unique integer q and unique real number r such that $b=aq+r$, where $0\leqq r<a$.*

LEMMA 2. $E^2=\bigcup_{g\in G} g[\bigcup\mathcal{A}]$ *where* $\bigcup\mathcal{A}=\{x: x\in A$ *for some* $A\in\mathcal{A}\}$.

Proof: Let $(a, b)\in E^2$. Now there exist q and p such that $b=3q+r_1$ and $a-2q=4p+r_2$ where $0\leqq r_1<3$ and $0\leqq r_2<4$. Then

$$L^{-p}U^{-q}(a, b) = (r_2, r_1) \in [0, 4) \times [0, 3).$$

As $[0, 4) \times [0, 3) \subseteq (\bigcup\mathcal{A}) \cup U^{-1}((3, 4) \times \{3\})$ where $(3, 4) \times \{3\} \in \mathcal{A}$, $E^2\subseteq\bigcup_{g\in G} g[\bigcup\mathcal{A}]$.

LEMMA 3. *If $l_1, l_2\in\mathcal{A}$ and $g_1, g_2\in G$, $l_1\neq l_2$ then $g_1(l_1)\cap g_2(l_2)=\phi$.*

Proof. Let $(a, b)\in g_1(l_1)\cap g_2(l_2)$. As in proof of Lemma 2, $g_1^{-1}=L^{-p}U^{-q}$ and $g_2^{-1}=L^{-p'}U^{-q'}$. But by Lemma 1, $p=p'$ and $q=q'$ or $g_1=g_2$ which is a homeomorphism.

THEOREM 4. *The collection $\{g[l]: g\in G$ and $l\in\mathcal{A}\}$ forms a disjoint cover of E^2 by closed line segments of length one and three.*

Proof. This is a direct corollary to Theorem 2 and 3.

Mail Address: Department of Mathematics, North Carolina State University, Raleigh, N. C. 27607.

EXPANSIVE MAPPINGS

R. K. WILLIAMS, Southern Methodist University

Let X be a metric space with metric d, and let f be a continuous multi-valued transformation of X onto itself, henceforth called a mapping.

DEFINITION 1. *Let $x\in X$. The orbit of x under f is defined by $O(x)=\bigcup_{-\infty}^{\infty} f^n(x)$.*

DEFINITION 2. *Let $x\in X$. A sub-orbit of x under f is a set of the form $\{x_i: x_0=x, x_{i+1}\in f(x_i)$ for each integer $i\}$.*

DEFINITION 3. *f is expansive on X with expansive constant $\delta>0$ if $x, y\in X$, $x\neq y$ implies for each sub-orbit A of x and for each sub-orbit B of y, there exist $x_n\in A$, $y_n\in B$ such that $d(x_n, y_n)>\delta$.*

DEFINITION 4. *x and y are positively (negatively) asymptotic under f if $x \neq y$ and if for each $\epsilon > 0$, there is an integer N such that $n > N (n < N)$ implies* $\inf\{d(a, b): a \in f^n(x), b \in f^n(y)\} < \epsilon$.

It is clear that these definitions reduce to the standard ones which occur in topological dynamics when f is a homeomorphism.

When X is compact, it is possible to construct a compact metric space S and a homeomorphism h of S onto itself with the property that h is an expansive homeomorphism on S if and only if f is an expansive mapping on X. Using S and h as the main tools, it can be shown that most of the standard theorems on expansive homeomorphisms and asymptoticity remain valid in this more general setting.

A sample theorem is the following:

THEOREM. *If x is compact and if f is an expansive mapping on X, and if g is a homeomorphism of X onto Y, then gfg^{-1} is an expansive mapping on Y.*

Mail Address: Department of Mathematics, Southern Methodist University, Dallas, TX 75222.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, UNIVERSITY OF CONNECTICUT

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

COINCIDENCES OF MAPS OF EUCLIDEAN SPACES

R. F. BROWN, University of California, Los Angeles

We will prove a generalization of the Brouwer fixed point theorem that includes results of Knaster, Kuratowski, and Mazurkiewicz [2] and Schirmer [3] as special cases. Since the proof depends only on the Eilenberg-Steenrod axioms of homology theory and their immediate consequences (see Hu [1], pp. 1–30), the theorem may be of use to instructors who wish to introduce applications at an early stage of an algebraic topology course.

We use the symbol R^n for n-dimensional Euclidean space. The norm of $x \in R^n$ is denoted by $|x|$. For $r > 0$, let

$$D_r^n = \{x \in R^n \mid |x| \leqq r\} \quad \text{and} \quad S_r^{n-1} = \{x \in R^n \mid |x| = r\}.$$

For $f: X \to Y$ a function and A a subset of X, the restriction of f to A is written $f|A$. The coefficient group for homology will be the integers.

THEOREM. *Let f and g be maps (continuous functions) from R^n to itself. If there exists $r>0$ such that $0 \leqq |g(x)| \leqq |f(x)| = r$ for all $x \in S_r^{n-1}$ and such that the induced homomorphism*

$$(f \mid S_r^{n-1})_* : H_{n-1}(S_r^{n-1}) \to H_{n-1}(S_r^{n-1})$$

is nontrivial, then there exists $x \in D_r^n$ for which $f(x) = g(x)$.

Proof. Let $f \times g$: $R^n \to R^n \times R^n$ be the map defined by $(f\times g)(x) = (f(x), g(x))$. The diagonal of $R^n \times R^n$ is denoted by $\Delta = \{(x, x) \in R^n \times R^n \mid x \in R^n\}$. For the case $n=1$, note that $R^1 \times R^1 - \Delta$ has two components $C_1 = \{(x, y) \in R^1 \times R^1 \mid x > y\}$ and $C_2 = \{(x, y) \in R^1 \times R^1 \mid x < y\}$. Assume that $f(x) \neq g(x)$ for all $x \in D_r^1$ and consider the path $P = (f \times g)(D_r^1) \subseteq R^1 \times R^1 - \Delta$. By hypothesis, the points $(-r, y)$ for some y, $-r < y \leqq r$, and (r, y) for some y, $-r \leqq y < r$, belong to P, so $P \cap C_1 \neq \varnothing$ and $P \cap C_2 \neq \varnothing$. We have arrived at a contradiction, since if $f(x) \neq g(x)$ for all $x \in D_r^1$, then we could express the connected set P as the union of two disjoint nonempty open subsets $P \cap C_1$ and $P \cap C_2$. Thus $f(x) = g(x)$ for some $x \in D_r^1$. For the case $n > 1$, we again assume $f(x) \neq g(x)$ for all $x \in D_r^n$ and seek to establish a contradiction. Consider the following diagram:

$$\begin{array}{ccc}
H_{n-1}(S_r^{n-1}) & \xrightarrow{(f \mid S_r^{n-1})_*} & H_{n-1}(S_r^{n-1}) \\
e_* \downarrow & & \uparrow p_* \\
H_{n-1}(D_r^n) & & H_{n-1}(S_r^{n-1} \times D_r^n - \Delta) \\
(f \times g \mid D_r^n)_* \downarrow & & \uparrow k_* \\
H_{n-1}(R^n \times R^n - \Delta) & \xrightarrow[h_*]{} & H_{n-1}(D_r^n \times D_r^n - \Delta).
\end{array} \tag{1}$$

The function e is the inclusion of S_r^{n-1} into D_r^n and p: $S_r^{n-1} \times D_r^n - \Delta \to S_r^{n-1}$ is projection. Note that our assumption that $f(x) \neq g(x)$ for all $x \in D_r^n$ implies that $(f \times g)(D_r^n) \subseteq R^n \times R^n - \Delta$. For $(x, y) \in R^n \times R^n$, let $m(x, y)$ be the maximum of $|x|$ and $|y|$. Define

$$h(x, y) = \begin{cases} (x, y) & \text{if } m(x, y) \leqq r \\ \left(\dfrac{r}{m(x, y)}\, x, \dfrac{r}{m(x, y)}\, y\right) & \text{if } m(x, y) \geqq r. \end{cases}$$

If $(x, y) \in D_r^n \times D_r^n - \Delta$, then $x \neq y$, so

$$L(x, y) = \{(1 - t)y + tx \mid t > 0\} \subseteq R^n$$

is a well-defined set which, since x and y are in D_r^n, intersects S_r^{n-1} at a single point. Define $k(x, y) = (L(x, y) \cap S_r^{n-1}, y)$; then $k(x, y) \notin \Delta$ because $y \notin L(x, y)$. Note that if $x \in S_r^{n-1}$, then $k(x, y) = (x, y)$. For $x \in S_r^{n-1}$,

$$(f \times g)e(x) = (f(x), g(x)) \in S_r^{n-1} \times D_r^n - \Delta$$

by hypothesis, so $m(f(x),\ g(x))=r$ and $h(f\times g)e(x)=(f(x),\ g(x))$. Next, since $f(x)\in S_r^{n-1}$, $kh(f\times g)e(x)=(f(x),\ g(x))$ and finally we have $pkh(f\times g)e(x)=f(x)$. Thus diagram (1) commutes, that is, $(f|S_r^{n-1})_*=p_*k_*h_*(f\times g|D_r^n)_*e_*$. But since $n>1$, $H_{n-1}(D_r^n)=0$, so e_* is the trivial homomorphism and therefore $(f|S_r^{n-1})_*$ is also. Our assumption that $f(x)\neq g(x)$ for all $x\in D_r^n$ has led to a contradiction of the hypothesis that $(f|S_r^{n-1})_*$ is nontrivial, so there must exist $x\in D_r^n$ such that $f(x)=g(x)$.

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References

1. S. T. Hu, Homology Theory, Holden-Day, San Francisco, 1966.
2. B. Knaster, K. Kuratowski, and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, Fund. Math., 14 (1929) 132–137.
3. H. Schirmer, A Brouwer type coincidence theorem, Canadian Math. Bull., 9 (1966) 443–446.

HADAMARD'S THREE CIRCLES THEOREM

E. C. MILNER, University of Calgary

An error found in some elementary texts on complex function theory is that theorems which are stated and proved for one valued functions are then applied to so-called multi-valued functions. The commonest example of this is in the evaluation of certain real integrals like

$$\int_0^\infty x^{\alpha-1}(1+x)^{-1}dx = \pi \operatorname{cosec} \pi\alpha \qquad (0<\alpha<1) \tag{1}$$

by means of contour integration. Of course, it is a simple matter to provide the missing justification in such cases and more carefully written texts either give the details or indicate their need. Perhaps particular mention should be made of the proof of (1) given by Estermann [1] which avoids any extraneous limit argument.

Another instance of the error referred to is to be found in some proofs of Hadamard's three circles theorem.

HADAMARD'S THEOREM. *Let* $0<r_1<r_2<r_3$. *If* $f(z)$ *is regular on the closed annulus* $r_1\leq |z| \leq r_3$ *and if* M_j *is the maximum value of* $|f(z)|$ *on the circle* $|z|=r_j$ $(j=1,2,3)$, *then*

$$M_2^{\log(r_3/r_1)} \leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}. \tag{2}$$

Moreover, there is strict inequality in (2) *unless* $f(z)$ *is a constant multiple of a power of* z.

The argument used to prove this (e.g. [2]) is essentially to assert that, by the maximum modulus principle, the maximum value of $|z^\lambda f(z)|$ in the closed

annulus $r_1 \leqq |z| \leqq r_3$ occurs on one of the bounding circles. The real constant λ is conveniently chosen so that $r_1^\lambda M_1 = r_3^\lambda M_3$, i.e.

$$\lambda = \log(M_3/M_1)/\log(r_1/r_3), \tag{3}$$

and the inequality (2) follows at once. This assertion about the maximum value of $|z^\lambda f(z)|$ is correct, but the fact seems to require a little more justification than we have just indicated. The trouble is that λ is not, in general, an integer and there is some ambiguity about the meaning of the symbol z^λ. Is it to be regarded as a multi-valued function, or is the principal value (which is one-valued but discontinuous on a half-line) intended? Whichever interpretation is chosen the maximum principle is not applicable since, in its usual form, this is a statement about (one-valued) regular functions:

THE MAXIMUM PRINCIPLE. *If $f(z)$ is regular and nonconstant on a closed bounded region, then the maximum value of $|f(z)|$ can only occur on the boundary of that region.*

One way of avoiding the difficulty just mentioned is to consider the more respectable function $z^p(f(z))^q$ where p, q are integers and then allow p/q to approach the value λ given by (3). This yields a correct proof of (2) but does not give the stronger result that there is strict inequality in all but the obvious cases.

The proof of Hadamard's theorem given here is in the same spirit as Estermann's proof of (1)—there is no limit argument and the maximum principle is only used in the form stated above. The symbol z^λ is unambiguously defined for $z \neq 0$ as $\exp(\lambda[\log|z| + i \arg z])$ with the usual convention that $-\pi < \arg z \leqq \pi$. Thus z^λ is regular at all points except possibly on the nonpositive part of the real axis.

Proof of Hadamard's Theorem. If $f(z)$ is a constant multiple of a power of z, then (2) holds with equality. We will assume that f is not a function of this kind and prove (2) with strict inequality.

For real α let L_α denote the open segment $z = \rho e^{i\alpha}$ $(r_1 < \rho < r_3)$. Let γ_1 and γ_2 be the closed contours shown in fig. 1, i.e. γ_1 is formed by the segments $L_{\pm 3\pi/4}$ and

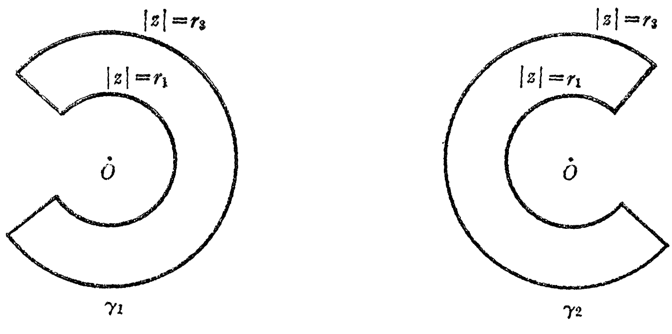


FIG. 1

the arcs $z=r_je^{i\theta}$ $(-3\pi/4\leqq\theta\leqq 3\pi/4;\, j=1,\, 3)$ and γ_2 is formed by the segments $L_{\pm\pi/4}$ and the arcs $z=r_je^{i\theta}$ $(\pi/4\leqq\theta\leqq 7\pi/4;\, j=1,\, 3)$.

Let λ be given by (3) so that $r_1^\lambda M_1=r_3^\lambda M_3=A$. Put $f_1(z)=z^\lambda f(z)$ and $f_2(z)=(-z)^\lambda f(z)$. Then f_j is regular and nonconstant on and within γ_j $(j=1,\, 2)$. By the maximum principle it follows that, if K_j is the maximum value of $|f_j(z)|$ on γ_j, then

$$|f_j(z)| < K_j \tag{4}$$

at interior points of γ_j $(j=1,\, 2)$. Suppose that $K_1>A$. Then $|f_1(z)|=K_1$ at some point on one of the open segments $L_{\pm 3\pi/4}$. Since these segments are in the interior of γ_2 and $|f_1(z)|=|f_2(z)|$ for all $z(\neq 0)$, it follows from (4) that $K_2>K_1>A$. A similar argument applied to f_2 now leads to the contradiction $K_1>K_2>A$. It follows that $K_1,\, K_2\leqq A$. Since each point on the circle $|z|=r_2$ is an interior point of either γ_1 or γ_2, we deduce from (4) that

$$r_2^\lambda M_2 < A = r_1^\lambda M_1 = r_3^\lambda M_3.$$

This implies that (2) holds with strict inequality.

References

1. T. Estermann, Complex Numbers and Functions, Oxford University Press, London, 1962.
2. R. L. Goodstein, Complex Functions, McGraw-Hill, New York, 1965.

UNIQUE FACTORIZATION IN THE INTEGERS MODULO n

M. BILLIS, University of Utah

Unique factorization is usually discussed only in the case of integral domains. However, the concept makes perfectly good sense even though zero divisors are present.

DEFINITION. *R is said to be a unique factorization ring if and only if every nonzero nonunit of R may be expressed uniquely, up to unit factors, as a product of primes.*

There are many well-known examples of integral domains which do not possess unique factorization. It is a fairly simple exercise to prove the following theorem which furnishes us with a wealth of examples of rings with zero divisors which do possess this property.

THEOREM. *The integers modulo n is a unique factorization ring if and only if $n=p^m$, where p is a prime.*

Reference

1. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, Macmillan, New York, 1953.

A PROOF THAT $z\Gamma(z)=\Gamma(z+1)$

D. A. Lee, Aerospace Research Laboratories

The fact that $z\Gamma(z)=\Gamma(z+1)$ for arbitrary complex z not a singularity of $\Gamma(z)$ is one of the most important properties of the gamma function. This useful result follows immediately from an integration by parts for z such that the Eulerian integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \tag{1}$$

converges. In several well-known texts [**1, 2, 3**] the general result is established from Euler's limit formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}, \qquad n = 1, 2, \cdots.$$

Artin's elegant monograph [4] shows that it is possible to incorporate the result in a definition of the gamma function for positive real arguments. Specifically, Artin proves that if a function $f(x)$ is defined for all real $x>0$ and satisfies the conditions

(a) $f(x+1)=xf(x)$
(b) f is log-convex for all $x>0$
(c) $f(1)=1$

then $f(x)$ is identical with the gamma function on $x>0$.

In this note we show that the general result can also be obtained directly from the definition

$$\Gamma(z) \equiv \sum_0^\infty \frac{(-1)^n}{n!(n+z)} + \int_1^\infty e^{-t}t^{z-1}dt \tag{2}$$

implicitly used by Copson (Reference 1, pages 206 and 207). This definition has the advantage of displaying the analytic structure of $\Gamma(z)$ clearly: the integral, which we shall call $\Omega(z)$, is an entire function, and the series represents a function $\Phi(z)$ which is analytic in the finite plane except for simple poles at zero and the negative integers.

It follows easily from an integration by parts that

$$z\Omega(z) = -e^{-1} + \Omega(z+1).$$

Considering $\Phi(z)$, one sees that

$$\Phi(z) \equiv \sum_0^\infty \frac{(-1)^n}{n!(n+z)} = \frac{1}{z} + \sum_1^\infty \frac{(-1)^n}{n(n+z)[(n-1)!]}.$$

But $1/[n(n+z)]=(1/z)[1/n-(1/n+z)]$, $n\neq 0$, so

$$\Phi(z) = \frac{1}{z} + \frac{1}{z}\sum_{1}^{\infty}\frac{(-1)^n}{n!} - \frac{1}{z}\sum_{1}^{\infty}\frac{(-1)^n}{(n+z)[(n-1)!]}$$
$$= \frac{1}{z}e^{-1} + \frac{1}{z}\sum_{0}^{\infty}\frac{(-1)^k}{(k+1+z)k!} = \frac{1}{z}e^{-1} + \frac{1}{z}\Phi(z+1),$$

whence $z\,\Phi(z) = e^{-1} + \Phi(z+1)$. Consequently

$$z\Gamma(z) = z\Phi(z) + z\Omega(z) = \Phi(z+1) + \Omega(z+1) = \Gamma(z+1),$$

which was to be proved.

Other properties of the gamma function can be deduced readily from definition (2). For example, one can establish condition (c) by observing that

$$\Omega(1) = \int_1^{\infty} e^{-t}dt = e^{-1},$$

and that

$$\Phi(1) = \sum_{0}^{\infty}\frac{(-1)^n}{(n+1)!} = -\sum_{1}^{\infty}\frac{(-1)^j}{j!} = 1 - e^{-1}.$$

One can demonstrate condition (b) by noting that since the series for $\Phi(z)$ converges absolutely, its terms may be grouped to give

$$\Phi(x) = \left[\frac{1}{x} - \frac{1}{1+x}\right] + \cdots + \frac{1}{n!}f_n(x) + \cdots,$$

$$f_n(x) = \frac{1}{n+x} - \frac{1}{(n+1)(n+1+x)}, \qquad n = 2, 4, \cdots.$$

Each $f_n(x)$ is log-convex for $x>0$, because clearly $f_n(x)>0$ if $x>0$, and it is easy to verify that

$$f_n(x)f_n''(x) - \{f_n'(x)\}^2 > 0 \quad \text{if } x > 0.$$

Therefore the sequence of partial sums of the series for $\Phi(x)$ is a sequence of log-convex functions, and since for $x>0$,

$$\Phi(x) > \frac{1}{x(1+x)} > 0,$$

$\Phi(x)$ is log-convex for $x>0$ by Theorem 1.6 of [4]. Now, $\Omega(x)$ is log-convex for $x>0$ by Theorem 1.9 of [4]. Thus $\Gamma(x)$ is the sum of two log-convex functions on $x>0$, and consequently $\Gamma(x)$ is log-convex for $x>0$.

I am indebted to Professor G. N. Raney and to the referee for suggesting the material of this last paragraph.

References

1. E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, London, 1935.
2. K. Knopp. Theory of Functions, Vol. II, Dover, New York, 1947.
3. E. D. Rainville, Special Functions, Macmillan, New York, 1960.
4. E. Artin, The Gamma Function, Holt, Rinehart and Winston, New York, 1964.

NEW PROOF OF A CLASSIC COMBINATORIAL THEOREM

S. W. GOLOMB, University of Southern California

THEOREM. *The number of samples of r objects from a set of k objects, allowing repetition but disregarding order, is*

$$\binom{k+r-1}{r}.$$

Proof. Consider the k objects to be cards numbered from 1 to k, and adjoin $r-1$ extra cards numbered from $k+1$ to $k+r-1$, and bearing the respective instructions "repeat lowest numbered card," "repeat 2nd-lowest numbered card," $\cdots$, "repeat $(r-1)^{st}$ lowest numbered card." Then a sample of size r without replacement from this enlarged ($k+r-1$ card) deck corresponds uniquely to a sample of size r from the original deck allowing replacement. The number of such samples is accordingly

$$\binom{k+r-1}{r}.$$

Examples:

1. To form a 5-card poker hand allowing repetition, it suffices to adjoin four "jokers" with the respective instructions: (a) repeat lowest card, (b) repeat 2nd-lowest card, (c) repeat 3rd-lowest card, and (d) repeat 4th-lowest card. We regard these four jokers as being the highest cards in the enlarged deck, with $a<b<c<d$.

A hand with no jokers is an ordinary poker hand (no repetition).

A hand with one joker has any one of its four ordinary cards repeated, depending upon which joker is held.

A hand with two jokers either has two ordinary cards duplicated (in the cases $a-b$, $a-c$, and $b-c$), or one ordinary card triplicated (in the cases $a-d$, $b-d$, $c-d$).

A hand with three jokers becomes one of the following: $xxyyy$, $xxxyy$, $xyyyy$, or $xxxxy$, where x and y are the two ordinary cards, depending on which three jokers are held.

A hand with all four jokers becomes a five-fold repetition of the ordinary card it contains.

2. If we form a 17-card deck with cards labeled A, 2, 3, 4, 5, 6, 7, 8, 9, 10,

J, Q, K, a, b, c, d, where a, b, c, d are as in the previous example, then 5-card hands have all the following possibilities: 1 pair, 2 pair, 3 of a kind, full house, 4 of a kind, 5 of a kind, straight, and "bust."

AN EXTENSION OF LIGHT'S ASSOCIATIVITY TEST

A. R. BEDNAREK, University of Florida

The purpose of this brief note is to observe that Light's associativity test (see, for example, Clifford-Preston [1], p. 7) can be extended, with only minor modifications, to systems in which finite groupoids "act" on finite sets. Such systems are of particular interest in the theory of automata, and are, indeed, sometimes called automata or abstract machines. It was for the purpose of constructing examples in this theory that the below-stated extension of the associativity test was noted.

Suppose $T=\{t_1, t_2, \cdots, t_n\}$ is a finite *groupoid* (set with a binary operation $\cdot$) and that the Cayley multiplication table for $(T, \cdot)$ is given. Let $X=\{x_1, x_2, \cdots, x_m\}$ be a finite set and let there be given a function $T\times X\rightarrow X$, the value at (t, x) being denoted by tx. We may say that T *acts on* X, or $(T, X, \rightarrow)$ is an *act*, if

$$(*) \qquad t(t'x) = (t\cdot t')x \quad \text{for all } t, t' \in T \text{ and for each } x \in X.$$

Of course when $T=X$ and the function "$\rightarrow$" is the multiplication on T, condition (*) is the associativity condition that makes T a semigroup.

We now assume that T and X are as above and that the function $T\times X\rightarrow X$ is given in tabular form

$$\begin{array}{c|c}
\rightarrow & x_1 \cdots x_j \cdots x_m \\
\hline
t_1 & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
t_i & \cdots\cdots t_i x_j \\
\cdot & \\
\cdot & \\
\cdot & \\
t_n &
\end{array}$$

To test (*) proceed as follows:

1. Fix a $t'\in T$.

2. For the t' in 1 construct a table (called the t'-table) in which the column indices are the elements of the t'-row $(t'x_1, \cdots, t'x_j, \cdots, t'x_m)$ of the function table and the row indices are the elements of the t'-column $(t_1\cdot t', \cdots, t_i\cdot t', \cdots, t_n\cdot t')$ of the T-table; i.e., of the Cayley table for the groupoid $(T, \cdot)$.

3. Fill in the t'-table by copying down for each new column index the column of the function table corresponding to that index.

4. For each row index of the constructed table check to see whether the row of the new table agrees with the row of the function table corresponding to the same index.

5. Repeat for each $t' \in T$. (Note: As in the case of semigroups it is easy to see that it is only necessary to carry out the procedure for a set of generators of T.)

Following step 3 of the procedure outlined above, the (i, j)th entry of the newly constructed table is $t_i(t'x_j)$. The check in step 4 insures that this entry is the same as $(t_i \cdot t')x_j$, so that repeating the procedure for each $t' \in T$ establishes (*). We illustrate the procedure with an example.

Example: Let $T = \{a, b, c\}$ and $X = \{1, 2, 3, 4\}$ with "$\cdot$" and "$\rightarrow$" defined by the tables below.

T-table

$\cdot$	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

function table

$\rightarrow$	1	2	3	4
a	2	2	2	4
b	3	2	2	4
c	2	2	3	4

a-table

	2	2	2	4
a	2	2	2	4
a	2	2	2	4
a	2	2	2	4

b-table

	3	2	2	4
a	2	2	2	4
a	2	2	2	4
b	3	2	2	4

c-table

	2	2	3	4
a	2	2	2	4
a	2	2	2	4
c	2	2	3	4

Reference

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Volume I, Amer. Math. Soc. Surveys, No. 7, 1961.

MATHEMATICAL EDUCATION NOTES

Edited by John R. Mayor, AAAS
Collaborating Editor: John A. Brown, University of Delaware

Material for this department should be sent to John G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706

GEOMETRY ACHIEVEMENT TESTS IN NLSMA

J. P. Becker, Rutgers—The State University

Purpose. The purpose of this paper is to provide an overview of the development of some of the geometry achievement tests used in the National Longitudinal Study of Mathematical Abilities. (NLSMA is a project within the School Mathematics Study Group and is supported by the National Science Founda-

tion.) Attention is paid to the content of tests and to a classification of mathematical abilities used in geometry.

Development of geometry achievement tests. During the summer, 1965, a writing team was convened by SMSG to develop geometry test item scales (a scale is defined to be a set of test items which measure the same mathematical content) along guidelines established by a NLSMA panel of consultants. The final goal was to develop enough good test items for one to one and one-half hours of testing. The writing team assumed students taking the tests had completed both elementary algebra and geometry in secondary school.

The following is a sample of the content covered by these tests:

1. Sets of points, lines, and planes and their configurations in two-and three-space: basic terms and properties.
2. Measurement: length, area, volume, degree and radian measure, and so forth.
3. Coordinates: choice of coordinates, representation of geometric ideas algebraically and vice versa.
4. Constructions: constructibility, etc.
5. Theorems dealing with congruence, symmetry, similarity, polygons, incidence, projection, circles, parallels, perpendiculars, and so forth.
6. Organization of a deductive system: intuitive ideas of implications, postulates, definitions, undefined terms, equivalences, and so forth.
7. Direct and indirect proofs (what they are and how they are used) and counter-examples.
8. Applications of geometry.
9. "Other" topics: spatial visualization, geometric transformations, nonplanar configurations, and so forth.

The writing team produced approximately fifty geometry achievement scales, ranging from two to seventeen items in length. Both multiple choice and nonmultiple choice scales were developed.

Before the close of the summer writing session, it became apparent that with nearly fifty geometry scales, a means had to be provided for systematically indicating what each scale was designed to measure. With this objective in mind, a subset of the writing team developed the following classification of mathematical abilities used in geometry:

A Classification of Mathematical Abilities Used in Geometry.

1. Ability to comprehend mathematical statements, including diagrams.
2. Ability to originate a line of reasoning.
3. Ability to detect a line of reasoning.
4. Ability to differentiate between possible and necessary conclusions.
5. Ability to solve locus problems.
6. Ability to recognize all possible cases of a situation.
7. Ability to visualize.
8. Ability to apply theorems.

9. Ability to carry out computations or algebraic manipulations.
10. Recall of theorems.
11. Knowledge of terminology and notation.
12. Ability to discriminate between relevant and irrelevant data.
13. Ability to recognize solvability.

It is important to point out here that the classification above was *not intended to be exhaustive nor are the categories intended to be mutually exclusive.*

Using the classification, a compilation was made of scales according to the mathematical abilities (used in geometry) that each was expected to measure. We note that each category was measured to some extent by three to thirteen different scales.

After three rounds of pilot testing and subsequent revision of both individual test items and scales, a total of twelve of the original fifty scales were used in NLSMA testing. The remaining items and scales and the accompanying pilot test data are in the test files at SMSG headquarters, for possible use in the future.

Summary of Scales Used in NLSMA Test Batteries.

Perhaps it would be valuable to look at some examples of scales used in the final NLSMA test battery. We list below the names of scales, a brief description of what they were designed to measure, and some examples.

Description of Point Sets (10 items). This scale was designed to measure a student's ability to solve locus problems and to translate between mathematical statements and diagrams.

In a given circle of radius r, let $p<2r$. The set of all *mid-points* of chords of length p is (A) two straight lines, (B) one straight line, (C) one point, (D) two points, (E) a circle.

Conclusions Drawn from Diagrams (7 items). This scale was designed to measure a student's ability to translate between a mathematical statement and a diagram, to differentiate between possible and necessary conclusions, and to recall theorems.

Suppose two circles with *equal* radii have centers at points A and C and that the two circles intersect in *exactly* two points, B and D. Answer the following problems about quadrilateral $ABCD$.

$ABCD$ contains exactly 4 right angles.

(A) always (B) sometimes (C) never

A is an acute angle.

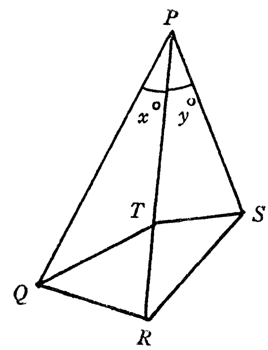
(A) always (B) sometimes (C) never.

NOTE: "Always" means the statement is true in *all* cases: "Sometimes" means the statement is true in some cases and false in others; "Never" means the statement is false in *all* cases.

Structure of Proof (2 items). This scale was designed to measure the student's ability to detect a line of reasoning and to recall theorems.

GIVEN: Each of triangles PQR and PTS is isosceles, and each has vertex P; $\triangle PQR \sim \triangle PTS$, with $\overline{QR}$ corresponding to $\overline{TS}$.

PROVE: $QT = RS$.



Statements in Proof: (Reasons omitted).

(1) $\triangle PQR$ and $\triangle PTS$ are isosceles with vertex P
(2) $\triangle PQR \sim \triangle PTS$, with $\overline{QR}$ corresponding to $\overline{TS}$
(3) $PQ = PR$ and $PT = PS$
(4) $x = y$
(5) $\triangle PTQ \sim \triangle PSR$
(6) $QT = RS$.

(A)
$$\begin{tikzcd}[row sep=small] (1) \arrow[r] & (3) \arrow[dr] & & \\ & & (5) \arrow[r] & (6) \\ (2) \arrow[r] & (4) \arrow[ur] & & \end{tikzcd}$$

(B)
$$\begin{tikzcd}[row sep=small] (1) \arrow[dr] & & & & \\ & (3) \arrow[r] & (4) \arrow[r] & (5) \arrow[r] & (6) \\ (2) \arrow[ur] & & & & \end{tikzcd}$$

(C)
$$\begin{tikzcd}[row sep=small] (1) \arrow[dr] & & & \\ & (4) \arrow[r] & (5) \arrow[r] & (6) \\ (3) \arrow[ur] & & & \end{tikzcd}$$

(D)
$$\begin{tikzcd}[row sep=small] & (1) \arrow[dr] & & \\ (2) \arrow[dr] & & (5) \arrow[r] & (6) \\ & (4) \arrow[ur] & & \\ (3) \arrow[ur] & & & \end{tikzcd}$$

(E)
$$\begin{tikzcd}[row sep=small] & & (3) \arrow[dr] & & \\ (1) \arrow[r] & (2) \arrow[dr] & & (5) \arrow[r] & (6) \\ & & (4) \arrow[ur] & & \end{tikzcd}$$

NOTE: A diagram like (3)→(5) means that statement (3) of the proof is needed to deduce statement (5). A diagram like

$$\begin{tikzcd}[row sep=small] (2) \arrow[dr] & \\ & (5) \\ (3) \arrow[ur] & \end{tikzcd}$$

means that *both* statement (2) and statement (3) are needed to deduce (5) and that neither statement (2) nor statement (3) alone is sufficient.

Simple Space Configurations (6 items). This scale was designed to measure a student's ability to visualize and to recognize all possible cases of a situation. It also measures his ability to comprehend mathematical statements, including diagrams.

Select the one FALSE statement.

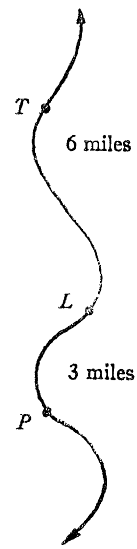
It is possible for two different straight lines in space

(A) each to be parallel to a third line
(B) each to be perpendicular to a third line at the same point on it
(C) to intersect and each be parallel to a plane
(D) to intersect and each be perpendicular to a plane
(E) to be neither parallel nor intersecting

Numerical Computation of Length (6 items). This scale was designed to measure a student's ability to carry out algebraic manipulations or computations and to originate a line of reasoning.

In the figure at the right T, L, and P are three towns on a road. Notice that L is twice as far from T as it is from P. There is another town on this road which is also twice as far from T as it is from P. How many miles is this town from L?

(A) 3
(B) 9
(C) 12
(D) 15
(E) 18

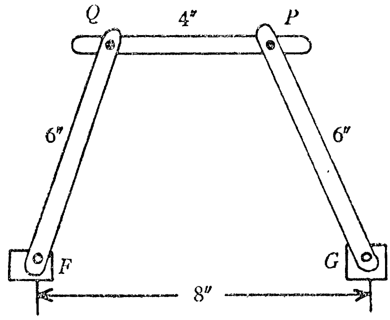


Constrained Motion (4 items). This scale was designed to measure a student's ability to solve locus problems and to visualize.

In the figure at the right, point P can be moved only

(A) along one or more straight line segments
(B) along an arc of a circle
(C) along a complete circle
(D) on and between two concentric circles
(E) in some other way than above

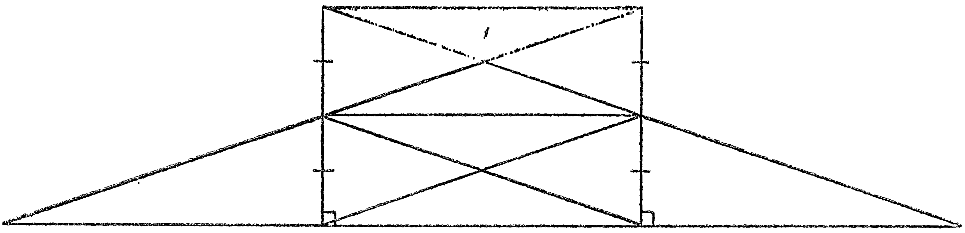
NOTE: Points F and G are fixed (stationary) points.



Similarity-Perception (3 items). This scale was designed to measure a student's ability to visualize.

How many triangles in the figure are similar to triangle I?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8



NOTE: Count triangle I as similar to itself.

Coordinate Geometry (10 items). This scale was designed to measure a student's knowledge of terminology and his ability to originate a line of reasoning and to carry out computations.

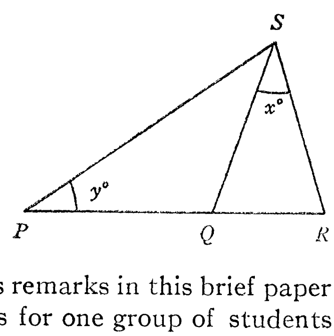
Which of the following is an equation of the straight line through the points $(0, -2)$ and $(3, 4)$?

(A) $y=-2x+2$
(B) $y=\frac{1}{2}x-2$
(C) $y=\frac{2}{3}x-2$
(D) $y=x-2$
(E) $y=2x-2$.

Geometry Applications (6 items). This scale was designed to measure the student's ability to originate a line of reasoning, apply theorems, and to carry out computations or algebraic manipulations.

As shown at the right, point Q is located on side $\overline{PR}$ of $\triangle PSR$ so that $PQ=QS=SR$. What is the value of x in terms of y?

(A) $x=(180-y)/4$
(B) $x=90-2y$
(C) $x=180-4y$
(D) $x=y$
(E) none of the above.



Concluding remarks. The author has confined his remarks in this brief paper to the development of geometry achievement scales for one group of students (those being tested in grades 7–11) at one point in the study, namely at the end of grade 10. This by no means represents the full scope of geometry testing in NLSMA.

Geometry achievement has been measured at various levels in each of three distinct populations in NLSMA. At the elementary and junior high school

levels, several measures of "informal" geometry have been taken. At the high school level, scales have been used which measure "informal" geometry, measurement, coordinate geometry, and "insightful" geometry.

For more information about the philosophy of test development, about pilot testing procedures, etc., in NLSMA, the reader may obtain an extended version of this paper by writing the author in the Graduate School of Education, Rutgers—The State University, New Brunswick, New Jersey, 08903.

The author was for three years on the staff of the Research and Analysis section of the School Mathematics Study Group, Stanford University, and participated in the development and analyses of achievement tests for NLSMA.

CAN WE BEAT THE BRITISH ON THEIR HOME GROUND—AND, IN MATHEMATICS?

NURA D. TURNER, State University of New York at Albany

The first attempt at an international mathematics competition in the "western" world will take place in London, Monday, May 20, 1968, when a team of secondary school students from the Upstate New York Contest Section of the MAA will compete in mathematics with a British team. Both teams will be composed of students who ranked high in their geographical areas in the 1968 Annual High School Mathematics Contest and who were of sophomore or junior status at the time of that Contest.

Some members of the British team were members of a team representing their country at the Ninth International Mathematical Olympiad of Soviet Satellite countries held in Yugoslavia in July, 1967. The British ranked fourth to the USSR, Hungary and East Germany at that IMO among thirteen competing countries. The USA was not represented. France, Italy and Sweden, in addition to England, sent teams. The tenth IMO will be held this summer in Moscow.

The six members of the Upstate New York Contest Section team and the two alternates are: Peter C. Ashbrook of Rochester; Robert E. Ergas of Rochester; Walter D. Lichtenstein of Schenectady; Peter A. Masters of Ithaca; Joel Nelson of Schenectady; Steven Seiler of Elnora; David J. Smith of Newport, and William E. Stevenson of Rochester. Intensive study on the part of these students has taken place during the current academic year under the supervision of Elmer E. Haskins, Professor of Mathematics, State University College at Potsdam.

The idea of this western competition originated in Moscow at the time of the International Congress of Mathematicians in August, 1966 when Mrs. Margaret Hayman of the Mayfield School, London, and the author met and discussed possibilities for it. Mrs. Hayman and her husband, Professor Walter Hayman, a mathematician at Imperial College, University of London, originated the British Mathematical Olympiad, patterned after the Olympiads of Soviet Satel-

lite countries. Participants in the British Olympiad are the 65 highest ranking students in the Annual High School Mathematics Contest in England.

The May 20th competition of the Upstaters of New York and the British will reflect procedures and activities of the International Mathematical Olympiads. The examination will take place from 9:30 A.M. to 12:30 P.M. It will be followed by luncheon and a press conference at the Imperial College. Our team members will spend the next day visiting the schools of their respective hosts. They will be house guests of the British team members for the week.

An Awards Ceremony, sponsored by Arthur Guinness Son and Co., Ltd., will be held on the 22nd. It will be a formal, dignified, and impressive affair, possibly at Saddlers' Hall where the Awards Ceremony for the Third British Olympiad was held. We have been told to expect that leading people in mathematics and science in England will be present. At this ceremony the Guinness Co. awards winners with attractive monetary prizes. During Thursday, Friday, and Saturday our team will be guests of the Guiness Co. in travel experience. This reflects the pattern of hospitality provided by the hosts of the International Mathematical Olympiads. Plans are for visiting Stratford-on-Avon and Cambridge University during this time.

Industries so far acting as sponsors underwriting costs other than flight costs are Gannett Newspapers, General Electric Company of Schenectady, Mohawk Data Sciences Corporation, and the Mathematical Association of America. Hungary credits her Olympiad with the fact that she is producing more mathematicians per unit of population than any other country in the world. Might not such competition as will take place in London in May, 1968 have some positive effect on mathematics in at least the Upstate New York Contest Section of the MAA? It is envisioned by the author as opening up the possibilities of a worldwide international competition in secondary school mathematics in the near future. Whether or not we beat the British this time on their own territory, we will be pioneering for the USA and the MAA.

A MATHEMATICS PROGRAM FOR ELEMENTARY SCHOOL TEACHERS

M. F. ROSSKOPF AND J. D. KAPLAN, Teachers College, Columbia University

One of the major issues confronting mathematics education today is the training of competent, imaginative teachers at all levels who are knowledgeable with respect to mathematics, methodology, and psychology. The reform movement in mathematics curricula and materials has underscored the inadequate training of many teachers of mathematics. Although qualified mathematics teachers are needed at all levels, it is at the elementary school level that there exists a serious shortage of teachers trained to teach mathematics.

With support from the Office of Education through the Prospective Teacher Fellowship Program the Department of Mathematical Education, Teachers College, Columbia University, developed a program in 1966–67 for novice ele-

mentary school teachers. The program was initially designed to train elementary school teachers to teach mathematics to disadvantaged children. Stimulated by the Prospective Teacher Fellowship Program, the Department has planned a permanent program to train mathematics specialists equipped with sufficient mathematical knowledge to teach modern school programs currently available in schools, or which are being proposed for the future.

Candidates for the program are selected from among those who offer these qualifications:

1. Certification as an elementary school teacher.
2. One year of college mathematics (this requirement has been increased to three semesters for 1968–69).

The Department, as a graduate department of mathematics education, has no responsibility for the initial training of elementary school teachers. The Department's goal is to offer a program of study which will get qualified teachers back into the classroom either as generalists or mathematics specialists.

A unique feature of the program is the close relationship between "theoretical" courses at Teachers College and practical classroom experience. This is achieved by means of a year-long internship, the first semester in observation of individual children in a laboratory classroom and the second semester full-time work with different classes of children. A sample of the courses available to students follows.

Modern Analysis: A development of the real number system; functions; Cauchy sequences.

Fundamental Concepts of Algebra: Linear transformations and matrices; groups; rings; fields.

Number Theory: Divisibility; primes; congruences; Fermat's theorems; Diophantine equations.

Fundamental Concepts of Geometry: Algebra of vectors; rotations, reflections, projections.

Teaching Mathematics in the Elementary School: Current experimental programs, such as Minnemast and the Madison Project.

Seminar and Practicum in Elementary Mathematics: Techniques and materials of newer programs such as the Madison and Nuffield Projects.

Evaluation in Elementary School Mathematics: Defining curriculum objectives and measurement of goals.

Internship in Elementary School Mathematics: By means of a mathematics laboratory in a near-by elementary school, students work with small groups of children during the first semester using a variety of different materials to teach mathematics; in the second semester, students are assigned full-time to the school to teach a number of mathematics lessons daily at different grade levels.

In addition to the above program in the Department of Mathematical Education, students also take courses in psychology, sociology, and curriculum and teaching. Students may choose electives either in the department or in other departments.

This program, as a graduate program, advances the mathematical knowledge of teachers, specialists and supervisors in the elementary field. The department views the program as offering a mathematical preparation for elementary school teachers that parallels in many respects the one outlined in *Goals for Mathemati-*

cal Education of Elementary School Teachers, a recent report of the Cambridge Conference. (Report of the Cambridge Conference on Teacher Training: *Goals for Mathematical Education of Elementary School Teachers*. Boston: Houghton Mifflin Company, 1967.)

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before September 30, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed postcards.

E 2057 [1968, 188]. **Correction.** *Proposed by R. S. Luthar, University of Wisconsin*

For $x_0 \in (a, b)$ read $x_0 \in [a, b]$. This change applies to both (1) and (2).

E 2085. *Proposed by J. H. Butchart, Northern Arizona University*

Find the unifying feature and prove the following three statements:

(a) The area bounded by two normals of one arch of a cycloid, the curve itself, and its evolute is divided in the ratio of 1:3 by the line on which the circle rolls.

(b) The area bounded by two normals to the right half of the catenary $y = \cosh x$, the x-axis, and the evolute is divided in the ratio of 1:3 by the catenary.

(c) The area bounded by two normals to the upper half of the parabola $y^2 = 4ax$, the directrix, and the evolute is divided in the ratio of 4:5 by the parabola.

E 2086. *Proposed by Beatriz Margolis, Fundación Bariloche, Buenos Aires, Argentina*

Show that the following is an increasing function:

$$f(t) = \ln\frac{t+2}{t+1}\Big/ \ln\frac{t+1}{t} \qquad (t > 1).$$

E 2087. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

For every nonzero integer k, show that $(2k)^{2^n}+1, n=1, 2, \cdots$, are relatively prime integers.

E 2088. *Proposed by H. E. Chrestenson, Reed College*

In problem 5014 [1963, 447] it was shown that a regular n-gon can be imbedded in a cubic lattice (of arbitrary dimension) only if $n=3$, 4 or 6, and in these cases dimension 3 suffices. Answer the analogous question for the five Platonic solids.

E 2089. *Proposed by J. F. Randolph, University of Rochester*

A roller is constructed from three circular arcs each having its center at one vertex of an equilateral triangle and joining the other two vertices. In the ordinary cycloid problem replace the circle by this roller and find the locus of a point on its boundary as it rolls without slipping along a straight line.

E 2090. *Proposed by J. C. Brooks, Georgia Institute of Technology*

In a triangle of sides a, b, c, prove that the distances from the centroid G to the incenter I and the excenters I', I'', I''' satisfy the relation

$$(s-a)GI'^2 + (s-b)GI''^2 + (s-c)GI'''^2 - (s)GI^2 = 2abc.$$

E 2091. *Proposed by J. O. Kiltinen and T. J. Grilliot, Duke University*

Consider the following two properties for a commutative ring with identity:

(i) If P is a monic polynomial over A and deg $P=n$, then P has at most n roots in A.

(ii) A is not an integral domain.

Are there any commutative rings with identity possessing both of these properties?

E 2092. *Proposed by A. Domergue, Paris, France*

Let a regular star polygon be constructed by dividing a circle into n $(n \geqq 5)$ equal parts and drawing the chords which join alternate points of division. Each of the n chords will carry 4 points of intersection. If distinct positive integers are assigned to each intersection in such a way that the sum of the 4 numbers on each chord is a constant, we have a magic number star. Such a number star may be characterized by Δ, the difference between the smallest and largest integers employed.

(a) Construct a magic star pentagon in which the constant is 1968 and such that Δ is smallest possible. (b) Do the same for a magic star hexagon. (c) Show that there is no magic star pentagon with constant sum 1967.

E 2093. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York*

Prove that there exist unique numbers c and d in the interval $[0, \frac{1}{2}\pi]$ such that $c<d$ and

$$\text{(A) } \sin\cos c = c, \quad \text{(B) } \cos\sin d = d$$

for each positive integer n.

E 2094. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York*

Define $\sin_n x = \sin(\sin_{n-1} x)$ and $\cos_n x = \cos(\cos_{n-1} x)$ with $\sin_1 x = \sin x$ and $\cos_1 x = \cos x$. For which values of n are there solutions to the equation $\cos_n x = \sin_n x$, $0 \leqq x \leqq \frac{1}{2}\pi$?

SOLUTIONS OF ELEMENTARY PROBLEMS

Multiplicative Function

E 1891 [1966, 538; 1967, 1267]. *Proposed by A. E. Livingston, Oregon State University*

Let f be a number-theoretic function, and set

$$F(n) = \sum_{d|n} f(d) f\left(\frac{n}{d}\right).$$

If $f(1)=1$, and F is multiplicative, then f is multiplicative.

Note by Sidney Spital, California State Polytechnic College. As pointed out by T. M. Apostol, Solution II is incorrect. This is perhaps best brought out by noting that if it were correct, one more inversion step would lead to $F(n) = \sum_{d|n} f(d)$, which is clearly a contradiction. The error stems from the requirement that a Moebius inversion of $F(n) = \sum_{d|n} g(d)$ requires that g be independent of n.

A Variation of Bernoulli's Inequality

E 1950 [1967, 76]. *Proposed by M. Marjanovic, University of Belgrade, Yugoslavia*

Prove the inequality

$$\left(1 + \frac{\alpha}{[\alpha+1]}\right)^{[\alpha+1]} \geqq 2^\alpha \geqq \left(1 + \frac{\alpha}{[\alpha]}\right)^{[\alpha]},$$

where $\alpha \geqq 1$ and $[\alpha]$ is the greatest integer not exceeding α.

Solution by Sidney Spital, California State Polytechnic College, Pomona. The requested result, rewritten as

$$\left(1 + \frac{\alpha}{[\alpha+1]}\right)^{[\alpha+1]/\alpha} \geqq 1 + 1 \geqq \left(1 + \frac{\alpha}{[\alpha]}\right)^{[\alpha]/\alpha}$$

follows directly from Bernoulli's inequality:

$$(1 + x)^r \gtreqless 1 + rx \quad \text{for (positive) } r \gtreqless 1 \quad \text{and} \quad x \geqq 0.$$

Also solved by Einar Andresen (Norway), Marcia Ascher, Günter Bach (Germany), Anders Bager (Denmark), Benedict Carlat, L. Carlitz, J. P. Celenza, G. C. Dodds, Ragnar Dybvik (Norway), R. B. Eggleton (Australia), M. A. Ettrick, Jerry Fischer, Michael Goldberg, D. Gootkind, M. G. Greening (Australia), J. E. Hafstrom, D. D. Huffman, R. W. Hurd, R. A. Jacobson, Donald Jeffords, Donald Kern, Bengt Klefsjö (Sweden), D. A. Marcus, D. C. B. Marsh, Norman Miller, Steven Minsker, C. B. A. Peck, D. H. Peterson, L. J. Pratte, Stanley Rabinowitz, Jürg Rätz (Switzerland), Simeon Reich (Israel), L. J. Schneider, Paul Sontag, Momir Stanojevic (Yugoslavia), Jože Vrečko (Yugoslavia), and the proposer.

A Pair of Arithmetic Identities

E 1951 [1967, 76]. *Proposed by R. Sivaramakrishnan, Engineering College, Trichur, India*

If $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ denotes the total number of prime factors of n (including repeated factors) and if $g(n) = 2^\nu$ where ν is the number of distinct prime factors of n, then prove

$$1. \quad \sum_{d|n} \lambda(d)\mu(d) = g(n), \qquad 2. \quad \sum_{d|n} \lambda(d)\mu(n/d) = g(n)\lambda(n),$$

where $\mu(n)$ is the Möbius function.

Solution by Donald Jeffords, Weedsport, N. Y. (1) Since $\mu(d) = 0$ for any divisor which is not square-free, we need sum over only the square-free divisors. Also, since $\mu(d) = \lambda(d)$ for these divisors, we are merely summing the number of square-free divisors, which is clearly 2^ν or $g(n)$.

(2) We have $\sum_{d|n} \lambda(d)\mu(n/d) = \sum_{d|n} \mu(d)\lambda(n/d)$, so again we sum over only the square-free divisors. Since $\lambda(d)\lambda(n/d) = \lambda(n)$, and $\mu(d) = \lambda(d)$, $\mu(d)\lambda(n/d) = \lambda(n)$ and upon factoring this out of the summation and summing as before, we obtain $\lambda(n)g(n)$.

Also solved by Einar Andresen (Norway), Anders Bager (Denmark), James Baumbach, J. F. Dillon, R. B. Eggleton (Australia), Jerry Fischer, Ray Glenn, Jerry Goodman, M. G. Greening (Australia), Emil Grosswald, E. S. Langford, C. P. Lawes, Douglas Lind, P. A. Lindstrom, D. C. B. Marsh, S. S. Muchnick, Stanley Rabinowitz, Simeon Reich (Israel), Lois J. Reid, Perry Scheinok, Al Somayajulu, D. R. Stark, L. J. Warren, Steven Weintraub, and the proposer.

A Factorization Containing Twin Primes

E 1952 [1967, 76]. *Proposed by Edgar Karst, University of Arizona*

(a) Find the smallest odd integer m greater than 1, such that $2^m - 1$ is divisible by a pair of twin primes (i.e. two primes p, q, with $q - p = 2$).

(b) Find the smallest positive integer n such that $2^n + 1$ is divisible by a pair of twin primes.

Solution by L. J. Warren, San Diego State College. (a) Let the order of 2 (mod p) be u and the order of 2 (mod q) be v with $q = p + 2$. Since $(p-1, q-1) = 2$,

$u|(p-1)$, and $v|(q-1)$, it follows that we must minimize the product uv with both u and v odd. Both p and q have 2 as a quadratic residue, hence $p\equiv -1 \pmod 8$. Now p is the first member of a twin prime pair, so $p\equiv -1 \pmod 6$; thus $p\equiv -1 \pmod{24}$. Armed with this information and a set of tables, m is found to be 315. The twin primes are 71 and 73.

(b) This follows by a similar analysis, but takes a little more time. The n must be odd and the smallest such n is 1645, and the twin primes are 281 and 283.

Also solved (part (a) only) by Michael Goldberg, and by Michael Jeffords. The proposer stated the answers without indicating his method.

On the Real Intersections of a Polynomial Curve with the Exponential Curve

E 1953 [1967, 77]. *Proposed by D. J. Newman, Yeshiva University*

What is the maximum possible number of real solutions for the system $y=e^x$, $P(x, y)=0$, if $P(x, y)$ is required to be a polynomial of degree n?

Solution by the proposer. The answer is $n(n+3)/2$. It is easy to see that this many solutions are possible. Indeed, given any $n(n+3)/2$ numbers x_i, we can determine coefficients $c_{i,j}$, not all 0, so that $P(x, y)=\sum c_{i,j}x^iy^j$ vanishes at all the points (x_i, e^{x_i}); we have, namely, $n(n+3)/2$ linear homogeneous conditions in $(n+1)(n+2)/2=n(n+3)/2+1$ unknowns.

On the other hand, note that the operator

$$Q(D) = D^{n+1}(D-1)^n(D-2)^{n-1}\cdots(D-n)$$

annihilates any one of the $P(x, e^x)$ so that, for some proper divisor $R(D)$ of $Q(D)$ we have

$$\text{(A)} \qquad R(D)P(x, e^x) = ce^{kx},\ c\neq 0,\ 0\leqq k\leqq n.$$

If $P(x, e^x)$ had more than $n(n+3)/2$ real zeros, however, then since $R(D)$ has degree $\leqq n(n+3)/2$, we would conclude, by an extension of Rolle's theorem, that

$$\text{(B)} \qquad R(D)P(x, e^x) \quad \text{has at least one real zero.}$$

Of course, (B) contradicts (A), and the proof is complete.

Also solved by L. Carlitz.

A Maximum Covering of Lattice Points by a Square

E 1954 [1967, 77]. *Proposed by D. J. Newman, Yeshiva University*

Prove that there is no position in which an $n\times n$ square can cover more than $(n+1)^2$ integral lattice points.

Solution by the proposer. Consider the square S in arbitrary position and let C be the convex hull of the lattice points which S contains. Clearly Area $(C)\leqq n^2$ and Perimeter $(C)\leqq 4n$. If I denotes the number of interior lattice points of C, and B the number of boundary lattice points of C, then, by Pick's theorem,

$$I + B/2 - 1 = \text{Area } (C) \leqq n^2. \tag{1}$$

Also, since the lattice points have mutual distances $\geqq 1$, a curve of length B can contain at most B lattice points, thus

$$B \leqq \text{Perimeter } (C) \leqq 4n. \tag{2}$$

From (1) and (2) it follows that $I+B \leqq n^2+2n+1$ and this is the required result.

Also solved by Anders Bager (Denmark), and by M. G. Greening (Australia). Several other solutions involved unproved assumptions.

For Pick's theorem see H. S. M. Coxeter, *Introduction to Geometry*, 1961, pp. 212–214.

Asymptotic Character of a Sequence

E 1955 [1967, 197]. *Proposed by D. J. Newman, Yeshiva University*

Suppose $x_{n+1} = x_n + 1/(x_1 + x_2 + \cdots + x_n)$ and that $x_1 = 1$ (thus $x_2 = 2$, $x_3 = 7/3$, $x_4 = 121/48, \cdots$). Prove that $x_n \sim \sqrt{2 \log n}$.

I. *Solution by R. K. Meany, Iowa State University.* It will be shown that for $n \geqq 2$,

$$\sqrt{1 + 2\log n + \log\log n} < x_n < \sqrt{5 + 2\log n + \log\log n}.$$

Let $S_n = x_1 + x_2 + \cdots + x_n$. For $n \geqq 2$ we have

$$\begin{aligned}(x_{n+1}^2 - 2\log S_n) - (x_n^2 - 2\log S_{n-1}) &= 2x_n/S_n + 1/S_n^2 + 2\log(1 - x_n/S_n)\\ &< 2x_n/S_n + 1/S_n^2 - 2x_n/S_n - (x_n/S_n)^2 < 0,\end{aligned}$$

so that the numbers $x_{n+1}^2 - 2\log S_n$ are decreasing. It follows that for $n \geqq 3$,

$$x_n^2 - 2\log S_n < x_{n+1}^2 - 2\log S_n \leqq x_4^2 - 2\log S_3 < 3,$$

and hence,

$$x_n < \sqrt{3 + 2\log S_n}. \tag{1}$$

Now $S_n \geqq 1$ and hence $x_{n+1} \leqq x_n + 1$, so that $x_n \leqq n$ and $S_n \leqq nx_n \leqq n^2$. Substitution of this inequality into the right side of (1) gives, for $n \geqq 3$,

$$S_n < nx_n < n\sqrt{3 + 4\log n} < n\sqrt{7\log n}.$$

If this last inequality is now substituted into the right side of (1), one then has, for $n \geqq 3$,

$$x_n < \sqrt{5 + 2\log n + \log\log n},$$

and it is easy to verify that this inequality holds also when $n = 2$.

On the other hand, we have for $n \geqq 1$,

$$\begin{aligned}(x_{n+1}^2 - 2\log S_{n+1}) - (x_n^2 - 2\log S_n) &= 2x_n/S_n + 1/S_n^2 - 2\log(1 + x_n/S_n)\\ &> 2x_n/S_n + 1/S_n^2 - 2x_n/S_n > 0,\end{aligned}$$

so that the numbers $x_n^2-2\log S_n$ are increasing. It follows that, for $n\geqq 3$,

$$x_n^2 - 2\log S_n \geqq x_3^2 - 2\log S_3 > 2,$$

and hence,

$$x_n > \sqrt{2 + 2\log S_n}. \tag{2}$$

Now $x_n\geqq 1$ and hence $S_n\geqq n$. Substitution of this inequality into the right side of (2) gives, for $n\geqq 3$,

$$x_n > \sqrt{2 + 2\log n},$$

and this inequality holds also when $n=2$. It follows that for $n\geqq 3$

$$S_n > \sum_{k=2}^{n} \sqrt{2 + 2\log k} > \sum_{k=[\frac{1}{2}n]+1}^{n} \sqrt{2 + 2\log(\tfrac{1}{2}n)}$$

$$> (\tfrac{1}{2}n)\sqrt{2 + 2\log(\tfrac{1}{2}n)} > n\sqrt{(\log n)/2}.$$

If this last inequality is now substituted into the right side of (2) one then has, for $n\geqq 3$,

$$x_n > \sqrt{1 + 2\log n + \log\log n},$$

and this inequality holds also when $n=2$.

II. *Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands.* Write $S_n=x_1+x_2+\cdots+x_n$. If for some k we have

$$\tfrac{1}{2}x_k^2 \geqq 1 + \tfrac{1}{2} + \cdots + \frac{1}{k} \quad\text{and}\quad S_k/x_k < k+1 \tag{A}$$

then

$$\frac{1}{2}x_{k+1}^2 = \frac{1}{2}\left(x_k + \frac{1}{S_k}\right)^2 > \frac{1}{2}x_k^2 + \frac{x_k}{S_k} > 1 + \frac{1}{2} + \cdots + \frac{1}{k+1}$$

and $S_{k+1}/x_{k+1}=S_k/x_{k+1}+1<S_k/x_k+1<k+2$.

Since (A) is true for $k=2$, it is true for $k\geqq 2$. From (A) it follows that

$$x_k > (2\log k)^{1/2} \quad\text{for}\quad k \geqq 1, \tag{B}$$

hence

$$S_n > \sum_{k=1}^{n} (2\log k)^{1/2} > \int_1^n (2\log x)^{1/2}dx = n(2\log n)^{1/2}\left(1 + O\left(\frac{1}{\log n}\right)\right).$$

Since by definition, $x_{n+1}=x_n+S_n^{-1}$ and

$$\sum_{n=2}^{k} n^{-1}(2\log n)^{-1/2} \sim (2\log k)^{1/2}$$

we have, for $\epsilon>0$,

(C) $$x_k < (1+\epsilon)(2\log k)^{1/2} \quad \text{if } k \text{ is sufficiently large.}$$

By (B) and (C) we have $x_k \sim (2\log k)^{1/2}$.

Also solved by N. J. Fine, L. Carlitz, and the proposer.

Editorial Note. An alternate derivation of (B) above follows by noting that $x_{n+1}>x_n+1/nx_n$ or $x_{n+1}^2>x_n^2+2/n$. Then by summing, we have

$$x_n^2 > 1+2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}\right) > 2\log n.$$

A Pair of Elementary Triangle Identities

E 1956 [1967, 197]. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India*

Let a, b, c be the sides of a triangle ABC, and let R, r, r_1, r_2, r_3 be the circumradius, inradius and exradii, and h_i $(i=1, 2, 3)$ be the altitudes from A, B, C to the opposite sides. Prove that

(1) $$\frac{h_1}{bcr_1}+\frac{h_2}{car_2}+\frac{h_3}{abr_3}=\frac{1}{2R\cdot r},$$

(2) $$\frac{h_1+r_1}{bcr_1}+\frac{h_2+r_2}{car_2}+\frac{h_3+r_3}{abr_3}=\frac{1}{R\cdot r}.$$

Solution by Earl A. Smith, Connetquot High School, Bohemia, N. Y. (1) Let K be the area of triangle ABC. Since $K=abc/4R$ and $1/r_1+1/r_2+1/r_3=1/r$, we have

$$\frac{h_1}{bcr_1}+\frac{h_2}{car_2}+\frac{h_3}{abr_3}=\frac{2K}{abcr_1}+\frac{2K}{abcr_2}+\frac{2K}{abcr_3}$$
$$=\frac{2K}{abc}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)=\frac{1}{2Rr}.$$

(2) is easily obtained by separating the fractions and using (1) together with $1/ab+1/bc+1/ca=1/2Rr$.

Also solved by A. N. Aheart, D. P. Ambrose, Einar Andresen (Norway), Anders Bager (Denmark), Leon Bankoff, M. T. L. Bizley (England), L. Carlitz, Ragnar Dybvik (Norway), Herta T. Freitag, Michael Goldberg, M. G. Greening (Australia), Louise S. Grinstein, A.S.A. Hunt, M. A. Laframboise, Ruth S. Lefkowitz, Andrzej Mąkowski (Poland), D. C. B. Marsh, Norman Miller, C. B. A. Peck, Edna M. Pratt, J. R. Purdy, J. M. Quoniam (France), Simeon Reich (Israel), Sister Stephanie Sloyan, H. E. Speece, Sidney Spital, Philip Trauber, Steven Weintraub, Gregory Wulczyn, C. C. Yalavigi (India), and the proposer.

An Impossible Group Table

E 1957 [1967, 198]. *Proposed by C. C. Lindner, Coker College, Hartsville, S.C.*

Can the accompanying table be completed to a group table without placing x_1 again on the principal diagonal, where the ith row and ith column are both

headed by x_i?

	x_1	x_2	x_3	$x_4 \cdots x_n$
x_1	x_1			
x_2		x_1	x_4	
x_3		x_5		
$\vdots$				
x_n				

Solution by Z. Z. Uoiea, Point of the Mountain, Utah. No. The partial table shows that x_2 has order 2 but does not commute with x_3. Hence, $x_3^{-1}x_2x_3$ must be an element of order 2 different from x_2.

Also solved by George Alberts, K. F. Bailie, Thomas Elsner, M. G. Greening (Australia), Erwin Just, M. W. Legg, D. A. Marcus, P. L. Montgomery, Barbara Z. Osofsky, Brian Parshall, Oswald Wyler, and the proposer.

Divisibility of a Numerator

1960 [1967, 198]. *Proposed by Erwin Just, Bronx Community College*

Prove that the numerator of the fraction determined by $\sum_{k=0}^{2^n-1} 1/(2k+1)^m$, where m is odd and $n=1, 2, \cdots$, is divisible by 2^{n+1}.

Solution by R. E. Shafer, Lawrence Radiation Laboratory, University of California. Since

$$\frac{1}{(2k+1)^m} + \frac{1}{(2^{n+1}-2k-1)^m} = \frac{(2^{n+1}-2k-1)^m + (2k+1)^m}{(2k+1)^m(2^{n+1}-2k-1)^m},$$

the numerator on the right is divisible by 2^{n+1}. Since there are an even number of rational numbers of the form $2^{n+1}(2K+1)/(2L+1)$ in the given sum, then the numerator of the given sum has at least one more factor of 2. Therefore $\sum_{k=0}^{2^n-1} 1/(2k+1)^m$, m an odd integer, has a numerator divisible by 2^{n+2} for $n>1$. (For $n=1$, the numerator is divisible by 2^{n+1}.)

Also solved by J. C. Abad, Einar Andresen (Norway), Anders Bager (Denmark), Peter Bundschuh (Germany), L. Carlitz, Stephen Clodman, N. J. Fine, Jerry Fischer, Michael Goldberg, M. G. Greening (Australia), Donald Jeffords, D. C. B. Marsh, Bohnslav Mišek, P. L. Montgomery, Robert Patenaude, C. B. A. Peck, L. J. Pratte, S. F. Robinson, P. A. Scheinok, R. L. Vogt, Charles Wexler, Gregory Wulczyn, and the proposer.

A Digit Problem

E 1961 [1967, 198]. *Proposed by L. M. Graves, Chicago, Ill.*

Note that $18 \cdot 6 = 108$, $15 \cdot 7 = 105$, $45 \cdot 9 = 405$. Find all bases α other than 10 for which there exist triples (a, b, c) of digits with a corresponding property: $(\alpha a+b)c = \alpha^2 a + b$, where $0<a<\alpha$, $1<b<\alpha$, $1<c<\alpha$. Also find, for each α, the number of such triples, or a lower bound for the number.

Solution by M. G. Greening, University of New South Wales, Australia. As $a\alpha(\alpha-c)=b(c-1)$, $\alpha=p$ would require $\alpha|b$ or $\alpha|(c-1)$; i.e., $\alpha\leqq b$ or $\alpha<c$, impossible. If, however, $\alpha=\lambda\mu$ where $\lambda\neq 1$, $\mu\neq 1$, then a solution is always provided by $b=(\lambda-1)\mu$, $c=(\mu-1)\lambda+1$, $a=\mu-1$. A lower bound to the number of triples is therefore $\tau(\alpha)-2$, where $\tau(\alpha)$ is the number of divisors of α; α itself can be any prime power or other composite number.

Also solved by Einar Andresen (Norway), L. Carlitz, Michael Goodman, G. A. Heuer, D. C. B. Marsh, Norman Miller, B. P. Sarkar (India), H. E. Thomas, Jr., Charles Wexler, Oswald Wyler, and the proposer.

A Simple Number-theoretic Inequality

E 1962 [1967, 198]. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India*

If $\phi(m)$ denotes Euler's phi-function and $\tau(n)$ is the number of divisors of n, prove that $\phi(n)\tau(n)\geqq n$.

I. *Solution by Sister Marion Beiter, Rosary Hill College, Buffalo, N.Y.* Let the prime factorization of n be $p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, with $a_i>0$. Then

$$\phi(n)\tau(n) = n\prod_{p_i}(1-1/p_i)\prod_{i=1}^{r}(a_i+1).$$

Now $1-1/p_i\geqq\frac{1}{2}$ for any prime p_i, and $a_i+1\geqq 2$ for any a_i. Hence $\phi(n)\tau(n)\geqq n(\frac{1}{2})^r\cdot 2^r=n$.

II. *Solution by M. G. Greening, University of New South Wales, Australia.* For $d|n$, $\phi(n)\geqq\phi(d)$. Then $\sum_{d|n}\phi(n)\geqq\sum_{d|n}\phi(d)$. So $\tau(n)\phi(n)\geqq\sum_{d|n}\phi(d)=n$.

For all $n>2$, strict inequality holds.

Also solved by J. C. Abad, George Alberts, Einar Andresen (Norway), Anders Bager (Denmark), W. T. Bailey, James Baumbach, A. R. Bolder, J. L. Brown, Jr., Peter Bundschuh (Germany), G. C. Dodds, H. M. Edgar, R. B. Eggelton (Australia), Henry Feldman, Jerry Fischer, Ray Glenn, Michael Goldberg, Morton Goldberg, Emil Grosswald, R. A. Herrmann, R. A. Hovis, D. G. Huffman, J. Y. Hung, Donald Jeffords, James Joiner, Erwin Just, John Kieffer, Lew Kowarski, W. B. Laffer II, C. P. Lawes, Douglas Lind, P. A. Lindstrom, Hymie London, Andrzej Makowski (Poland), D. A. Marcus, D. C. B. Marsh, C. C. McBride, Charles McCracken, M. J. Merscher, P. L. Montgomery, J. E. Morrill, Michael Motto, W. L. Mrozek, D. A. Myers, Brian Parshall, Robert Patenaude, C. B. A. Peck, D. H. Peterson, James Poggione, Bob Prielipp, J. R. Purdy, Simeon Reich (Israel), S. F. Robinson, Perry Scheinok, R. E. Shafer, E. A. Smith, Al Somayajulu, Alan Stang, D. R. Stark, Michael Stolnicki, Hugo Sun, H. E. Thomas, Jr., H. H. Thoyre, Stephen Tice, Philip Trauber, Richard Vogt, Steven Weintraub, Charles Wexler, and the proposer.

A Greatest Power of 2

E 1963 [1967, 199]. *Proposed by Jack C. Abad, City College of San Francisco*

For whole numbers q and n define $S_q(0)=1$, $S_q(n+1)=q^{S_q(n)}$. Find the greatest power of 2 that will divide $S_3(n+1)-S_3(n)$.

Solution by Oswald Wyler, Carnegie-Mellon Institute of Technology. Little additional effort is required to solve the problem for arbitrary odd q. If $q=q_0 2^k\pm 1$, with q_0 odd and $k>1$, then $q^{2^r}=2^{k+r}q_r+1$ for $r>0$, with q_r odd. This is easily

verified by induction. Now let $S_q(n)-S_q(n-1)=2^rQ$, with Q odd. Then

$$S_q(n+1) - S_q(n) = q^{S_q(n-1)}(q^{2^rQ} - 1) = R(q^{2^r} - 1) = 2^{r+k}R'$$

with R and R' odd.

Thus if r_n is the exponent of 2 in the prime factorization of $S_q(n+1)-S_q(n)$, then $r_n=r_{n-1}+k=kn+r_0$ for all n. If $q\equiv 1 \pmod 4$, then $r_0=k$. If $q\equiv 3 \pmod 4$, then $r_0=1$.

For $q=3$, we have $r_0=1$ and $k=2$, hence $r_n=2n+1$ for all n.

Also solved by Einar Andresen (Norway), Anders Bager (Denmark), James Baumbach, Peter Bundschuh (Germany), L. Carlitz, Michael Goldberg, M. G. Greening (Australia), Emil Grosswald, Donald Jeffords, Kenneth Kramer, D. A. Marcus, Norman Miller, P. L. Montgomery, C. B. A. Peck, D. H. Peterson, L. J. Pratte, Kao-Hwa Sze, H. E. Thomas, Jr., Philip Trauber, and the proposer.

Jeffords also notes the extension to general q.

A Non-Integral Sum

E 1964 [1967, 199]. *Proposed by Erwin Just, Bronx Community College*

Let $\{m_i\}$ be the set of $\phi(m)$ positive integers which are less than m and relatively prime to m. If $m>2$, prove that $\sum_{i=1}^{\phi(m)}1/m_i$ is not an integer.

Solution by D. A. Marcus, Rutgers—The State University. For $m>2$, it is well known (Bertrand's postulate) that there exists a prime p for which $m/2<p<m$. If we write

$$\sum_{i=1}^{\phi(m)} \frac{1}{m_i} = \left\{\sum_{i=1}^{\phi(m)} \frac{1}{m_i} \prod_{k=1}^{\phi(m)} m_k\right\} \Big/ \prod_{k=1}^{\phi(m)} m_k,$$

it is clear that p divides the denominator (being one of the m_k) but not the numerator (since it divides every term except one). Thus the denominator cannot divide the numerator.

Also solved by Einar Andresen (Norway), Anders Bager (Denmark), James Baumbach, J. L. Brown, Jr., L. Carlitz, G. C. Dodds, R. B. Eggleton (Australia), Jerry Fischer, Ray Glenn, M. G. Greening (Australia), D. G. Huffman, R. A. Jacobson, Donald Jeffords, Douglas Lind, D. C. B. Marsh, C. B. A. Peck, D. H. Peterson, L. J. Pratte, Bob Prielipp, S. F. Robinson, R. E. Shafer, Al Somayajulu, Howard Thoyre, R. L. Vogt, Charles Wexler, and the proposer.

Brown shows that if $\{m_i\}$ is any increasing sequence of positive integers which contains all the primes ≥ 2, then $\sum_{i=1}^{n}1/m_i$ is not an integer for any $n\geq 2$.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate signed sheets and should be mailed before November 30, 1968. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed postcards.

5590. *Proposed by E. C. Milner and A. Oppenheim, University of Reading, England*

For any integer $n\geq 1$, let $f(n)$ denote the largest integer m such that $\sigma(m)$

$=\sum_{d|m} d \leqq n$. Prove that, for fixed $k \geqq 1$, the equation $n-f(n)=k$ has infinitely many solutions.

5591. *Proposed by A. Oppenheim, University of Reading, England*

Suppose that $n>1$, $\sigma(n)$ is the sum of divisors of n, $\phi(n)$ the totient of n (Euler's function). Prove the inequality

$$\phi\left(n\left[\frac{\sigma(n)}{n}\right]\right) < n.$$

5592. *Proposed by W. M. Myers and D. R. Arterburn, University of Montana*

Let E be a measurable subset of the line and $\{f_n\}$ a sequence of functions summable on E with the property that $\int_F f_n \to 0$ for all measurable subsets F of E. Prove or disprove that $\int_E |f_n| \to 0$.

5593. *Proposed by David Shelupsky, The City College of New York*

Let (U_{rs}), $1 \leqq r$, $s < \infty$, be a unitary operator on l^2, and let $\{z_r\}$, $1 \leqq r < \infty$, be an arbitrary sequence of complex numbers. Prove that if $\sum_{s=1}^{\infty} U_{rs} z_s = 0$, $1 \leqq r < \infty$ (that is, if formally we have $Uz=0$), then $z_r = 0$ for each r.

5594. *Proposed by W. G. Dotson, Jr., North Carolina State University*

Suppose F is a field with a rational subfield R. Suppose V is an Abelian group and that $\cdot$ and $*$ are functions from $F \times V$ into V which satisfy the usual scalar multiplication axioms. (1) Show that if F is a prime field, then $\cdot$ and $*$ coincide. (2) Give an example in which $\cdot$ and $*$ do not coincide.

5595. *Proposed by Fred Gross, Bell Communications, Inc., New York*

Let $f(z)$ be an entire function and let $M(r)$ and $M'(r)$ denote the maximum modulus function of f and f' respectively. If for some $\sigma > 0$, $M(r) < M'(r) = e^{\sigma r}$ for sufficiently large r, then $\sigma > e^{-1}$.

5596. *Proposed by J. D. Dixon, University of New South Wales, Australia*

Let f be a real-valued function defined on a real interval I. We shall call f *quasi-periodic* on I if, for each $x \in I$, there exists a rational number $r \neq 0$, depending on x, such that $x+r \in I$ and $f(x+r)=f(x)$. Prove that if f is analytic and quasi-periodic on I, then f is periodic on I.

Is this result still true if we define quasi-periodic as above except that we permit r to be irrational?

5597. *Proposed by Joseph Arkin, Suffern, New York*

If t, x, y, u, v, and w are nonzero distinct integers, solve the simultaneous Diophantine equations

$$v^2 + w^2 = (u^2 - t^2)(y^2 + x^2),$$
$$v^2 - w^2 = (u^2 + t^2)(y^2 - x^2).$$

5598. *Proposed by S. J. Metz, Fort MacArthur, California*

Does there exist a real function whose set of discontinuities has measure zero but has an uncountable intersection with every open interval?

5599. *Proposed by Howard Kleiman, Queensborough Community College, Bayside, N.Y.*

Let R be a field of characteristic zero. Let $x \equiv \psi_1(x), \psi_2(x), \cdots, \psi_n(x)$ be distinct polynomials (with coefficients in R) of degree less than n, and define $\psi_j^{(1)}(x) = \psi_j(x), \psi_j^{(2)}(x) = \psi_j(\psi_j(x)), \cdots$. Show that there exists at most one monic irreducible polynomial $f(x)$ over R of degree n such that for any fixed positive k, $(k, n) = 1$, there is a permutation π_k of the integers $\{1, 2, \cdots, n\}$ such that

$$\psi_j^{(k)}(x) = \psi_{\pi_k(j)}(x) \pmod{f(x)}, j = 1, 2, \cdots, n.$$

Furthermore, any irreducible $f(x)$ satisfying the above conditions is also normal over R.

SOLUTIONS OF ADVANCED PROBLEMS

Irreducible R-modules for Commutative Rings

5480 [1967, 446]. *Proposed by Barbara L. Osofsky, Rutgers—The State University*

Let R be a ring with unity 1. Prove that a unital right R-module M is injective if and only if every R-homomorphism from a right ideal of R into M can be lifted to an R-homomorphism from R into M. If R is commutative, show that every indecomposable R-module is irreducible if and only if every indecomposable R-module is injective.

Solution by G. T. Georgantas, State University of New York at Buffalo. The first part of the problem is well known as Baer's condition. See, for example, Lambek, *Lectures on Rings and Modules*, p. 88.

The second part is true in the more general form where R may be any ring. We offer a proof in the category of right R-modules.

(A)$\Rightarrow$. Let M be indecomposable, and let $\overline{M}$ denote the injective hull of M. Since M is irreducible, $\overline{M}$ is indecomposable and hence $\overline{M}$ is irreducible. Thus $\overline{M} = M$.

(B)$\Leftarrow$. Let M be indecomposable and let $(0) \neq N$ be any submodule of M. $\overline{N}$ cannot be a proper submodule of M since the injectivity of $\overline{N}$ would make $\overline{N}$ a nontrivial direct summand of M. Hence $\overline{N} = M$. Now if $N \neq M$, we have $N \neq \overline{N}$ so N is not injective; then by assumption $N = N_1 \oplus N_2$ nontrivially. Therefore $\overline{N} = \overline{N}_1 \oplus \overline{N}_2$ and $\overline{N}_1$ (likewise $\overline{N}_2$) becomes a nontrivial direct summand of M. Thus $N = M$.

Also solved by V. C. Cateforis, Brian Parshall, Qazi Zameeruddin (India), and the proposer.

Functions in L_p

5490 [1967, 598]. *Proposed by J. T. Darwin, Jr., Auburn University, Alabama*

Let L_1 and L_2 denote those sets of measurable real-valued functions which are Lebesgue integrable and square integrable, respectively, on the interval $[0, 1]$. Prove that $f \in L_2$ if and only if (1) $f \in L_1$ and (2) there exists a nondecreasing function h such that for every $[r, s] \subset [0, 1]$,

$$\left|\int_r^s f(x)dx\right|^2 \leqq [h(s) - h(r)][s - r].$$

I. *Solution by B. S. Lalli and R. Singh, University of Saskatchewan.* We generalize slightly by replacing in the statement of the problem, L_1 by L_p, L_2 by L_{p+1}, and $[s-r]$ by $[s-r]^p$.

Proof. Suppose f is in L_{p+1}. Then obviously f is in L_p. Define the nondecreasing function h by $h(x) = \int_0^x |f(t)|^{p+1}dt$, x in $[0, 1]$. By Holder's inequality, for every $[r, s] \subset [0, 1]$,

$$\text{(a)} \qquad \int_r^s |f(x)|\,dx \leqq \left(\int_r^s |f(x)|^{p+1}dx\right)^{1/(p+1)}\left(\int_r^s 1^q\cdot dx\right)^{1/q},$$

where $1/(p+1)+1/q=1$. From (a) it follows that

$$\left(\int_r^s |f(x)|\,dx\right)^{p+1} \leqq \left(\int_r^s |f(x)|^{p+1}dx\right)[s - r]^{(p+1)/q} \leqq [h(s) - h(r)][s - r]^p.$$

Conversely, for $f \in L_p$, (2) implies for every $[r, s] \subset [0, 1]$,

$$\text{(b)} \qquad \left|\frac{1}{s - r}\int_r^s f(x)dx\right|^{p+1} \leqq \frac{h(s) - h(r)}{s - r}$$

which becomes, on letting $s \to r$,

$$\text{(c)} \qquad |f(x)|^{p+1} \leqq h'(x) \text{ almost everywhere on } [0, 1].$$

It now follows that $\int_0^1 |f(x)|^{p+1}dx \leqq \int_0^1 h'(x)dx < \infty$.

II. *Solution by P. R. Chernoff, Harvard University.* The necessity follows from the Schwarz inequality by taking $h(x) = \int_0^x f^2$. To prove the sufficiency, consider $g = \sum_i a_i \chi_{[r_i, s_i]}$, a linear combination of characteristic functions of disjoint intervals. We then have

$$\begin{aligned}
\left|\int_0^1 fg\right|^2 &= \left|\sum_i a_i \int_{r_i}^{s_i} f\right|^2 \\
&\leqq \sum_i |a_i|^2 \left|\int_{r_i}^{s_i} f\right|^2 \\
&\leqq \sum_i |a_i|^2 (s_i - r_i)(h(s_i) - h(r_i)) \\
&\leqq (h(1) - h(0)) \sum_i |a_i|^2 (s_i - r_i) \\
&= (h(1) - h(0)) \int_0^1 |g|^2.
\end{aligned}$$

Since the functions g are dense in L^2 it follows that f induces a bounded linear functional on L^2 and hence that $f \in L^2$.

III. *Solution by J. A. Dyer, Iowa State University.* It is a well-known result that $f \in L_2$ if and only if $f \in L_1$ and the Hellinger integral

$$\int_0^1 \frac{\left| d \int_0^x f(\xi) d\xi \right|^2}{dx}$$

exists. The condition stated in the problem is a necessary and sufficient condition for the existence of this Hellinger integral; see, e.g., H. S. Wall, *Creative Mathematics*, p. 167.

Also solved by Stephen Berman, N. J. Fine, G. J. Foschini, H. A. Guess, D. A. Hejhal, William Kelly, Jr., M. D. Mavinkurve (India), J. P. Williams, and the proposer.

A Singular Triangular Integral

5491 [1967, 598]. *Proposed by Roy O. Davies and Roy Buckley, The University, Leicester, England*

Determine the coefficients $C_{n\nu}$ in the expansion

$$\int_\alpha^{\pi/2} \frac{\sin (2n+1)\psi}{(\cos^2\alpha - \cos^2 \psi)} d\psi = \sum_{\nu=0}^n C_{n\nu} \cos^{2\nu}\alpha \qquad (0 < \alpha < \tfrac{1}{2}\pi).$$

I. *Solution by L. E. Ward, Sr., Escondido, California.* Use of the well-known formula

$$\frac{\sin (2n+1)\psi}{\sin \psi} = \sum_{\nu=0}^n (-1)^{n-\nu} \frac{(n+\nu)!}{(n-\nu)!(2\nu)!} (2 \cos \psi)^{2\nu}$$

enables us to write

$$\int_\alpha^{\pi/2} \frac{\sin (2n+1)\psi}{\sqrt{(\cos^2 \alpha - \cos^2 \psi)}} d\psi = \sum_{\nu=0}^n (-1)^{n-\nu} \frac{(n+\nu)!\, 2^{2\nu}}{(n-\nu)!\,(2\nu)!} \int_\alpha^{\pi/2} \frac{\cos^{2\nu}\psi \sin \psi \, d\psi}{\sqrt{(\cos^2 \alpha - \cos^2 \psi)}}.$$

The change of variable $\cos \psi = \cos \alpha \cos \theta$ reduces the integral in the right-hand member of this equation to

$$\cos^{2\nu} \alpha \int_0^{\pi/2} \cos^{2\nu} \theta \, d\theta = \frac{\pi}{2} \frac{(2\nu)!}{2^{2\nu}(\nu!)^2} \cos^{2\nu} \alpha.$$

Thus the integral of the problem becomes

$$\sum_{\nu=0}^n (-1)^{n-\nu} \frac{(n+\nu)!}{(n-\nu)!\,(\nu!)^2} \frac{\pi}{2} \cos^{2\nu} \alpha.$$

It follows that

$$C_{n\nu} = (-1)^{n-\nu} \frac{(n+\nu)!}{(n-\nu)!(\nu!)^2} \frac{\pi}{2}, \qquad \begin{pmatrix} n = 0, 1, 2, \cdots \\ \nu = 0, 1, \cdots, n \end{pmatrix}.$$

II. *Solution by H. E. Fettis, Aerospace Research Laboratories, Wright-Patterson AF Base.* Set $\psi=\theta/2$, $\cos^2\alpha=\frac{1}{2}(1+\cos 2\alpha)$, $\cos^2\psi=\frac{1}{2}(1+\cos\theta)$. The left side becomes

$$\frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{\sin(n+\frac{1}{2})\theta\,d\theta}{\sqrt{(\cos 2\alpha-\cos\theta)}}=\tfrac{1}{2}\pi P_n(\cos 2\alpha)$$

from a well-known representation of the Legendre polynomials. (See Magnus & Oberhettinger, *Special Functions of Mathematical Physics*, p. 72.)

Also, using a relation between the Legendre polynomials and the hypergeometric function (loc. cit. p. 69) the result can be written

$$\int_\alpha^{\pi/2}\frac{\sin(2n+1)\psi\,d\psi}{\sqrt{(\cos^2\alpha-\cos^2\psi)}}=\frac{\pi}{2}(-1)^n\,{}_2F_1(-n,n+1,1,\cos^2\alpha)$$

$$=\frac{\pi}{2}(-1)^n\left[1+\frac{(-n)(n+1)}{1\cdot 1!}\cos^2\alpha\right.$$

$$\left.+\frac{(-n)(-n+1)(n+1)(n+2)}{1\cdot 2\cdot 2!}\cos^4\alpha+\cdots\right].$$

Hence

$$\frac{2}{\pi}C_{n\nu}=\frac{(-1)^{n+\nu}(n+\nu)(n+\nu-1)\cdots(n-\nu+1)}{(\nu!)^2}=\frac{(-1)^{n+\nu}(n+\nu)!}{(n-\nu)!\,(\nu!)^2}.$$

Also solved by A. S. Adikesavan (India), Robert Breusch, L. Carlitz, Donald Chand & S. S. Kapur, Naoki Kimura & Tetsundo Sekiguchi, J. Koekoek (Netherlands), E. L. Koh, R. Manohar and B. S. Lalli, R. E. Shafer, Franklin C. Smith, and the proposer.

A Symmetric Random Walk

5492 [1967, 599]. *Proposed by S. B. Akers and R. D. Berlin, General Electric Company, Syracuse, N.Y.*

A $4k-1$ step symmetric random walk is performed by A along the x-axis starting at the origin. He is accompanied by B through the first $2k-1$ steps; thereafter each of B's steps is taken opposite in direction to the corresponding step taken by A. What is the probability that at the termination of the walk A and B will be on opposite sides of the origin?

I. *Solution by J. W. Grossman, Stanford University.* Let $D(W_i)$ be the net distance traveled in a one-dimensional random walk W_i. The given problem is easily seen to be equivalent to the following: If W_1 is a symmetric random walk of length $2k-1$ and W_2 is a symmetric random walk of length $2k$, what is $\text{Prob}\{D(W_2)>D(W_1)\}$?

Let W_2^* be a random walk of length $2k-1$. Let $p=\text{Prob}\{D(W_2^*)>D(W_1)\}$ $=\text{Prob}\{D(W_2^*)<D(W_1)\}$ and $q=1-2p=\text{Prob}\{D(W_2^*)=D(W_1)\}$. Also note that, to preserve parity, $D(W_2^*)>D(W_1)$ implies $D(W_2^*)>D(W_1)+1$. Now W_2

is W_2^* followed by one more step. If $D(W_2^*)>D(W_1)$, then $D(W_2)>D(W_1)$; if $D(W_2^*)<D(W_1)$, then $D(W_2)<D(W_1)$. Finally, if $D(W_2^*)=D(W_1)$, then the symmetry of the walk insures that $\text{Prob}\{D(W_2)>D(W_1)\}=\frac{1}{2}q$. The total probability that $D(W_2)>D(W_1)$ is thus one-half.

II. *Solution by Steven Weintraub, Oceanside, N.Y.* The probability is $\frac{1}{2}$, regardless of the length of the walk. This can be shown by induction. For $k=1$, a 3-step random walk, the probability is easily calculated to be $\frac{1}{2}$. Assume that the probability for a $4k-1$ step random walk is $\frac{1}{2}$. A $4(k+1)-1$ step random walk may be regarded as a 2-step random walk, in which B walks with A, followed by a $4k-1$ step random walk under the conditions of the problem, followed by a 2-step random walk in which B walks in the opposite direction to A. On the average, the first and last random walks will cancel out, (their net effect will be to leave A and B both at the origin). Thus the probability for this walk will be the same as the probability for a $4k-1$ step walk, which is, by the induction assumption, $\frac{1}{2}$. We now conclude that the probability that A and B will end up on opposite sides of the origin is $\frac{1}{2}$ for all walks.

III. *Solution by M. A. Bershad, Washington, D.C.* Let $P(u, v; n)$ denote the probability that the symmetric random walk of n steps starting at point u will terminate at point v. The problem calls for the evaluation of

$$C = 2\sum_{x=1}^{2k-1} P(0,\ x;\ 2k-1)\left[2\sum_{y=-1}^{-(2k-x)} P(x, y; 2k)\right].$$

Since $P(x, y; 2k)=P(0, x-y; 2k)=\frac{1}{2}P(0, x-y-1; 2k-1)+\frac{1}{2}P(0, x-y+1;\ 2k-1)$, we have

$$\begin{aligned} C=2 &\sum_{x=1}^{2k-1} P(0, x; 2k-1) \sum_{y=1}^{2k-x} [P(0, x+y-1; 2k-1) \\ &+P(0, x+y+1; 2k-1)] \\ &= 2\sum_{x=1}^{2k-1} P(0, x; 2k-1)\left[P(0, x; 2k-1) + 2\sum_{z=x+1}^{2k-1} P(0, z; 2k-1)\right] \\ &= 2\left[\sum_{x=1}^{2k-1} P(0, x; 2k-1)\right]^2 = \tfrac{1}{2}, \quad \text{since} \quad \sum_{x=1}^{2k-1} P(0, x; 2k-1) = \tfrac{1}{2} \end{aligned}$$

IV. *Comment by D. Ž. Djoković, University of Waterloo, Ontario.* Using the result of the problem, we can get the following identity

$$\sum_{i=0}^{k-1}\binom{2k-1}{i}\left[\binom{2k}{0}+\binom{2k}{1}+\cdots+\binom{2k}{i}\right] = 2^{4(k-1)}, \tag{1}$$

where $k=1, 2, \cdots$. Let x_r be the abscissa of A after the rth step and let

$$U = |x_{2k-1}|,\quad V = |x_{4k-2} - x_{2k-1}|,\quad W = |x_{4k-1} - x_{2k-1}|.$$

Then

$$p(U = 2r - 1) = 2\binom{2k-1}{k-r}2^{-2k+1} \qquad (r = 1, \cdots, k),$$

$$p(W > 2r - 1 \mid U = 2r - 1) = 2\sum_{p=0}^{k-r}\binom{2k}{p}2^{-2k},$$

$$p(W > U) = \sum_{r=1}^{k} p(U = 2r - 1)\cdot p(W > U \mid U = 2r - 1)$$

$$= 2^{-4k+3}\sum_{r=1}^{k}\sum_{p=0}^{k-r}\binom{2k-1}{k-r}\binom{2k}{p} = \frac{1}{2},$$

which is equivalent to (1).

Also solved by N. J. Fine, Michael Goodman, A. J. Keeping, M. D. Mavinkurve (India), Steven Mensker, Walter Reichert, Perry Scheinok, R. E. Shafer, J. C. Tanner (England), J. H. van Lint (Netherlands), C. P. Wood, Jr., and the proposers.

The Extremal Points of a Cubic Polynomial

5493 [1967, 599]. *Proposed by H. S. Shapiro, New York University*

Let $f(x, y)$ denote a cubic polynomial in x, y (i.e. a combination of monomial terms x^iy^j with $i+j\leqq 3$). Suppose the surface $z=f(x, y)$ has three saddle points and a local maximum at points S_1, S_2, S_3, and M respectively. Then M lies inside the triangle $S_1S_2S_3$.

I. *Solution by the proposer.* According to a formula of classical algebra (See Kronecker, *Werke*, Bd. I, p. 133), for any two quadratic polynomials P, Q and any linear polynomial L we have

$$(*) \qquad \sum_{i=1}^{4}\frac{L(x_i, y_i)}{J(x_i, y_i)} = 0,$$

where J is the Jacobian of P and Q, providing the equations $P=0$, $Q=0$ have four complex roots (x_i, y_i) at each of which the Jacobian is nonzero.

Applying this to our problem with $P=\partial f/\partial x$, $Q=\partial f/\partial y$, we have

$$J = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y} \\ \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2}\end{vmatrix},$$

which is positive at M and negative at S_1, S_2 and S_3. If now M lies outside the triangle $S_1S_2S_3$ we can draw a line $L(x, y)=0$ separating M from this triangle. For this function L the summands in (*) are all of the same sign, and we have a contradiction.

II. *Solution by P. G. Combea, IBM Corporation, New York City.* The following more general statement will be proved: If $f(x, y)$ denotes a nontrivial cubic polynomial in x, y, and the surface $z=f(x, y)$ has saddle points at three noncollinear points S_1, S_2, S_3, then there is a point M inside the triangle $S_1S_2S_3$ where there is a local maximum or minimum. This property of the surface $z=f(x, y)$ is invariant under nonsingular affine transformations in the xy-plane, which preserve ratios of areas. Hence it is sufficient to prove it when the saddle points occur at $(0, 0)$, $(1, 0)$, $(0, 1)$. Given

$$f(x, y) = a_1 + b_1x + b_2y + \tfrac{3}{2}(c_1x^2 + 2c_2xy + c_3y^2) + d_1x^3 + 3d_2x^2y + 3d_3xy^2 + d_4y^3,$$

this requires $f_x(0, 0)=f_y(0, 0)=f_x(1, 0)=f_y(1, 0)=f_x(0, 1)=f_y(0, 1)=0$; hence $b_1=b_2=0$, $d_1=-c_1$, $d_2=d_3=c_2$, $d_4=-c_3$, and

$$f(x, y) = a_1 + \tfrac{3}{2}(c_1x^2 + 2c_2xy + c_3y^2) - (c_1x^3 + 3c_2x^2y + 3c_2xy^2 + c_3y^3).$$

Let $H(x, y)=f_{xx}(x, y)f_{yy}(x, y)-f_{xy}^2(x, y)$. Then

$$H(x, y) = 9\{(c_1 - 2(c_1x + c_2y))(c_3 - 2(c_2x + c_3y)) - (c_2(1 - 2x - 2y))^2\}.$$

For the stationary points $(0, 0)$, $(1, 0)$, $(0, 1)$ to be saddle points, the following are necessary conditions:

$$(\ast_\ast)\qquad \begin{cases} H(0, 0) < 0 & \text{or} \quad f_{xx}(0, 0) = f_{xy}(0, 0) = f_{yy}(0, 0) = 0 \\ H(1, 0) < 0 & \text{or} \quad f_{xx}(1, 0) = f_{xy}(1, 0) = f_{yy}(1, 0) = 0 \\ H(0, 1) < 0 & \text{or} \quad f_{xx}(0, 1) = f_{xy}(0, 1) = f_{yy}(0, 1) = 0. \end{cases}$$

In each case the second alternative in $(\ast_\ast)$ implies $c_1=c_2=c_3=0$, hence $f(x, y)=a_1$. Therefore, the inequalities must hold. They are equivalent to

$$A \equiv c_2^2 - c_1c_3 > 0, \quad B \equiv c_2^2 + c_1c_3 - 2c_1c_2 > 0, \quad C \equiv c_2^2 + c_1c_3 - 2c_2c_3 > 0.$$

The fourth stationary point may be obtained from the equations

$$\tfrac{1}{3}f_x = c_1x + c_2y - (c_1x^2 + 2c_2xy + c_2y^2) = 0,$$
$$\tfrac{1}{3}f_y = c_2x + c_3y - (c_2x^2 + 2c_2xy + c_3y^2) = 0,$$

by taking linear combinations with multipliers $(c_3, -c_2)$ and $(c_1, -c_2)$ respectively, obtaining

$$x[-(c_2^2 - c_1c_3) + (c_2^2 - c_1c_3)x + 2c_2(c_2 - c_3)y] = 0,$$

$$y[-(c_2^2 - c_1c_3) + 2c_2(c_2 - c_1)x + (c_2^2 - c_1c_3)y] = 0.$$

Equating to zero the expressions in brackets, and noting that $2c_2(c_2-c_3)=A+C$ and $2c_2(c_2-c_1)=A+B$, this gives

$$Ax + (A + C)y = A, \quad (A + B)x + Ay = A,$$

with solution $\bar{x} = AC/(AB+AC+BC)$, $\bar{y} = AB/(AB+AC+BC)$. Hence $\bar{x} > 0$, $\bar{y} > 0$, $\bar{x} + \bar{y} < 1$, and the point $(\bar{x}, \bar{y})$ is inside the unit triangle. By direct substitution one can verify that

$$H(x, y) = 9\{A(1 - x - y) - 2A(1 - x - y)^2 + Bx(1 - 2x) + Cy(1 - 2y)\}$$

and $H(\bar{x}, \bar{y}) = 9ABC/(AB+AC+BC) > 0$. This is a sufficient condition for $(\bar{x}, \bar{y})$ to be a relative maximum or minimum.

Similar Permutations with Fixed Elements

5494 [1967, 599]. *Proposed by József Dénes and Raisz Klára, Central Research Institute for Physics, Budapest, Hungary*

If two permutations of degree n ($n \geqq 3$) having at least two fixed elements are similar then they belong to the same conjugate class of the alternating group of degree n.

I. *Solution by Walter Stromquist, Student, University of Kansas.* If δ and γ are similar, there exists $\alpha \in S_n$ such that $\alpha^{-1}\delta\alpha = \gamma$. If α is even, we are done; otherwise, let $\beta = (x_i x_j)\alpha$ where x_i and x_j are fixed by δ. Then β is even, and $\beta^{-1}\delta\beta = \gamma$.

II. *Solution by William Scott, University of Utah.* This is a very special case of the following general criterion, found, for example, in Boerner, *Darstellungen von Gruppen*, Satz 2.1.

Let K be a conjugate class of elements from the symmetric group S_n, and suppose that $K \subset A_n$, the alternating group. Each permutation in K is a product of i_1 1-cycles, i_2 2-cycles, $\cdots$, i_n n-cycles. Then K is the union of two conjugate classes in A_n if $i_{2j} = 0$ for all j and $i_{2j+1} \leqq 1$; and otherwise K is a single conjugate class in A_n.

In the problem $i_1 > 1$ by assumption, hence K is a single conjugate class in A_n, hence the given permutations are conjugate in A_n.

Also solved by L. Carlitz, D. Ž. Djoković, N. J. Fine, M. G. Greening (Australia), C. C. Lindner, W. C. Waterhouse, and the proposers.

NOTE. Dénes informs us that the following theorem will appear in a paper coauthored with P. Erdös and P. Turán in *Acta Mathematica Hungarica:* If α is an even permutation of degree n and $N(\alpha)$ denotes the normalizer subgroup of α then all the similar permutations belong to the same conjugate class of the alternating group of degree n if and only if $N(\alpha)$ contains at least one odd permutation.

Linear Combinations of Functions with a Single Zero

5495 [1967, 599]. *Proposed by Erwin Just, Bronx Community College, New York*

Let f and g be real-valued continuous functions defined on $[0, 1]$. If no linear combination of f and g has more than one zero in $(0, 1)$, and $f(0)g(1) \neq g(0)f(1)$, prove that some linear combination of f and g does not have any zeros in $(0, 1)$.

I. *Solution by Max Jodeit, Jr., Rice University.* The hypotheses regarding the endpoints are unnecessary. If f, g are continuous functions defined on $(0, 1)$ and each nontrivial linear combination of them has at most one zero, then some linear combination has no zero. Consider three cases: (a) either f or g has no zeros; (b) each has a zero but neither has a sign change; (c) each has a zero, and at least one has a sign change.

The result is immediate in cases (a) and (b), since the zeros of f, g must be different.

In case (c) we may assume (by changing the names, multiplying either by -1, or making the change of variable $x \to 1-x$) that there exist numbers $0<x_1<x_0<1$ such that $g(x)<0$ for $x<x_1$, $g(x)>0$ for $x>x_1$, and $f(x)>0$ for $x<x_0$. Set $F(x)=-f(x)/g(x)$, $x\neq x_1$. The range of F contains all y such that $f+yg$ has a zero. F is one-to-one since $F(u)=F(v)$ implies $f+F(u)g$ has zeros at u and v. F is continuous on $(0, x_1)$ and $(x_1, 1)$, hence is strictly monotone on each piece. Moreover $\lim_{x\to x_1^-} F(x) = -\lim_{x\to x_1^+} F(x) = +\infty$. Hence

$$F(0^+) = b \equiv \inf_{x<x_1} F(x) \geqq \sup_{x>x_1} F(x) \equiv a = F(1^-),$$

so, since neither bound is assumed by F, $f+yg$ has no zero in $(0, 1)$ if $y\in[a, b]$.

II. *Solution by R. J. Driscoll, Loyola University.* Let $\mathcal{L}$ be the set of nontrivial linear combinations of f and g. We first note that we cannot have $f(x_0)=0=g(x_0)$ for some $x_0\in(0, 1)$, for, if so, then $f(x_1)g-g(x_1)f$, where x_1 is some other point in $(0, 1)$, would be a member of $\mathcal{L}$ that vanished at x_0 and x_1. It follows that no member of $\mathcal{L}$ can vanish at a point in $(0, 1)$ without changing sign at that point, for if $h\in\mathcal{L}$, $x_0\in(0, 1)$, $h(x_0)=0$, and h is positive in a deleted neighborhood of x_0, we choose $v\in\mathcal{L}$ with $v(x_0)>0$ and note that for sufficiently small positive ϵ, $h-\epsilon v$ changes sign twice. The boundary condition permits us to choose a (unique) function, F, in $\mathcal{L}$ with $F(0)=1=F(1)$; the preceding comments show that F is positive on $[0, 1]$.

We note that the boundary condition is equivalent to the statement that no element of $\mathcal{L}$ can vanish at both 0 and 1; we can then infer that no element of $\mathcal{L}$ can vanish twice on $[0, 1]$. Indeed, if $h\in\mathcal{L}$ and $h(0)=0=h(x_0)$ for some $x_0\in(0, 1)$, then we can choose c sufficiently small and of suitable sign so that $h+cF$ changes sign twice on $(0, 1)$.

Also solved by D. Borwein, D. R. Chand & S. S. Kapur, P. R. Chernoff & W. C. Waterhouse, Robert Coen, T. J. Cullen, Roy O. Davies (England), Mary R. Embry, M. A. Ettrick, N. J. Fine, M. G. Greening (Australia), C. P. Gupta, D. A. Hejhal, Kenneth Kramer, J. R. Kuttler, D. S Lawrence, O. P. Lossers (Netherlands), M. D. Mavinkurve (India), A. Meir, M. E. Muldoon, D. J. Pierce, Jernej Polajnar (Yugoslavia), Frederick Sipinen, Marlow Sholander, R. M. Warten, Steven Weintraub, J. B. Wilker, and the proposer.

Editorial Note. Although the boundary conditions are not needed if only the zeros in $(0, 1)$ are counted, the condition $f(0)g(1)\neq f(1)g(0)$ is needed if $(0, 1)$ is replaced by the closed interval $[0, 1]$ in the conclusion.

Subdeterminants of Adjoint Matrices over the Integers

5496 [1967, 599]. *Proposed by R. H. Shudde and E. S. Langford, U. S. Naval Postgraduate School*

Let B be any $m\times m$ matrix over the integers, with B^* the adjoint matrix of B. Prove that every $s\times s$ subdeterminant of B^* is divisible by $(\det B)^{s-1}$.

I. *Solution by R. Manohar and B. S. Lalli, University of Saskatchewan.* Without loss of generality we can take any $s\times s$ submatrix P of B^* to be the principal submatrix. Let

$$B^* = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} P & 0 \\ R & I \end{pmatrix},$$

where 0 is the $s\times(m-s)$ zero matrix and I the $(m-s)\times(m-s)$ identity matrix. Note that $\det C = \det P$. It follows easily that $\det BC = (\det B)^s\cdot d$, where d is a nonzero integer. Hence $(\det B)(\det P) = (\det B)^s\cdot d$.

II. *Solution by Marvin Marcus, University of California, Santa Barbara.* Let $C_s(B)$ and $C^s(B)$ denote the sth compound of B and the sth supplementary compound of B, respectively. Formula (20), p. 67 in J. H. M. Wedderburn, *Lectures on Matrices*, A.M.S. Colloq. Publ., v. 17 states that

$$C_s(B^*) = \det (B)^{s-1}C^{n-s}(B').$$

The entries of $C_s(B^*)$ are the s-square subdeterminants of B^*.

Also solved by A. S. Adikesavan (India), P. L. Claypool, D. Ž. Djoković, Ralph Freese, M. G. Greening (Australia), T. L. Markham, M. D. Mavinkurve (India), and the proposers.

For other well-known identities solving the problem, Djoković refers to Kowalewski, *Einführung in die Determinantentheorie*, 1954, p. 80; Markham refers to F. R. Gantmacher, *Theory of Determinants*, v. 2, p. 21.

Groups from Punctured Semigroups

5498 [1967, 599]. *Proposed by J. A. Hildebrandt, Louisiana State University*

Let S be a semigroup with at least three elements and $x\in S$ such that $S\backslash x$ is a group. Prove that S is regular if and only if x is a zero or an identity.

Solution by S. J. Pierce, University of California, Santa Barbara. We replace *identity* with *idempotent* in the statement of the problem.

If x is a zero or an idempotent, then $x^2=x$ and $xxx=x$, so that x is regular. Since $S\backslash x=G$ is a group, all other elements of S are also regular.

Suppose now that S is regular. For any $a\in G$, $ax=xa=x$. If $ax\neq x$, then $ax\in G$ and hence $x\in G$. Now, by the regularity of S, there exists $a\in S$ such that $xax=x$. If $a\in G$, then we have $x^2=x$ and x is zero or idempotent. If $a=x$, then $x^3=x$. Suppose in this case that $u=x^2\neq x$. Then $u\in G$. Since S has at least three elements, pick $v\in G$, $v\neq u$. Then the equation $zu=v$ has a solution $z\in G$. But if $z\in G$, then $zu=zx^2=x^2=u\neq v$. Thus x^2 cannot be in G and hence $x^2=x$, and we achieve the same result. Actually, x can only be a zero.

Also solved by C. W. Austin, Ralph Bennett, Ralph Freese, M. G. Greening (Australia), K. M. Kapp, Kwangil Koh, C. C. Lindner, Howard Penn, John Shafer, Kermit Sigmon, and D. P. Sumner.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

COLLABORATING EDITOR: SEYMOUR SCHUSTER, University of Minnesota

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

Numerical Integration. By Philip J. Davis (Brown University) and Philip Rabinowitz (Weizmann Institute). Blaisdell, Waltham, Mass., 1967. ix+230 pp. $7.50 (Telegraphic review, Jan. 1968).

This book is written for the mathematician, scientist, engineer, or layman who wants to find out quickly the present state of the art in Numerical Integration. The mathematical background required is perhaps Sophomore or Junior College level. The scope of the book is adequately indicated by the chapter headings. These are: Approximate integration over a finite interval; . . . over an infinite interval; Error Analysis. Approximate integration in two or more dimensions; Automatic Integration. This is the first non-cursory treatment in book form of either of these latter two topics.

The book has two features which distinguish it from other books on comparable topics. One concerns the arrangement of the material. Each chapter forms an independent unit and within each chapter the sections are almost independent of each other. Consequently the book may be used as a compendium or a small encyclopedia. The reader may look up a particular topic in the index, and find himself able to read the section in which this topic occurs without having to read any of the previous sections.

The second feature concerns the presentation within each section. Large numbers of small and unimportant details which use up a lot of space in conventional text books are simply omitted. Instead adequate references to accessible articles are given. The emphasis is focused on the underlying mathematical ideas on which the particular method is based. The reader is not left with the feeling that the subject is completely worked out. Instead he is inspired to read more and develop further some of these ideas. The numerical examples illustrate both how well and how badly particular methods work in different circumstances.

While these features are virtues from the point of view of the bright student and the professional scientist or mathematician, this reviewer would not recommend this attitude in dealing with the mediocre student, or a large inhomogeneous class of students. The not-so-bright student often equates brevity with unimportance and usually needs a more pedestrian and less elegant approach than he is likely to encounter here.

However it is for the scientist who is about to embark on a series of integrations using a computer that this book should be really useful. He will find in Appendix 1 a reprint of the article "On the practical evaluation of integrals" by the late Milton Abramowitz. In other appendices he will find Fortran programs and a bibliography of Algol procedures. But more important, using the index, he will be able to find out quickly what this book has to offer towards his particular problem, and if this is insufficient he will be directed to the appropriate references in the literature for further information.

J. N. LYNESS, Argonne National Laboratory

University Algebra. By Richard E. Johnson. Prentice-Hall, Englewood Cliffs, N. J., 1966. xii+271 pp. $8.25.

A text for an undergraduate first course in abstract algebra, written, the author says in his preface, as a response to recent university and high school curricula changes which have resulted in a student 'much more aware of algebraic structure.' The list of contents of the outcome is: the real number system, Abelian groups, commutative rings, integral domains and fields, polynomial rings, algebraic extensions of a field, factorization in integral domains, vector spaces, groups, rings, linear equations and determinants, lattices. The material is presented straightforwardly enough, but is given little motivation—even the theorems follow their proofs nearly as often as not. This may be in anticipation of more sophisticated students; but I am sure the early, basic chapters on commutative groups and rings would be improved by more discussion of the relation between the abstract concepts defined and 'elementary' mathematics, and of the relative importance of the topics introduced, just so as to orient the beginner, irrespective of whether he has had the "New Math."

The exercises are good; as also are the "Theoretical Projects," which are brief surveys of topics not covered in the text, set out as (difficult) exercises. Professor K. K. Norton has brought to my attention the existence of numerous misprints.

Altogether the text seems neither very good nor very bad, providing an adequate first course in abstract algebra in a rather lacklustre way.

J. G. ANDERSON, University of New Brunswick

The Programming Language LISP: Its operation and Applications. Edited by Edmund C. Berkeley and Daniel G. Bobrow. M.I.T., Cambridge, Mass., 1966. ix+382 pp. (paper) $5.00. (Teleg. Rev. March 1967)

The LISP language is a very powerful programming language, especially when a recursive algorithm is needed. The users of LISP are very devoted to it, and find it very useful for their purposes. Unfortunately, the notation, while elegant and precise, is not very easy to use. This book attempts to make the language more easily understood and more available to mathematicians and programmers. It is not a very easy book to read, however. It appears to this reviewer that some sophistication in programming (at least) and preferably some familiarity with LISP itself, is almost necessary to read beyond the first few pages of this book. For those who have the minimal background, however, it should be very useful in understanding what LISP is about and how it can be used. A new version of LISP, called LISP 2, is being developed at this time, and most of the notational inconveniences of LISP should disappear. This book is recommended for those with some background, since the LISP language should be a tool in everyone's kit.

B. A. GALLER, University of Michigan

Topics from the Theory of Numbers. By Emil Grosswald. Macmillan, New York, 1966. xiv+299 pp. $8.50 (Teleg. Rev. May 1967)

If one regards the 14-page Part I as an introduction, the remainder of this book is divided into two sections: Part II, containing most of the standard topics for a one-semester undergraduate course, and Part III, furnishing a graduate-level introduction to the methods of analytic and algebraic number theory. Part III is supported by appendices and an enumeration of needed results from analysis and algebra, and is accessible in considerable degree to the industrious undergraduate student. For full appreciation of this material, however, previous course work in modern algebra and complex variables would be very desirable.

Parts II and III invite comparison with Volumes I and II, respectively, of Le Veque's "Topics in Number Theory." Grosswald's book is shorter and his choice of topics is considerably more restricted. Part II of Grosswald covers essentially the same material as

Volume I of Le Veque, except that rational approximations, sums of squares, and nonlinear Diophantine equations, other than the Fermat equation, are omitted.

Part III proves the Prime Number Theorem (following Titchmarsh) analytically without an error term; Le Veque gets an error term. Grosswald derives the functional equation for the Riemann ζ-function non-rigorously, using Fourier series. The discussions in both books of algebraic number fields are similar; both lead to a proof of Fermat's Last Theorem for regular primes; both assume "Kummer's lemma." Grosswald concludes with a section on the theory of partitions, using generating functions. This is technically elementary and self-contained.

The book is carefully and interestingly written throughout. There is a discussion of the historical background of each topic. This reviewer feels that the use of symbolism is excessive in the early chapters, an example being the definition of greatest common divisor: $d=(a, b)\Leftrightarrow d\in Z$, $d>0$, $d|a$, $d|b$, and $c\in Z$, $c|a$, $c|b\Rightarrow c|d$. Each chapter is supplied with an adequate problem set and a bibliography.

DAVID REARICK, University of Colorado

Nonlinear Differential Equations. By T. V. Davies and Eleanor M. James. Addison-Wesley, Reading, Mass., 1966. ix+274 pp. $13.75. (Teleg. Rev. February 1967)

By its very nature a book which is a collection of research results obtained by different authors over a period of 75 years will run into notational difficulties and organizational problems. Also, unless considerable additional detail is furnished in the text the reader who is not already an expert in the subject area of a cited paper will find the material difficult. In the book of Davies and James these plaguing problems have not been completely overcome. For instance the symbol 0 has one meaning in (2.2) and another meaning elsewhere, e.g. in (3.13). The symbols x_i and y_i are used in three or four different senses in the text and the introduction of the letter d in (2.3) has led to the unfortunate equation (2.4). Some of the derivations are so sketchy that digestion of the proofs and techniques by the novice in differential equations might take a considerable amount of time.

In the reviewer's opinion this book would therefore be more suitable as a reference book than as a classroom text. It is not a comprehensive treatise. Its value lies in the fact that it presents under a single cover a number of important results and techniques for handling systems of nonlinear ordinary differential equations. It should therefore be a worthwhile addition to the bookshelf of a research engineer or physicist.

L. E. PAYNE, Cornell University

Problems for Computer Solution. By Fred Gruenberger and George Jaffray. Wiley, New York, 1966. 401 pp. $4.50.

The book is intended for use with students who already know what a computer can do. Ninety-two problem areas are presented. Each is introduced by a brief problem statement and one or more exercises are given. The first problem area is entitled "Evaluating a Polynomial"; it contains eight exercises, each of which requires a new computer program or a modification in a previous one. The problem areas are coded into the following subject categories (the number of problem areas is given in brackets): Numerical analysis (11), geometry (6), more geometry (2), combinational problems (6), probability (14), number theory (14), puzzles (5), miscellaneous problems (7), business arithmetic (4), operations research (3), business data processing (4), statistics (6), IBM 1620 computer related problems (10).

The author states that "The problems cover also a wide range of difficulty. Some are little more than exercises, while others could lead to term projects extending over a semester's time." Since there is no mention of the existence of an instructor's manual,

each instructor is on his own in evaluating the difficulty of a problem area before he assigns it.

Occasionally, the text contains remarks such as: "Back in Middle Ages of computing (that is, the middle 1940's) . . ." which have questionable value. In spite of these interjections and the lack of an instructor's manual, the book should be of considerable value in introductory computer courses.

DANIEL TEICHROEW, Case Institute of Technology

Introduction to Algol Programming. By Torgil Ekman and Carl-Erik Froberg. Studentlitteratur, Lund, Sweden, 1965, 123 pp. (*Editorial Note:* We are informed that a 2nd ed. of 172 pages was published in 1967 with an appendix covering the ALGOL 60 report.)

Algol is a very general and flexible programming language for scientific problems. It has found widespread use, especially in Europe, at universities and technical institutes. If one knows Algol, one has available a great many algorithms published and certified in technical journals. It follows that a good introductory text to Algol is of great value. Several such texts have been published, but the one by Ekman and Froberg compares favourably with any of them.

The book treats adequately the different concepts of Algol such as Algol numbers, identifiers, arithmetic, Boolean and designational expressions, the different types of statements, e.g. the assignment, go to, dummy, conditional, for and procedure statements. The different declarations such as type, array, switch and procedure declarations are well treated. Excellent discussion is given of the block concept, call by value and call by name and even of some peculiarities of Algol like side effects. The different structures are well illustrated with a great number of examples. Most chapters are followed by exercises, the solutions of which are given at the back of the book. This is an excellent introductory text to Algol.

I. FARKAS, University of Toronto

Tensor Calculus through Differential Geometry. By John Abram. Butterworths, London 1965. v +170 pp. $7.50.

This book is primarily an introduction to tensor calculus and to the differential geometry of curves and surfaces in Euclidean 3-space. It also contains chapters on curvilinear coordinates, Riemannian geometry and applications of tensor methods to elasticity, hydrodynamics and classical dynamics. Within the narrow confines of this slim volume, a considerable amount of territory is covered economically and quickly.

On the assumption of an elementary knowledge of vector and linear algebra, the formalism of tensor analysis is derived in connection with the differential geometry of a surface. Many details of the surface geometry are then developed in rapid sequence, using vectors in the enveloping Euclidean 3-space and tensors on the surface itself. Each chapter concludes with a series of relevant problems.

However, the exposition is marred by a number of shortcomings. The principal objection is that portions of the book would be comprehensible only to someone already familiar with the subject. Thus terms such as "flat" and "Riemannian space" appear in the book but are never defined. One section is entitled "Dupin's Indicatrix" but the term is not defined or even mentioned in the text itself. References to Cartan's transported reference frames and the embedding problem at the beginning of the book may be obscure to the novice. Gaps occur in some proofs as, for example, the assumption that $EG-F^2>0$ in surface theory without deriving the conditions under which this is true and the absence of any reference to the fundamental lemma of the calculus of variations in the derivation of Euler's equations for the geodesics.

In summary, the volume is a concise introduction to tensor calculus and its applications, whose usefulness would be increased by greater attention to details.

AARON FIALKOW, Polytechnic Institute of Brooklyn

Theory of Groups in Classical and Quantum Physics. By T. Kahan in collaboration with P. Cavailles, R. Gouarne, T. D. Newton, G. Rideau, B. Lochak, R. Nataf. English translation edited by A. R. Edmonds. Vol. I. Mathematical Structures and the Foundations of Quantum Theory. Translated by H. Ingram. American Elsevier, New York, 1965. xxiii+566 pp. $37.50.

This book is designed to provide physicists with an introduction to modern mathematical structures, a primary aim being to give a comprehensive view of the mathematics of group theory as applied to theoretical physics and chemistry (e.g., crystallography, spectroscopy, chemical bonds, theory of metals, particle theory).

Group theory has become a powerful instrument which has been applied to whole classes of difficult problems. In the study of physical structures the wise use of symmetry principles restricts the number of possible situations which need be considered, hence the use of groups. Without group theory the discovery of the connections between theory and experimental results in a field such as complex spectra becomes practically impossible, and, elementary particle theory would be most bewildering. Group theory is now considered to be at the heart of the theory of elementary particles.

The book is divided into seven parts. Part 1 is concerned with group theory in the framework of axiomatized mathematics. Included are chapters on sets, algebraic and topological structures, varieties, analytic groups, Lie algebras and spinor representations. This part is fundamental to the appreciation and understanding of modern physical theories. Parts 2 to 5 treat the inhomogeneous Lorentz group, abstract group theory, group representations and the permutation group. Parts 6 and 7 treat the axiomatics of quantum mechanics in relation to group theory; and rotation groups.

The approach reflects the influence of the Bourbaki team, and the pioneering works of Wigner, Heisenberg, De Broglie, Schroedinger, Weyl, Dirac, Corson and Chevalley. Many illustrations of the principles used are given, but the book does not include exercises for the reader to work out as is typical of American texts.

This book will provide a useful addition to libraries and will be helpful to specialists in theoretical physics. It should prove to be a source of inspiration to theorists. Serious students of mathematics should find in it things to stimulate their interest and enrich their understanding of the wealth of material furnished by the physical sciences. Some parts are quite readable, others are challenging.

Volume II, in preparation, will be concerned with applications of groups to theoretical physics and chemistry. This reviewer will look forward with interest to the extent and detail to which the authors illustrate the ways the mathematical structures of Volume I are applied to physical theories.

R. S. BURINGTON, Arlington, Virginia

Calculus on Manifolds. By Michael Spivak. Benjamin, New York, 1965. 158 pp. $8.00 (hardbound), $3.95 (paperback).

This is a young and enthusiastic book by a young and enthusiastic lecturer at Brandeis. One can almost hear him saying to a colleague, "This semester I'm going to do Stokes' theorem *right!*" Given the author's skill as an expositor, one would prophesy a good deal of excitement—and that some students would get lost along the way.

The book begins with a treatment of continuity, differentiation, and integration for functions of several variables; standard theorems such as the implicit function theorem appear along the way, neatly done. This material constitutes half the book. The other half is devoted to introducing differential forms, first in euclidean space and then on

differentiable manifolds, and to proving the modern version of Stokes' theorem, which states that the integral of a form ω over the boundary of a manifold equals the integral of its differential $d\omega$ over the manifold itself. A final section is devoted to the special cases of this result to which are usually attached the names of Green, Stokes, and Gauss.

The treatment is thoroughly modern and very concise; too concise, in my opinion, for use of the book as a text. As a supplement to a standard course in advanced calculus, however, it would be ideal for the able student. Alternatively, it might be used as a basis for a senior seminar for mathematics majors. In any case, the student should have a semester of rigorous calculus under his belt first, as well as the linear algebra and set theory which the author suggests.

Finally, any mathematician whose studies have bypassed differential forms should find this little book pleasant reading. It is well done, and it has personality.

JAMES MUNKRES, Massachusetts Institute of Technology

Catégories et Structures. By Charles Ehresmann. Collection Travaux et Recherches Mathématiques, Vol. 10. Dunod, Paris, 1965. xvii+358 pp. NF 84.

This is the first full book on categories in French. It differs appreciably from the extant books in English. Most obviously, it totally ignores questions of abelian categories. Nearly as noticeable is the dearth of examples. But most characteristic of the difference is the expository approach. The principle that category theory should illuminate mathematics is waived in favor of the principle asserting mathematics is correct formalism. The resulting nearly impenetrably austere text is thereby forced to rely solely on its chapter synopses and bibliographic notes for conceptual enlightenment. The forthcoming companion volume advertised in the preface will, hopefully, remedy this defect.

The book starts with a detailed analysis of composition, and 2-categories appear under the name double category. The next two chapters deal with species of structures and contain the shadowy traces of fibered categories and presheaves as well as an interesting generalization of crossed homomorphisms. The last two chapters present fiber products and related constructions. Three short appendices (one giving a very idiosyncratic development of adjoint functors), a hundred-plus item errata sheet, and a highly personal and illuminating sonnet and couplet pair round out the book.

F. E. J. LINTON, Wesleyan University and E. T. H. Zürich

Seminar on the Atiyah-Singer Index Theorem. By Richard S. Palais. Annals of Mathematics Studies, No. 57. Princeton University Press, 1965. x+366 pp. $7.50.

The work under review contains the notes of a seminar held at the Institute for Advanced Study in 1963–1964 on the Atiyah-Singer Index Theorem. This theorem asserts the equality of the difference in dimension between the kernel and cokernel of an elliptic operator d on an oriented compact manifold X and the value on the orientation class of a certain total cohomology class of X associated to d. The text concentrates on the analytical difficulties in the proof.

F. E. J. LINTON, Wesleyan University and E. T. H. Zürich

Mathematical Analysis of Observations. By B. M. Shchigolev. Translated by Scripta Technica Inc. Edited by H. Eagle. American Elsevier, New York, 1965. xv+350 pp. $12.50.

This is a textbook on the processing of numerical observations and on some numerical methods in general. It was written as a text for courses in "Mathematical Analysis of Observations" for students of astronomy.

One may wish to compare this text with the classical treatise by Whittaker and Robinson ("The Calculus of Observations"). The latter (first published in 1924) is a collection

of many topics in Numerical Mathematics including such varied subjects as interpolation, practical Fourier analysis, the method of least squares, and the numerical solution of algebraic, transcendental as well as differential equations. The text under review has a much more limited scope, and is mainly directed to the needs of scientists who have to process numerical data and to deal with tables of functions. A reader interested in these subjects will find Shchigolev's text a good source of information, showing him clearly how to apply mathematical theories to his concrete problems.

OVED SHISHA, Wright-Patterson AFB, Ohio

Lectures on Invariant Subspaces. By Henry Helson. Academic Press, New York, 1964. xi+130 pp. $5.00.

This is a beautifully written account of recent developments in an area of much current interest, with a bearing on a variety of fields: harmonic analysis, classical function theory, prediction theory, and operators on Hilbert space.

The focus of these developments is a pair of classical results due to Beurling and Wiener. In L^2 of the unit circle a closed subspace M is called *invariant* if it is closed under multiplication by the function $\chi(e^{i\theta})=e^{i\theta}$ (so $\chi\cdot M\subset M$); M is *doubly invariant* if $\chi^{-1}\cdot M\subset M$ as well, and *simply invariant* otherwise. (An example of a nontrivial invariant subspace is the Hardy class H^2—the closed span of the nonnegative powers of χ—which may be viewed as boundary value functions of a corresponding class of analytic functions on $|z|<1$, providing the connection with classical function theory.)

The cited theorems of Beurling and Wiener neatly characterize invariant subspaces of L^2 of the circle: any doubly invariant subspace consists of all functions vanishing off a fixed set (Wiener) and conversely; simply invariant subspaces are those of the form $E\cdot H^2$, where E is measurable and $|E|\equiv 1$ (Beurling).

In 1958 the late David Lowdenslager and the author showed how a whole cycle of classical results, including Beurling's, followed from a simple Hilbert space variational argument; somewhat later they found a simple projection argument which led to extremely elegant proofs of many of the same results, and in particular to those of Beurling and Wiener. A complete development of these results begins the Lectures, which for the most part are an account of where the projection argument leads when one moves on to Hilbert space valued functions (and which cannot be detailed briefly).

Despite the author's claim that the book is written for a graduate student who knows a little, but not a lot, about analytic functions and Hilbert space, a good deal more is required of the reader, who should be fairly well acquainted with analytic functions (inner functions in particular) and elementary functional analysis. But for the appropriately armed reader the author has provided an admirable service in making these results available from one source. (There are few misprints, the book is handsomely printed, but sadly there is no index.)

IRVING GLICKSBERG, University of Washington

Mathematical Introduction to Celestial Mechanics. By Harry Pollard. Prentice-Hall, Englewood Cliffs, N.J., 1966. x+111 pp. $5.95. (Teleg. Rev. February. 1967)

The mathematics of this little book is correct. The vector methods used by the author permit a concise arrangement of the material and give a classical topic a modern approach. A student with a knowledge of vector analysis, partial differentiation, and elementary differential equations will find the text fairly easy reading. However, he will find a number of typographical errors, such as omission of subscripts or exponents, which lead to confusion. In some passages the same symbol is used for different quantities without clarification. The derivations of certain fundamentally important laws such as Kepler's third law are left as exercises. Perhaps the author has tried to cover too much in such a short book.

Unfortunately the presentation is not consistent with accepted astronomical usage. The author uses awkward notation or symbols not ordinarily used by astronomers who specialize in celestial mechanics. For example, for the eccentric anomaly the author uses "u" instead of the conventional "E". One finds rather quaint terminology for well-known quantities such as the longitude of perihelion. On page 28 one reads "The origin is taken as the Sun, the plane of the earth's orbit is the XY-plane. This orbit is known as the *ecliptic*, the XY-Plane is the *plane of the ecliptic*." Astronomers define the ecliptic as the great circle on the celestial sphere which is the apparent path of the Sun. The orbit of the earth is an ellipse which lies in the plane of the ecliptic. There are several other debatable if not incorrect statements.

If an astronomer, trained in celestial mechanics, attempted to discuss the classical topics covered in this text with a student whose only acquaintance with these topics had been gained through a study of this book, then the two would have to spend considerable time in translating terminology and symbols before they could carry on a fruitful conversation. The author should have had an astronomer, trained in celestial mechanics review the manuscript.

R. T. MATHEWS, Goodsell Observatory, Carleton College

Entire Functions. A. I. Markushevich. Translated by Scripta Technica, Inc. Translation Editor Leon Ehrenpreis. American Elsevier, New York, 1966. iv+105 pp. $6.50. (Teleg. Rev. April, 1967)

The purpose of the book seems to be mainly to make clear the difference between an algebraic and a transcendental function. The author restricts himself mostly to the consideration of entire, rather than meromorphic functions, mainly because the corresponding proofs are easier. As a matter of fact, the choice of material seems to have been dictated predominantly by the wish to give very elementary proofs. It is actually surprising how many rather deep results can be proven elegantly by extremely simple considerations. Hardly anything is assumed beyond a certain familiarity with complex numbers (not even the concept of a holomorphic function).

After a leisurely introduction (with many examples) to the concept of an entire function, Chapter 2 gives the proof of Cauchy's inequality for the coefficients, Liouville's theorem and the definition of order. An entire function is described loosely as ". . . a sort of polynomial of infinitely high degree." The shortcomings of this statement are, however, clearly pointed out in relation to some pertinent topics (e.g. Picard exceptional values). Chapter 3 discusses the zeros of entire functions. (Main results: the zeros are isolated, there are only a finite number of zeros in any bounded region and, if $f(z)$ has no zeros, then $f(z)=e^{g(z)}$, with $g(z)$ an entire function.) In Chapter 4, Liouville's theorem is used to prove the "fundamental theorem of algebra" and Picard's (little) theorem is stated in the form: If $f(z)$ is entire transcendental, then the equation $f(z)-AP(z)=0$ has infinitely many roots for every polynomial $P(z)$ and every value of the constant A, with one possible exception $A(f,P)$. The theorem is proven directly for some particular functions, such as e^z, $\sin z$, etc. Some mention is made of meromorphic functions and the existence of two exceptional values. The fifth (last) chapter discusses functional equations satisfied by entire functions and addition theorems. Both topics are treated in the same general spirit, with important simple examples completely (and leisurely) worked out. The Appendix consists of two somewhat more sophisticated sections. In the first, Picard's (little) theorem is proven for functions of finite order; in the second, Weierstrass' theorem (entire transcendental functions, satisfying an algebraic addition formula are trigonometric polynomials) is proven.

The most surprising omission from a monograph on entire functions is probably the absence of Weierstrass' product representation. The reason is presumably that the reader is not expected to be familiar with infinite products. As in many European texts,

there are no exercises or problems, but there is a short index. The translation is adequate, but the number of printing errors seems excessive (e.g., on p. 51, last line, the equation "$\sin z=0$" is omitted altogether; p. 69, line 19, "irrational" should be "rational," and on p. 94, last line, "$\mathrm{Ln}\, t=2\pi i\omega/z$ and hence $z=(z/2\pi i)\mathrm{Ln}\, t$" should read "$\mathrm{Ln}\, t=2\pi iz/\omega$ and hence $z=(\omega/2\pi i)\mathrm{Ln}\, t$").

It does not seem likely that the book could be used as a textbook in college. Its original purpose was for the supplementary training of teachers and it could well serve the same purpose in the USA.

EMIL GROSSWALD, University of Pennsylvania

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses; T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year). A boldface star (★) marks a notable book that might be overlooked.

Algebra

Algebraic Numbers. By L. E. Dickson, et al. Chelsea, Bronx, New York, 1967. 211 pp. \$4.95. This is an unaltered reprint of the classic report of 1928. P, L.

Lectures on Modern Algebra. By P. Dubreil and M. L. Dubreil-Jacotin (Both of Univ. of Paris). Translated by A. Geddes (Univ. of Glasgow). Hafner, New York, 1967. xii +364 pp. \$16.80. This is a translation of the second edition of *Leçons d'Algèbre Moderne* (Dunod, Paris, 1961). Topics include semigroups, generators, lattices, Zorn's axiom, Noetherian rings, vector spaces, fields and algebraic equations. T (16–17), P, L.

Linear Algebra. By Richard E. Johnson. (Univ. of New Hampshire). Prindle, Weber & Schmidt, Boston, Mass. 1967. ix+223 pp. \$7.95. This brief introduction intended for a one semester course, gives the usual introductory topics. T (14–15).

Analysis

Modern Elementary Differential Equations. By Richard Bellman (Univ. of Southern California). Addison-Wesley, Reading, Mass., 1968. xii+196 pp. \$8.95. The "modern" in the title seems to be more than a mere bow to fashion. There are many up to date applications. Emphasis is given to numerical solution using both desk and digital computers (with FORTRAN). T (14–15), S.

Tensor Analysis on Manifolds. By Richard L. Bishop and Samuel I. Goldberg (both of Univ. of Illinois). Macmillan, New York, 1968. viii+280 pp. \$11.95. Tensor analysis treated as a continuation of advanced calculus and as part of differential geometry, bringing together manifold theory and multilinear algebra with emphasis on global and conceptual aspects rather than manipulation and notation. The last three chapters are on integration theory, Riemannian and semi-Riemannian Manifolds, and application to classical mechanics. T (16), S, P, L.

Selected Problems on Exceptional Sets. By Lennart Carleson (Univ. of Uppsala). Van Nostrand, Princeton, N. J., 1968. v+152 pp. \$2.75 (paper). A selection of problems of personal interest to the author and related to those parts of potential theory that are important for applications. There are a few historical references and a bibliography (1049 titles drawn from Mathematical Reviews 1940–1965) that adds substantially to the value of the book. S, P, L.

Topics in the Theory of Functions of One Complex Variable. By W. H. J. Fuchs (Cornell Univ.). With the collaboration of Alan Schumitsky. Van Nostrand, Princeton, N. J., 1967. vi+193 pp. $3.25 (paper). These lecture notes offer a "second course" in complex variables. Topics include subharmonic functions, reflection principle, Dirichlet problem, Green's functions, Mergelyan's Solution, subordination, harmonic measures, extremal length, potential theory, the first and second fundamental theorem of Nevanlinna theory. T (17), S, P.

Linear Algebra and Analysis. By Andre Lichnerowicz. Translated by Alison Johnson. Holden-Day, San Francisco, 1968. xv+304 pp. $10.00. Topics include Hermitian spaces, matrices, forms, tensors, exterior algebra, exterior differential forms, series expansions, linear operators, and integral equations. Coordinates rather than abstract methods are used. T (15–16), S, P, L.

Lectures on the Theory of Functions of a Complex Variable. By George W. Mackey (Harvard Univ.). Van Nostrand, Princeton, N. J., 1967 iv+266 pp. $3.95 (paper). These lecture notes (only lightly edited from the form in which they were written down and accompanied by a few summaries) were used in an introductory course at Harvard in 1959–1960. Topics include entire and meromorphic functions, conformal mapping, analytic continuation and Riemann surfaces, and algebraic functions and their integrals. There are no excercises, but there is a good deal for students to do in completing the exposition and proofs. T (15–16), S.

Elements of Approximation Theory. By Leopoldo Nachbin (Instituto de Matematica Pura e Aplicada, Brazil) and George Eastman (Univ. of Rochester). Van Nostrand, Princeton, N. J., 1967. xii+119 pp. $2.75 (paper). This volume developed from courses taught at the Universities of Paris and Rochester. "Approximation theory is concerned with the problem of describing the elements of a topological space E that may be approximated by those of a subset X of E, that is, of characterizing the closure of X in E." Included are classical results and recent work by the author. There is a six page bibliography. S, L.

Integrals and Operators. By Irving Segal (MIT) and Ray Kunze (Washington Univ.). McGraw-Hill, New York, 1968. xi+308 pp. $10.50. Intended for a first graduate course in contemporary real analysis, the book covers integration theory, real variable theory, and elementary functional analysis T (17), L.

Partial Differential Equations of Mathematical Physics. By A. N. Tychonov and A. A. Samarski. Two volumes. Translated by S. Radding. Holden-Day, San Francisco, 1964, 1967. Vol. I. 380 pp. $11.75. Vol. II. x+621 pp. $10.75. The first volume treats hyperbolic, parabolic and elliptic differential equations in the two dimensional case. The second covers the generalizations to three dimensions (including spatial wave and heat propagation) and has a long appendix on special functions of mathematical physics. There are problems, references to the literature by the author and at the end of Volume I a bibliography by the editor. T (15–17), S, P, L.

Applications

Handbook of Physics. By E. U. Condon (Univ. of Colorado) and H. Odishaw (National Academy of Sciences). Second edition. McGraw-Hill, New York, 1967. xxix+1680 pp. $32.50. There are 200 pages devoted to mathematics, but the main value of the book to mathematicians is the enormous amount of information about mathematical applications to physics. The book should be in the reference section of every serious mathematical library. P, L.

Foundations of Optimal Control Theory. By Bruce Lee (Univ. of Minnesota) and L. Markus (Univ. of Minnesota). Wiley, New York, 1967. x+576 pp. $17.95. The authors characterize optimal control theory historically as a development of a special topic within the discipline of differential equations. They approach it through the qualitative theory of differential systems, while recognizing that it can also be treated within the framework of the calculus of variations. They believe that the theory has "reached a certain degree of stability and perfection" and intend to give a careful presentation of its current status. The exposition is fairly formal. There are exercises, references and a substantial bibliography. T (16–17), S, P, L.

An Introduction to Linear Programming and the Theory of Games. By S. Vajda. Methuen London, 1966. Distributed in the U.S.A. by Barnes and Noble, New York. 76 pp. $1.25 (paper), $2.50 (cloth). A paperback reprint of this well-known elementary exposition of 1960. Notable for the reasonable price. S, P.

Computers etc.

Cybernetics and Forecasting Techniques. By A. G. Ivakhnenko and V. G. Lapa (both of the Institute of Cybernetics, Ukrainian Academy of Sciences, Kiev). Translation edited by Robert N. McDonough. American Elsevier, New York, 1967. xxvii+168 pp. $13.75. It is interesting that the Russians are explicitly recognizing phenomena that involve "a deterministic part," "a probabilistic part," and "a purely random part, which in principle cannot be predicted." S, P.

Introduction to Automata. By R. J. Nelson (Case Inst. of Technology). Wiley, New York, 1968. xii+400 pp. $12.95. This is an introduction to the main formal mathematical ideas underlying the still amorphous discipline associated with the logic of digital computers, programming, translations, nerve network and so on, called cybernetics by Wiener. The author presupposes some knowledge of switching and sequential circuits as well as propositional logic or Boolean algebra. There are exercises, a bibliography, historical remarks at the end of each chapter. T (16–17), S, P.

Computer Facilities for Mathematics Instruction. By Computer-Oriented Mathematics Committee of the National Council of Teachers of Mathematics. NCTM, Washington, D.C. 1967. v+47 pp. 90¢. This pamphlet refers to secondary schools, but it is relevant for those interested in both computers and teacher training. There is a chapter on computer systems, and another on sample problems. TT, P.

A Comparative Study of Programming Languages. By Bryan Higman. American Elsevier, New York, 1967. 164 pp. $8.50. This may be the first book on programming languages that takes a linguistic approach and analyzes their general properties rather than concentrating on any one language. It may serve as an introduction to specific programming languages, and should be of interest to those concerned with computers, foundations, and linguistics. S, P, L.

Elementary Theory and Application of Numerical Analysis. By David G. Moursund (Michigan State Univ.) and Charles Duris (Michigan State Univ.). McGraw-Hill, New York, 1967. xi+297 pp. $8.95. Designed for a sophomore-junior course for engineers and physical scientists who have had elementary calculus to differential equations, the text assumes some knowledge of FORTRAN (possibly given in a few lectures at the beginning of the course) and covers solution of equations by fixed-point iteration matrix computations, iterative solution of systems of equations, polynomials, series, and interpolations, errors and floating-point arithmetic, numerical differentiation and integration, numerical solution of ordinary differential equations. T (14–15), S.

Education

Qualifications for a College Faculty in Mathematics. Report of the Ad Hoc Committee on the Qualifications of College Teachers of the Committee on the Undergraduate Program in Mathematics. Mathematical Association of America, 1967. 16 pp. Copies may be obtained without charge from CUPM, P.O. Box 1024, Berkeley, California 94701. This report faces the impossibility of staffing undergraduate colleges entirely with Ph. D.'s. It argues that the Ph.D. is not essential in any case and proposes criteria for teacher evaluation and for composition of undergraduate mathematics departments. It should be read by all college teachers of mathematics and by administrators responsible for choosing staff. TT, P, L.

★*The Changing Curriculum: Mathematics.* By Robert B. Davis (Syracuse Univ. and Webster College). Prepared for the Association for Supervision and Curriculum Development, National Education Association, Washington, D.C. 1967. vi+80 pp. $2.00 (paper). This analysis by one of the pioneers and leaders in the reform movement in mathematical education, is concerned primarily with teaching systems and the broader context of mathematical education. It proceeds from the author's view that the new mathematics "thus far has constituted an entirely inadequate response" to contemporary needs. The emphasis is on new methods, and there is thoughtful discussion of contemporary activity. TT, T, L.

Mathématiques Générales. Algèbre-Analyse. By Charles Pisot and Marc Zamansky (both of Faculty of Science of the Univ. of Paris). Dunod, Paris, 1966. Distributed by Gordon and Breach, New York. xxiv+648 pp. $12.00. This text for the first year of university mathematics embodies the Bourbaki program. Its four parts are general notions (logic, sets, functions, relations); algebra (operations, polynomials, vector spaces, matrices); analysis (real numbers, derivative, integral, vector functions, double integrals); mathematical instruments and methods (numerical methods, series, differential equations). P, L.

Geometry and Topology

Complex Manifolds without Potential Theory. By S. S. Chern (Univ. of California, Berkeley). Van Nostrand, Princeton, N. J., 1967. iii+96 pp. $2.25 (paper). This is probably the first published introductory account in English to the theory of complex manifolds where results on elliptic operators are not needed. S,P.

(Note: As often happens nowadays with publications close to the research boundary, classification under algebra, analysis, geometry, or topology would be equally reasonable. We have placed this book here simply because the geometric aspect tends to be overlooked.)

Initiation to Combinatorial Topology. By Maurice Frechet and Ky Fan. Translated from the French, with some notes, by Howard W. Eves. Prindle, Weber & Schmidt, Boston, Mass. 1967. xii+124 pp. $2.95 (paper). Howard Eves is to be congratulated for making available this little classic that appeared in 1946 but is now virtually unavailable. His notes run to 44 pages and include a bibliography of text books on combinatorial topology. S, P, L.

Simplicial Objects in Algebraic Topology. By J. Peter May (Univ. of Chicago). Van Nostrand, Princeton, N. J., 1968. vi+162 pp. $2.95 (paper). Notes derived from a second year graduate course given at Yale in 1965, concentrating on "an investigation of the category of simplicial sets." T (18), S, P.

College Geometry. By Lawrence A. Ringenberg (Eastern Illinois Univ.). Wiley, New York, 1968. xvi+308 pp. $8.95. A well written survey of the foundation ideas and formula-

tions now found in most high school geometry texts. Coordinates in the plane and space are treated briefly. Also included are topics such as cross ratio, pole and polar with regard to circles, and compass and straight edge constructions. T (14).

A Background (Natural, Synthetic and Algebraic) to Geometry. By T. G. Room (Univ. of Sydney). Cambridge Univ. Press, New York, 1968. viii+342 pp. \$10.50. Evolved out of courses given over the last twenty years, this book is by no means another conventional introduction to geometry. The author describes it as woven of three strands: "making congruence respectable," geometrical construction and the theorems corresponding to operations of an algebraic field, and the consequences of prescribing only finitely-many points. The interrelations between observations, model building, and deductive mathematical systems are discussed explicitly (there is a nice diagram, which is reproduced on the dust jacket) and kept in mind throughout. T (15), TT, S, P, L.

History

Ninth Bridgewater Treatise. A Fragment. By Charles Babbage. Second Edition (1838). Reprinted with an index. Frank Cass, London, 1967. xviii+273 pp. £4.10s. This treatise appears quaint to modern eyes, but it is an important document of the times. Among many topics considered are immortality, calculating engines, and probability. P, L.

Travaux sur les Fonctions Réelles et sur les Séries Orthogonales. By Stefan Banach. *Oeuvres*, Volume I. Éditions Scientifiques de Pologne, Warsaw, 1967. 382 pp. \$10.00. The works will be completed by a second volume covering all topics other than real functions and othogonal series. Most of the papers are in French, a few in English or German. P, L.

★*A History of Vector Analysis.* The evolution of the Idea of a Vectorial System. By Michael J. Crowe (Univ. of Notre Dame). Univ. of Notre Dame Press, 1967. xvii +270 pp. \$12.95. This is the most substantial effort anyone has made to write the detailed history of an important branch of modern mathematics. It covers the background in mechanics and complex numbers, the work of Hamilton, Grassman, Gibbs and Heaviside, the "struggle for existence" between quaternions and vectors in the 1890's and the emergence of modern vector analysis early in the 20th century. There is some statistical analysis of the quaternion literature, a detailed chronology, and numerous references to primary sources. P, L.

Ibn al-Muthannâ's Commentary on the Astronomical Tables of al-Khwârizmi. Two Hebrew versions, edited and translated with an astronomical commentary. By Bernard R. Goldstein. Yale University Press, New Haven, 1967. x+408 pp. \$17.50. This commentary was written in the tenth century and describes tables that have since been lost. (Reviewed by H. Woolf in *Science* 5, Jan. 1968). P, L.

Lexicon Technicum or an Universal English Dictionary of Arts and Sciences. By John Harris. A Facsimile of the London Edition of 1704. The Sources of Science, No. 28. Johnson Reprint, New York, 1966. Two volumes. \$98.00. This is an invaluable reference for the state of mathematics and science in general in England at the time. The author attempted to present the latest theories, including those of Newton on calculus and the classification of cubic curves. L.

Scales and Weights. A Historical Outline. By Bruno Kisch. Yale Univ. Press, New Haven, Conn. 1965. xxi+297 pp. \$15.00. This history of weighing and its instruments is beautifully illustrated and covers three millennia up to the present time. P, L.

Technology in Western Civilization. Edited by Melvin Kranzberg, and Carroll W. Pursell. Oxford, New York, 1967. Vol I: xii+802 pp. Vol. II: xii+772 pp. $27.50 (two volumes boxed). There are a few casual references to mathematics in volume I, and in volume II numerous references to the computer and a thirteen page chapter on its origin, but the existence of mathematical technology in a broader sense (including both hardware and software for calculations and solution of problems) is not noticed in spite of the very important role it has played in the history of technology. P, L.

Congruence of Sets and other Monographs. On the Congruence of Sets and their Equivalence by Finite Decomposition. By W. Sierpinski. The Mathematical Theory of the Top. By F. Klein. Graphical Methods. By C. Runge. Introduction to the Theory of Algebraic Equations. By L. E. Dickson. Chelsea, Bronx, New York, 1967. 464 pp. $6.50. Reprints of monographs published originally in 1954, 1897, 1912, and 1903 respectively. This is the fourth time that Chelsea has published a volume containing four unrelated monographs. The practice is excused by claims of greater economy, but there seems to be no reason why such monographs could not be published as small paperbacks for a total price no more than the collection. Moreover, collections of this kind serve to bury the monographs in a library, since there is no unifying theme and one forgets the contents. Leaving aside this regrettable throwing together of unconnected papers, each paper is of both mathematical and historical interest. S, P, L.

History of the Inductive Sciences. By William Whewell. Facsimile reproduction of the third edition of 1857 with an index. Three volumes. Frank Cass, London, 1967. xxxii+394, pp., xii+565 pp., xiv+614 pp. £15.15.0d. This great classic is extraordinary for its conscious neglect of mathematics and the role of deduction in science. The separation of inductive from deductive sciences is about as pernicious as the related separation of pure and applied mathematics. P, L.

Mathematical Concepts, A Historical Approach. By Margaret F. Willerding. Prindle, Weber & Schmidt, Boston, Mass. 1967. ix+126 pp. $2.95 (paper). This is a very brief collection of historical sketches intended for supplementary reading by teachers and teacher trainees learning about the "new" mathematics. It can be useful for this purpose but it is not suitable for the general mathematics major who would do better to read more solid material such as the *Concise History* by Struik. TT, S.

Logic and Foundations

★*Foundations of Constructive Analysis*. By Errett Bishop (Univ. of California, San Diego). McGraw-Hill, New York, 1967. xiii+370 pp. $12.00. The author describes this lively looking book as "a piece of constructivist propaganda, designed to show that there does exist a satisfactory alternative" to the classical foundations of mathematics which were criticized by the intuitionists but not successfully replaced "with something better." He attempts "to develop a large portion of abstract analysis within a constructive framework" according to a simple goal: "to give numerical meaning to as much as possible of classical abstract analysis." This appears to be an important book. T (16–17, for teachers with the pioneering spirit), S, P, L.

Foundations of Real Numbers. By Claude W. Burrill (New York Univ.) McGraw-Hill, New York, 1967. 163 pp. $6.95. The author has written for "the mature undergraduate or beginning graduate student" in order to provide him with "the necessary and sufficient information: a concise development of the real numbers from a foundation of set theory, a proof that the real numbers are unique, and a discussion of the two classical definitions due to Dedekind and to Cantor." The author defines the real numbers as decimals directly in terms of the integers. T (16), S.

Probability and Statistics

Markov Processes and Potential Theory. Proceedings of a Symposium conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison, May 1–3, 1967. Edited by Joshua Chover (Univ. of Wisconsin). Wiley, New York, 1967, x+235 pp. $7.95. P.

Statistical Communication and Detection with Special Reference to Digital Data Processing of Radar and Seismic Signals. By Enders A. Robinson. With foreword by Markus Bath and Anthony F. Gangi. Hafner, New York, 1967. xvii+362 pp. $14.75. P.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Mr. A. F. Baylock, Saint Francis College, has been named Chairman of the Mathematics Department.

Assistant Professor B. A. Fusaro, University of South Florida, has been appointed Professor and Chairman of the Mathematics Department at Queens College, Charlotte.

Associate Professor W. J. Hardell, Worcester Polytechnic Institute, has been promoted to Professor.

Professor S. H. Kimball, University of Maine, died on December 25, 1967. He was a member of the Association for thirty-five years.

Professor Emeritus Rudolph E. Langer, University of Wisconsin, died on March 11, 1968. He was a member of the Association for 41 years and was President for 1949–50. He was author of the first Slaught Paper, "Fourier's Series, The Genesis and Evolution of a Theory."

Professor Charles Loewner, Stanford University, died on January 8, 1968. He was a member of the Association for twenty-two years.

ADVANCED PLACEMENT CONFERENCE IN MATHEMATICS AT LAKE FOREST COLLEGE

The Advanced Placement Conference in Mathematics will be held at Lake Forest College, June 20, 21, and 22, 1968. The featured speakers will be Mr. Charles O'Connell, Dean of Students, University of Chicago; Professor Daniel Finkbeiner, Kenyon College; Professor Ernst Snapper, Dartmouth College; and Professor Gail Young, Tulane University. Much of the program will be devoted to the new syllabi for the advanced placement courses. There will also be panels and small group discussions relating to many phases of the Advanced Placement Program. Inquiries for information about the conference should be addressed to the Department of Mathematics, Lake Forest College, Lake Forest, Illinois 60045.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE NEW JERSEY SECTION

The twelfth annual meeting of the New Jersey Section of the Mathematical Association of America was held at Saint Peter's College on November 4, 1967. Dean A. E. Meder, Jr., Chairman of the section, presided at the morning session and F. A. Brooks, Jr., presided at the afternoon session. One hundred and one persons attended the meeting including 62 members of MAA.

At the business meeting F. A. Brooks, Jr., of Mutual Benefit Life Insurance Co. was elected Chairman and Professor S. L. Grietzer of Rutgers—The State University was elected Member at Large of the Executive Committee ('70).

The following papers were presented during the morning session:

1. *Not quite, but almost*, by T. H. Slook of Temple University (by invitation).

This paper briefly reviews the developments made by J. Bohr, W. Stepanoff, N. Weiner, H. Weyl, A. Besicovitch, J. von Neumann and L. Loomis in the field of almost periodic functions on *CLC* groups and summarizes this research into the structure theorem and the approximation theorem. The research of K. Deleeuw and I. Glicksberg is particularly emphasized in the introduction because this team demonstrated that the above theorems (1) do not in general hold for almost periodic functions defined on *CLC* semigroups and (2) do hold for almost periodic functions defined on certain classes of *CLC* semigroups having an identity. This author shows that the structure and approximation theorems hold for almost periodic functions defined on certain *CLC* semigroups having no identity.

2. *A New Algorithm for Problems of Choice*, by Carlos Fallon of Radio Corporation of America (by invitation).

A row of additive requirements to attain a given objective is related to a column of available courses of action in the form of two orthogonal check lists which constitute the framework of a matrix. Ratings of the contribution, made by each course of action to each choice, constitute the body of the matrix which is then multiplied by a row vector made up of the weighting factors corresponding to each of the requirements. The weighted contributions constitute a system of linear equations which yield a column that rates the relative worth of each course of action. These measures of worth are then multiplied by the nonadditive benefits whose relationship to each other and to the measures of worth is that of a product. The resulting combined measures of worth—additive and nonadditive—are then normalized into shadow dollars and related to their respective cost in order to yield an index of value and a measure of gain or loss for each course of action.

The following paper was presented at the afternoon session:

1. *On Representation of positive integers as sums of arithmetic progressions*, by J. W. Andrushkiw of Seton Hall University (by invitation).

F. A. VARRICHIO, *Secretary-Treasurer*

MATHEMATICAL SCIENCES EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register will again schedule interviews during the summer meeting at The University of Wisconsin. The Register will be located in the Wisconsin Center and will be open from Tuesday, August 27, through Thursday, August 29, from 9:00 A.M. to 5:00 P.M. on each of the three days.

In response to many requests, the Register, which is sponsored by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will have a literature display for employers wishing

to have recruitment literature available to interested applicants. The charge for this service is $15.00 for a single poster or five hundred informational brochures.

Registration for the Employment Register is separate and apart from meeting registration. It is, therefore, most important that both applicants and employers sign in at the Employment Register desk as early as they can on Tuesday morning. A separate visual index will be maintained for Employment Register use only. Appointments will be scheduled *only* for applicants and employers who have actually signed in at the Register.

There is no charge for registration except when the late registration fee of $5.00 is applicable. Provision will be made for anonymity of applicants upon payment of $5.00 to defray the cost involved in handling such a listing.

Applicants and employers who wish to be listed with the Employment Register should write to the Mathematical Sciences Employment Register, Post Office Box 6248, Providence, Rhode Island 02904, for either applicant qualification forms or position description forms. These forms must be completed and returned to the Register not later than July 15, 1968, in order to be included in the August lists. Those forms which arrive too late to be included in the printed lists are taken to the meeting where they may be seen by applicants and/or employers who are interested in them. The printed lists will be mailed to subscribers during the first week in August. Lists can be ordered from the Register office in Providence. They will also be available at the meeting.

A subscription to the lists, which includes three issues (January, May, and August) of both the applicants list and the positions list, is available for $25.00 a year; the individual issues of both lists may be purchased in January, May, and August for $12.50. A subscription to the applicants list alone or single copies of that list are not available. Copies of the positions list only may be purchased for $3.00. Checks should be made payable to the American Mathematical Society and sent to the address given above. The 1968 List of Retired Mathematicians is free upon request and can be obtained from the Employment Register office.

USE OF CEM MATERIALS

The primary purpose of the Committee on Educational Media is the improvement of mathematics in colleges and schools. Therefore the wide propagation of the ideas in CEM materials, especially when thoughtfully reworked by competent authors, is clearly desirable. We encourage the free and widespread use of these ideas in the production of new textbooks. However, because the materials were produced with public funds, the use of CEM materials should be supervised.

To implement these principles, CEM continues to urge authors to improve, adapt, expand on and draw from materials prepared by CEM. Permission to make verbatim use of current and projected CEM materials must be secured from the Treasurer of the MAA. Permission will be granted except in unusual circumstances.

A copy of the regulations governing the use of CEM materials may be obtained from the Treasurer, Professor E. A. Cameron, University of North Carolina, Chapel Hill, N.C. 27514.

CALENDAR OF FUTURE MEETINGS

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

Fifty-second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
FLORIDA
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY, Rutgers—The State University, New Brunswick, November 2, 1968.
NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.
NORTHERN CALIFORNIA, University of Santa Clara, February 8, 1969.
OHIO
OKLAHOMA-ARKANSAS
PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, University of Wisconsin, Madison, August 27–30, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, Las Vegas, Nevada, August 27–29, 1968.

ASSOCIATION FOR SYMBOLIC LOGIC, Warsaw, Poland, August 30–31, 1968.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS, University of Wisconsin, Madison, August 27–28, 1968.

MU ALPHA THETA, Trinity University, San Antonio, Texas, August 11–14, 1968.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cedar Rapids, Iowa, August 22–24, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, November 6–9, 1968.

PI MU EPSILON, University of Wisconsin, Madison, August 27–28, 1968.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS. King Edward Sheraton Hotel, Toronto, Canada, June 11–14, 1968. (Symposium on optimization.)

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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CONTENTS

JUNE–JULY 1968

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor-Elect, HARLEY FLANDERS, American Mathematical Monthly, Purdue University, Lafayette, IN 47907.

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

Articles should be typewritten on good quality paper, triple spaced with wide margins. Submit the original (and a duplicate if convenient) keeping a complete copy for your protection against loss. Please follow the format used in current issues of the MONTHLY.

The manuscript must make the author's intent clear to the printer. A paper which is accepted for publication by the editor may not be acceptable to the printer and will be returned to the author for retyping. Please use the American Mathematical Society *Manual for Authors of Mathematical Papers* as your guide and please conform to correct English usage.

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A PROOF OF MINKOWSKI'S INEQUALITY FOR CONVEX CURVES

HARLEY FLANDERS, Purdue University

1. Introduction. There is a beautiful proof of the isoperimetric inequality in the plane by L. A. Santaló [7, pp. 38–39]. This is based on the following ideas. Given an oval in the plane, move a circle of suitable radius, counting for each position how many times it intersects the oval. The average number of intersections can be computed in two ways, and the result is the isoperimetric inequality with an explicit error term.

This proof of Santaló has been reproduced several times and is fairly well known. What is not so widely known is that the proof may be modified to yield a proof of the Minkowski inequality on mixed areas, and actually yields an improvement of this inequality which is due to Bonnesen. See Blaschke [2, pp. 33–36].

The purpose of this paper is to give an exposition of the theory of closed convex plane curves, mixed area, and the Minkowski inequality. The prerequisites are a slight acquaintance with convex bodies in the plane and the beginnings of the differential geometry of plane curve theory now included in most vector oriented calculus books. In the next section we review the elementary differential geometry of closed convex curves.

In the last section we state the form taken by the Minkowski inequality in three-space. Alas, no analogue of the Santaló proof is known and each of the known proofs is much harder than the one we give shortly for the plane case. Those wishing to explore the subject further will find plenty in the list of the references at the end which includes more than the works cited here.

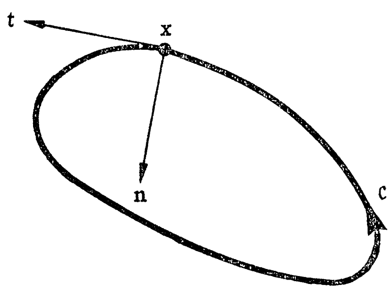


FIG. 1

2. Preliminaries. Let $\mathfrak{c}$ be a smooth closed convex curve in E^2 with positive curvature. Let s be the arc length, $\mathbf{x}=\mathbf{x}(s)$ the moving point in $\mathfrak{c}$, $\boldsymbol{t}=\boldsymbol{t}(s)$ the moving unit tangent, and $\boldsymbol{n}=\boldsymbol{n}(s)$ the moving unit normal (Fig. 1). The Frenet formulas are

$$\frac{d\mathbf{x}}{ds} = \boldsymbol{t}, \qquad \frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{n}, \qquad \frac{d\boldsymbol{n}}{ds} = -\kappa\boldsymbol{t}. \tag{2.1}$$

Here $\kappa=\kappa(s)$ is the curvature which we are assuming satisfies $\kappa>0$ everywhere. If the total length of the curve is L, then x is a periodic vector function of s with fundamental period L and the other functions $\boldsymbol{t}$, $\boldsymbol{n}$, κ also have period L, although not necessarily as a fundamental period. Note that $\boldsymbol{n}$ is the inward drawn normal so that $\boldsymbol{t}$, $\boldsymbol{n}$ is a right-handed frame.

Of course we have

$$L = \oint_{\mathfrak{c}} ds. \tag{2.2}$$

There is also a line integral for the area A enclosed by $\mathfrak{c}$. If we write $d\mathbf{x}=(dx, dy) = \boldsymbol{t}ds$, then a rotation by angle $\pi/2$ leads to

$$\begin{aligned} (-dy,\ dx) &= \boldsymbol{n}ds, \\ (x,\ y)\cdot(-dy,\ dx) &= \mathbf{x}\cdot\boldsymbol{n}ds, \\ y\ dx - x\ dy &= -\ \mathbf{x}\cdot\boldsymbol{n}\ ds. \end{aligned}$$

Because of the well-known relation

$$A = \frac{1}{2}\oint_{\mathfrak{c}} (ydx - xdy),$$

an immediate consequence of Green's Theorem, we have

$$A = -\frac{1}{2}\oint_{\mathfrak{c}} \mathbf{x}\cdot\boldsymbol{n}ds. \tag{2.3}$$

This formula is valid for any simply closed curve and has nothing whatever to do with the convexity. We exploit the fact that $\mathfrak{c}$ is convex by introducing the parameter θ, the angle the outward drawn normal $-\boldsymbol{n}$ makes with the fixed x-axis. It is convenient to take the origin inside $\mathfrak{c}$. Since the curve $\mathfrak{c}$ is turning continuously ($d\boldsymbol{t}/ds=\kappa\boldsymbol{n}$, $\kappa>0$), each point $\mathbf{x}$ of the curve has a unique θ (modulo 2π) associated with it and θ makes a complete circuit, $0\leqq\theta\leqq 2\pi$ as $0\leqq s\leqq L$. (Fig. 2). The *support function* p is the distance of the tangent line at $\mathbf{x}$ from 0.

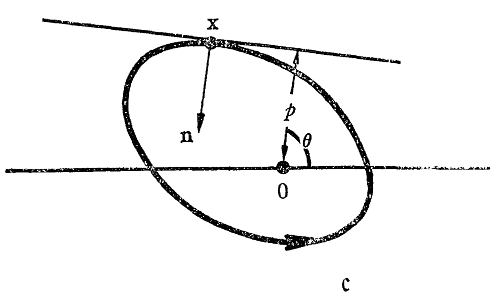


FIG. 2

We write $p=p(\theta)$ and

$$p = -\mathrm{x}\cdot\boldsymbol{n}. \tag{2.4}$$

The function $p=p(\theta)$ has period 2π.

Analytically (this means as usual that, picturesque drawings to the contrary, all vectors start at 0) we have

$$\begin{aligned} \boldsymbol{t} &= (-\sin\theta, \cos\theta), \\ \boldsymbol{n} &= (-\cos\theta, -\sin\theta). \end{aligned} \tag{2.5}$$

The second formula is a direct consequence of the definition of θ and the first follows from this by rotating $\pi/2$.

After this we shall always denote $d/d\theta$ by ($'$). Clearly

$$\boldsymbol{t}' = \boldsymbol{n}, \qquad \boldsymbol{n}' = -\boldsymbol{t}. \tag{2.6}$$

Thus $d\boldsymbol{t}=\boldsymbol{n}d\theta$. But $d\boldsymbol{t}=\kappa\boldsymbol{n}ds$ by (2.1). Hence

$$d\theta = \kappa ds. \tag{2.7}$$

Of course this is the standard interpretation of the curvature at the rate of turning of the tangent. Next we differentiate (2.4): $p'=-\mathrm{x}'\cdot\boldsymbol{n}+\mathrm{x}\cdot\boldsymbol{t}$. Since x' is parallel to $\boldsymbol{t}$ ($\mathrm{x}'=s'd\mathrm{x}/ds=s'\boldsymbol{t}$), $\mathrm{x}'\cdot\boldsymbol{n}=0$ and so

$$\mathrm{x}\cdot\boldsymbol{t} = p'. \tag{2.8}$$

Differentiating again, $\mathrm{x}'\cdot\boldsymbol{t}+\mathrm{x}\cdot\boldsymbol{n}=p''$. But $\mathrm{x}\cdot\boldsymbol{n}=-p$ and $\mathrm{x}'\cdot\boldsymbol{t}=(s'\boldsymbol{t})\cdot\boldsymbol{t}=s'$, hence

$$s' = p + p''. \tag{2.9}$$

But (2.7), $s'=1/\kappa=\rho$, the radius of curvature so this formula may be rewritten

$$\rho = p + p''. \tag{2.9$'$}$$

We now transform the integrals (2.2), (2.3):

$$L = \oint ds = \int_0^{2\pi} \rho d\theta = \int_0^{2\pi} (p + p'')d\theta.$$

Since $p''d\theta=d(p')$ is exact and p' has period 2π, $\oint p''d\theta=0$ and we have $L=\int_0^{2\pi} pd\theta$. For the area we have

$$A = -\frac{1}{2}\oint \mathrm{x}\cdot\boldsymbol{n}ds = \frac{1}{2}\int_0^{2\pi} p(p + p'')d\theta.$$

Since $d(pp')=pp''d\theta+p'^2d\theta$ we have $\int_0^{2\pi} pp''d\theta=-\int_0^{2\pi} p'^2d\theta$. We have obtained

$$L = \int_0^{2\pi} pd\theta, \qquad A = \frac{1}{2}\int_0^{2\pi} (p^2 - (p')^2)d\theta. \tag{2.10}$$

It is worth noting that p determines $\mathfrak{c}$ completely. In fact, by (2.4), (2.8), and (2.5),

$$\mathbf{x} = p'\boldsymbol{t} - p\boldsymbol{n} = (-p' \sin \theta + p \cos \theta, p' \cos \theta + p \sin \theta).$$

Geometrically, the tangents to $\mathfrak{c}$ envelop $\mathfrak{c}$. To say it another way, the convex set K whose boundary is $\mathfrak{c}$ is the intersection of the half-planes including K determined by the lines of support (tangents). This means

$$K = \{v \in E^2 \mid -v\cdot\boldsymbol{n}(\theta) \leqq p(\theta) \text{ for all } \theta\}. \tag{2.11}$$

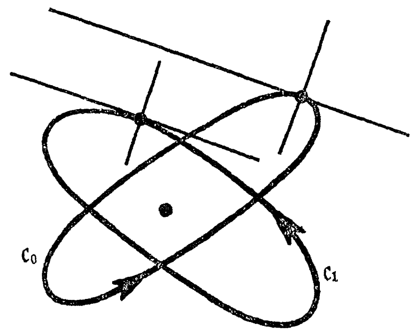


FIG. 3

MIXED AREAS. For this concept we work with two closed convex curves $\mathfrak{c}_0$ and $\mathfrak{c}_1$. Thus we have two vector functions $\mathbf{x}_0 = \mathbf{x}_0(\theta)$, $\mathbf{x}_1 = \mathbf{x}_1(\theta)$ of θ. By writing them this way we automatically set up a one-one correspondence between the curves whereby corresponding points have the same normal $\boldsymbol{n} = \boldsymbol{n}(\theta)$. (See Fig. 3.) The support functions are

$$p_0(\theta) = -\mathbf{x}_0\cdot\boldsymbol{n}, \qquad p_1(\theta) = -\mathbf{x}_1\cdot\boldsymbol{n}.$$

We now study a new convex curve obtained by translating $\mathfrak{c}_1$ to all possible positions such that the origin of $\mathfrak{c}_1$ is on $\mathfrak{c}_0$. To understand this (Fig. 4) we think of $\mathfrak{c}_1$ as the boundary of a rigid lamina which we are free to slide by translations only over the plane in which $\mathfrak{c}_0$ is fixed. We let the origin of $\mathfrak{c}_1$ in this lamina slide along $\mathfrak{c}_0$. Then $\mathfrak{c}_1$ envelops a curve (two curves actually; we take the outside one as illustrated). It is clear that the moving point of contact of this new curve $\mathfrak{c}$ with a particular translate of $\mathfrak{c}_1$ has the same direction as its moving origin has on $\mathfrak{c}_0$. We immediately conclude that our new curve $\mathfrak{c}$ has support function

$$p = p_0 + p_1.$$

Another way to look at this is through the Minkowski sum of convex sets. Let $\mathfrak{c}_0 = \partial K_0$, $\mathfrak{c}_1 = \partial K_1$ so that K_0, K_1 are convex regions bounded by $\mathfrak{c}_0$ and $\mathfrak{c}_1$ respectively. The Minkowski sum is the set K defined by

$$K = K_0 + K_1 = \{v_0 + v_1 \mid v_0 \in K_0, v_1 \in K_1\}. \tag{2.12}$$

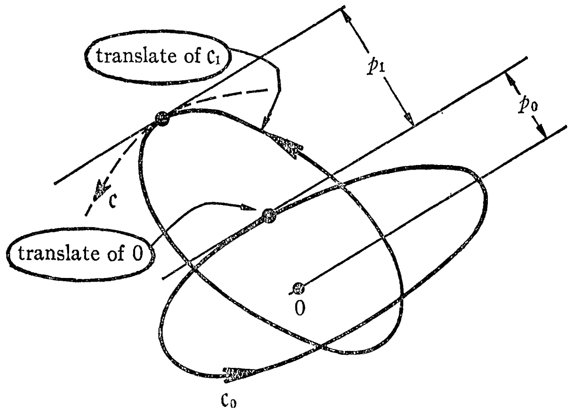


FIG. 4

By (2.11) applied to both K_0 and K_1, if $v_0 \in K_0$ and $v_1 \in K_1$, then

$$-(v_0 + v_1) \cdot n(\theta) = -v_0 \cdot n(\theta) - v_1 \cdot n(\theta) \leqq p_0(\theta) + p_1(\theta).$$

But $\mathrm{x}_0(\theta) \in K_0$, $\mathrm{x}_1(\theta) \in K_1$, hence $\mathrm{x}_0(\theta) + \mathrm{x}_1(\theta) \in K$ and

$$-[\mathrm{x}_0(\theta) + \mathrm{x}_1(\theta)] \cdot n(\theta) = p_0(\theta) + p_1(\theta).$$

This shows that K is the intersection of all the half-planes

$$\{v \mid -v \cdot n(\theta) \leqq p_0(\theta) + p_1(\theta)\}$$

so that K is convex and its boundary $\mathfrak{c} = \partial K$ has support function $p = p_0 + p_1$.

(2.13) LEMMA. *The length and area of* $\mathfrak{c}$ *are given by*

$$L = L_0 + L_1, \qquad A = A_0 + 2A_{01} + A_1,$$

where L_i, A_i *are the length and area of* $\mathfrak{c}_i$ $(i = 0, 1)$ *and* $A_{01} = \frac{1}{2}\int_0^{2\pi}[p_0 p_1 - p_0' p_1']d\theta$ *is the* mixed area *of* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$.

Proof.

$$L = \int_0^{2\pi} p\,d\theta = \int_0^{2\pi} (p_0 + p_1)d\theta = \int_0^{2\pi} p_0\,d\theta + \int_0^{2\pi} p_1\,d\theta = L_0 + L_1.$$

$$\begin{aligned} A &= \tfrac{1}{2}\int_0^{2\pi} (p^2 - p'^2)d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} [(p_0^2 + 2p_0p_1 + p_1^2) - (p_0'^2 + 2p_0'p_1' + p_1'^2)]d\theta \\ &= \tfrac{1}{2}\left\{\int_0^{2\pi} (p_0^2 - p_0'^2)d\theta + 2\int_0^{2\pi} (p_0p_1 - p_0'p_1')d\theta + \int_0^{2\pi} (p_1^2 - p_1'^2)d\theta\right\} \\ &= A_0 + 2A_{01} + A_1. \end{aligned}$$

It is clear that everything in sight is symmetric. If we slide the origin of $\mathfrak{c}_0$ along $\mathfrak{c}_1$ we get the same curve $\mathfrak{c}$ simply because $p_1+p_0=p_0+p_1$. Also $A_{10}=A_{01}$. We may obtain an unsymmetric formula for A_{01} by integrating the exact differential

$$d(p_0p_1') = (p_0'p_1' + p_0p_1'')d\theta.$$

This yields

$$\oint p_0'p_1'\,d\theta = -\oint p_0p_1''\,d\theta.$$

Thus

$$A_{01} = \tfrac{1}{2}\int_0^{2\pi} p_0(p_1 + p_1'')d\theta. \tag{2.14}$$

By (2.9) and (2.9′) we have

$$A_{01} = \tfrac{1}{2}\int_0^{2\pi} p_0\rho_1 d\theta = \tfrac{1}{2}\oint p_0 ds_1, \tag{2.15}$$

and similarly

$$A_{01} = \tfrac{1}{2}\int_0^{2\pi} p_1\rho_0 d\theta = \tfrac{1}{2}\oint p_1 ds_0. \tag{2.16}$$

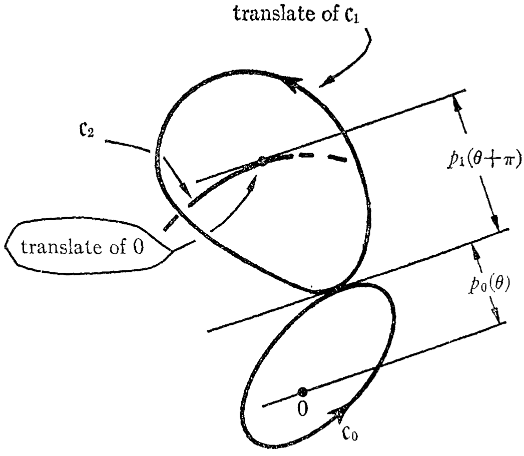


FIG. 5

Instead of translating $\mathfrak{c}_1$ so that the origin moves along $\mathfrak{c}_0$, let us try the following. We translate $\mathfrak{c}_1$ so that it is in contact with $\mathfrak{c}_0$ externally. Then the locus of the translated origin traces a new curve $\mathfrak{c}_2$ (Fig. 5). From the figure, this is a convex curve with support function p_2 given by

$$p_2(\theta) = p_0(\theta) + p_1(\theta + \pi).$$

Now $p_1(\theta+\pi)$ is the support function of the curve $\mathfrak{c}_1^*$ obtained by reflecting $\mathfrak{c}_1$ in the origin or, equivalently, by rotating $\mathfrak{c}_1$ through angle π. Thus the curve $\mathfrak{c}_2$ is obtained simply by applying our previous construction to $\mathfrak{c}_0$ and $\mathfrak{c}_1^*$ and (2.13) applies.

(2.17) LEMMA. *The length and area of* $\mathfrak{c}_2$ *are given by*

$$L_2 = L_0 + L_1, \qquad A_2 = A_0 + 2\overset{*}{A}_{01} + A_1$$

where

$$\overset{*}{A}_{01} = \tfrac{1}{2}\int_0^{2\pi} [p_0(\theta)p_1(\theta+\pi) - p_0'(\theta)p_1'(\theta+\pi)]d\theta$$

$$= \tfrac{1}{2}\oint p_1(\theta+\pi)ds_0(\theta).$$

Proof. It is clear both geometrically and analytically that $\mathfrak{c}_1^*$ and $\mathfrak{c}_1$ have the same length and same area so the formula is a consequence of (2.13). The last expression comes from (2.16) applied to $p_1^*(\theta) = p_1(\theta+\pi)$.

3. The Minkowski inequality. The main result we are after is the Minkowski inequality which is the two-dimensional version of the Brunn-Minkowski inequality. We refer to Bonnesen-Fenchel [**3**, Sect. 49, 51] and Hadwiger [**6**, Chapt. 4] for other treatments of this and more general results.

(3.1) MINKOWSKI INEQUALITY. *If* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$ *are closed convex curves with areas* A_0 *and* A_1 *respectively and mixed area* A_{01}, *then* $A_{01}^2 \geqq A_0A_1$.

Note that in case $\mathfrak{c}_1$ is the unit circle we have $A_1 = \pi$, $p_1 = 1$, and by (2.16)

$$A_{01} = \tfrac{1}{2}\oint ds_0 = \tfrac{1}{2}L_0,$$

so the Minkowski inequality specializes to

$$\tfrac{1}{4}L_0^2 \geqq \pi A_0 \tag{3.2}$$

which is the classical isoperimetric inequality.

We also can state precisely when there is equality in (3.1).

(3.3) SUPPLEMENT. *If* $A_{01}^2 = A_0A_1$ *then* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$ *are homothetic, i.e., they differ by a dilatation and translation.*

Our proof will yield the following stronger form of (3.1) which makes (3.3) obvious. To state it we need the ideas of relative inradius and relative circumradius. The curves $\mathfrak{c}_0$ and $\mathfrak{c}_1$ are given and we write as before $\mathfrak{c}_0 = \partial K_0$, $\mathfrak{c}_1 = \partial K_1$.

The *inradius of* $\mathfrak{c}_0$ *relative to* $\mathfrak{c}_1$ is the largest real number r_0 such that a translate of r_0K_1 is in K_0 (Fig. 6). The *circumradius of* $\mathfrak{c}_0$ *relative to* $\mathfrak{c}_1$ *is the* smallest real number R_0 such that a translate of R_0K_1 contains K_0. Obviously $R_0 \geqq r_0$ with equality if and only if K_0 is a translate of r_0K_1 or, to say it another way, K_0 and K_1 are similar and similarly placed. Note that if $\mathfrak{c}_1$ is the unit circle then r_0 and R_0 are the ordinary inradius and circumradius of $\mathfrak{c}_0$.

(3.4) THEOREM. *Let* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$ *be closed convex curves with areas* A_0 *and* A_1 *respectively and mixed area* A_{01}. *Let* r_0, *resp.* R_0, *be the inradius, resp. circumradius, of* $\mathfrak{c}_0$ *relative to* $\mathfrak{c}_1$. *Then*

$$A_{01}^2 - A_0A_1 \geqq \frac{A_1^2}{4}(R_0 - r_0)^2.$$

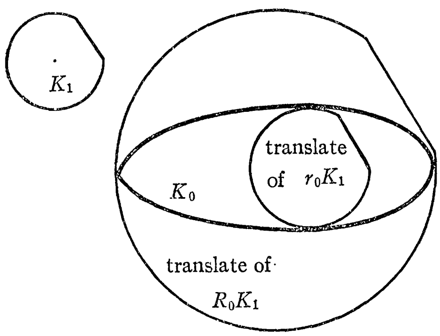
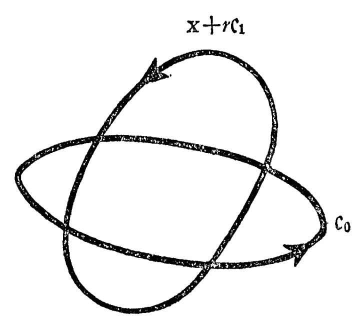


FIG. 6



Example: $N(\mathrm{x})=4$.

FIG. 7

4. The proof. We fix a number r with $r_0 \leqq r \leqq R_0$. For each point $\mathrm{x}=(x, y)$ of the plane we consider the translate $\mathrm{x}+r\mathfrak{c}_1$ of $r\mathfrak{c}_1$. We are interested in the number $N(\mathrm{x})$ of points of intersection of this translate with $\mathfrak{c}_0$. (See Fig. 7.) Thus

$$N(\mathrm{x}) = |(\mathrm{x} + r\mathfrak{c}_1) \cap \mathfrak{c}_0|. \tag{4.1}$$

We shall average this. Precisely, we set

$$I = \iint_{E^2} N(\mathrm{x})dxdy. \tag{4.2}$$

The function $N(\mathrm{x})$ may be infinite for certain values of x such that the curves $\mathfrak{c}_0$ and $\mathrm{x}+r\mathfrak{c}_1$ have a point of tangency. Now the points of x for which there is such a point of tangency lie on two curves which constitute a set of zero area. Consequently it makes no difference what $N(\mathrm{x})$ is equal to on these curves, the integral I is insensitive to these values.

The outside curve (Fig. 8) is precisely the curve $\mathfrak{c}_2$ we studied in Section 2 (see Fig. 5) but with $\mathfrak{c}_1$ replaced by $r\mathfrak{c}_1$. If $\mathbf{x}$ lies outside of this curve then $N(\mathbf{x})=0$. But if $\mathbf{x}$ lies inside of this curve, $N(\mathbf{x})>0$. For otherwise either $\mathbf{x}+r\mathfrak{c}_1$ entirely surrounds $\mathfrak{c}_0$, a contradiction to $r\leqq R_0$, or $\mathfrak{c}_0$ entirely surrounds $\mathbf{x}+r\mathfrak{c}_1$, a contradiction to $r_0\leqq r$.

Actually if we ignore the boundary $\mathfrak{c}_2$ and the other curve ($\mathfrak{c}_0$ and $\mathbf{x}+r\mathfrak{c}_1$ tangent internally) where $N(\mathbf{x})$ may have nasty values we may assert that $N(\mathbf{x})\geqq 2$.

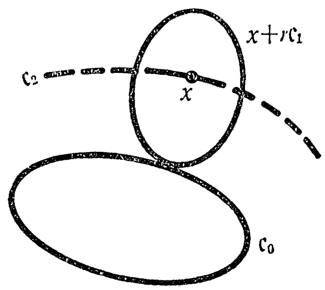


FIG. 8

For when two closed curves intersect at all (and really cross, no common tangents) then they intersect an even number of times. What comes in must go out.

According to (2.17) applied to $\mathfrak{c}_0$ and $r\mathfrak{c}_1$ we have for the area of K_2 where $\mathfrak{c}_2=\partial K_2$,

$$A_2 = A_0 + 2rA_{01}^* + r^2A_1.$$

From these remarks we have

$$I = \iint_{E^2} N(\mathbf{x})dxdy = \iint_{K_2} N(\mathbf{x})dxdy \geqq \iint_{K_2} 2dxdy \geqq 2A_2,$$

$$I \geqq 2(A_0 + 2rA_{01}^* + r^2A_1). \tag{4.3}$$

One should worry why the discrete valued function $N(x)$ has an integral. It is easiest to ignore it, but a word to the wise in the ways of measure theory will be sufficient: the sets where $n(\mathbf{x})=2m$ are open and E^2 is the join of these and a couple of piecewise smooth curves.

The relation (4.3) gives a lower bound for I. We now go after a precise evaluation of I. This is done in three steps. Step 1: we find all translates of $r\mathfrak{c}_1$ passing through a fixed point. Step 2: we find all translates of $r\mathfrak{c}_1$ which intersect a fixed short segment. Step 3: we break up $\mathfrak{c}_0$ into many short segments (or use an inscribed polygon approximation) and sum. Now the details!

If $\mathbf{x}_0$ is a fixed point, the moving origin of $r\mathfrak{c}_1$ traces the oval $\mathbf{x}_0-r\mathfrak{c}_1$ as $r\mathfrak{c}_1$ is translated to all positions such that it passes through $\mathbf{x}_0$, i.e., as $\mathbf{x}_1(\theta)$ is translated

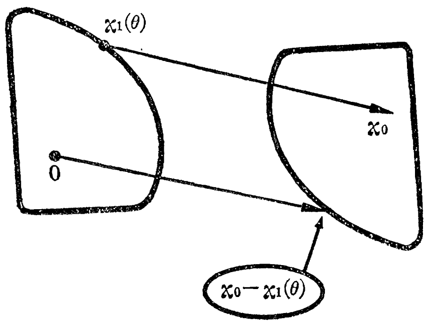


FIG. 9

to $\mathbf{x}_0$, the origin gets translated to $\mathbf{x}_0-\mathbf{x}_1(\theta)$. See Fig. 9.

Now consider a short segment of length ds_0. Think of this as a part of the fixed curve $\mathfrak{c}_0$. Which translates of $r\mathfrak{c}_1$ intersect this segment? Those for which the center of the translate lies in the thin strip swept out by $\mathbf{x}_0-r\mathfrak{c}_1$ as $\mathbf{x}_0$ moves its distance ds_0 along the segment (Fig. 10). Ignoring the small shaded area at

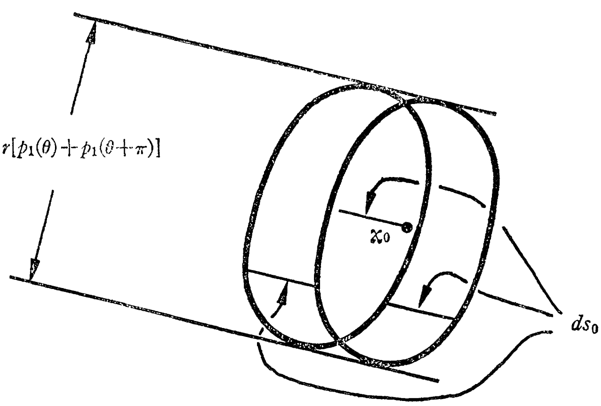


FIG. 10

the end, which is of smaller order of magnitude, the area of each half of this strip is obtained by multiplying its base $r[p_1(\theta)+p_1(\theta+\pi)]$ by its constant height ds_0. Now except for the shaded areas, each point of this strip is the center of a translate of $r\mathfrak{c}_1$ which intersects the segment once. Thus the contribution of this segment to the integral I is

$$2r[p_1(\theta) + p_1(\theta + \pi)]ds_0 + O(ds_0)^2.$$

Summing and passing to the limit we have

$$I = 2r\int_0^{2\pi}[p_1(\theta) + p_1(\theta+\pi)]ds_0. \tag{4.4}$$

By (2.16) and (2.17) we may write this as

$$I = 4r(A_{01} + \overset{*}{A}_{01}), \tag{4.5}$$

which completes our exact evaluation of I.

By (4.3) and (4.5) we have

$$2rA_{01} \geqq A_0 + r^2A_1 \tag{4.6}$$

for all r such that $r_0 \leqq r \leqq R_0$.

Now the existence of a single real number r such that the polynomial $x^2A_1 - 2xA_{01} + A_0$ is less than or equal to zero at $x = r$ implies this polynomial has real roots and hence positive discriminant $A_{01}^2 - A_0A_1 \geqq 0$. This is the Minkowski inequality which is good enough. However, we learn more by exploiting the fact that the polynomial is nonpositive on the whole interval $r_0 \leqq r \leqq R_0$. Simplest is to complete the square to rewrite (4.6) as

$$A_{01}^2 - A_0A_1 \geqq (rA_1 - A_0)^2.$$

We substitute for r the two extreme values and slyly change the sign inside the square:

$$A_{01}^2 - A_0A_1 \geqq (R_0A_1 - A_{01})^2,$$

$$A_{01}^2 - A_0A_1 \geqq (A_{01} - r_0A_1)^2.$$

We average these inequalities and use the elementary fact that the average of the squares is greater than or equal to the square of the average:

$$A_{01}^2 - A_0A_1 \geqq \left\{\tfrac{1}{2}[(R_0A_1 - A_{01}) + (A_{01} - r_0A_1)]\right\}^2,$$

so finally $A_{01}^2 - A_0A_1 \geqq A_1^2(R_0 - r_0)^2/4$.

5. Remarks.

1. If $\mathfrak{c}_1$ is a circle of radius r then $L_1 = 2\pi r$, $A_1 = \pi r^2$, $A_{01} = \frac{1}{2}rL_0$. The curve $\mathfrak{c}$ of (2.13) is the oval parallel to $\mathfrak{c}_0$ at distance r and (2.13) yields Steiner's formulas

$$L = L_0 + 2\pi r, \qquad A = A_0 + rL_0 + \pi r^2.$$

2. If $\mathfrak{c}_1$ is the unit circle, then (3.4) specializes to

$$L_0^2 - 4\pi A_0 \geqq \pi^2(R_0 - r_0)^2$$

which is the inequality of Bonnesen mentioned in the Introduction. This is a striking relation between the length, area, circumradius and inradius of an oval.

3. Of course (3.4) implies $A_{01}^2 \geqq A_0 A_1$. It also implies that if $A_{01}^2 = A_0 A_1$, then $R_0 = r_0$ and (3.3) follows.

4. The estimate (4.3) may be replaced by the more precise

$$I = 2(A_0 + 2r\overset{*}{A}_{01} + r^2 A_1) + 2 \sum_{m=2}^{\infty} (m - 1) f_m,$$

where f_m is the area of the set on which $N(\mathrm{x}) = 2m$. This follows from

$$I = \sum_1^{\infty} 2mf_m \quad \text{and} \quad \sum_1^{\infty} f_m = A_2.$$

It may be used to improve (4.6) and the subsequent inequalities.

6. The situation in space. Let K be a compact convex body in E^3 whose boundary $\mathfrak{S} = \partial K$ is a smooth closed convex surface with positive total curvature at each point. (This means that if n is the outward unit normal at x, then $\mathrm{x} \to n$ is a one-one smooth mapping of $\mathfrak{S}$ onto the unit sphere whose inverse map is also smooth.) In addition to the volume V of K and the surface area A of $\mathfrak{S}$ there is another invariant,

$$M = \iint_S H dA, \tag{6.1}$$

where H is the mean curvature. The isoperimetric inequality in E^3, due to H. A. Schwarz, is

$$A^3 \geqq 36\pi V^2. \tag{6.2}$$

This may be sharpened to

$$A^3 - 36\pi V^2 \geqq [\sqrt{A} - \sqrt{4\pi}\, r_0]^6, \tag{6.3}$$

where r_0 is the inradius. This is enough to show that equality holds in (6.2) only for a sphere. Another refinement of (6.2) consists of the pair of inequalities

$$A^2 \geqq 3MV, \tag{6.4}$$

$$M^2 \geqq 4\pi A. \tag{6.5}$$

These imply both (6.2) and the further result

$$M^3 \geqq 48\pi^2 V. \tag{6.6}$$

These and (6.3) may all be summarized in the assertion that the function

$$f(t) = (V + At + Mt^2 + \tfrac{4}{3}\pi t^3)^{1/3} \tag{6.7}$$

is concave ($f'' \leqq 0$).

Now let K_0, K_1 be two convex bodies of the type considered. The first remarkable fact is that the volume of a linear combination $\lambda K_0 + \mu K_1 (\lambda, \mu \geqq 0)$ is given by a polynomial

$$(6.8) \qquad |\lambda K_0 + \mu K_1| = V_0\lambda^3 + 3V_{001}\lambda^2\mu + 3V_{011}\lambda\mu^2 + V_1\mu^3.$$

This defines the *mixed* volumes V_{001}, V_{011}. The function $f(t)^3$ from (6.7) is the special case $K_0 = K$, $K_1 =$ sphere of radius t. The Brunn-Minkowski inequality is the assertion that

$$(6.9) \qquad |K_0 + tK_1|^{1/3}$$

is a concave function of t, $0 \leqq t$. This has the consequences

$$(6.10) \qquad V_{001}^2 \geqq V_0 V_{011},$$

$$(6.11) \qquad V_{011}^2 \geqq V_{001} V_1,$$

which generalize (6.4) and (6.5) and furthermore

$$(6.12) \qquad V_{001}^3 \geqq V_0^2 V_1,$$

$$(6.13) \qquad V_{011}^3 \geqq V_0 V_1^2$$

which generalize (6.4) and (6.5).

This provides a glimpse into a large and fascinating part of geometry.

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References

1. W. Blaschke, Kreis und Kugel, Veit, Leipzig, 1916 and Chelsea paperback reprint, New York 1949.

2. ———, Vorlesungen über Integralgeometrie I, II, Hamb. Math. Einzel. 20, 22, Teubner, Leipzig, 1936/37 and Chelsea reprint, New York, 1949.

3. T. Bonnesen, W. Fenchel, Theorie der konvexen Körper, Ergeb. der Math., 3 (1934) Springer, Berlin, and Chelsea reprint, New York, 1948.

4. H. Busemann, Convex Surfaces, Interscience, New York, 1958.

5. H. Hadwiger, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer, Berlin, 1957.

6. ———, Altes und Neues über konvexe Körper, Birkhäuser, Basel, 1955.

7. L. A. Santaló, Introduction to integral geometry, A.S.I. 1198, Hermann, Paris, 1953.

8. ———, Integral geometry, M.A.A. Studies in Math., S.S. Chern ed., Studies in global geometry and analysis, 4 (1967) 147–193.

ON A CLASS OF MONOTONE FUNCTIONS GENERATED BY ERGODIC SEQUENCES

HARRY DYM, Rockefeller University

1. Introduction. In this note we shall use some ideas from Information Theory to classify certain monotone functions which arise in the study of D-ary sequences. Our interest in these sequences stems from their role in probability theory as models for games of chance. Thus, for example, each binary sequence may be identified with a sequence of outcomes of a spun coin. In the most interesting cases the monotone functions are singular continuous, akin to the Cantor function. We defer, however, for the time being a more complete discussion of results in order to develop first some notation and to summarize briefly some relevant background material.

Let Ω denote the collection of all sequences $\omega=(\omega_1, \omega_2, \cdots)$ whose entries ω_i are drawn from the finite set of nonnegative integers $\{0, 1, \cdots, D-1\}$. Let $\mathcal{B}$ denote the Borel field generated by sets of the form $(\omega\colon \omega_1=a_1, \omega_2=a_2, \cdots, \omega_n=a_n)$, $n=1, 2, \cdots$ where the $a_i \in \{0, 1, \cdots, D-1\}$ and let P be measure on $\mathcal{B}$ such that $P(\Omega)=1$. It is then customary to refer to P as a probability measure, to call $P(B)$ the probability of the event (i.e. measurable set) B, and to term the sequence of measurable functions

$$X_n\colon \omega \in \Omega \to \omega_n \in \{0, 1, \cdots, D-1\}, \qquad n = 1, 2, \cdots,$$

as random variables. To avoid trivialities we shall assume that $D \geqq 2$ and $P(\omega\colon X_1(\omega)=i)>0$ for $i=0, 1, \cdots, D-1$.

We now introduce the so called shift transformation

$$T\colon (\omega_1, \omega_2, \cdots) \in \Omega \to (\omega_2, \omega_3, \cdots) \in \Omega,$$

that is to say $(T\omega)_k=\omega_{k+1}$, $k=1, 2, \cdots$. Note that $T^{-1}B=(\omega\colon T\omega \in B)$ belongs to $\mathcal{B}$ if B does. We shall assume that $P(T^{-1}B)=P(B)$ for every $B\in\mathcal{B}$. This is equivalent to assuming that the sequence of random variables X_n is stationary, i.e.,

$$\begin{aligned} P(\omega\colon X_{1+k}(\omega) &= a_1, X_{2+k}(\omega) = a_2, \cdots, X_{n+k}(\omega) = a_n) \\ &= P(\omega\colon X_1(\omega) = a_1, X_2(\omega) = a_2, \cdots, X_n(\omega) = a_n) \end{aligned}$$

for every choice of the positive integers k, n. A set $B\in\mathcal{B}$ is said to be invariant (relative to T) if B and $T^{-1}B$ differ by at most a set of P measure zero. We shall also make the assumption that T is metrically transitive, i.e., that the only invariant sets of $\mathcal{B}$ have either P measure zero or P measure one. This is equivalent to assuming that the stationary sequence of random variables X_n is also ergodic. For more details on these and related questions the reader is referred to Rosenblatt [10] which is a good reference book for the level of probability theory used in this paper.

The random variable

$$Z = \sum_{i=1}^{\infty} X_i D^{-i}$$

may be used to transfer the probability structure of the sequence of X_i to the unit interval; each point $\omega \in \Omega$ is mapped by Z into a point $Z(\omega)$ satisfying $0 \leqq Z(\omega) \leqq 1$. In this note we study the distribution function of Z, $F(a) = P(\omega\colon Z(\omega) < a)$. We show (Theorem 1) that either F is a step function with k jumps of height $1/k$, or F is continuous and purely singular, or else $F(a) = a$; $F(a) = a$ if and only if the random variables X_i are independent and uniformly distributed. Harris [5] has proved an analogous theorem using different techniques. Our method of proof utilizes Information Theory and illustrates some interesting relationships between the entropy rate H of the random sequence and the form of the distribution function. In particular, H is maximal; that is $H = \log D = 1$ (all logarithms will be taken to the base D), if and only if $F(a) = a$. If $0 < H < 1$ then F is purely singular and continuous. If F is a step function then $H = 0$. There exist ergodic sequences for which $H = 0$ and F is continuous and purely singular as the following example (which the reader may defer), suggested by L. Shepp, illustrates:

Let $\theta_n = (\theta_0 + \beta_n)$ (modulo 1), $n = 1, 2, \cdots$, where β is an irrational number and θ_0 is uniformly distributed over $[0, 1)$. Let I be a subinterval of $(0, 1)$ and let $\psi_n = 1$ if $\theta_n \in I$ and 0 otherwise. Then ψ_n is a stationary ergodic sequence. Moreover, since the number of distinct sequences of length n $(\psi_1, \cdots, \psi_n)$ which can occur is no greater than $2n$ it follows that the entropy rate of the random ψ_n sequence is zero.

In the proof of Theorem 1 we define a set E which is in some sense the support of the distribution function F when it is regarded as a measure. Thus if F is a step function E consists of the points of discontinuity of F; if $F(a) = a$ then $E = [0, 1]$; if F is continuous and purely singular $\mu(E) = 0$ but $\mu(F(E)) = 1$ (μ is Lebesgue measure).

In section 4 we establish Holder conditions on F and show that the Hausdorff dimension of the set E is equal to H. The basic ideas here are due to Kinney [7] who proved similar results under the extra assumption that the ergodic chain is also a Markov chain. Indeed this whole paper arose out of my own attempts to gain a better understanding of [7]. The fact that the Hausdorff dimension of E is equal to H is also a consequence of a general theorem due to Billingsley [2].

2. Some information theoretic preliminaries. We review briefly some pertinent facts from Information Theory. It will be convenient to introduce the symbol $P(a_1, \cdots, a_n)$ as a shorthand notation for $P(\omega\colon X_1(\omega) = a_1, \cdots, X_n(\omega) = a_n)$ where each $a_i \in \{0, 1, \cdots, D-1\}$ for $i = 1, \cdots, n$. Also, as noted earlier, all logarithms will be taken to the base D.

To the process $[P, X_n: n \geqq 1]$ we associate a set of nonnegative numbers:

$$H_n = -\sum_{a_1,\cdots,a_n}^{*} P(a_1, \cdots, a_n) \log P(a_1, \cdots, a_n).$$

We mean

$$\sum_{a_1,\cdots,a_n}^{*}$$

to signify that the summation is carried out only over those n-tuples which have a positive probability of occurrence; the number of summands is clearly $\leqq D^n$. The numbers H_n which are termed (n-fold) entropies satisfy the following simple inequalities:

(a) $H_1 \leqq \log D = 1$
$H_1 = 1$ if and only if $P(\omega: X_1(\omega) = j) = 1/D$ for $j = 0, \cdots, D-1$.

(b) $H_1 \geqq H_n - H_{n-1} \geqq H_{n+1} - H_n \quad n = 2, 3, \cdots$
$H_1 = H_n - H_{n-1}$ for all $n \geqq 2$ if and only if the random variables $X_1, X_2, \cdots$ are independent.

(c) $H_{n+1} - H_n \geqq 0$
$H_{n+1} - H_n = 0$ if and only if $P(a_1, \cdots, a_{n+1}) \neq 0$ implies that $P(a_1, \cdots, a_{n+1}) = P(a_1, \cdots, a_n)$.

These inequalities may be derived by judicious application of the following inequality which is valid for $x \geqq 0$: $\log x \leqq (x-1)/\ln D$; $\log x = (x-1)/\ln D$ if and only if $x = 1$ where $\ln = \log_e$. For purposes of illustration we sketch the proof of (b).

$$\begin{aligned} H_{n+1} + H_{n-1} - 2H_n &= \sum_{a_1,\ldots,a_{n+1}}^{*} P(a_1, \cdots, a_{n+1}) \log \frac{P(a_1, \cdots, a_n)P(a_2, \cdots, a_{n+1})}{P(a_1, \cdots, a_{n+1})P(a_2, \cdots, a_n)} \\ &\leqq \frac{1}{\ln D} \sum_{a_1,\cdots,a_{n+1}}^{*} P(a_1, \cdots, a_{n+1}) \left[\frac{P(a_1, \cdots, a_n)P(a_2, \cdots, a_{n+1})}{P(a_1, \cdots, a_{n+1})P(a_2, \cdots, a_n)} - 1\right] \\ &= \frac{1}{\ln D}\left[\sum_{a_2,\cdots,a_{n+1}}^{*} \frac{P(a_2, \cdots, a_{n+1})}{P(a_2, \cdots, a_n)} \sum_{a_1}^{*} P(a_1, \cdots, a_n) - 1\right] \\ &= \frac{1}{\ln D}\left[\sum_{a_2,\cdots,a_{n+1}}^{*} P(a_2, \cdots, a_{n+1}) - 1\right] = 0. \end{aligned}$$

In order for equality to hold throughout it is necessary and sufficient that $P(a_1, \cdots, a_{n+1}) \neq 0$ imply

$$P(a_1, a_2) = P(a_1)P(a_2)$$
$$P(a_1, \cdots, a_{n+1})P(a_2, \cdots, a_n) = P(a_1, \cdots, a_n)P(a_2, \cdots, a_{n+1}) (n \geqq 2).$$

This is, however, a necessary and sufficient condition for the random variables $X_1, X_2, \cdots$ to be independent.

From (a), (b) and (c) we deduce that the sequence $\{(H_n - H_{n-1})\}$ is monotone

nonincreasing, and bounded between 0 and 1. Thus the sequence has a limit which we designate by H:

$$H = \lim_{n\to\infty} (H_n - H_{n-1}).$$

Since

$$\lim_{n\to\infty} (H_n - H_{n-1}) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=2}^{n} (H_j - H_{j-1})$$

it follows readily that $H = \lim_{n\to\infty} H_n/n$. Accordingly we term H the entropy rate of the process.

REMARKS: (1) The results of this section are valid for sequences which are stationary but not ergodic.

(2) Observe that $0 \leqq H \leqq 1$; $H=1$ if and only if the random variables X_i are both independent and uniformly distributed.

3. The form of the distribution function.

LEMMA 1. *Suppose there exists a point $u \in \Omega$ such that $P(u) = \epsilon > 0$. Then $\epsilon = 1/k$ for some positive integer k, F is a step function with k jumps of height $1/k$ and $H=0$.*

Proof. If $P(u) = \epsilon > 0$ then $P(T^{-1}u) = P(T^{-2}u) = \cdots = \epsilon$ also. Thus, as $P(\Omega) = 1$ the sequence of sets $u, T^{-1}u, T^{-2}u, \cdots$ must contain some overlaps. In fact there must be a smallest integer $k \geqq 1$ such that $u \in T^{-k}u$. But this implies that the set $(u, T^{-1}u, \cdots, T^{-(k-1)}u)$ is invariant and so must have probability one. Moreover the entries in the sequence u have periodicity k: $u_{n+k} = u_n$, $n = 1, 2, \cdots$. It follows readily that $\epsilon = 1/k$, that the distinct points in Ω with nonzero probability are $u, Tu, \cdots, T^{k-1}u$, and that $P(u) = P(Tu) = \cdots = P(T^{k-1}u) = 1/k$. For $n \geqq k$ $H_n = \log k$ therefore

$$H = \lim_{n\to\infty} \frac{H_n}{n} = 0.$$

LEMMA 2. *If $P(\omega) = 0$ for every $\omega \in \Omega$ then F is continuous.*

Proof. A probability measure is countably additive. Consequently $P(\omega\colon Z(\omega) \leqq a)$ is a right continuous function of a and $P(\omega\colon Z(\omega) < a)$ is a left continuous function of a. Thus $P(\omega\colon Z(\omega) \leqq a) - P(\omega\colon Z(\omega) < a) = P(\omega\colon Z(\omega) = a) = 0$ since there are at most two points in Ω which can be mapped by Z into a and each of these has 0 probability.

REMARK (3) F will be strictly monotone increasing if and only if $P(a_1, \cdots, a_n) > 0$ for every finite sequence $a_1, \cdots, a_n$ of elements chosen from $\{0, 1, \cdots, D-1\}$.

LEMMA 3. *The random variables X_i are independent and uniformly distributed if and only if $H=1$. In this case $F(a) = a$.*

Proof. If the random variables X_i are independent and uniformly distributed then $F(g)=g$ at each D-adic rational point, g. The D-adic rational points are dense in the unit interval. Hence, as F is monotone, $F(a)=a$ for all $0\leqq a\leqq 1$. It was observed in Remark 2 that $H=1$ if and only if the X_i are independent and uniformly distributed.

THEOREM 1. *The distribution function F is one of the following 3 types:*
(1) *a step function with k jumps of height $1/k$ where k is a positive integer;*
(2) *continuous and purely singular;*
(3) $F(a)=a,\quad 0\leqq a\leqq 1$.
F is of type (3) if and only if $H=1$. If F is type (1) then $H=0$. If $H>0$ then F is continuous. (Thus if $0<H<1$ then F is type (2).)

Proof. It follows from Lemmas 1 and 2 that either F is type (1), in which case $H=0$, or F is continuous. If F is continuous and $H=1$ then, by Lemma 3, $F(a)=a$. It remains to show that if F is continuous and $0\leqq H<1$ then F is purely singular. It suffices to exhibit a set E for which $\mu(E)=0$ and $\mu(F(E))=1$.

Given any $x\in[0,1)$ and any $i\geqq 1$ we choose the unique pair of numbers $\zeta_i=\zeta_i(x)>0$ and $\eta_i=\eta_i(x)\geqq 0$ such that $\zeta_i(x)+\eta_i(x)=D^{-i}$ and $D^i(x-\eta_i(x))$ and $D^i(x+\zeta_i(x))$ are both nonnegative integers.

Consider the set

$$E=\left\{x:\lim_{i\to\infty}\frac{\log[F(x+\zeta_i)-F(x-\eta_i)]}{\log[\zeta_i+\eta_i]}=H\right\}$$

and note that for each $x\in E$ and each $\epsilon>0$ there exists a number $N(x,\epsilon)$ such that if $i>N(x,\epsilon)$ then

$$D^{-i(H+\epsilon)}\leqq F(x+\zeta_i)-F(x-\eta_i)\leqq D^{-i(H-\epsilon)}.$$

If $H<1$ we can choose $\epsilon>0$ so that $H+\epsilon<1$.

Clearly then for each $x\in E$ and $i>N$

$$\frac{F(x+\zeta_i)-F(x-\eta_i)}{\zeta_i+\eta_i}\geqq D^{i(1-H-\epsilon)}\to\infty.$$

Since F is a bounded monotone increasing function it must have a finite derivative almost everywhere in the sense of Lebesgue. (See McShane [**9**], Thm 34.2, p. 202.) Thus $\mu(E)=0$.

For $u\in\Omega$ set $[u]_n=\{\omega\in\Omega:\omega_1=u_1,\cdots,\omega_n=u_n\}$ and let

$$\Omega^*=\left\{\omega:\lim_{n\to\infty}\frac{-\log P([\omega]_n)}{n}=H\right\}.$$

By the strong form of McMillan's theorem (Breiman [**3**] and [**4**]) $P(\Omega^*)=1$. The identity

$$F(Z(\omega)+\zeta_i)-F(Z(\omega)-\eta_i)=P([\omega]_i)$$

implies that if $\omega \in \Omega^*$ then $Z(\omega) \in E$. Thus

$$\mu(F(E)) = P(\omega: Z(\omega) \in E) \geqq P(\Omega^*) = 1.$$

McMillan's theorem also implies that if $H>0$ then $P(\omega)=0$ for every $\omega \in \Omega$. In this case then, by Lemma 2, F must be continuous.

4. On the Hausdorff Dimension of E. In this section we show that the Hausdorff dimension of the set E constructed in Theorem 1 is equal to H. This is in fact an immediate consequence of a general theorem due to Billingsley [2, Thm. 2.4]. We shall give a direct proof, however, establishing in the process Holder conditions on F. The techniques used are similar to those of Kinney [7] and Kinney and Pitcher [8].

Actually Lemmas 4 and 5 may be classified as technical in the sense that their implications seem, at first glance, almost obvious from the definitions of the set E; yet the proofs, particularly that of Lemma 5, are somewhat involved. The difficulty arises from the fact that in the original definition of the set E the quantities ζ_i and η_i do not approach zero symmetrically.

LEMMA 4. *If $x \in E$ then for any $\alpha>0$*

$$\lim_{h\to 0} [F(x+h) - F(x-h)](2h)^{-(H+\alpha)} = \infty.$$

Proof. Let $\delta>0$ be chosen smaller than α. Then, if $D^{-(i+1)} \leqq h \leqq D^{-i}$, $[F(x+h) - F(x-h)](2h)^{-(H+\alpha)} \geqq 2^{-(H+\alpha)}[F(x+\zeta_{i+1}) - F(x-\eta_{i+1})]h^{-(H+\alpha)}$ which, for small enough h,

$$\begin{aligned} &\geqq 2^{-(H+\alpha)} D^{-(i+1)(H+\delta)} h^{-(H+\alpha)} \\ &= 2^{-(H+\alpha)} D^{-(H+\delta)} h^{-(\alpha-\delta)} \to \infty \quad \text{as} \quad h \to 0. \end{aligned}$$

LEMMA 5. *For every $\alpha>0$ there exists a set $A \subset E$ such that $\mu(F(A))=1$ and for every x in A*

$$\lim_{h\to 0} [F(x+h) - F(x-h)](2h)^{-H+\alpha} = 0.$$

Proof. It suffices, given an arbitrary $\epsilon>0$, to exhibit a set A with $\mu(F(A)) > 1-\epsilon$ whose elements satisfy a Holder condition of the above type. Choose a positive $\delta<\alpha$ and let B_n denote the set of points $x \in [0, 1]$ for which

$$F(x+\zeta_i) - F(x-\eta_i) > (\zeta_i+\eta_i)^{H-\delta}$$

for some choice of $i>n$. Then $B=\bigcap B_n$ is precisely the set of points x for which this inequality holds for infinitely many i. Let C_n be a covering of B_n made up of a countable collection of intervals each of length $<3D^{-n}$, containing at least one point of B_n, and so chosen that for every $x \in B_n$ the interval

$$[x - 2\eta_i - \zeta_i, x + \eta_i + 2\zeta_i] \subset C_n$$

where i is the smallest integer exceeding n for which

$$F(x+\zeta_i)-F(x-\eta_i)>(\zeta_i+\eta_i)^{H-\delta}.$$

Since $\mu(F(B))=0$ we can choose n so large that $\mu(F(C_n))<\epsilon$. Set $A=E-C_n$. Clearly $\mu(F(A))>1-\epsilon$. Moreover, for $x\in A$ there exists an integer $t>n$ such that for all $m\geqq t$

$$F(x+\zeta_m)-F(x-\eta_m)\leqq(\zeta_m+\eta_m)^{H-\delta}$$

and $F(x+2\zeta_m+\eta_m)-F(x-\zeta_m-2\eta_m)\leqq 3(\zeta_m+\eta_m)^{H-\delta}$. The desired Holder condition for such points x follows easily since $\delta<\alpha$.

REMARK (4). Let $\{\alpha_j\}$ be a sequence of positive numbers which decrease monotonically to zero. Let A_j be the set of $x\in E$ for which $F(x+h)-F(x-h)<(2h)^{H-\alpha_j}$ for all sufficiently small positive h. By Lemma 5, $\mu(F(A_j))=1$. Clearly the sequence of sets A_j decrease monotonically to a set A with $\mu(F(A))=1$ and if $x\in A$ then $F(x+h)-F(x-h)<(2h)^{H-\gamma}$ for any $\gamma>0$ and all sufficiently small $h>0$.

We now recall some definitions. The γ dimensional Hausdorff measure of a set $A\subset[0, 1]$ is defined by $\Gamma(\gamma; A)=\lim_{\delta\to 0}\{\inf\sum\mu(I_j)^\gamma\}$ where the inf is taken over all countable sets of intervals $\{I_j\}$ such that $\bigcup I_j\supset A$ and $\mu(I_j)<\delta$.

It is easy to check that $\Gamma(\gamma, A)$ is monotone nonincreasing in γ and moreover that if $\Gamma(\gamma, A)<\infty$ then $\Gamma(\gamma+\epsilon, A)=0$ for every $\epsilon>0$. The Hausdorff dimension of the set A, $\beta(A)$, is the number with the property that for any $\epsilon>0$

$$\Gamma[\beta(A)-\epsilon, A]=\infty \qquad \Gamma[\beta(A)+\epsilon, A]=0.$$

That is to say

$$\beta(A)=\inf\{\gamma:\Gamma(\gamma, A)=0\}=\sup\{\gamma:\Gamma(\gamma, A)=\infty\}.$$

Roughly speaking, the Hausdorff dimension is useful as a measure (in the heuristic sense) of the size of noncountable sets which have Lebesgue measure zero. Thus β is a monotone set function, i.e. if $A_2\supset A_1$ then $\beta(A_2)\geqq\beta(A_1)$, such that if A is countable $\beta(A)=0$ whereas if $\mu(A)>0$ then $\beta(A)=1$. The Hausdorff dimension of the Cantor set, for example, is equal to log 2/log 3.

We now prove

THEOREM 2. *If $E^*\subset E$ and $\mu(F(E^*))>0$ then $\beta(E^*)=H$.*

Proof. Given any fixed $\epsilon>0$ there exists for each $x\in E^*$ a smallest $i\geqq n$ such that $F(x+\zeta_i)-F(x-\eta_i)>[\zeta_i+\eta_i]^{H+\epsilon}$. We can thus construct a countable set of mutually disjoint intervals $\{I_i^{(n)}\}$ $i=1, 2, \cdots$ each of length $\leqq D^{-n}$ and so chosen that

$$\bigcup_i I_i^{(n)}\supset E^* \quad\text{and}\quad \mu(F(I_i^{(n)}))\geqq\mu(I_i)^{H+\epsilon}.$$

Thus

$$1\geqq\mu\left(F\left(\bigcup_i I_i^{(n)}\right)\right)\geqq\sum_i(\mu(I_i^{(n)}))^{H+\epsilon}.$$

Hence $\Gamma(H+\epsilon, E^*) \leqq 1$ and $\beta(E^*) \leqq H$.

To prove the inequality in the other direction fix $\epsilon > 0$ and choose $h_0 > 0$ and a set $A(h_0) \subset E^*$ so that $\mu(F(A(h_0))) \geqq \delta > 0$ and for each $x \in A(h_0)$ and every $h \leqq h_0$ $F(x+h) - F(x-h) \leqq (2h)^{H-\epsilon}$. Now cover $A(h_0)$ with a set of intervals I_i, each chosen with length $< h_0$. We can assume that each interval I_i contains at least one point, $x_i \in A(h_0)$. Let J_i be the smallest interval symmetric about x_i which contains I_i. Then $\mu(J_i) \leqq 2\mu(I_i)$ and correspondingly

$$\sum (2\mu(I_i))^{H-\epsilon} \geqq \sum (\mu(J_i))^{H-\epsilon} \geqq \sum \mu(F(J_i)) \geqq \mu(F(A(h_0))) \geqq \delta.$$

Since the chosen covering of $A(h_0)$ was arbitrary, aside from the fact that the length of each interval was constrained to be smaller than $h_0, \Gamma(H-\epsilon, A(h_0)) \geqq \delta/2$. Hence $\beta(A(h_0)) \geqq H$. But $E^* \supset A(h_0)$ thus $\beta(E^*) \geqq H$.

REMARK (5). The proof as presented is valid for $0 \leqq H \leqq 1$. The effort expended is, however, only really needed for distribution functions of the second type. For if F is type (1) the set E consists of the k jump points and if F is type (3) $E = [0, 1]$. The Hausdorff dimension of a finite or even countable set of points is 0, whereas the Hausdorff dimension of a bounded set of positive Lebesgue measure is 1.

The reader who finds this material of interest will surely enjoy the recent book by Billingsley *Ergodic Theory and Information*, Wiley, 1965.

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References

1. P. Billingsley, Hausdorff Dimension in probability theory, Illinois J. Math., 4 (1960) 187–209.
2. ———, Hausdorff Dimension in probability theory, II, *Ibid.*, 5 (1961) 291–298.
3. L. Breiman, The individual ergodic theorem of information theory, Ann. Math. Statist., 28 (1957) 809–811.
4. ———, Correction to the individual ergodic theorem of information theory, Ann. Math. Statist., 31 (1960) 809–810.
5. T. E. Harris, On chains of infinite order, Pacific J. Math., 5 (1955) 707–724.
6. P. Hartman and R. Kershner, The structure of monotone functions, Amer. J. Math., 59 (1937) 809–822.
7. J. R. Kinney, Singular functions associated with Markov Chains, Proc. Amer. Math. Soc., 9 (1958) 603–608.
8. J. R. Kinney and T. S. Pitcher, The dimension of the support of a random distribution function, Bull. A.M.S., 70 (1964) 161–164.
9. E. J. McShane, Integration, Princeton Univ. Press, 1944.
10. M. Rosenblatt, Random Processes, Oxford Univ. Press, 1962.

NOTES ON PERFECTNESS AND TOTAL DISCONNECTEDNESS

H. E. LACEY and JACK HARDY, University of Texas

The purpose of this paper is to study the concepts of total disconnectedness and perfectness in the light of a common third property: collections of disjoint pairs of closed sets. Throughout this paper X represents a compact Hausdorff space and A and B represent disjoint closed subsets of X.

1. DEFINITION. Let X be a compact Hausdorff space.

(a) A family $\{A_i, B_i\}_{i\in I}$ of nonempty closed sets in X with $A_i\cap B_i=\varnothing$ for each $i\in I$ is said to be a *separation* if $X=A_i\cup B_i$ for all $i\in I$.

(b) A family $\{A_i, B_i\}_{i\in I}$ of nonempty closed sets in X with $A_i\cap B_i=\varnothing$ for each $i\in I$ is said to *separate points* if for each pair of distinct points x, y in X there is an $i\in I$ with $x\in A_i$ and $y\in B_i$ or $x\in B_i$ and $y\in A_i$.

(c) A family $\{A_i, B_i\}_{i\in I}$ of nonempty closed sets in X with $A_i\cap B_i=\varnothing$ for each $i\in I$ is said to be *interlocking* if for each finite set $J\subseteq I$ and each collection $\{\epsilon_j\}_{j\in J}$ where $\epsilon_j\in\{-1, 1\}$, $\bigcap_{j\in J}\epsilon_jA_j\neq\varnothing$, $1\cdot A_j=A_j$ and $-1\cdot A_j=B_j$ [**3**, p. 39].

Clearly, a space X has a separation if and only if it is disconnected.

2. PROPOSITION. *Let X be a compact Hausdorff space. Then X has a separation which separates the points of X if and only if it is totally disconnected.*

Proof. If X is totally disconnected, then it is zero dimensional, and thus there is a neighborhood base at each point consisting of closed-open sets. If $\{A_{i,x}\}_{i\in I_x}$ is an indexing of such a neighborhood base at X then

$$B_{i,x} = X\backslash A_{i,x}, \quad \text{and} \quad \{A_{i,x}, B_{i,x}\}_{x\in X, i\in I_x}$$

is a separation of X which separates the points of X.

Conversely, if $\{A_i, B_i\}_{i\in I}$ is a separation of X that separates the points of X and C is a connected set in X, then C is a single point.

3. PROPOSITION. *Let X be a compact Hausdorff space. Then X has an interlocking separation if and only if there is a continuous mapping of X onto $\{-1, 1\}^I$ for some nonempty set I.*

Proof. Suppose f is a continuous mapping of X onto $\{-1, 1\}^I$ for some nonempty set I. Let $\{A_i, B_i\}$ be the standard interlocking separation of $\{-1, 1\}^I$. Then $A_i'=f^{-1}(A_i)$ and $B_i'=f^{-1}(B_i)$ clearly defines an interlocking separation of X.

Conversely, suppose that $\{A_i, B_i\}_{i\in I}$ is an interlocking separation of X. A mapping f on X to $\{-1, 1\}^I$ is defined by $f(x)=\{\epsilon_i\}_{i\in I}$ where $\epsilon_i=1$ if $x\in A_i$, $\epsilon_i=-1$ if $x\in B_i$. Clearly, f is continuous since A_i, B_i are disjoint closed-open sets. Let $\{\epsilon_i\}_{i\in I}$ be an element of $\{-1, 1\}^I$. Since $\{\epsilon_iA_i\}_{i\in I}$ has the finite intersection property and X is compact, $\bigcap_{i\in I}\epsilon_iA_i\neq\varnothing$. For any x in the intersection, $f(x)$

$=\{\epsilon_i\}_{i\in I}$. Hence f is onto $\{-1, 1\}^I$. Combining the above ideas, one obtains the following.

4. PROPOSITION. *Let X be a compact Hausdorff space. Then X is homeomorphic to $\{-1, 1\}^I$ for some nonempty set I if and only if there is an interlocking separation of X which separates the points of X.*

Proof. If X is homeomorphic to $\{-1, 1\}^I$ for some nonempty set I, then by the above remarks X has an interlocking separation which separates the points of X.

Suppose X has an interlocking separation which separates the points of X then by Proposition 3 there is a mapping of X onto $\{-1, 1\}^I$. Since the separation separates points of X, the above defined mapping is clearly one-to-one. Since X is compact, it is a homeomorphism.

The above ideas are used to give a simple proof of the well-known result that every perfect totally disconnected compact metric space is homeomorphic to the Cantor set. For a more sophisticated proof of this classical result see [1, p. 97].

5. DEFINITION. Let X be a compact Hausdorff space. An n-separation of X is a collection $(A_1, \cdots, A_n)$ of nonempty pair-wise disjoint closed sets such that $X=\bigcup_{i=I}^n A_i$.

6. LEMMA. *Let X be a perfect, totally disconnected, compact metric space. Let K be a nonempty, compact-open subset of X. Then for each $\epsilon>0$ there is a positive integer N such that if $n\geqq N$ there is an n-separation $(A_1, \cdots, A_n)$ of K with $d(A_i)<\epsilon$ for $i=1, \cdots, n$, where $d(A_i)=$ diameter of the set A_i with respect to the metric d on X.*

Proof. Since X is totally disconnected, each $x\in X$ has a neighborhood base of compact-open sets. Thus if $x\in K$, there is a compact-open set C_x with $x\in C_x$ and $d(C_x)<\epsilon$. By the compactness of K, there are $x_1, \cdots, x_k$ in K with $K\subset\bigcup_{i=1}^k C_{x_i}$ where C_{x_i} is compact-open and $d(C_{x_i})<\epsilon$, $i=1, \cdots, k$. Let

$$B_1 = C_{x_1} \quad\text{and}\quad B_i = C_{x_i}\setminus \bigcup_{j=1}^{i-1} C_{x_j}$$

for $i=2, \cdots, k$. Choose N to be the largest integer such that $K\cap B_N\neq\emptyset$. Then $A_i=B_i\cap K$, $i=1, \cdots, N$, is an N-separation of K and $d(A_i)\leqq d(B_i)<\epsilon$, $i=1, \cdots, N$. Since each A_i is compact-open and X perfect implies A_i is infinite, one can obtain an n-separation of K for $n\geqq N$ by suitably refining the A_i's.

7. PROPOSITION. *A topological space X is homeomorphic to the Cantor set if and only if it is compact, totally disconnected, metrizable, and perfect.*

Proof. Suppose X is compact, totally disconnected, metrizable, and perfect and let d be a metric for X. By the lemma there is a 2^{N_1}-separation $(A_1, \cdots, A_{2^{N_1}})$ of X with $d(A_i)<1$ for $1\leqq i\leqq 2^{N_1}$. For each $1\leqq i\leqq 2^{N_1}$ there is a 2^{N_2}-separation $(A_{i_1 1}, \cdots, A_{i_1 2^{N_2}})$ of A_{i_1} with $d(A_{i_1, i_2})<1/2$ for $1\leqq i_2\leqq 2^{N_2}$. In general, for each

$1 \leqq i_1 \leqq 2^{N_1}, \cdots, 1 \leqq i_{k-1} \leqq 2^{N_{k-1}}$, there is a 2^{N_k}-separation $(A_{i_1} \cdots i_{k-1}1, \cdots, A_{i_1 \cdots i_{k-1}}2^{N_k})$ of $A_{i_1 \cdots i_{k-1}}$ with $d(A_{i_1 \cdots i_k}) < 1/k$ for $1 \leqq i_k \leqq 2^{N_k}$.

Define a family $\{(\tilde{A}_n, \tilde{B}_n)\}$ of separations of X as follows:

$$\tilde{A}_1 = \bigcup_{i_1=1}^{1/2(2^{N_1})} A_{i_1} \qquad \tilde{B}_1 = \bigcup_{i_1=1/2(2^{N_1})+1}^{2^{N_1}} A_{i_1}$$

$$\tilde{A}_2 = \left(\bigcup_{i_1=1}^{1/4(2^{N_1})} A_{i_1} \right) \cup \left(\bigcup_{i_1=1/2(2^{N_1})+1}^{3/4(2^{N_1})} A_{i_1} \right)$$

$$\tilde{B}_2 = \left(\bigcup_{i_1=1/4(2^{N_1})+1}^{1/2(2^{N_1})+1} A_{i_1} \right) \cup \left(\bigcup_{i_1=3/4(2^{N_1})+1}^{2^{N_1}} A_{i_1} \right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tilde{A}_{N_1} = \bigcup \{ A_{i_1} \mid 1 \leqq i_1 \leqq 2^{N_1}, i_1 \text{ odd} \}$$

$$\tilde{B}_{N_1} = \bigcup \{ A_{i_1} \mid 1 \leqq i_1 \leqq 2^{N_1}, i_1 \text{ even} \}$$

$$\tilde{A}_{N_1+1} = \bigcup_{i_1=1}^{2^{N_1}} \bigcup_{i_2=1}^{1/2(2^{N_2})} A_{i_1 i_2}$$

$$\tilde{B}_{N_1+1} = \bigcup_{i_1=1}^{2^{N_1}} \bigcup_{i_2=1/2(2^{N_2})+1}^{2^{N_2}} A_{i_1 i_2}$$

Continuing in this fashion one obtains a sequence $\{\tilde{A}_n, \tilde{B}_n\}$ which forms an interlocking separation which separates the points of X. To see that it separates the points of X, let x, y be distinct points of X. Choose k so that $1/k < d(x, y)$. Then one of $(\tilde{A}_1, \tilde{B}_1), \cdots, (\tilde{A} \sum_{i=1}^k N_i, \tilde{B} \sum_{i=1}^k N_i)$ must separate x and y since otherwise x and y would lie in some $A_{i_1}, \cdots, i_k$ which has diameter less than $1/k$. By the proof of proposition 4, X is homeomorphic to $\{-1, 1\}^I$ where $I = \{1, 2, 3, \cdots\}$. This is just the Cantor set.

The following proposition is well known, but the authors have never seen proof of it.

8. Proposition. *Let X be an uncountable compact metric space, then X contains a set homeomorphic to the Cantor set.*

Proof. Since X is uncountable, X contains a nonvoid perfect set P. In the relative topology, P is a perfect metrizable compact space. Let d be a metric for P. Let A_1, B_1 be nonvoid open subsets of P such that $\overline{A}_1 \cap \overline{B}_1 = \varnothing$ and $d(\overline{A}_1) < 1$, $d(\overline{B}_1) < 1$. Suppose an interlocking family $\{(A_1, B_1), \cdots, (A_n, B_n)\}$ of sets has been defined such that for $1 \leqq i \leqq n$,

(1) A_i, B_i are nonempty open subsets of P,

(2) $\overline{A}_i \cap \overline{B}_i = \varnothing$,

(3) if $\{\epsilon_j\}_{j=1}^n$ is a sequence in $\{-1, 1\}$,

$$1 \cdot \overline{A}_j = \overline{A}_j, \; -1 \cdot \overline{A}_j = \overline{B}_j, \qquad \text{then } d\left(\bigcap_{j=1}^{n} \overline{A}_j\right) < 1/n.$$

For each sequence $\{\epsilon_{j,i}\}_{j=1}^{n}$ in $\{-1, 1\}$, let $C_i = \bigcap_{j=1}^{n} \epsilon_{j,i} A_j$. Then C_i is a nonempty given set in P for $1 \leqq i \leqq 2^n$. Since P is perfect, C_i has more than one point. Thus for each $1 \leqq i \leqq 2^n$, there are nonempty open sets U_{i_1}, U_{i_2} of P such that

(4) $\overline{U}_{i_1} \cap \overline{U}_{i_2} = \varnothing$,

(5) $\overline{U}_{i_1} \cup \overline{U}_{i_2} = C_i$,

(6) $d(\overline{U}_{i_j}) < 1/n+1$ for $j = 1, 2$.

Let

$$A_{n+1} = \bigcup_{i=1}^{2n} U_{i_1}, \quad \text{and} \quad B_{n+1} = \bigcup_{i=1}^{2n} U_{i_2}.$$

Then $\{(A_1, B_1), \cdots, (A_{n+1}, B_{n+1})\}$ is an interlocking family satisfying 1, 2, 3, above. Thus by induction there is an interlocking sequence $\{(A_n, B_n)\}$ satisfying 1, 2, 3 above. Let $C = \bigcap_{n=1}^{\infty} (\overline{A}_n \cup \overline{B}_n)$. Now $\{\overline{A}_n \cap C, \overline{B}_n \cap C\}$ forms an interlocking sequence in C which separates the points of C. Thus by the proof of proposition 4 again, C is homeomorphic to $\{-1, 1\}^I$, $I = \{1, 2, 3, \cdots\}$.

The ideas inherent in the last proof will be developed below to show the relationship between perfectness and interlocking sequences. These ideas are mainly due to Peter Morris and appear in [2].

9. Proposition. *Let X be a compact Hausdorff space. Then X contains a perfect set if and only if there is an interlocking sequence $\{(A_n, B_n)\}$ in X of closed sets such that $A_n \cap B_n = \varnothing$, $n = 1, 2, \cdots$.*

Proof. Suppose $\{(A_n, B_n)\}$ is such a sequence in X. Then for $C = \bigcap_{n=1}^{\infty} (A_n \cup B_n)$, C is compact and the map $f(x) = \{\epsilon_n\}$, $\epsilon_n = 1$ if $x \in A_n$, $\epsilon_n = -1$ if $x \in B_n$ is continuous and onto the Cantor set from C. For f is continuous since $A_n \cap C$, $B_n \cap C$ are disjoint and compact-open in C and onto since $\{(A_n, B_n)\}$ is interlocking. By Zorn's lemma there is a minimal set $P \subseteq C$ such that $f(P)$ is onto. It follows readily that P is perfect.

Conversely, let P be a nonvoid perfect subset of X. Thus in the relative topology, P is regular and dense-in-itself. We restrict our attention to P. Let x, y be distinct points of P, then there are open sets U_1, V_1 in P with $x \in U_1$, $y \in V_1$ and $\overline{U}_1 \cap \overline{V}_1 = \varnothing$. Let N be a positive integer and suppose that pairs $\{U_n, V_n\}_{n=1}^{N}$ of open sets have been chosen so that $\overline{U}_n \cap \overline{V}_n = \varnothing$, $n = 1, \cdots, N$ and for each sequence $\{\epsilon_n\}_{n=1}^{N}$ in $\{-1, 1\}$, $\bigcap_{n=1}^{N} \epsilon_n U_n \neq \varnothing$ where $1 \cdot U_n = U_n$, $-1 \cdot U_n = V_n$. Let $\{C_n, n = 1, \cdots, 2^N\}$ be the family of all intersections obtained from all sequences $\{\epsilon_n\}_{n=1}^{N}$ in $\{-1, 1\}$. The sets C_n are pair-wise disjoint and infinite. Thus for each $n = 1, \cdots, 2^N$ there are nonempty open sets O_{n_1}, O_{n_2} with $O_{n_1} \subseteq C_n$, $O_{n_2} \subseteq C_n$, $\overline{O}_{n_1} \cap \overline{O}_{n_2} = \varnothing$. Let

$$U_{N+1} = \bigcup_{i=1}^{2N} O_{n_1}, \qquad V_{N+1} = \bigcup_{n=1}^{2N} O_{n_2}.$$

Then $\{(U_1, V_1), \cdots, (U_{N+1}, V_{N+1})\}$ is an interlocking sequence of open sets with $\overline{U}_n \cap \overline{V}_n = \varnothing$ for $n=1, \cdots, N+1$. Thus by induction such an infinite sequence exists and $\{\overline{U}_n, \overline{V}_n\}$ is the required interlocking sequence of closed sets.

References

1. John G. Hocking and Gail S. Young, Topology, Addison-Wesley, New York, 1961.
2. Peter D. Morris, Spaces of Continuous Functions on Dispersed Sets, Ph.D. Dissertation, University of Texas, 1966.
3. Roman Sikorski, Boolean Algebras, Academic Press, New York, 1964.

FACTORIAL EXPRESSIONS AND QUASI-ORTHOGONAL NUMBER SETS

P. P. ROWE, University of Alaska

1. Introduction. The factorial expression

$$x^{(n)} = x(x-1)(x-2)\cdots(x-n+1) \tag{1}$$

can be related to ordinary powers of x by Stirling numbers of the first kind, $S_{n,i}$. The relation is

$$x^{(n)} = \sum_{i=1}^{n} S_{n,i}x^i.$$

Similarly one can express x^n in terms of $x^{(i)}$ by

$$x^n = \sum_{i=1}^{n} \mathcal{S}_{n,i}x^{(i)}$$

where $\mathcal{S}_{n,i}$ is a Stirling number of the second kind.

The factorial expression in equation (1) has many applications in the difference calculus, since the first difference of $x^{(n)}$ with respect to the forward difference operator $\triangle$ is $\triangle x^{(n)} = nx^{(n-1)}$. Therefore, the Stirling numbers of the first and second kind are important in applied mathematics. See Richardson [3] and Miller [2]. The difference operator $\triangle$ can be defined as follows: $\triangle f(x) = f(x+h) - f(x)$ where $h = \triangle x$. In much of this paper $h=1$.

The Stirling numbers also have the property of being quasi-orthogonal. That is, the two sets of numbers $S_{n,i}$ and $\mathcal{S}_{n,i}$ are quasiorthogonal number sets. See Gould [1] and Tauber [4]. Two sets of numbers $A_{s,n}$ and $B_{m,s}$ are said to be quasi-orthogonal if

$$\sum_{s=m}^{n} A_{s,n}B_{m,s} = \delta_n^m, \tag{2}$$

where δ_n^m is the Kronecker-δ. The following factorial expressions are also of interest

$$x^{[n]} = x(x+1)(x+2)\cdots(x+n-1), \tag{3}$$

$$x^{\langle n\rangle} = \left(x+\frac{n-1}{2}\right)\left(x+\frac{n-3}{2}\right)\cdots\left(x-\frac{n-3}{2}\right)\left(x-\frac{n-1}{2}\right). \tag{4}$$

The expressions in equations (3) and (4) have the properties that $\nabla x^{[n]} = nx^{[n-1]}$ and $\delta x^{\langle n\rangle} = nx^{\langle n-1\rangle}$ where ∇ is the backward and δ the central difference operators. Similar to the definition of $\triangle$, $\nabla f(x) = f(x) - f(x-h)$ and $\delta f(x) = f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)$.

The factorial expressions in equations (1), (3) and (4) can each be generalized in at least two important ways. The expressions given are generalizations of those equations.

$$(a+bx)^{(n)} = (a+bx)(a+\overline{bx-1})\cdots(a+\overline{bx-n+1}) \tag{5}$$

$$(a+bx)^{[n]} = (a+bx)(a+\overline{bx+1})\cdots(a+\overline{bx+n-1}) \tag{6}$$

$$\begin{aligned}(a+bx)^{\langle n\rangle} = {} & (a+b(\overline{x+(n-1)/2}))(a+b(\overline{x+(n-3)/2}))\cdots \\ & \cdot(a+b(\overline{x-(n-3)/2}))(a+b(\overline{x-(n-1)/2})).\end{aligned} \tag{7}$$

If the increment h is unspecified, one more generalization is obtained for each factorial expression and one gets:

$$(a+bx)_h^{(n)} = (a+bx)(a+\overline{bx-h})\cdots(a+\overline{bx-nh+h}) \tag{8}$$

$$(a+bx)_h^{[n]} = (a+bx)(a+\overline{bx+h})\cdots(a+\overline{bx+nh-h}) \tag{9}$$

$$\begin{aligned}(a+bx)_h^{\langle n\rangle} = {} & (a+b(\overline{x+(nh-h)/2}))(a+b(\overline{x+(nh-3h)/2}))\cdots \\ & \cdot(a+b(\overline{x-(nh-3h)/2}))(a+b(\overline{x-(nh-h)/2})).\end{aligned} \tag{10}$$

In these equations $\overline{bx\pm c} = bx \pm bc$.

It is the purpose of this paper to exhibit pairs of quasiorthogonal number sets relating each of the factorial expressions in equations (3), (4) and (5)—(10) to ordinary powers of x and to other factorial expressions and to explore some of the properties of these pairs of number sets. Such pairs of number sets are important in applied mathematics. Pairs of number sets for more general functions than the factorial expressions are discussed briefly.

2. Pairs of number sets for ordinary powers of x and the factorials of equation (3). This section will be used to show the general method of obtaining the partial difference equations that are used to generate pairs of number sets. Write

$$x^{[n]} = \sum_{j=0}^{n} A_{n,j}x^j. \tag{11}$$

Multiply by $x+n$ to get

$$x^{[n+1]} = \sum_{j=0}^{n+1} A_{n+1,j}x^j = \sum_{j=0}^{n} A_{n,j}x^{j+1} + n\sum_{j=0}^{n} A_{n,j}x^j.$$

From this, $A_{n+1,j}=A_{n,j-1}+nA_{n,j}$. By inspection one finds

$$A_{n,n} = 1 \quad \text{and} \quad A_{n,m} = 0 \qquad \text{for } m > n. \tag{12}$$

To obtain the other number set of the pair of number sets write

$$\begin{aligned} x^n &= \sum_{j=0}^{n} \mathfrak{a}_{n,j}x^{[j]}, \\ x^{n+1} &= \sum_{j=0}^{n+1} \mathfrak{a}_{n+1,j}x^{[j]} = \sum_{j=0}^{n} \mathfrak{a}_{n,j}x^{[j+1]} - j\sum_{j=0}^{n} \mathfrak{a}_{n,j}x^{[j]}. \end{aligned} \tag{13}$$

Equating coefficients of like powers, $\mathfrak{a}_{n+1,j}=\mathfrak{a}_{n,j-1}-j\mathfrak{a}_{n,j}$

$$\mathfrak{a}_{n,n} = 1, \qquad \mathfrak{a}_{n,m} = 0 \quad \text{for } m > n. \tag{14}$$

From equation (11), $x^{[j]}=\sum_{m=0}^{j} A_{j,m}x^m$. Substitute this into equation (13) to get

$$x^n = \sum_{j=0}^{n} \mathfrak{a}_{n,j}\sum_{m=0}^{j} A_{j,m}x^m,$$

$$x^n = \sum_{j=0}^{n} \mathfrak{a}_{n,j}\sum_{m=0}^{n} A_{j,m}x^m = \sum_{m=0}^{n} x^m \sum_{j=m}^{n} \mathfrak{a}_{n,j}A_{j,m}.$$

Therefore,

$$\sum_{j=m}^{n} \mathfrak{a}_{n,j}A_{j,m} = \delta_m^n. \tag{15}$$

For a similar development see Tauber [4]. By a similar process one can obtain pairs of sets of numbers for the more general factorial expressions.

THEOREM 1. *Let $\phi_i(x)$, $\theta_i(x)$, $i=1, 2, \cdots$ represent two sets of independent functions. If the relations*

$$\phi_n(x) = \sum_{j=0}^{n} L_{n,j}\theta_j(x) \quad \text{and} \quad \theta_j(x) = \sum_{k=0}^{j} \mathfrak{L}_{j,k}\phi_k(x)$$

hold and if $L_{n,j}=0$, $\mathfrak{L}_{n,j}=0$, $n<0$, $j<0$, $j>n$ then

$$\sum_{j=k}^{n} L_{n,j}\mathfrak{L}_{j,k} = \delta_k^n. \tag{16}$$

Proof: The proof is similar to the development by which equation (15) was established, except that one obtains the equation

$$\phi_n(x) = \sum_{k=0}^{n} \phi_k(x) \sum_{j=k}^{n} L_{n,j}\mathfrak{L}_{j,k}.$$

Since the sets are sets of independent functions, equation (16) is true. This completes the proof of Theorem 1.

3. More general factorial expression. For the factorial in equation (5) write

$$(a + bx)^{(n)} = \sum_{j=0}^{n} B_{n,j}\, x^j \quad \text{and} \quad x^n = \sum_{j=0}^{n} \mathfrak{B}_{n,j}(a + bx)^{(j)}.$$

From these equations the partial difference equations obtained are

$$B_{n+1,j} = bB_{n,j-1} + (a - bn)B_{n,j} \quad \text{and} \quad \mathfrak{B}_{n+1,j} = 1/b(\mathfrak{B}_{n,j-1} + (bj - a)\mathfrak{B}_{n,j}).$$

For $a=0$, $b=1$, the recurrence equations for the Stirling numbers are obtained. Since $B_{n,j}=0$, $\mathfrak{B}_{n,j}=0$, for $n<0, j<0$, and $j>n$, an equation similar to equation (15) is obtained:

$$\sum_{j=m}^{n} \mathfrak{B}_{n,j}B_{j,m} = \delta_m^n.$$

This result also follows from Theorem 1.

Since $(-a+bx')^{(n)}=(-1)^n(a+bx)^{[n]}$, $x'=-x$, the following equations can be obtained:

$$C_{n+1,j} = bC_{n,j-1} + (a + bn)C_{n,j}, \qquad \mathfrak{C}_{n+1,j} = 1/b(\mathfrak{C}_{n,j-1} - (a + bj)\mathfrak{C}_{n,j})$$

and

$$\sum_{j=m}^{n} C_{n,j}\mathfrak{C}_{j,m} = \delta_m^n.$$

The $C_{n,j}$ and $\mathfrak{C}_{n,j}$ represent number sets relating powers of x to the factorial in equation (6). Let $D_{n,j}$ and $\mathfrak{D}_{n,j}$ represent the number sets obtained by expressing the factorials in equation (7) to sums of powers of x and also expressing powers of x to sums of those factorials. By methods used previously in this paper one obtains

$$D_{n+2,j} = b^2D_{n,j-2} + 2abD_{n,j-1} + \left[a^2 - \frac{(n+1)^2}{4}b^2\right]D_{n,j}$$

$$\mathfrak{D}_{n+2,j} = \frac{-2a}{b}\mathfrak{D}_{n+1,j} + \frac{1}{b^2}\mathfrak{D}_{n,j-2} - \frac{1}{b^2}\left[a^2 - \frac{(j+1)^2}{4}b^2\right]\mathfrak{D}_{n,j}$$

and $\sum_{j=m}^{n} D_{n,j}\mathfrak{D}_{j,m} = \delta_m^n$.

The following relations are very useful:

$$(a + bx)_h^{(n)} = h^n(a' + bx')^{(n)}, \qquad (a + bx)_h^{[n]} = h^n(a' + bx')^{[n]}$$

and $(a+bx)_h^{(n)}=h^n(a'+bx')^{(n)}$ where $a=a'h$, $x=x'h$. Since these equations relate formulas (5) and (8), (6) and (9), (7) and (10), and since quasi-orthogonal number sets have been obtained using equations (5), (6), and (7), it is obvious that quasiorthogonal number sets can be found using equations (8), (9) and (10).

4. Relations between two factorial expressions. Relationships between the factorial expressions in equations (1) and (3) could be obtained by using:

$$x^{[n]} = \sum_{j=0}^{n} G_{n,j}\, x^{(j)} \quad \text{and} \quad x^{(n)} = \sum_{j=0}^{n} \mathcal{G}_{n,j}\, x^{[j]}.$$

However, it is easier to use $(-x)^{(n)}=(-1)^n x^{[n]}$. The equations obtained are

$$G_{n+1,j} = G_{n,j-1} + (n+j)G_{n,j}, \qquad \mathcal{G}_{n+1,j} = \mathcal{G}_{n,j-1} - (n+j)\mathcal{G}_{n,j}$$

and by Theorem 1,

$$\sum_{j=m}^{n} G_{n,j}\, \mathcal{G}_{j,m} = \delta_m^n.$$

It is apparent that sets of numbers could be obtained that relate any two of the factorial expressions. The general method for doing this is the same as used in obtaining the relations in section 2 and 3.

5. Some generalizations of the concept of quasi-orthogonal number sets. Theorem 1 can be extended to triple products or greater by using three or more sets of independent functions where the functions of one set are suitably related to the functions of the other sets. Theorem 1 can also be extended to obtain quasi-orthogonal number sets relating functions of two or more variables. An example of a triple product can be obtained from

$$x^n = \sum_{j=0}^{n} \mathcal{A}_{n,j}\, x^{[j]}, \qquad x^{[j]} = \sum_{k=0}^{j} G_{j,k}\, x^{(k)} \quad \text{and} \quad x^{(k)} = \sum_{i=1}^{k} S_{n,i}\, x^i.$$

The three sets of numbers satisfy the quasi-orthogonality condition

$$\sum_{j=k}^{n} \sum_{k=i}^{j} \mathcal{A}_{n,j} G_{j,k} S_{k,i} = \delta_i^n.$$

As an example of powers and factorials in two variables, from

$$x_1^m x_2^n = \sum_{k=0}^{m} \sum_{j=0}^{n} A_{m,k;n,j} x_1^{(k)} x_2^{(j)} \quad \text{and} \quad x_1^{(k)} x_2^{(j)} = \sum_{p=0}^{k} \sum_{q=0}^{j} B_{k,p;j,q} x_1^p x_2^q$$

can be obtained

$$\sum_{k=p}^{m} \sum_{j=q}^{n} A_{m,k;n,j} B_{k,p;j,q} = \delta_p^m \delta_q^n.$$

The extension to functions of several variables is obvious. Also the use of other factorials is possible. Several other generalizations should be possible. The following difference equations that generate these number sets are

$$A_{m,k;n,j} = A_{m-1,k-1;n-1,j-1} + jA_{m-1,k-1;n-1,j} + kA_{m-1,k;n-1,j-1} + kjA_{m-1,k;n-1,j}$$

and

$$B_{m,k;n,j} = B_{m-1,k-1;n-1,j-1} - nB_{m-1,k-1;n-1,j} - mB_{m-1,k;n-1,j-1} + mnB_{m-1,k;n-1,j}.$$

THEOREM 2. *The difference equations that generate the number sets relating monomials and factorials in two or more variables can be expressed in factored form. The factors will be the same as the difference equations relating powers and factorials in one variable.*

A proof of this theorem will not be given here. However, this result can be obtained for any particular case by replacing the coefficients of the powers or factorials of two or more dimensions by products of coefficients. For example, let ${}_1S_{k,p}$ and ${}_2S_{j,q}$ be the Stirling numbers of the first kind associated with x_1 and x_2, then the product of these two numbers could be used in the defining equation. Since the number will be the same $B_{k,p;j,q} = {}_1S_{k,p}\,{}_2S_{j,q}$. More interesting examples can be found without too much difficulty.

6. Conclusion. Many more relations between the various sets of numbers exist, not only for the sets of numbers obtained from functions of a single variable, but for the sets of numbers obtained from functions of two or more variables.

References

1. H. W. Gould, The construction of orthogonal and quasi-orthogonal number sets, this MONTHLY, 72 (1965) 591–602.

2. K. S. Miller, An Introduction to the calculus of finite differences and difference equations, Holt, Rinehart and Winston, New York, 1960.

3. C. H. Richardson, An introduction to the calculus of finite differences, Van Nostrand, Princeton, N. J., 1954.

4. Selmo Tauber, On two classes of quasi-orthogonal numbers, this MONTHLY, 72 (1965) 602–607.

NIL ALGEBRAS AND PERIODIC GROUPS

IRWIN FISCHER AND RUTH R. STRUIK,
University of Colorado

Introduction. The purpose of this paper is to give an account of some recent work ([2], [3]) in groups and rings which the authors hope will be comprehensible to a person with the background of a standard first year graduate course in modern algebra.

We discuss two questions which have turned out to be more closely related

than appeared originally. The first was originally asked by Burnside in 1902 [1], and can be stated as follows:

Let G be a finitely generated, periodic group. Is G finite? *Finitely generated* means that G contains a finite set of elements, $g_1, g_2, \cdots, g_r$ (called its generators) such that every element can be expressed as a finite product of the generators and their inverses. A group, G, is *periodic* if for every element $g \in G$, there exists some integer n (which may depend on g) such that $g^n = 1$. If $g^n = 1$ with n fixed, for all $g \in G$, and n is the smallest positive integer for which this is true, then n is called the *exponent* of G.

Suppose G is of exponent 3 and generated by two elements a and b, then $1, a, b, a^2, b^2, ab, ba, a^2b, ab^2, aba, bab, \cdots$ may or may not be distinct as elements of G. If an infinite number of them are distinct, then G would be an infinite group. In this particular case, G is finite, but this requires proof.

The other question was originally asked by Kuroš in 1941 [14]: Let A be a finitely generated, algebraic algebra. Is A finite-dimensional (as a vector space)? An *algebra*, A, is a ring which is simultaneously a vector space over a field K. In addition, the following holds:

$$\alpha(ab) = (\alpha a)b = a(\alpha b) \qquad \text{for all } \alpha \in K,\ a, b \in A.$$

A is *finitely-generated* if there is a finite subset $a_1, \cdots, a_r$ (called its generators) such that every element of A can be obtained from the generators by a finite number of additions, multiplications and/or scalar multiplications. An algebra A is *algebraic* if every $a \in A$ satisfies an equation of the form

$$k_n a^n + k_{n-1} a^{n-1} + \cdots + k_0 = 0, \tag{1}$$

where $k_i \in K$. The equation may differ for different $a \in A$.

Since each of these questions was originally proposed, a great many attempts to solve them have occurred. Most of the results obtained indicated that the answer is "yes" if other hypotheses were added. For example, Burnside [1] considered the following special cases:

(1) G of exponent 2,

(2) G of exponent 3,

(3) G of exponent 4, $r = 2$ (i.e., G with two generators).

The answer in these cases is "yes." Then in 1940, Sanov [19] obtained an affirmative answer for exponent 4 and an arbitrary (but finite)number of generators. Marshall Hall Jr. in 1958 [6] similarly disposed of groups of exponent 6. If $g^5 = 1$ for all $g \in G$, the answer is still unknown. Then Novikov in 1959 [18] announced that the answer is "no" if $n \geqq 72$, and $r \geqq 2$.

The work on the second question has had a similar history. Kuroš [14] discussed several special cases, all with affirmative answers. In [10], [15] and [16] Jacobson and Levitzki settled the question affirmatively for algebras of bounded degree. In an algebraic algebra A the *degree* of $a \in A$ is the least degree of the polynomials of which a is a root. If the degrees of all $a \in A$ have a maximum,

then A is of *bounded degree*. Since these papers, many special cases have been studied. (See, e.g., [4], [7], [8], [11], [12], [20].)

Then in 1964, Golod [3] announced that the answer to both questions is "no." In particular, he constructed a finitely generated nil algebra which is infinite-dimensional. In a *nil algebra*, $a^n=0$ for all $a\in A$ (n might depend on a). (A nil algebra is obviously algebraic: put $k_n=1$; $k_i=0$, $1\leqq i\leqq n-1$ in (1).) Using this nil algebra, Golod constructed a finitely generated group which is periodic and infinite. This settles negatively both Burnside's problem and Kuroš' problem.

In view of the many previous attempts to solve these two problems, the Golod-Šafarevič construction is remarkable for its simplicity and the small amount of background needed to understand it. The details of their method will be given in subsequent sections of this exposition.

Even with Golod's results, much still remains unknown. For example, given Novikov's result, it is still unknown whether finitely generated groups of exponent 5, 7, 8, 9, $\cdots$, 71 are finite or infinite. Also, the details of Novikov's proof have not yet been published as of the writing of this paper, and the proof, when it appears, will very likely be long and difficult. Hence the Golod-Šafarevič results are still very significant.

In Section 1 we will give a sufficient condition for an algebra to be infinite-dimensional. In Section 2 we will apply this method to settle Burnside's and Kuroš' questions.

1. In this section we give a sufficient condition for an algebra A to be infinite-dimensional. Before the main theorem can be stated, a few terms will be defined.

Let K be any field and let $R_d=K[x_1, x_2, \cdots, x_d]$ be the polynomial ring over K in the noncommuting indeterminates $x_1, \cdots, x_d$. This means that the x_i do not commute with each other, but they do commute with the elements of K. Then

$$R_d = T_0 \oplus T_1 \oplus \cdots \oplus T_n \oplus \cdots, \tag{2}$$

where $T_0=K$ and T_n is the vector space over K spanned by the d^n monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$. In (2), $\oplus$ means direct sum: that is, every $u\in R_d$ can be written uniquely as a sum

$$u = u_0 + u_1 + \cdots + u_k, \quad \text{where} \quad u_i \in T_i. \tag{3}$$

The elements of T_i are said to be *homogeneous* of degree i. For example, $x_1^3+x_1x_2x_3$ is homogeneous of degree 3, but $x_1^4+x_1x_2x_3$ is not homogeneous. If u is a homogeneous polynomial of degree i, we will denote by $\partial(u)=i$ the degree of u.

Let F be a set of nonzero homogeneous polynomials, $f_1, f_2, \cdots$ such that $2\leqq\partial(f_i)\leqq\partial(f_{i+1})$ and such that the number, r_n, of polynomials of degree n is finite (possibly zero). We will be studying the ideal, I, generated by $f_1, f_2, \cdots$. This ideal is made up of polynomials whose summands are of the form

$$uf_jv, \qquad u, v \in R_d. \tag{4}$$

If u and v are written as sums of homogeneous polynomials, this means that (4) can be written as a sum of homogeneous polynomials of the form $u_i f_j v_k$, where u_i and v_k are homogeneous polynomials of R_d. We now form the factor algebra

$$A = R_d/I. \tag{5}$$

If $r = u_2 + u_3 + \cdots + u_s \in I$, $u_i \in T_i$, then each $u_i \in I$, since I is generated by homogeneous polynomials. This means that if

$$v_1 + \cdots + v_s \equiv w_1 + \cdots + w_t \pmod{I},\ v_i,\ w_i \in T_i,$$

then $v_i \equiv w_i \pmod{I}$ for each i. Hence,

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_n \oplus \cdots, \tag{6}$$

where $A_n = (T_n + I)/I$. In particular,

$$A_0 \simeq T_0 \simeq K \quad \text{and} \quad A_1 \simeq T_1,$$

since $\partial(f_i) \geqq 2$. Since A_n is a vector space over K, it makes sense to talk about $b_n = \dim A_n$. Obviously $\dim T_n = d^n$.

We will also be considering formal power series such as

$$1 - dt + \sum_{i=2}^{\infty} r_i t^i \tag{7}$$

and

$$\left(1 - dt + \sum_{i=2}^{\infty} r_i t^i\right)^{-1} = 1 + \sum_{i=1}^{\infty} x_i t^i. \tag{8}$$

Here a formal power series is simply a function from the positive integers into the real numbers, and the coefficient of t^n is merely the value of the function at n; convergence is irrelevant. The "inverse" (8) is obtained by solving the following equations:

$$\begin{aligned} x_1 &= d \\ x_2 &= dx_1 - r_2 = d^2 - r_2 \\ x_3 &= dx_2 - r_2 x_1 - r_3 = d(d^2 - r_2) - r_2 d - r_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= dx_{n-1} - \sum_{i=2}^{n} r_i x_{n-i}, \quad \text{where} \quad x_0 = 1. \end{aligned} \tag{9}$$

We can now state the Golod-Šafarevič theorem ([2] and [3]). The proof we give here is a modification of one appearing in [21].

THEOREM 1.1. *Let $R_d = K[x_1, \cdots, x_d]$ be the polynomial ring over K in the noncommuting indeterminates $x_1, \cdots, x_d$. Let $f_1, f_2, \cdots, \in F$ be a set of homogeneous polynomials of R_d, and let the number of polynomials of degree i be r_i. Let*

$2 \leqq \partial(f_i) \leqq \partial(f_{i+1})$. *Let I be the ideal generated by $f_i \in F$. Let*

$$R_d/I = A = A_0 \oplus A_1 \oplus \cdots \oplus A_n \oplus \cdots \tag{10}$$

If all the coefficients of

$$\left(1 - dt + \sum_{i=2}^{\infty} r_i t^i\right)^{-1} \tag{11}$$

are nonnegative, then A is infinite-dimensional.

We first prove a lemma:

LEMMA 1.2. *Under the conditions of Theorem* 1.1, *let $b_n = \dim A_n$. Then*

$$b_n \geqq d b_{n-1} - \sum_{i=2}^{n} r_i b_{n-i}, \qquad n \geqq 2. \tag{12}$$

(Note the similarity between (12) and the last equation of (9).)

Proof. Let $I_n = I \cap T_n$, i.e., I_n is the vector space spanned by the homogeneous polynomials of I of degree n. By an elementary theorem on vector spaces, I_n has a complementary subspace; call it S_n. Then

$$T_n = I_n \oplus S_n.$$

Obviously,

$$\dim T_n = \dim I_n + \dim S_n = \dim I_n + b_n. \tag{13}$$

For the case $n = 2$, I_2 is generated by the $f_i \in F$ such that $\partial(f_i) = 2$. Hence $\dim I_2 \leqq r_2$. (If the f_i of degree 2 were linearly independent, then $\dim I_2 = r_2$.) Hence

$$d^2 = \dim T_2 = \dim I_2 + b_2 \leqq r_2 + b$$

i.e., $b_2 \geqq d^2 - r_2 = d b_1 - r_2 b_0$, since $b_1 = d$ and $b_0 = 1$, which is (12) for $n = 2$.

Let now $T_{n-1} = S_{n-1} \oplus I_{n-1}$, and let $s_1, s_2, \cdots, s_{b_{n-1}}$ be a basis for S_{n-1} and $g_1, \cdots, g_m$ a basis for I_{n-1}, both considered as vector spaces. Then the elements

$$s_i x_j, \qquad (i = 1, 2, \cdots, b_{n-1} \text{ and } j = 1, \cdots, d)$$

together with the elements

$$g_k x_j, \qquad (k = 1, 2, \cdots, m \text{ and } j = 1, \cdots, d)$$

obviously form a basis for T_n. Let J be the vector space spanned by the $g_k x_j$. Let $v_1, v_2, \cdots$ be a set of homogeneous polynomials of degrees up to $n-1$, which forms a basis for $S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}$. Let L be the vector space spanned by all elements of degree n of the form $v_i f_j, f_j \in F$. We now show that if $u \in I_n$, then $u = v + w$, $v \in L$ and $w \in J$. Since $u \in I_n$, it can be written as a sum of polynomials of the form $u_i f_j u_k$, where u_i, u_k are homogeneous polynomials and $\partial(u_i f_j u_k) = n$.

Case I. $\partial(u_k) \geqq 1$. Then $u_i f_j u_k \in J$. Let us indicate the proof of this by an

example. Let $u_i = x_1x_2$, $u_k = x_3x_4$. Then

$$x_1x_2f_jx_3x_4 = (x_1x_2f_jx_3)x_4.$$

But $x_1x_2f_jx_3 \in I_{n-1}$ and therefore $u_if_ju_k \in I_{n-1}x_4 \subseteq J$, since $J = I_{n-1}x_1 \oplus I_{n-1}x_2 \oplus \cdots \oplus I_{n-1}x_d$.

Case II. $\partial(u_k) = 0$, i.e., $u_if_ju_k = u_if_j$. If $u_i \in T_k$, then $u = v' + w'$, where $v' \in S_k$, $w' \in I_k$. Then $u_if_j = v'f_j + w'f_j$. By the argument given for Case I, $w'f_j \in J$. By definition of $v_1, v_2, \cdots, v'f_j = \sum_k c_{i_k}v_{u_k}f_j \in L$, $c_{i_k} \in K$.

Hence $u \in I_n$ implies $u = v + w$, $v \in L$, $w \in J$. This means that $\dim I_n \leqq \dim L + \dim J$. (We cannot say $\dim I_n = \dim L + \dim J$, for it may be that nonzero members of I_n are contained in $L \cap J$.) Hence

$$\dim T_n = \dim I_n + \dim S_n \leqq \dim J + \dim L + b_n. \tag{14}$$

But $\dim L \leqq \sum_{i=2}^n b_{n-i}r_i$, i.e.,

$$\dim T_n \leqq \dim J + \sum_{i=2}^{n} b_{n-i}r_i + b_n. \tag{15}$$

Now the s_ix_j along with g_kx_j form a basis for T_n. This means

$$\dim T_n = db_{n-1} + \dim J. \tag{16}$$

Putting (15) and (16) together gives (12).

Proof of Theorem. Consider

$$\left(1 + \sum_{i=1}^{\infty} b_it^i\right)\left(1 - dt + \sum_{j=2}^{\infty} r_jt^j\right) = A(t). \tag{17}$$

These are just formal power series. If one multiplies the left hand side of (17) and compares with (12), one obtains that the coefficients of $A(t)$ are all nonnegative. Multiplying on both sides by

$$\left(1 - dt + \sum_{j=2}^{\infty} r_jt^j\right)^{-1} = C(t),$$

one obtains

$$1 + \sum_{i=1}^{\infty} b_it^i = A(t)C(t). \tag{18}$$

By hypothesis the coefficients of $C(t)$ are nonnegative. By definition of b_i, $b_i \geqq 0$ for $i \geqq 1$. We want to make sure that the left hand side of (18) is *not* a polynomial. Since $A(t)$ has nonnegative coefficients, it is sufficient to show that $C(t)$ is *not* a polynomial. Suppose $C(t)$ is a polynomial: then $(1 - dt + \sum_{j=2}^{\infty} r_jt^j)\ C(t) = 1$, and hence

$$\left(1+\sum_{j=2}^{\infty} r_j t^j\right) C(t) = 1 + dtC(t). \tag{19}$$

Comparing the two sides of (19), it is impossible for $C(t)$ to be a polynomial. This completes the proof of Theorem 1.

COMMENT: In [21], Vinberg shows that if (11) is replaced by

$$(1-t)\left(1 - dt + \sum_{i=2}^{\infty} r_i t^i\right)^{-1},$$

then the requirement that the f_i be homogeneous can be removed. In this case, the degree of the polynomial is defined to be the least degree of its homogeneous components. Recently, R. H. Bruck has shown that for $d \geqq 2$, $r_i \geqq 0$, any formal power series satisfying the requirements of Theorem 1.1 also satisfies Vinberg's requirements and vice-versa. R. H. Bruck has also obtained necessary and sufficient conditions for such formal power series to satisfy these requirements.

COROLLARY 1.3. *Under the hypotheses of Theorem* 1.1, *let* $r_i \leqq s_i$. *If*

$$\left(1 - dt + \sum_{i=2}^{\infty} s_i t^i\right)^{-1} \tag{20}$$

has all its coefficients nonnegative, then A is infinite dimensional.

Proof of Corollary 1.3. Let

$$H = 1 - dt + \sum_{n=2}^{\infty} r_n t^n,$$

$$G = 1 - dt + \sum_{n=2}^{\infty} s_n t^n,$$

$$U = \sum_{n=2}^{\infty} (s_n - r_n) t^n,$$

then $U+H=G$. We are given that G^{-1} has all its coefficients nonnegative, and we want to prove that the same is true for H^{-1}. Now $H=G-U$ and hence,

$$H^{-1} = [G(1 - UG^{-1})]^{-1} = G^{-1}(1 - UG^{-1})^{-1}.$$

Since U and G^{-1} have all their coefficients nonnegative, so does UG^{-1}. Then all the coefficients of

$$(1 - UG^{-1})^{-1} = 1 + \sum_{n=1}^{\infty} (UG^{-1})^n$$

are nonnegative, and hence the same is true for $G^{-1}(1-UG^{-1})^{-1}$. This shows that all the coefficients of H^{-1} are nonnegative, and we can now use Theorem 1.1.

In order to apply Theorem 1.1 or Corollary 1.3 in a particular case, we need some expressions for possible r_i:

THEOREM 1.4 (Golod [3]). *Let r_i and A be as in Theorem* 1.1. *If*

$$r_i \leqq \epsilon^2(d-2\epsilon)^{i-2},$$

where ϵ is any positive number such that $(d-2\epsilon)>0$, then A is infinite-dimensional.

Proof. It is sufficient to examine the coefficients of

$$\left(1 - dt + \sum_{i=2}^{\infty} \epsilon^2(d-2\epsilon)^{i-2}t^i\right)^{-1}. \tag{21}$$

To do this, we make use of the following manipulations with formal series:

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots = \sum_{i=0}^{\infty} a^i; \tag{22}$$

$$(1-a)^2(1 + 2a + 3a^2 + \cdots) = 1, \text{ i.e.,} \tag{23}$$

$$\frac{1}{(1-a)^2} = \sum_{n=1}^{\infty} na^{n-1}. \tag{24}$$

Hence

$$\begin{aligned}1 - dt + \sum_{i=2}^{\infty} \epsilon^2(d-2\epsilon)^{i-2}t^i &= 1 - dt + \epsilon^2t^2[1 + (d-2\epsilon)t + \cdots + (d-2\epsilon)^n t^n + \cdots] \\ &= 1 - dt + \frac{\epsilon^2t^2}{1-(d-2\epsilon)t} = \frac{1 + (2\epsilon - 2d)t + (d-\epsilon)^2t^2}{1-(d-2\epsilon)t} \\ &= \frac{[1-(d-\epsilon)t]^2}{1-(d-2\epsilon)t}.\end{aligned}$$

Hence (21) becomes

$$\frac{1-(d-2\epsilon)t}{[1-(d-\epsilon)t]^2}. \tag{25}$$

Now use (24) with $a=(d-\epsilon)t$:

$$\begin{aligned}\frac{1-(d-2\epsilon)t}{[1-(d-\epsilon)t]^2} &= [1-(d-2\epsilon)t]\left[1 + \sum_{n=1}^{\infty}(n+1)(d-\epsilon)^n t^n\right] \\ &= 1 + \sum_{n=1}^{\infty}[(n+1)(d-\epsilon)^n - (d-2\epsilon)n(d-\epsilon)^{n-1}]t^n \qquad (26) \\ &= 1 + \sum_{n=1}^{\infty}(d-\epsilon)^{n-1}[d+(n-1)\epsilon]t^n.\end{aligned}$$

Since $d-2\epsilon>0$, $d-\epsilon>\epsilon>0$, and all the coefficients of (26) are nonnegative.

COROLLARY 1.5. *In Theorem* 1.4, *let* $d=2$ *and* $r_i=0$ *for* $i=2, 3, \cdots, 9$ *and* $r_i=0$ *or* 1 *for* $i\geqq 10$. *Then* A *is infinite-dimensional.*

Proof. Use Theorem 1.4 with $\epsilon=1/4$ and $d=2$. Then

$$\frac{(2-1/2)^8}{16}=\frac{(1+1/2)^8}{16}.$$

Expand $(1+\frac{1}{2})^8$ using the binomial theorem. From the first four terms, $(1+\frac{1}{2})^8>16$. Hence $\epsilon^2(d-2\epsilon)^8>1$. Since $(d-2\epsilon)^i<(d-2\epsilon)^{i+1}$ if $d-2\epsilon>1$, this is sufficient to prove Corollary 1.5.

COMMENT. Actually $(1.5)^7>16$ and $r_i=0$, $i=1, 2, \cdots, 7, 8$; $r_i=0$ or 1, $i\geqq 9$ will be sufficient to make A infinite-dimensional.

2. We can now start constructing examples of nil algebras and periodic groups. The first two examples, simple ones, are modifications of those that appear in [8].

THEOREM 2.1. *Let K be any finite or countable field. Then there exists an infinite-dimensional nil algebra over K generated by 2 elements.*

Proof. Consider $K[x_1, x_2]$, the polynomial ring over K generated by the noncommuting variables x_1 and x_2. Consider all polynomials with constant term equal to zero. These can be enumerated: $u_1, u_2, \cdots$. Then

$$u_1^{10}=s_{11}+s_{12}+\cdots+s_{1k_1},$$

where s_{1j} are homogeneous polynomials and $10\leqq\partial(s_{1j})<\partial(s_{1,j+1})$. Let $\partial(s_{1k_1})=N_1$. Then

$$u_2^{N_1+1}=s_{21}+\cdots+s_{2k_2},$$

where $N_1+1\leqq\partial(s_{2j})<\partial(s_{2,j+1})$. Continue in this way. This gives a collection of homogeneous polynomials, s_{ij}, all of degree $\geqq 10$, and for each degree, there is at most one polynomial. We now use Corollary 1.5 and Theorem 1.1 to obtain an infinite-dimensional algebra,

$$A=A_0\oplus A_1\oplus\cdots\oplus A_n\oplus\cdots.$$

Consider $B=A_1\oplus A_2\oplus\cdots\oplus A_n\oplus\cdots$. B is an algebra generated (as an algebra) by x_1+I and x_2+I, and every element $b\in B$ is such that $b^m=0$ for some m. This is our example.

THEOREM 2.2. *For every prime p, there is an infinite group generated by two elements in which every element has order a power of p.*

Proof. Let K be the field of integers modulo p. Let A be the algebra con-

structed in the proof of Theorem 2.1. Then if $1+u\in A$, with $u\in B$, there exists an m such that $u^m=0$. Let $p^j>m$. Then

$$(1+u)^{p^j}=1+u^{p^j}=1, \tag{27}$$

since if one uses the binomial theorem, the coefficients of u^i, $1\leqq i\leqq p^j-1$ will be divisible by p. Let

$$a=1+x_1+I \quad\text{and}\quad b=1+x_2+I.$$

Consider the subset G of A consisting of all finite power products of a and b, with nonnegative exponents. Since a and b are of finite multiplicative order, G is a group, the multiplication of A being the operation of G. By (27) every element of G has order a power of a prime.

We want to show that G is infinite. Suppose not. Each element of G is a coset of the form $1+u_1+u_2+\cdots+u_s+I$, u_i homogeneous polynomials. If G is finite, we can choose coset representatives so that there is a maximum M to the degrees of the homogeneous components u_i. Since A, the algebra of Theorem 2.1, is infinite-dimensional, there is a monomial $x_{i_1}x_{i_2}\cdots x_{i_{M+1}}\notin I$. Then

$$\begin{aligned} g &= (1+x_{i_1})(1+x_{i_2})\cdots(1+x_{i_{M+1}})+I \\ &= 1+v_1+\cdots+v_M+x_{i_1}x_{i_2}\cdots x_{i_{M+1}}+I\neq 1+I, \end{aligned}$$

v_i homogeneous of degree i. g cannot be any of the elements of G already enumerated. Hence G must be infinite.

Theorem 2.1 assumed that K was finite or countable. This is not necessary, and we now give Golod's construction which holds for an arbitrary field:

THEOREM 2.3. (Golod [3]). *Let K be any field. Then there exists a nil algebra over K with $d\geqq 2$ generators which is infinite-dimensional.*

Proof. Let $R_d=K[x_1,\cdots,x_d]$ as previously. We want to construct a set of homogeneous polynomials $f_1,f_2,\cdots\in F$ such that the number of polynomials of degree $i\leqq\epsilon^2(d-2\epsilon)^{i-2}$ for some $\epsilon>0$, and such that if $u\in R_d$ has its constant term 0, then $u^m\in$ ideal generated by $f_1,f_2,\cdots$ for some m. As usual, we denote the ideal generated by $f_1,f_2,\cdots$ by I. We will construct a sequence of ideals, $I_1,I_2,\cdots$ such that $I_j\subseteq I_{j+1}$ and such that if $u\in R_d$, with constant term 0, and the highest degree of the monomials appearing in u is j, then $u^m\in I_j$ for some m. Then we will choose

$$I=I_1\cup I_2\cup\cdots\cup I_n\cup\cdots \qquad \text{(set-theoretic union).}$$

We start with I_1. Consider an arbitrary monomial of degree 1 with constant term 0: $u=c_1x_1+c_2x_2+\cdots+c_dx_d$, $c_i\in K$. Then

$$u^M=(c_1x_1+c_2x_2+\cdots+c_dx_d)^M. \tag{28}$$

If one expands the right hand side of (28), one obtains a sum of monomials of

degree M. Consider the c_i as "variables" with the x_j as "coefficients." For example, the "coefficient" of $c_1c_2 \cdots c_M$ (let's say $d \geqq M$) would be

$$\sum x_{i_1}x_{i_2} \cdots x_{i_M}$$

where the summation is over all permutations, $(i_1, i_2, \cdots, i_M)$ of $(1, 2, \cdots, M)$. Similarly the "coefficient" of c_3^M would be x_3^M. For any $c_{i_1}c_{i_2} \cdots c_{i_M}$, the "coefficient" is a homogeneous polynomial in $x_{i_1}, \cdots, x_{i_M}$. These "coefficients" will be our f_i. We have still to decide what M will be. Each of the "coefficients" is a homogeneous polynomial of degree M. How many are there? It is the number of *commutative* monomials in d variables of degree M. This is equal to the binomial coefficient

$$\binom{M+d-1}{d-1}. \tag{29}$$

(If this is not "well-known" to you, (29) can be obtained by the following argument: Put $M+d-1$ points on a horizontal line, and color $d-1$ of them blue. The number of points from the left end to the first blue point is the exponent of x_1, the number of points between the first and second blue points is the exponent of x_2, etc. This gives a monomial of degree M in $x_1, \cdots, x_d$. To each selection of $d-1$ points, there corresponds a unique monomial and vice-versa.) But

$$\binom{M+d-1}{d-1} \leqq (M+d-1)^{d-1}. \tag{30}$$

We want to choose an M such that

$$(M+d-1)^{d-1} \leqq \epsilon^2(d-2\epsilon)^{M-2}. \tag{31}$$

Pick any $\epsilon>0$, such that $d-2\epsilon>1$, say $\epsilon=\frac{1}{3}$. Then (31) becomes

$$(M+d-1)^{d-1} \leqq \frac{(d-2/3)^{M-2}}{9}. \tag{32}$$

Consider (32) as an inequality with M as the "variable" and d as a constant. Considered as a function of M, $(M+d-1)^{d-1}$ is a polynomial, while the right-hand side of (32) is an exponential function of M. This means that there exists an M_1 (M_1 a positive integer) such that

$$(M_1+d-1)^{d-1} \leqq \frac{(d-2/3)^{M_1-2}}{9},$$

and (32) holds for all $M \geqq M_1$. The corresponding

$$\binom{M_1+d-1}{d-1}$$

homogeneous polynomials will be used to generate I_1. Evidently for every linear homogeneous polynomial, u, $u^{M_1} \in I_1$.

Suppose we now assume that $I_1, I_2, \cdots, I_{k-1}$ have been formed from the homogeneous polynomials $f_1, f_2, \cdots, f_{m_{k-1}}$, with $\partial(f_i) \leqq \partial(f_{i+1})$. Let $\partial(f_{m_{k-1}}) = M_{k-1}$. We now form a typical polynomial of degree k with constant term 0:

$$u = \overset{(1)}{c_1} x_1 + \cdots + \overset{(1)}{c_d} x_d + \overset{(2)}{c_1} \overset{2}{x_1} + \cdots + \overset{(k)}{c_{d^k}} \overset{k}{x_d}. \tag{33}$$

Again we look at the expansion of u^M, and consider the x_i as "coefficients" of the $c_i^{(j)}$. Repeating the argument used in obtaining I_1, we find that we will have to count the number of *commutative* monomials in $q = d + d^2 + \cdots + d^k$ variables. This is

$$\binom{M+q-1}{q-1} \leqq (M+q-1)^{q-1}.$$

Again we look for an M such that

$$(M+q-1)^{q-1} \leqq \frac{(d-2/3)^{M-2}}{9}\,. \tag{34}$$

The M we choose in addition to satisfying (34) must also satisfy the requirement

$$M > M_{k-1} = \partial(f_{m_{k-1}}), \tag{35}$$

i.e., to form I_k, we use all the $f_1, \cdots, f_{m_{k-1}}$, and only add homogeneous polynomials whose degree is greater than that of any of the preceding f_i. Once an M is found satisfying (34) and (35), we can then add the corresponding homogeneous polynomials to $f_1, \cdots, f_{m_{k-1}}$ to generate I_k. This gives us $f_1, \cdots, f_{m_k}$. Evidently for all $u \in R_d$, u with zero constant term and of degree k, $u^m \in I_k$ for some m.

This gives a sequence of homogeneous polynomials, $f_1, f_2, \cdots$ and a sequence of ideals, $I_1 \subseteq I_2 \subseteq \cdots$ which have the desired properties.

COMMENT. Golod, in addition, shows that the group G of Theorem 2.2 has another property which he calls "finitely-approximable." This means that the intersection of all normal subgroups of G of finite index is the identity. Novikov's [18] infinite groups cannot have this property. It is beyond the scope of this exposition to prove these things. Some further interesting constructions are given in [17].

The authors wish to thank those who read a preliminary version of this paper for their helpful comments. In particular, the authors are grateful to J. T. McCall for his careful reading of that version and his many thoughtful suggestions. The second author wishes to thank the Council on Research and Creative Work at the University of Colorado for a grant which supported some of her work on this paper.

Added in proof. The details of the proof announced in [18] are now appearing: *On infinite periodic groups, I,* Izv. Akad. Nauk SSSR, Ser. Mat. 32 (1968) 212–244 by P. S. Novikov and S. I. Adyan begins this proof. $n \geqq 72$ has been replaced by odd $n \geqq 4381$.

References

1. W. Burnside, On an unsettled question in the theory of discontinuous groups, Quart. J. Pure and Applied Math., 33 (1902) 230–238.
2. E. S. Golod and I. R. Šafarevič, On class field towers, Izv. Akad. Nauk SSSR. Ser. Mat., 28(1964) 261–272. AMS Translations, Ser. 2, 48 (1965) 91–102.
3. E. S. Golod, On nil-algebras and finitely-approximable p-groups, Izv. Akad. Nauk SSSR. Ser. Mat., 28(1964) 273–276. AMS Translations. Ser. 2, 48 (1965) 103–106.
4. M. P. Drazin, A generalization of polynomial identities in rings, Proc. AMS, 8 (1957) 352–361.
5. Marshall Hall, Jr., The Theory of Groups, Macmillan, New York, 1959.
6. ———, Solution of the Burnside problem for exponent 6, Illinois J. of Math., 2 (1958) 764–786.
7. I. N. Herstein, Theory of Rings (1961), mimeographed notes obtainable from the University of Chicago Math Dept.
8. ———, Topics in Ring Theory (1965), mimeographed notes obtainable from the University of Chicago Math Dept.
9. Graham Higman, On a conjecture of Nagata, Proc. Camb. Phil. Soc., 52 (1956) 1–4.
10. N. Jacobson, Structure theory for algebraic algebras of bounded degree, Ann. Math., 46(1945) 695–707.
11. ———, Structure of Rings, Revised edition, 1964, American Mathematical Society.
12. Irving Kaplansky, Rings with a polynomial identity, Bull. AMS, 54 (1948) 575–580.
13. ———, Topological representation of algebras II, Trans. AMS, 68 (1950) 62–75.
14. A. G. Kuroš, Problems in ring theory, connected with the Burnside problem in periodic groups, Izv. Akad. Nauk SSSR Ser. Mat., 5 (1941) 233–241.
15. Jakob Levitzki, On the radical of a general ring, Bull. AMS, 49 (1943) 462–466.
16. ———, On a problem of A. Kurosch, Bull. AMS, 52 (1946) 1033–1035.
17. M. F. Newman, A theorem of Golod-Šafarevič and an application in group theory, preprint obtainable from author at Dept. of Pure Math., SGS, Australian National Univ, GPO Box 4, Canberra ACT, Australia.
18. P. S. Novikov, On periodic groups, Dokl. Akad. Nauk SSSR, 127 (1959) 4, 749–752. AMS Translations, Ser. 2, 45 (1965) 19–22.
19. I. N. Sanov, Solution of Burnside's problem for exponent 4. Leningrad State Univ. Ann., 10 (1940) 166–170.
20. A. I. Širšov, On a problem of Levitzki, Dokl. Akad. Nauk SSSR, 120 (1958) 41–42.
21. E. B. Vinberg, On the theorem concerning the infinite-dimensionality of an associative algebra, Izv. Akad. Nauk SSSR Ser. Mat., 29 (1965) 209–214.

Correction: In the paper "A Mixed Doubles Tournament Problem" by C. C. Yalavigi (this MONTHLY, 74(1967) 926–933) Parts (i) and (ii) of Theorem 1, on page 927 should read as follows:

(i) *no two of the* $2t$ *MM differences are equal,*

(ii) *no two of the* $2t$ *WW differences are equal.*

Parts (i) and (ii) of Generalized Theorem 1, on page 933 should be replaced by

(i) *the* MM_t *differences are in* $2t$ *distinct sets of* 2^s-1 *equal differences except* $s=0$ *and* 1 *where no two differences are equal for* $s=1$,

(ii) *the* WW_t *differences are in* $2t$ *distinct sets of* 2^s-1 *equal differences except* $s=0$ *and* 1 *where no two differences are equal for* $s=1$.

MATHEMATICAL NOTES

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, In 47907.

ON INEQUALITIES GENERALIZING A FUNCTIONAL EQUATION CONNECTED WITH IVORY'S THEOREM

HIROSHI HARUKI, University of Waterloo

1. Introduction. In previous papers (see [1], [2], [3]) we solved the following functional equation:

$$|f(x+y)-f(x-y)| = |f(x+\bar{y})-f(x-\bar{y})|, \tag{1}$$

where $f(z)$ is an entire function of z and x, y are complex variables.

It was proved in these papers [1], [2], [3] that the solutions of (1) are the following and only these:

$$f(z) = a \sin \alpha z + b \cos \alpha z + c,$$

or

$$f(z) = a \sinh \alpha z + b \cosh \alpha z + c,$$

or

$$f(z) = az^2 + bz + c,$$

where a, b, c are arbitrary complex constants and α is an arbitrary real constant.

(1) is connected with the following Ivory's Theorem in geometry:

For a family of confocal conics, let P, Q, R, S be the four vertices of a curvilinear rectangle formed by any four conics arbitrarily chosen. Then $\overline{PR}=\overline{QS}$ holds. (See [1].)

Geometrically speaking, (1) is the above Ivory's property and the three solutions of (1) characterize confocal conics.

Now, we consider the following four functional inequalities where x, y are complex variables:

$$|f(x+y)-f(x-y)| \leqq |f(x+\bar{y})-f(x-\bar{y})| \quad \text{with } |x| < +\infty \text{ and } \operatorname{Im}(y^2) \geqq 0, \tag{2}$$

$$|f(x+y)-f(x-y)| \geqq |f(x+\bar{y})-f(x-\bar{y})| \quad \text{with } |x| < +\infty \text{ and } \operatorname{Im}(y^2) \geqq 0, \tag{3}$$

$$|f(x+y)-f(x-y)| \leqq |f(x+\bar{y})-f(x-\bar{y})| \quad \text{with } |x| < +\infty \text{ and } \operatorname{Im}(y^2) \leqq 0, \tag{4}$$

$$|f(x+y)-f(x-y)| \geqq |f(x+\bar{y})-f(x-\bar{y})| \quad \text{with } |x| < +\infty \text{ and } \operatorname{Im}(y^2) \leqq 0. \tag{5}$$

We can easily prove that (2) and (5) are mutually equivalent and that (3) and (4) are mutually equivalent.

In this paper we shall solve (2), (3) which are extensions of (1) and then by the results obtained we shall solve (1).

THEOREM. *If $f(z)$ is an entire function of z, then* (i) *the solutions of* (2) *are the following and only these:*

$$(z) = a \sinh((\beta + i\gamma)z) + b \cosh((\beta + i\gamma)z) + c,$$

or

$$f(z) = az^2 + bz + c,$$

where a, b, c are arbitrary complex constants and β, γ are arbitrary real constants with $\beta \geqq 0$, $\gamma \geqq 0$,

(ii) *the solutions of* (3) *are the following and only these:*

$$f(z) = a \sinh((\beta - i\gamma)z) + b \cosh((\beta - i\gamma)z) + c,$$

or

$$f(z) = az^2 + bz + c,$$

where a, b, c are arbitrary complex constants and β, γ are arbitrary real constants with $\beta \geqq 0$, $\gamma \geqq 0$.

To prove the Theorem we shall use the following two lemmas:

LEMMA 1. *Suppose that $H(z)$ is an entire function of z. If $A(t) = |H(te^{i\phi})|^2$ where t, ϕ are real and ϕ is fixed, then we have*

$$\text{(6)} \quad \begin{aligned} &\text{(i)} \ A''(0) = 2\,\mathrm{Re}(e^{2i\phi}H''(0)\overline{H(0)}) + 2|H'(0)|^2, \\ &\text{(ii)} \ A^{(4)}(0) = 2\,\mathrm{Re}(e^{4i\phi}H^{(4)}(0)\overline{H(0)} + 4e^{2i\phi}H'''(0)\overline{H'(0)}) + 6|H''(0)|^2. \end{aligned}$$

Proof. Since it is easy, we omit it.

LEMMA 2. *Suppose that $p(t)$ is a one-valued real function of t and is four times differentiable in $|t| < +\infty$. If $p(t)$ has a minimum at $t=0$ with $p'(0)=0$, $p''(0)=0$, $p'''(0)=0$, then $p^{(4)}(0) \geqq 0$.*

Proof. Since this lemma is familiar, we omit the proof.

2. Proof of the Theorem.

Proof of (i). Squaring both sides of (2) and putting $y = te^{i\phi}$ where t, ϕ are real and ϕ is fixed with $0 < \phi < \pi/2$ (this y satisfies $\mathrm{Im}(y^2) \geqq 0$), we have

$$\text{(7)} \qquad |f(x + te^{i\phi}) - f(x - te^{i\phi})|^2 \leqq |f(x + te^{-i\phi}) - f(x - te^{-i\phi})|^2.$$

Fixing x (complex), ϕ and putting

$$p(t) = |f(x + te^{-i\phi}) - f(x - te^{-i\phi})|^2 - |f(x + te^{i\phi}) - f(x - te^{i\phi})|^2,$$

$p(t)$ is a one-valued real function of t and of course is four times differentiable in $|t| < +\infty$. Further $p(t)$ is an even function of t. Hence we have

$$p'(0) = 0, \qquad p'''(0) = 0. \tag{8}$$

By Lemma 1 we have

$$p''(0) = 0 \qquad (\text{by } (6)(\text{i})), \tag{9}$$

$$p^{(4)}(0) = -64 \sin 2\phi \, \mathrm{Re}(if'''(x)\overline{f'(x)}) \qquad (\text{by } (6)(\text{ii})). \tag{10}$$

Since $0 < \phi < \pi/2$, we have

$$\sin 2\phi > 0. \tag{11}$$

By (2) $p(t)$ has a minimum at $t = 0$. Hence, by (8), (9) and by Lemma 2 we have

$$p^{(4)}(0) \geqq 0. \tag{12}$$

By (10), (11), (12) we have in $|x| < +\infty$

$$\mathrm{Im}(f'''(x)\overline{f'(x)}) \geqq 0. \tag{13}$$

We may assume that $f(x) \not\equiv \text{const.}$ Hence, by (13) we have in the domain where $f'(x) \neq 0$

$$\mathrm{Im}\left(\frac{f'''(x)}{f'(x)}\right) \geqq 0. \tag{14}$$

We put

$$F(x) = \exp\left(i \frac{f'''(x)}{f'(x)}\right). \tag{15}$$

If $f'(x_0) \neq 0$, by (14), (15) we have $|F(x_0)| \leqq 1$.

Next, if $f'(x_0) = 0$, by $f'(x) \not\equiv \text{const.}$ there exists a positive constant δ such that we have $f'(x) \neq 0$ in $0 < |x - x_0| < \delta$. Hence, by (14), (15) we have $|F(x)| \leqq 1$ in $0 < |x - x_0| < \delta$. So, by Riemann's theorem concerning a removable singularity, $F(x)$ is regular at $x = x_0$ with $|F(x_0)| \leqq 1$. Hence, $F(x)$ is an entire function of x and $|F(x)| \leqq 1$ in $|x| < +\infty$. Hence, by Liouville's theorem $F(x)$ is a complex constant K with $|K| \leqq 1$. Hence we have $f'''(x)/f'(x) = A$, where A is a complex constant with $\mathrm{Im}(A) \geqq 0$ (by (14)).

Solving this differential equation, we have

$$f(z) = a \sinh((\beta + i\gamma)z) + b \cosh((\beta + i\gamma)z) + c,$$

or $f(z) = az^2 + bz + c$, where a, b, c are arbitrary complex constants and β, γ are arbitrary real constants with $\beta \geqq 0$, $\gamma \geqq 0$.

Conversely, it is easily checked that these two functions satisfy our original inequality (2).

Proof of (ii). Putting $f^*(z)=\overline{f(\bar{z})}$, $f^*(z)$ is an entire function of z and by (3) $f^*(z)$ satisfies the following functional inequality (2′)

$$(2') \qquad |f^*(x+y)-f^*(x-y)| \leqq |f^*(x+\bar{y})-f^*(x-\bar{y})|,$$

where x, y are complex variables with $|x|<+\infty$ and $\mathrm{Im}(y^2)\geqq 0$.

Hence, by (2′) and by Theorem (i), Theorem (ii) is proved.

COROLLARY. *If $f(z)$ is an entire function of z and satisfies the preceding functional equation* (1), *then the solutions of* (1) *are the following and only these:*

$$f(z) = a\sin\alpha z + b\cos\alpha z + c,$$

or $f(z)=a\sinh\alpha z+b\cosh\alpha z+c$, or $f(z)=az^2+bz+c$, where a, b, c are arbitrary complex constants and α is an arbitrary real constant.

Proof. By the Theorem, the proof is clear.

I wish to thank the referee for his many helpful suggestions.

References

1. H. Haruki, On Ivory's Theorem, Mathematica Japonicae, No. 4, 1 (1949) 151.
2. ———, Studies on certain functional equations from the standpoint of analytic function theory, Science Reports, Osaka University, No. 1, 14 (1965).
3. ———, On the functional equations $|f(x+iy)|=|f(x)+f(iy)|$ and $|f(x+iy)|=|f(x)-f(iy)|$ and on Ivory's Theorem, Canad. Math. Bull., No. 4, 9 (1966).

A GENERATING FUNCTION

H. L. MANOCHA, Panjab Engineering College, Chandigarh (India)

1. Carlitz [1] has proved

$$(1) \quad \sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n}(x-1)^n(y-1)^n t^n P_n^{(\alpha-n,\beta-n)}\left(\frac{x+1}{x-1}\right)P_n^{(\beta-n,\alpha-n)}\left(\frac{y+1}{y-1}\right)$$

$$= (1-xt)^{\alpha}(1-yt)^{\beta}{}_2F_1\left[-\alpha,-\beta;-\alpha-\beta;\frac{(x-1)(y-1)t}{(1-xt)(1-yt)}\right],$$

where the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is defined as [3, pp. 254–255]

$$(2) \quad P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n,-\beta-n;1+\alpha;\frac{x-1}{x+1}\right)$$

$$(3) \qquad = \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n,-\beta-n;-\alpha-\beta-2n;\frac{2}{x+1}\right).$$

The aim of this note is to generalize (1) into

$$\sum_{n=0}^{\infty}\frac{(m+n)!}{(-\alpha-\beta-m)_n}P_{m+n}^{(\alpha-n,\beta-n)}(x)P_{m+n}^{(\alpha-n,\beta-n)}(y)t^n$$

$$(4)\quad = \frac{1}{m!}(1+\alpha)_m(1+\alpha+\beta+m)_m[\tfrac{1}{4}(x+1)(y+1)]^m[1-\tfrac{1}{4}(x-1)(y-1)t]^{\beta+m}$$

$$\cdot F_2\Big[-\alpha;\ 1,\ -\beta-m;\ 1+m,\ -\alpha-\beta-2m;$$

$$\tfrac{1}{4}(x+1)(y+1)t,\ \frac{-t}{1-\tfrac{1}{4}(x-1)(y-1)t}\Big],$$

where

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n}x^my^n.$$

2. At the outset we observe that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\alpha)_n}F_4(1-\alpha-n,\beta;1-\lambda-n,\mu;x,y)t^n$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\lambda)_{n-i}(\beta)_{i+j}}{i!j!(\alpha)_{n-i-j}(\mu)_j}t^nx^i(-y)^j$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_{n+k}(\beta)_k}{k!(\alpha)_n(\mu)_k}t^n(-yt)^k\sum_{i=0}^{\infty}\frac{(\beta+k)_i}{i!}(xt)^i$$

from which it follows that

$$(5)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\alpha)_n}F_4(1-\alpha-n,\beta;\ 1-\lambda-n,\mu;\ x,y)t^n = (1-xt)^{-\beta}F_2\left(\lambda;\ 1,\beta;\ \alpha,\ \mu;\ t,\frac{-yt}{1-xt}\right).$$

In (5) we replace α, β, λ, μ, x and y by $1+m$,, $-\beta-m$, $-\alpha$, $-\alpha-\beta-2m$, $x(1-y)$ and $y(1-x)$, respectively. Then using [2, p. 238]

$$(6)\quad F_4[\alpha,\gamma+\gamma'-\alpha-1;\gamma,\gamma';x(1-y),y(1-x)] = {}_2F_1(\alpha,\gamma+\gamma'-\alpha-1;\gamma;x)\,{}_2F_1(\alpha,\gamma+\gamma'-\alpha-1;\gamma';y).$$

we get

$$\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{(1+m)_n}\,{}_2F_1(-m-n,-\beta-m;1+\alpha-n;x)$$

$$(7)\quad \cdot{}_2F_1(-m-n,-\beta-m;-\alpha-\beta-2m;y)t^n = [1-x(1-y)t]^{\beta+m}$$

$$\cdot F_2\Big[-\alpha;1,-\beta-m;1+m,-\alpha-\beta-2m;t,\frac{-y(1-x)t}{1-x(1-y)t}\Big].$$

Now, with the help of (2) and (3) we set x, y and t in (7), so that we arrive at

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{(-\alpha-\beta-m)_n} P_{m+n}^{(\alpha-n,\beta-n)}(x) P_{m+n}^{(\alpha-n,\beta-n)}(y) t^n$$

$$= \frac{1}{m!}(1+\alpha)_m(1+\alpha+\beta+m)_m[\tfrac{1}{4}(x+1)(y+1)]^m[1-\tfrac{1}{4}(x-1)(y-1)t]^{\beta+m}$$

$$\cdot F_2\Big[-\alpha; 1, -\beta-m; 1+m, -\alpha-\beta-2m;$$

$$\cdot \tfrac{1}{4}(x+1)(y+1)t, \frac{-t}{1-\tfrac{1}{4}(x-1)(y-1)t}\Big].$$

This completes the proof of (4).

Putting $m=0$ and using the relation [2, p. 238]

$$F_2(\alpha; \beta, \beta'; \beta, \gamma'; x, y) = (1-x)^{-\alpha} {}_2F_1\left(\alpha, \beta'; \gamma'; \frac{y}{1-x}\right), \tag{8}$$

(4) reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x) P_n^{(\alpha-n,\beta-n)}(y) t^n$$

$$= [1-\tfrac{1}{4}(x+1)(y+1)t]^{\alpha}[1-\tfrac{1}{4}(x-1)(y-1)t]^{\beta} \tag{9}$$

$$\cdot {}_2F_1\left[-\alpha, -\beta; -\alpha-\beta; \frac{-t}{(1-\tfrac{1}{4}(x+1)(y+1)t)(1-\tfrac{1}{4}(x-1)(y-1)t)}\right].$$

Now, if we replace x, y and t by $(x+1)/(x-1)$, $-(y+1)/(y-1)$ and $-(x-1)(y-1)t$, respectively, and apply the relation [3, p. 256]

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \tag{10}$$

we get (1).

I wish to record my sincere thanks to the referee for suggestions which led to a better presentation.

References

1. L. Carlitz, A bilinear generating function for the Jacobi polynomials, Boll. Un. Mat. Ital. (3), 18 (1963) 87–89.
2. A. Erdelyi, Higher Transcendental Functions, vol. I, McGraw-Hill, New York, 1954.
3. E. D. Rainville, Special Functions, Macmillan, New York, 1960.

ON LINEAR DIFFERENCE EQUATIONS

K. S. MILLER, Columbia University Electronics Research Laboratories

1. Introduction. In this note we shall adumbrate some portions of the theory of linear difference equations analogous to those in the field of linear differential equations.

Let $y(x)$ be a real-valued function of the real variable x defined on $(-\infty, \infty)$. We shall use the difference operator Δ to mean $\Delta y(x) = y(x+1) - y(x)$ for all x, the translation operator E to mean $Ey(x) = y(x+1)$, and the identity operator I to mean $Iy(x) = y(x)$. If $p_i(x)$, $0 \leqq i \leqq n$, are defined for all x, then

$$L(x) = p_0(x)\Delta^n + p_1(x)\Delta^{n-1} + \cdots + p_n(x)I \tag{1.1}$$

is a *linear difference operator*. We may also write $L(x)$ as

$$L(x) = P_0(x)E^n + P_1(x)E^{n-1} + \cdots + P_n(x)I. \tag{1.2}$$

Equation (1.1) or (1.2) is said to be of the nth order if $P_0(x)P_n(x) \neq 0$ for all x, [1; page 116]. In this case we may construct the one-sided Green's function $H(x, \xi)$, [2; page 146], for $L(x)$.

2. The adjoint operator. Let $L(x)$ be as above. Then the *adjoint operator* $L^*(x)$ is defined by the form

$$L^*(x)[y(x)] = (-1)^n\Delta^n[p_0(x)y(x)] + (-1)^{n-1}\Delta^{n-1}[p_1(x+1)y(x+1)] + \cdots + p_n(x+n)y(x+n),$$

[see 1; page 245]. An equivalent expression for $L^*(x)$ is

$$L^*(x) = \overset{*}{P_0}(x)E^n + \overset{*}{P_1}(x)E^{n-1} + \cdots + \overset{*}{P_n}(x)I.$$

The adjoint, as we have just defined it, has many of the properties analogous to those of the adjoint *differential* operator. We shall exhibit some of these results in the remainder of this paper.

The relation between $P_i^*(x)$ and $P_i(x)$ is readily obtained as $P_j(x) = P_{n-j}^*(x-j)$, $0 \leqq j \leqq n$. In particular one may conclude that $P_0(x)P_n(x) \neq 0$ for all x implies $P_0^*(x)P_n^*(x) \neq 0$ for all x. Thus if $L(x)$ is an nth order linear difference operator, then so is $L^*(x)$.

Using the above results we may prove the following theorems:

THEOREM 1. *Let $u(x)$ and $v(x)$ be defined for all x, and let $L(x)$ and $L^*(x)$ be defined as above. Then Lagrange's identity is*

$$v(x+n)L(x+n)[u(x)] - u(x)L^*(x)[v(x)] = \Delta\pi[u(x), v(x)]$$

where $\pi[u(x), v(x)]$, the Lagrange bilinear concomitant, is given by

$$\pi[u(x), v(x)] = \sum_{k=1}^{n}\sum_{j=0}^{k-1}(-1)^{k-j-1}\Delta^j u(x)\Delta^{k-j-1}[p_{n-k}(x+n+j-k)v(x+n+j-k)].$$

THEOREM 2. *Let* $L(x)$, [*eq.* (1.1)], *be an n-th order linear difference operator. Let* $L^*(x)$ *be the adjoint of* $L(x)$. *Let* $H(x, \xi)$ *be the one-sided Green's function for* $L(x)$ *and* $H^*(x, \xi)$ *the one-sided Green's function for* $L^*(x)$. *Let* $\xi \equiv \zeta$ (mod 1). *Then*

$$H^*(\xi - 1, \zeta) = -H(\zeta + n - 1, \xi).$$

If $L^{**}(x)$ is the adjoint of $L^*(x)$, then from the above equation we may deduce that $H^{**}(\xi, \zeta) = H(\xi+n, \zeta+n)$ where $H^{**}(\xi, \zeta)$ is the one-sided Green's function for $L^{**}(x)$.

3. Fundamental sets of solutions. Let $L(x)$ be a linear difference operator of the nth order. Let $\phi_i(x)$, $1 \leqq i \leqq n$, be n solutions of $L(x)[\phi(x)] = 0$ with the property that their Casorati is unequal to zero for all x. Then $\{\phi_i(x) \mid 1 \leqq i \leqq n\}$ is called a *fundamental set of solutions*. Every fundamental set of solutions is linearly independent [2; page 141]; the converse is not necessarily true.

From Theorem 2 of the previous section we may prove:

THEOREM 3. *Let* $L(x)$ *be an n-th order linear difference operator. Let* $H(x, \xi)$ *be its one-sided Green's function. Let* ξ_0 *be any constant. Then* $\{\psi_i(x) \mid 1 \leqq i \leqq n\}$ *forms a fundamental set of solutions of* $L(x)[\psi(x)] = 0$ *where*

$$\psi_{k+1}(x) = E_\xi^k H(x, \xi_0), \qquad 0 \leqq k \leqq n - 1.$$

(The subscript on E indicates the variable on which the operator acts.)

4. Composition of operators. If $L(x)$ and $M(x)$ are linear difference operators, then we define the composition of $M(x)$ with $L(x)$ as $LM(x)$ where

$$LM(x)[y(x)] = L(x)\{M(x)[y(x)]\}$$

for all $y(x)$, $-\infty < x < \infty$. Then:

THEOREM 4. *Let* $L(x)$ *be an n-th order and* $M(x)$ *an m-th order linear difference operator. Let* $LM(x)$ *be the composition of* $M(x)$ *with* $L(x)$. *Then*

$$H_{LM}(x, \xi) = \sum_{\zeta=\xi+1}^{x} H_L(\zeta - 1, \xi) H_M(x, \zeta)$$

where H_L, H_M, *and* H_{LM} *are the one-sided Green's functions for* $L(x)$, $M(x)$ *and* $LM(x)$ *respectively*; *and* $x \equiv \xi$ (mod 1).

5. Conclusion. Using the above results one may deduce various other properties of linear difference operators. For example, we may solve $L(x)[u(x)] = M(x)[v(x)]$ for $u(x)$; and establish certain properties of self-adjoint difference operators. (We say that $L(x)$ is formally self-adjoint if $L(x) = (-1)^n L^*(x)$.)

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References

1. T. Fort, Finite differences and difference equations in the real domain, Oxford University Press, Oxford, 1948.
2. K. S. Miller, An introduction to the calculus of finite differences and difference equations, Dover, New York, 1966.

TWO RESULTS CONCERNING DOUBLY STOCHASTIC MATRICES

RICHARD SINKHORN, University of Houston

1. Introduction. An $n\times n$ matrix $A=(a_{ij})$ is said to be doubly stochastic if

$$a_{ij} \geqq 0 \quad \text{and} \quad \sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} = 1 \qquad \text{for } i, j = 1, \cdots, n.$$

The set of $n\times n$ doubly stochastic matrices is denoted by Ω_n. The particular member of Ω_n whose elements are all $1/n$ is denoted by J_n. The $n\times n$ identity matrix is denoted by I_n.

An $n\times n$ matrix A with nonnegative elements is said to be reducible if there exists a permutation matrix P such that P^TAP has the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B and D are square. Otherwise A is said to be irreducible.

It is the intent of this paper to give a characterization of the idempotent members of Ω_n, i.e. those doubly stochastic A such that $A^2=A$, and to determine necessary and sufficient conditions for reducibility in Ω_n. Both results are consequences of the following version of the Perron-Frobenius theorem [1, pp. 53–54, 78–79].

THEOREM 1. *Let A be an $n\times n$ matrix with nonnegative elements and let r be the maximal characteristic root of A. There exists an n dimensional vector x and an n dimensional vector y with positive components such that $Ax=rx$ and $A^Ty=ry$ if and only if there exists a permutation matrix P such that*

$$P^TAP = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{bmatrix}$$

where each A_k is square and irreducible with maximal root r. The number s is the multiplicity of the root r in the characteristic equation of A. A is irreducible if and only if r is a simple root and x and y exist.

2. Idempotents in Ω_n.

THEOREM 2. *$A \in \Omega_n$ is idempotent if and only if there exist positive integers $n_1, \cdots, n_s$ with sum n and a permutation matrix P such that*

$$A = P \begin{bmatrix} J_{n_1} & 0 & \cdots & 0 \\ 0 & J_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_s} \end{bmatrix} P^T.$$

Proof: The sufficiency is clear since $J_{n_k}^2 = J_{n_k}$ for $k=1, \cdots, s$. The proof of necessity follows.

Suppose $A \in \Omega_n$ is idempotent, i.e. $A^2 - A = 0$. If $m(x)$ is the minimum function of A, $m(x) \mid x^2 - x$. Thus the eigenvalues of A are 1 and/or 0 with appropriate multiplicities and A is similar to a diagonal matrix. Suppose the multiplicity of the root 1 is s. Since every $A \in \Omega_n$ has 1 as a root, $s \geqq 1$. Also 1 is the maximal root for any $A \in \Omega_n$. Thus by Theorem 1 there is a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{bmatrix}$$

where the A_k are each square and irreducible, and thus each has a simple root of 1. The other roots of each A_k must be 0, and therefore, since A is similar to a diagonal matrix, it follows that each A_k has rank one. But P and $P^T \in \Omega_n$ and thus so does $P^T A P \in \Omega_n$. This means that if A_k is $n_k \times n_k$, $A_k \in \Omega_{n_k}$. There is only one member of Ω_{n_k} of rank one, namely J_{n_k}. Thus $A_k = J_{n_k}$, $k=1, \cdots, s$, and the proof is finished.

3. A characterization of reducibility in Ω_n.

THEOREM 3. *A necessary and sufficient condition that $A \in \Omega_n$ be reducible is that $(A - I_n - J_n)$ be singular. If the nullity of $(A - I_n - J_n)$ is $s-1$, then there exists a permutation matrix P such that*

$$P^T A P = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{bmatrix}$$

where the A_k, $k=1, \cdots, s$, are irreducible and doubly stochastic.

Proof: Since

$$(I_n - J_n)(A - I_n - J_n) = A - I_n$$

the null space of $A-I_n-J_n$ is contained in the null space of $A-I_n$. Since the nullity of I_n-J_n is one, $\dim \text{Null}(A-I_n) \leqq 1+\dim \text{Null}(A-I_n-J_n)=s$ by Sylvester's law of nullity. However, the n dimensional vector

$$e = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

belongs to Null $(A-I_n)$ and not to Null $(A-I_n-J_n)$. Whence dim Null $(A-I_n) \geqq 1+\dim$ Null $(A-I_n-J_n)=s$. It follows that dim Null $(A-I_n)=s$.

It is known, e.g. see [1, p. 84], that the elementary divisors corresponding to the root 1 of a stochastic matrix (and therefore to a doubly stochastic matrix) are linear. Thus the multiplicity of the root 1 in the characteristic equation of A is dim Null $(A-I_n)=s$.

By Theorem 1 there is a permutation matrix P such that

$$P^TAP = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & A_s \end{pmatrix}$$

where each A_k is square and irreducible. Each A_k is doubly stochastic since P and $P^T \in \Omega_n$.

A is reducible if and only if $s>1$ and this holds if and only if dim Null $(A-I_n-J_n)>0$, i.e. if and only if $(A-I_n-J_n)$ is singular.

Reference

1. F. R. Gantmacher, The Theory of Matrices, vol. 2, Chelsea, New York, 1959.

SOLUTIONS OF $X^2=I$ FOR MATRICES OVER FINITE RINGS WITH UNITY

R. R. KORFHAGE, Purdue University

The matrix equation $X^2=I$ has been investigated by Hodges [1] and Reiner [5] for matrices over finite fields, and by Levine and Korfhage [2, 3, 4] for matrices over the ring of integers modulo n. In each of these papers the solutions to this equation are enumerated and characterized. In this paper we present a discussion of the solutions of this equation over an arbitrary finite ring with unity, which is presented as a direct product of such rings. In particular, we apply our results to a finite Boolean algebra, obtaining an enumeration of the solutions to $X^2=I$ in this case.

DEFINITION. *Let $R_1, \cdots, R_m$ be finite rings with unity, and for $k=1, \cdots, m$, let $M_k=(a_{ij}^k)$ be an $n \times n$ matrix over R_k, n fixed. Let $M=(a_{ij})$ be an $n \times n$ matrix*

over $R = R_1 \otimes R_2 \otimes \cdots \otimes R_m$. *We say that* M *is a direct product of* $M_1, \cdots, M_m$ *if for each entry* a_{ij} *of* M, $a_{ij} = (a_{ij}^1, a_{ij}^2, \cdots, a_{ij}^m)$. *We write* $M = M_1 \otimes M_2 \otimes \cdots \otimes M_m$.

THEOREM 1. *Let* $R_1, R_2, \cdots, R_m$ *be finite rings with unity, and let* $N_i(n)$ *be the number of solutions of the matrix equation* $X^2 = I$ *for* $n \times n$ *matrices over* R_i. *Then the number of solutions of this equation for* $n \times n$ *matrices over* $R = R_1 \otimes R_2 \otimes \cdots \otimes R_m$ *is* $N(n) = \prod_{i=1}^m N_i(n)$. *Moreover, if* $S_{i,j}$, $j = 1, 2, \cdots$, *are the similarity classes of solutions over* R_i, $i = 1, 2, \cdots, m$ *(for* $n \times n$ *matrices) then each similarity class of solutions over* R *is determined uniquely as the class of all direct products of matrices in some fixed similarity classes* $S_{1,j_1}, S_{2,j_2}, \cdots, S_{m,j_m}$.

Proof. The theorem follows directly from the fact that if A and B are matrices over R such that $A = A_1 \otimes \cdots \otimes A_m$ and $B = B_1 \otimes \cdots \otimes B_m$, then $AB = A_1B_1 \otimes \cdots \otimes A_mB_m$.

REMARK. The primary method of characterization of the solutions to $X^2 = I$ in the aforementioned papers is that of similarity classes: if a matrix A is a solution of the equation, then so is any matrix which is similar to A.

Now a Boolean algebra of 2^m elements is a direct product of m copies of the 2-element Boolean algebra. For $n \times n$ matrices over the 2-element algebra, it is known [1, 2] that the number of solutions of $X^2 = I$ is

$$T_{n,2} = g_n \sum_{t=0}^{[n/2]} \frac{2^{-t(2n-3t)}}{g_t g_{n-2t}},$$

where g_t is defined by

$$g_0 = 1$$

$$g_t = \prod_{k=0}^{t-1} (2^t - 2^k), \qquad t \geqq 1.$$

Moreover, the similarity classes are those defined by the matrices

$$H_t = \operatorname{diag}(I_{n-2t}, E_1, \cdots, E_t), \quad \text{where} \quad E_i = E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

each containing

$$S(n, t) = \frac{g_n}{g_t g_{n-2t}} 2^{-t(2n-3t)}$$

matrices.

Hence, applying our general result to this particular case, we may easily determine the solutions of this equation over a finite Boolean algebra.

THEOREM 2. *Let* B *be a Boolean algebra of* 2^m *elements. Then the number of* $n \times n$ *matrices over* B *satisfying* $X^2 = I$ *is*

$$T_n = \left(g_n \sum_{t=0}^{[n/2]} \frac{2^{-t(2n-3t)}}{g_t g_{n-2t}}\right)^m.$$

Each similarity class is the class of matrices similar to $H_{t_1} \otimes H_{t_2} \otimes \cdots \otimes H_{t_m}$ *for some fixed* $t_1, t_2, \cdots, t_m (t_i = 0, 1, \cdots, [n/2]; i = 1, 2, \cdots, m)$, *and contains exactly* $\prod_{i=1}^{m} S(n, t_i)$ *matrices.*

References

1. John H. Hodges, The matrix equation $X^2 - I = 0$ over a finite field, this MONTHLY, 65 (1958) 518–520.
2. Jack Levine and Robert R. Korfhage, Automorphisms of abelian groups induced by involutory matrices, Duke Math. J., 29 (1962) 631–646.
3. ———, and ———, Automorphisms of abelian groups induced by involutory matrices modulo $p > 2$, *ibid.*, 30 (1963) 161–170.
4. ———, and ———, Automorphisms of abelian groups induced by involutory matrices, general modulus, *ibid.*, 31 (1964) 631–654.
5. Irma Reiner, The matrix congruence $X^2 \equiv I (\mathrm{mod}\ p^a)$, this MONTHLY, 67 (1960) 773–775.

ON PRIMITIVITY OF MATRIX RINGS

R. N. GUPTA, Ramjas College, Delhi, India

E. C. Posner [2] has proved that a ring R is (right) primitive if and only if R_n, the ring of all $n \times n$ matrices over R is primitive. This result of Posner is proved in this note in a different way.

DEFINITION. *A ring* R [1, page 8] *is said to be (right) primitive iff there exists a maximal right ideal* I *of* R *such that* $(I: R) = (0)$.

THEOREM. *A ring* R *is primitive iff* R_n, *the ring of all* $n \times n$ *matrices over* R *is primitive.*

LEMMA. *If* I *be a maximal right ideal of a ring* R *such that* $(I: R) \neq R$ *and* $x \notin I$, *then* $I + xR = R$.

Proof: If $I + xR \neq R$, then $I + xR = I$, because I is maximal right ideal of R. Therefore $xR \subset I$. Consider the set $J = \{y: yR \subset I\}$. Clearly J is a right ideal of R containing I. The element $x \in J$ and $x \notin I$, therefore $J = R$, giving $R^2 \subset I$, which contradicts $(I: R) \neq R$.

Proof of the Theorem. Let R be primitive. There exists a maximal right ideal I of R such that $(I: R) = (0)$. We show that

$$J = \{[x_{ij}]: x_{1j} \in I, x_{ij} \in R \ \forall\, i \geqq 2\}$$

is a maximal right ideal of R_n and $(J: R_n) = (0)$. Let $[r_{ij}] \notin J$. We shall prove that $J + [r_{ij}]R_n = R_n$ and this proves that J is a maximal right ideal of R_n. Clearly $r_{1k} \notin I$ for some k. Let $Y = [y_{ij}]$ be any matrix of R_n. By the lemma, there exist $s_{11}, s_{12}, \cdots, s_{1n}$ in I and $t_{k1}, t_{k2}, \cdots, t_{kn}$ in R such that $s_{1i} + r_{1k}t_{ki} = y_{1i}$ for all

$i=1, 2, 3, \cdots, n$. Let S denote the matrix having its first row as $(s_{11}, s_{12}, \cdots, s_{1n})$ and the remaining elements zero. Then $S \in J$. Let T be the matrix whose kth row is $(t_{k1}, t_{k2}, \cdots, t_{kn})$ and the remaining elements zero. The first row of $Y' = S + [r_{ij}]T$ is $(y_{11}, y_{12}, \cdots, y_{1n})$. Therefore $Y - Y' \in J$, since its first row is zero. Also $Y' \in J + [r_{ij}]R_n$. Therefore $Y \in J + [r_{ij}]R_n$. Now we show that $(J: R_n) = (0)$. Let $[s_{ij}] \in (J: R_n)$, which gives $rE_{1k}[s_{ij}] \subset J$ for all k and for all r in R. Therefore $rs_{kl} \subset I$ for all l. Now $Rs_{kl} \subset I$ for all k, l. Hence $s_{kl}=0$ for all k, l because $(I: R) = (0)$. Therefore $[s_{ij}] = 0$.

Conversely, let R_n be primitive. There exists a maximal right ideal J of R_n such that $(J: R_n) = (0)$. Then $rE_{kl} \notin J$ for some r in R and some k, l. Let $I = \{x : x \in R,\ r \times E_{kl} \in J\}$. We assert that $(I: R) = (0)$. For if $s \in (I: R)$, then $Rs \subset I$. We shall prove that $R_n s E_{ll} \subset J$, which implies $sE_{ll} = 0$, $s = 0$. Let $[x_{ij}] \in R_n$. By the lemma there exist $[y_{ij}]$ in J and $[r_{ij}]$ in R_n such that $[y_{ij}] + rE_{kl}[r_{ij}] = [x_{ij}]$. Post multiplying by sE_{ll} we get $[x_{ij}]sE_{ll} = [y_{ij}]sE_{ll} + rE_{kl}[r_{ij}]sE_{ll}$, giving $[x_{ij}]sE_{ll} = [y_{ij}]sE_{ll} + rr_{ll}sE_{kl}$. Now since $Rs \subset I$, $rr_{ll}sE_{kl} \in J$. Therefore $[x_{ij}]sE_{ll} \in J$. Since $R_n sE_{ll} \subset J$, $(I: R) = (0)$. Clearly therefore $I \neq R$. Now we show that I is a maximal right ideal of R. Let $x \notin I$ so that $r \times E_{kl} \notin J$. Again by the lemma, there exist $[z_{ij}]$ in J and $[t_{ij}]$ in R_n such that $rE_{kl} = [z_{ij}] + r \times E_{kl}[t_{ij}]$. Let $y \in R$. Postmultiplying by yE_{ll}, we get $ry\,E_{kl} = r \times t_{ll} y E_{kl} + [z_{ij}]yE_{ll}$, so that $r(y - xt_{ll}y)E_{kl} \in J$. Therefore $y - x(t_{ll}y) \in I$. $y \in I + xR$. $I + xR = R$. Thus I is maximal. Hence R is primitive.

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References

1. N. Jacobson, On the theory of primitive rings, Ann. Math., 48 (1947) 8–21.
2. E. C. Posner, Primitive matrix rings, Arch. Math., 12 (1961) 97–101.

GENERAL LINEAR SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

J. W. Evans, Melpar, Falls Church, Va.

Rainville [1] gives the following technique for solution of nth order linear differential equations. Let

$$D^n x = \sum_{k=0}^{n-1} a_k(t) D^k x + f(t), \qquad D = d/dt. \tag{1}$$

Suppose y is any particular solution of the corresponding homogeneous system

$$D^n y = \sum_{k=0}^{n-1} a_k(t) D^k y. \tag{2}$$

Let $x = yz$ define a new variable $z(t)$. Substitution into (1), followed by application of (2), yields a linear differential equation of order $n-1$ in the variable Dz.

Our purpose is to generalize this technique to nth order systems of first order

linear differential equations. Note that the smallest such is merely a single first order general linear differential equation, and so is easily solved.

Given an nth order matrix, $A(t)=(a_{ij}(t))$, and n-vector functions $\phi(t)$ and $y(t)$ such that $Dy=Ay$, find $x(t)$ such that

$$Dx = Ax + \phi. \tag{3}$$

Without loss of generality suppose that y_1 is almost everywhere nonzero. Consider the change of variables defined by the vector equation

$$x = z_1 y + z^*, \tag{4}$$

where $z^*=\text{col}\,(0, z_2, \cdots, z_n)$. Equating the expressions for Dx obtained from (3) and from differentiation of (4) results in the equation

$$(Dz_1 + z_1 Dz)y + Dz^* = A(z_1 y + z^*) + \phi.$$

Since $Dy=Ay$, this reduces to

$$(Dz_1)y + Dz^* = Az^* + \phi. \tag{5}$$

The first of the scalar equations described by (5),

$$(Dz_1)y_1 = \sum_{q=2}^{n} a_{1q}z_q + \phi_1, \tag{6}$$

may be solved for Dz_1. Substitution of this expression into the remainder of equations (5) yields the reduced system

$$y_1 Dz_p = y_1\left(\sum_{q=2}^{n} a_{pq}z_q + \phi_p\right) - y_p\left(\sum_{q=2}^{n} a_{1q}z_q + \phi\right), \tag{7}$$

where $2\leqq p\leqq n$. Substitution of the solution of this $(n-1)$-st order system in equation (6) provides a single first order linear equation. Equation (5) then determines the desired general solution of (3).

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Reference

1. Earl D. Rainville, Intermediate Course in Differential Equations, Wiley, New York, 1943.

POLYNOMIAL INTERPOLATION OVER COMMUTATIVE RINGS

ROBERT SPIRA, Michigan State University

The question of the existence of an interpolating polynomial is of interest not only for integral domains such as the integers, but also for rings with zero-divisors such as the integers mod m (Carlitz [1]). In this note, a necessary and sufficient condition for the existence of such a polynomial in $R[x]$ is found for a general commutative ring R.

The topic is connected with divided differences, but since we are in a ring, some care must be taken in defining the exact expressions for the divided differences. Moreover, in a general commutative ring, divided differences may not be unique. For instance, if $a_0, a_1, b_0, b_1 \in R$, $b_0 \neq b_1$, if $b_0 - b_1 | a_0 - a_1$, we can define a first divided difference as some $c \in R$ such that $a_0 - a_1 = c(b_0 - b_1)$, but c is not necessarily unique. It may even happen that some choices of c permit higher divided differences to be defined, while other choices of c may not.

Let $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_n$ be elements of R, the b_j's distinct. We define *a set of divided differences* of the a_j's by the b_j's as a set of pairs $\langle I, P(I)\rangle$, where $P(I)$is a ring element defined below and I is an ordered k-tuple ($k \geqq 1$) of distinct integers from $0, 1, \cdots, n$. $P(I)$ is defined sequentially for $k = 1, 2, \cdots, n$ as follows:

For $k=1$, $P(I) = a_j$, where j is the single element of I.

For $k>1$, let I^* be I except for its last element k and I_* be I except for its first element j. If $P(I^*)$ and $P(I_*)$ have been defined, we choose, if possible, $P(I)$ to satisfy $P(I^*) - P(I_*) = (b_j - b_k) \cdot P(I)$.

It is clear that a set of divided differences is never empty, and that in an integral domain it is unique, since there is cancellation. We say a set of divided differences of the a_j's by the b_j's is *complete* if it is defined for all I with elements taken from $\{0, \cdots, n\}$. We now state the main result.

THEOREM. *If R is a commutative ring, and $a_0, \cdots, a_n, b_0, \cdots, b_n$ are in R, the b_j's distinct, then there is a polynomial $f(x)$ in $R[x]$ such that $f(b_j) = a_j$, $j = 0, \cdots, n$, if and only if there is a complete set of divided differences of the a_j's by the b_j's.*

Proof. Let $f(x) = A_0 + A_1 x + \cdots$ be a polynomial in $R[x]$, such that $f(b_j) = a_j$, $j = 0, \cdots, n$. In Steffensen [2, pp. 16–19] (and in other books on interpolation) it is shown that the divided differences can be expressed as polynomials symmetric in the b_j's, and this proof carries directly over into the ring situation (using commutativity). For example, we have

$$\begin{aligned} a_j - a_k &= P(\langle j\rangle) - P(\langle k\rangle) \\ &= (b_j - b_k)[A_1 + A_2(b_j + b_k) + A_3(b_j^2 + b_j b_k + b_k^2) + \cdots] \end{aligned}$$

and we can take $P(\langle j, k\rangle)$ as the polynomial $A_1 + A_2(b_j + b_k) + \cdots$. In general, if I has t elements, we can choose $P(I)$ as

$$A_{t-1} + A_t\left(\sum_{j\in I} b_j\right) + \cdots.$$

Now suppose we are given a complete set of divided differences of the a_j's by the b_j's, $\{\langle I, P(I)\rangle\}$. If I is a k-tuple which is a naturally ordered segment of integers with first element j and last element k, we write $P(I) = P_{j,k}$. We form the polynomial:

$$f(x) = a_0 + P_{0,1}(x - b_0) + P_{0,2}(x - b_0)(x - b_1) + \cdots$$
$$+ P_{0,n}(x - b_0)(x - b_1) \cdots (x - b_{n-1}).$$

That this polynomial satisfies the conditions of the theorem requires a complicated algebraic proof, which we only describe. One takes the last term of $f(b_k)$ and expands it into two terms by the identity

$$P_{r,k}(b_k - b_r) = - P_{r,k-1} + P_{r+1,k}$$

and combines the $P_{r,k-1}$ term with the previous term, where a cancellation will occur giving a term $P_{r,k-1}(b_{k-1}-b_r)$. One repeats this process all the way down to the term which originally had one (b_s-b_t) factor. Starting back with the new last term, one chains down again, etc. Finally what remains is $a_0+(a_1-a_0) + \cdots +(a_k-a_{k-1})=a_k$. We illustrate the process for $k=2$:

$$\begin{aligned} f(b_2) &= a_0 + P_{0,1}(b_2 - b_0) + P_{0,2}(b_2 - b_0)(b_2 - b_1) \\ &= a_0 + P_{0,1}(b_2 - b_0) - P_{0,1}(b_2 - b_1) + P_{1,2}(b_2 - b_1) \\ &= a_0 + P_{0,1}(b_1 - b_0) + P_{1,2}(b_2 - b_1) \\ &= a_0 + (a_1 - a_0) + (a_2 - a_1) = a_2. \end{aligned}$$

The proof also shows that if any such polynomial exists, the Newton interpolating polynomial will work, and that only the existence of the differences $P_{j,k}$ is sufficient.

References

1. L. Carlitz, Functions and polynomials (mod p^n), Acta Arithmetica, 9 (1964) 67–78.
2. J. F. Steffensen, Interpolation, 2nd ed., Chelsea, New York, 1950.

ON A LEMMA OF SUGAWARA

R. W. WEST, University of California, Los Angeles

One of the main concepts in homotopy theory is that of an H-space, a generalization of a topological group, and is defined as follows. If X is a topological space with base point $*$, let i_1: $X\to X\times X$ be the usual injection map given by $i_1(x)=(x, *)$, $x\in X$, and define i_2 similarly. A (base point preserving) multiplication map μ: $X\times X\to X$ on X is said to be an *H-structure* if the compositions μi_1 and μi_2 are homotopic (relative to the base point) to the identity map 1: $X\to X$; the pair (X, μ) is then called an *H-space*. For some standard examples and properties of H-spaces, see Section 5 of Chapter I of [1].

The following result, Lemma (1.2) of [3], is a useful criterion for a space to be an H-space.

PROPOSITION (Sugawara). *Let ν be a multiplication map on a connected CW complex X. If the maps f_k: $X\times X\to X\times X$ defined by $f_k(x_1, x_2)=(\nu(x_1, x_2), x_k)$, k*

$=1, 2$, *are homotopy equivalences, then* X *admits a multiplication map* μ *such that* (X, μ) *is an H-space.*

Sugawara proved this by using almost everything established in his earlier paper [2]. The purpose of our note is to present an easy direct proof of this result.

REMARK 1. The proposition does *not* assert that ν itself is an H-structure. To demonstrate this take (X, ν) to be a *weak* H-space which is not an H-space with X a CW complex; such a space exists by Theorem (1.1) of [4]. In this case the maps f_k will induce isomorphisms of all homotopy groups and hence will be homotopy equivalences by [1, Cor. 24, p. 405].

REMARK 2. It is natural to ask about the nonconnected case. For concreteness suppose X has two components X_0 and X_1, the base point belonging to X_0. Then any H-structure μ on X_0 extends to an H-structure on X as follows: let $\mu: X_1 \times X_0 \to X_1$ and $\mu: X_0 \times X_1 \to X_1$ be the projection maps, and let $\mu: X_1 \times X_1 \to X$ be completely arbitrary. Therefore the question of existence of H-structures on X is the same as that of H-structures on the component of the base point. It is also worthwhile to notice that if $\mu(X_1 \times X_1) \subset X_1$ then $f_k(X \times X)$ does not meet $X_0 \times X_1$, $k=1, 2$, and hence *neither* f_1 nor f_2 are homotopy equivalences.

Proof of Proposition. The multiplication ν induces a product in the homotopy groups

$$\nu_*: \pi_q(X) \times \pi_q(X) = \pi_q(X \times X) \to \pi_q(X)$$

and we write $\nu_*(\alpha, \beta) = \alpha\beta$. It follows that

$$f_{2*}: \pi_q(X) \times \pi_q(X) \to \pi_q(X) \times \pi_q(X)$$

is the homomorphism given by $f_{2*}(\alpha, \beta) = (\alpha\beta, \beta)$. Let $p_1: X \times X \to X$ be projection onto the first factor. Since $\nu i_1 = p_1 f_2 i_1: X \to X$ the following diagram is commutative:

$$\begin{tikzcd}
\pi_q(X) \times \pi_q(X) \arrow[r, "f_{2*}"] & \pi_q(X) \times \pi_q(X) \arrow[d, "p_{1*}"] \\
\pi_q(X) \arrow[u, "i_{1*}"] \arrow[r, "(\nu i_1)_*"] & \pi_q(X)
\end{tikzcd}$$

Suppose now that f_2 is a homotopy equivalence. Then f_{2*} is an isomorphism and we assert that $(\nu i_1)_*$ is too. Let $0 \in \pi_q(X)$ be the group identity, and let $\beta \in \pi_q(X)$. Then there exists $(\alpha, \gamma) \in \pi_q(X) \times \pi_q(X)$ such that

$$(\beta, 0) = f_{2*}(\alpha, \gamma) = (\alpha\gamma, \gamma).$$

Hence $\gamma = 0$ and $(\nu i_1)_*\alpha = p_{1*} f_{2*} i_{1*}(\alpha) = \beta$, so $(\nu i_1)_*$ is onto. If $(\nu i_1)_*\alpha = 0$, then $p_{1*} f_{2*}(\alpha, 0) = 0$. It follows that $f_{2*}(\alpha, 0) = (0, 0) = f_{2*}(0, 0)$. Hence $\alpha = 0$ and $(\nu i_1)_*$ is an isomorphism.

Since X is a CW complex, νi_1 is a homotopy equivalence by the equivalence theorem of J. H. C. Whitehead, Corollary 24, page 405, of [1]. A similar argument shows that if f_1 is a homotopy equivalence then so is νi_2. Let $g_k: X \to X$ be a homotopy inverse to νi_k, $k=1, 2$, and define μ: $X \times X \to X$ by $\mu = \nu(g_1 \times g_2)$. Then

$$\mu i_k = \nu(g_1 \times g_2) i_k = \nu i_k g_k, \qquad k = 1, 2,$$

is homotopic to 1 and (X, μ) is an H-space.

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References

1. E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.
2. M. Sugawara, On a condition that a space is an H-space, Math. J. Okayama Univ., 6 (1957) 109–129.
3. ———, On the homotopy-commutativity of groups and loop spaces, Mem. Coll. Sci. Univ. Kyoto, ser. A, 33 (1960) 257–269.
4. R. W. West, Weak H-spaces, Jour. Math. and Mechs., 17 (1967) 421–432.

EXTENDING A COMPLETE METRIC

PHILIP BACON, University of Florida

Independently, Hausdorff [2] and Bing [1] showed that if X is a metrizable topological space, if A is a closed subset of X and if d is a metric for A (compatible with the relative topology for A), there is an extension of d to a metric on all of X. We use this extension theorem here to prove a similar result. Call a metrizable space X *topologically complete* if there exists a complete metric for X (every Cauchy sequence converges to a point of X).

THEOREM. *If X is a topologically complete metrizable space, if A is a closed subset of X and if d_1 is a complete metric for A, then there is a complete metric for X that is an extension of d_1.*

Proof. By the Hausdorff-Bing extension theorem, there is a metric d_2 (not necessarily complete) on X that is an extension of d_1. Let d_3 be a complete metric for X. Define d_4: $X \times X \to R$ by

$$d_4(x, y) = d_2(x, y) + d_3(x, y), \qquad x, y \in X.$$

It is easily seen that d_4 is a complete metric for X and that $d_4(a, b) \geqq d_1(a, b)$ whenever $a, b \in A$. Define d_5: $X \times X \to R$ by

$$d_5(x, y) = \min\{d_4(x, y), \inf\{d_4(x, a) + d_1(a, b) + d_4(b, y): a, b \in A\}\}.$$

The proof that d_5 is a metric for X that extends d_1 is essentially a repetition of an argument given by Bing [1, p. 518]. It remains to be shown that d_5 is complete.

Suppose x is a point sequence that is Cauchy in the d_5 metric. It is known (see [3] p. 82, Theorem A) that if the collection of all 2-element subsets of the

positive integers is partitioned into two disjoint collections C_1 and C_2, there is an $i \in \{1, 2\}$ and an infinite subset Δ of the positive integers such that each 2-element subset of Δ is in C_i. Therefore there is a subsequence y of x such that one of the following holds.

(a) $d_5(y_j, y_k) = d_4(y_j, y_k)$ for all positive integers j, k;

(b) $d_5(y_j, y_k) = \inf \{d_4(y_j, a) + d_1(a, b) + d_4(b, y_k) : a, b \in A\}$ for all positive integers j, k.

If (a) holds, the completeness of d_4 guarantees the existence of a point p that is a sequential limit point of y and accordingly a sequential limit point of x. If (b) holds, let z be a subsequence of y such that, for each positive integer n, $d_5(z_n, z_{n+1}) < (1/2)^{n+2}$. For each positive integer n, let $a_n, b_n \in A$ be such that

$$d_4(z_n, a_n) + d_1(a_n, b_n) + d_4(b_n, z_{n+1}) < d_5(z_n, z_{n+1}) + (1/2)^{n+2} < (1/2)^{n+1}.$$

Then

$$\begin{aligned} d_1(a_n, a_{n+1}) &\leqq d_1(a_n, b_n) + d_1(b_n, a_{n+1}) \\ &\leqq d_1(a_n, b_n) + d_4(b_n, a_{n+1}) \\ &\leqq d_1(a_n, b_n) + d_4(b_n, z_{n+1}) + d_4(z_{n+1}, a_{n+1}) \\ &\leqq (1/2)^{n+1} + (1/2)^{n+2} \\ &< (1/2)^n. \end{aligned}$$

Thus a is a Cauchy sequence in the complete metric d_1 and accordingly converges to a point p of A. Since the sequence whose nth term is $d_4(z_n, a_n)$ tends to 0, z converges to p, as does x.

References

1. R. H. Bing, Extending a metric, Duke Math. J., 14 (1947) 511–519.
2. F. Hausdorff, Erweiterung einer Homöomorphie, Fund. Math., 16 (1930) 353–360.
3. F. P. Ramsey, The Foundations of Mathematics, Humanities Press, New York, 1950.

A NOTE ON DERIVING THE EXPANSION OF x^k IN TERMS OF POLYNOMIALS

D. W. SIDA, Carleton University, Canada

The following simple method for obtaining the expansion of a function as a finite series of special functions uses the differential equation, linear independence and the highest coefficient in that special function. It is applicable especially to the problem of expressing x^k as a finite series of *orthogonal polynomials* defined by certain differential equations and is demonstrated here by using Legendre polynomials since this is one of the special functions most commonly dealt with. However, the orthogonal property is not actually used.

Let x^k be expressed by the series, valid in the range $(-1, +1)$,

$$x^k = \sum_{n=0}^{k} a_n^k P_n(x), \tag{1}$$

where $P_n(x)$ is the polynomial satisfying the equation

$$\frac{d}{dx}\left[(1 - x^2)\frac{dP_n}{dx}\right] + n(n+1)P_n(x) = 0.$$

Multiply the differential equation by a_n^k and sum over $n = 0, 1, \cdots, k$;

$$\frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}(x^k)\right] + \sum_{n=0}^{k} a_n^k n(n+1)P_n(x) = 0,$$

where x^k has replaced the series in the first term. Therefore

$$k(k-1)x^{k-2} - k(k+1)x^k + \sum_{n=0}^{k} a_n^k n(n+1)P_n(x) = 0.$$

Let x^{k-2}, x^k be expressed by similar series to (1), hence

$$\sum_{n=0}^{k-2} k(k-1)a_n^{k-2}P_n(x) = \sum_{n=0}^{k}(k-n)(k+n+1)a_n^k P_n(x).$$

Since the $P_n(x)$ are linearly independent in the range $(-1, +1)$ we can equate corresponding coefficients,

$$a_{k-1}^k = 0,$$

$$a_n^k = \frac{k(k-1)}{(k-n)(k+n+1)} a_n^{k-2}. \tag{2}$$

The nonzero coefficients are thus given by $n = k-2, k-4, \cdots 0$ or 1 according to whether k is even or odd. The highest one a_k^k is the reciprocal of the coefficient of x^k in $P_k(x)$ and is given by

$$a_k^k = \frac{2^k k!\,k!}{(2k)!}.$$

Using the recurrence relation (2) and introducing $n = k - 2r$ we obtain

$$a_{k-2r}^k = 2^{k-2r}(1 + 2k - 4r)\frac{k!(k-r)!}{r!(2k-2r+1)!},$$

$$x^k = \sum_{r=0}^{k/2 \text{ or } (k-1)/2} a_{k-2r}^k P_{k-2r}(x).$$

Further Applications. This method applies equally well to Laguerre, Hermite,

Tchebishev and other polynomials but is inapplicable to the general Fourier expansions in the form of infinite series, which are not uniformly convergent. However, when a finite number of terms is required, it is valid, as in the case of the trigonometric series of the form

$$\cos^k (x) = \sum_{n=0}^{k} a_n^k \cos nx.$$

ON SEQUENCES

JULIO CANO, Ball State University

For a given sequence of real numbers $\{a_k\}$ the associated sequence of ratios $\{a_{k+1}/a_k\}$ may not converge. In this connection we prove the following two propositions:

I. *If* $\{a_k\}$ *is a bounded sequence such that* $a_k \neq 0$, *then there is a subsequence* $\{b_n\}$ *of* $\{a_k\}$ *such that* $\{b_{n+1}/b_n\}$ *converges.*

II. *If* $\{a_k\}$ *is a sequence such that for every subsequence* $\{b_n\}$ *of* $\{a_k\}$, $\lim_{n\to\infty} \cdot |b_{n+1}/b_n| \leqq 1$, *then* $\{a_k\}$ *has at most two limit points. If they are not equal and one is* t, *the other is* $-t$.

Proof of I. We consider two cases:

CASE 1: There is a positive ϵ such that for infinitely many k, $|a_k| \geqq \epsilon$. Let $\{b_n\}$ be the subsequence of $\{a_k\}$ consisting of the a_k with $|a_k| \geqq \epsilon$. Then $|b_{n+1}/b_n| \leqq (\mathrm{Sup}_k |a_k|)/\epsilon$ for all n.

CASE 2. For every $\epsilon > 0$, ultimately $|a_k| < \epsilon$. Then there is a subsequence $\{b_n\}$ of $\{a_k\}$ such that $|b_{n+1}| < |b_n|$. Now in both cases the sequence of ratios $\{b_{n+1}/b_n\}$ is bounded, and thus by the Bolzano-Weierstrass theorem there exists a convergent subsequence.

Proof of II: $\{a_k\}$ is bounded. Now suppose that $\{a_k\}$ has at least two distinct limit points s and t, where $s \neq -t$. Then there are subsequences $\{b_n\}$ and $\{c_m\}$ of $\{a_k\}$ converging to s and t respectively. Let $\{d_i\}$ be the following subsequence of $\{a_n\}$

$$d_i = \begin{cases} b_n \text{ if } i \text{ is odd (i.e., } d_1 = b_1,\ d_3 = b_2, \cdots) \\ c_m \text{ if } i \text{ is even (i.e., } d_2 = c_1,\ d_4 = c_2, \cdots). \end{cases}$$

But then clearly $\lim_{i\to\infty} |d_{i+1}/d_i|$ does not exist. This contradiction establishes II.

Reference

1. G. H. Hardy, Divergent Series, Clarendon Press, Oxford, 1963.

A CHARACTERIZATION OF l_p NORMS ON E_n

T. G. NEWMAN, Texas Technological College

For any real number $p \geqq 1$ the formula

$$\|u\|_p = \left(\sum |u_i|^p\right)^{1/p}, \qquad u = (u_1, u_2, \cdots, u_n) \tag{1}$$

defines a norm on real Euclidean space E_n. One naturally wonders if a larger class of norms may be obtained by replacing the map $x \rightarrow x^p$ by a more general mapping. To be precise we pose the question as follows:

PROBLEM: Let R_+ denote the nonnegative real numbers and let ϕ be a one-to-one continuous mapping of R_+ onto R_+ and let ψ denote the inverse of ϕ. Find the necessary and sufficient conditions that, for all n, the formula

$$\|u\| = \psi\left(\sum \phi(|u_i|)\right) \tag{2}$$

defines a norm on E_n. Assume without any loss of generality that $\phi(1) = 1$.

Clearly it is sufficient that $\phi(x) = \alpha x^p$, $\alpha > 0$, $p \geqq 1$ and the normalizing condition allows us to choose $\alpha = 1$. In the following we will show that this is also a necessary condition. Let us then assume that ψ and ϕ satisfy the conditions of the problem.

THEOREM 1. *For all $x, y \in R_+$, $\phi(xy) = \phi(x)\phi(y)$ and $\psi(xy) = \psi(x)\psi(y)$.*

Proof: It will suffice to prove the latter equality. For any integer $n > 0$ and $x \in R_+$ we have

$$\begin{aligned} \psi(n)\psi(x) &= \psi(x)\|(1, 1, \cdots, 1)\| = \|\psi(x)(1, 1, \cdots, 1)\| \\ &= \psi(n\phi\psi(x)) = \psi(nx). \end{aligned}$$

In particular, $\psi(n)\psi(1/n) = \psi(1) = 1$ so that $\psi(1/n) = 1/\psi(n)$. Now $\psi(x) = \psi(n)\psi(x/n)$ from which $\psi(x/n) = \psi(x)\psi(1/n)$. We conclude that for any rational number $r \geqq 0$, $\psi(rx) = \psi(r)\psi(x)$. Continuity of ψ completes the proof.

THEOREM 2. *$\phi(x) = x^p$ for all $x \in R_+$. Moreover $p = \log_2 \phi(2) \geqq 1$.*

Proof: Define $f(z) = \log_2 \phi(2^z)$ for all real numbers z. We have $f(z+w) = f(z) + f(w)$. Since f is continuous it is not difficult to show that

$$f(z) = zf(1) = z \log_2 \phi(2) = pz.$$

Solving for ϕ we have $\phi(x) = x^p$ for all $x > 0$. Also $\phi(0) = 0 = 0^p$. It remains only to show $p \geqq 1$. Note that

$$\psi(2) = \|(1, 1)\| \leqq \|(1, 0)\| + \|(0, 1)\| = 2.$$

Since ϕ must be an increasing function we have $\phi(2) \geqq 2$, from which $p = \log_2 \phi(2) \geqq 1$.

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the Monthly. Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.

ON WHO FIRST PROVED THE IMPOSSIBILITY OF CONSTRUCTING CERTAIN REGULAR POLYGONS WITH RULER AND COMPASS ALONE

N. D. KAZARINOFF, University of Michigan

Carl Friedrich Gauss published in 1801 in his *Disquisitiones Arithmeticae* exquisite arithmetic yielding the constructibility of regular p-gons for p a prime of the form $1+2^k$. Closing his discussion, he asserted he had proved that no other regular p-gons were constructible. Gauss never published a proof of this assertion, nor did he ever outline one in his correspondence or notes. Yet Felix Klein in his *Famous Problems of Elementary Geometry* defers to Gauss. Practically all those who have written since on this subject follow Klein in stating that Gauss proved the impossibility theorem in the *Disquisitiones Arithmeticae.* R. C. Archibald partially corrected the record in an article in this MONTHLY in 1914. But he incorrectly asserted that James Pierpont gave the first proof.

In my opinion, Professors Gauss, Klein, Pierpont, and Archibald should each have credited Pierre L. Wantzel (1814–1848), who proved the impossibility of constructing non-Gaussian regular n-gons with ruler and compass alone in 1837 in the very article where he solved the problems of angle trisection and cube duplication. Interestingly, Wantzel is cited by many authors, but only for solving the latter two problems.

Should one believe Gauss? There are aspects of this question to be considered aside from Gauss apparently never writing so much as an outline of a proof anywhere. Abel wrote to Holmboe from Paris in 1826 and referred to the "mystery that has reigned over the theory of Mr. Gauss on the division of the circle into equal parts." Gauss in 1796 did not have the bright light of Ruffini's, Abel's, and Galois' researches to guide him. Lastly, it was the general belief in 1796 that the classical ruler and compass construction problems were soluble in the negative, so that Gauss should have been more timid about publishing proofs of constructibility than proofs of nonconstructibility.

Mail Address: Dept. of Math., Angell Hall, The University of Michigan, Ann Arbor, MI 48104.

ON THE LARGEST ODD DIVISOR OF n

R. A. MACLEOD, University of Victoria

Let $\delta(n)$ denote the largest odd divisor of n. St. Guzel [1] showed

$$g(x) = \sum_{n \leq x} \delta(n) = \frac{x^2}{3} + O(x), \tag{1}$$

$$f(x) = \sum_{n \leq x} \frac{\delta(n)}{n} = \frac{2}{3} x + O(1). \tag{2}$$

We replace these by 'best possible' estimates for integer x, viz.

$$\frac{2}{3} x + \frac{1}{3} \frac{1}{x} \leqq f(x) \leqq \frac{2}{3} x + \frac{2}{3} - \frac{2}{3} \frac{1}{x+1}, \tag{3}$$

with equality on the left for $x = 2^k$ and on the right for $x = 2^k - 1$, and

$$\frac{1}{3} x^2 + \frac{2}{3} \leqq g(x) \leqq \frac{1}{3} x^2 + \frac{2}{3} x. \tag{4}$$

The main tool is the following lemma,

LEMMA:

$$\text{(i)} \quad \frac{\delta(2^n + k)}{2^n + k} = \frac{\delta(k)}{k}, \qquad 1 \leqq k \leqq 2^n - 1, \; n = 1, 2, \cdots, \tag{5}$$

$$\text{(ii)} \quad \frac{\delta(2^{n+1})}{2^{n+1}} = \frac{1}{2} \frac{\delta(2^n)}{2^n}, \qquad n = 1, 2, \cdots. \tag{6}$$

Statement (i) is proved by induction on n, while (ii) is immediate. From the lemma (3) easily follows; (4) follows from the observations that $g(4n) = 5n^2 + g(n)$ and that $g(2^n + k) = g(2^n) + 2^n f(k) + g(k)$, $0 \leqq k \leqq 2^n - 1$.

Mail Address: Dept. of Math., University of Victoria, Victoria, B.C., Canada.

Reference

1. St. Guzel, On the greatest odd divisor of a number (Polish), Wiadom. Mat., Warsaw, 14 (1910) 171–180.

A MATRIX IDENTITY

D. E. CRABTREE, Amherst College

THEOREM. *Let M be a matrix of order $n = mk$, partitioned into the form $M = [A_{ij}]_1^k$, where each A_{ij} is an $m \times m$ matrix and A_{kk} is nonsingular. Let $\phi(M)$ denote the matrix $[B_{ij}]_1^{k-1}$, where $B_{ij} = A_{ij} - A_{ik} A_{kk}^{-1} A_{kj}$. Let $\psi(M)$ denote the matrix $[d_{ij}]_1^{n-m}$ with elements $d_{ij} = \det M(i, n-m+1, \cdots, n; j, n-m+1, \cdots, n) / \det A_{kk}$. (The notation $M(i_1, \cdots, i_p; j_1, \cdots, j_p)$ indicates the submatrix of M formed using rows $i_1, \cdots, i_p$ and columns $j_1, \cdots, j_p$.) Then $\phi(M) = \psi(M)$.*

Proof. It is clear that $\phi(M) = \phi(P)$, where P is the matrix $[C_{ij}]_1^k$ with blocks $C_{ij} = A_{ij}$ if $j \neq k$, and $C_{ik} = A_{ik} A_{kk}^{-1}$. Moreover, $\phi(P)$ has as a typical element $\phi(P)_{\lambda,\mu} = (C_{ij} - C_{ik} C_{kj})_{\alpha,\beta}$ (where $\lambda = (i-1)m + \alpha$ and $\mu = (j-1)m + \beta$), which can be written as the determinant of the matrix

$$S = \begin{bmatrix} (C_{ij})_{\alpha,\beta} & (C_{ik})_{\alpha,1} \cdots (C_{ik})_{\alpha,m} \\ (C_{kj})_{1,\beta} & \\ \vdots & I \\ (C_{kj})_{m,\beta} & \end{bmatrix}.$$

In order to evaluate this determinant, let G denote the matrix obtained by substituting A_{kk} for the last m rows and columns of the identity matrix of order $m+1$. Then $(\det S)(\det G) = \det SG$, which equals

$$\det \begin{bmatrix} (A_{ij})_{\alpha,\beta} & (A_{ik})_{\alpha,1} \cdots (A_{ik})_{\alpha,m} \\ (A_{kj})_{1,\beta} & \\ \vdots & A_{kk} \\ (A_{kj})_{m,\beta} & \end{bmatrix}.$$

Thus

$$\begin{aligned}\phi(M)_{\lambda,\mu} &= \det S = \frac{\det SG}{\det G} = \frac{\det M(\lambda, n-m+1, \cdots, n; \mu, n-m+1, \cdots, n)}{\det A_{kk}} \\ &= \psi(M)_{\lambda,\mu},\end{aligned}$$

which completes the proof.

Details on the construction of $\phi(M)$ and $\psi(M)$ can be found in [1] and [2].

Mail Address: Dept. of Math., Amherst College, Amherst, MA 01002.

References

1. Douglas E. Crabtree, Applications of M-matrices to nonnegative matrices, Duke Math. J., 33 (1966) 197–208.

2. ———, Characteristic roots of M-matrices, Proc. Amer. Math. Soc., 17 (1966) 1435–1439.

CLASSROOM NOTES

Edited by George Raney, University of Connecticut

Material for this department should be sent to David Drasin, Div. of Math. Sci., Purdue University, Lafayette, IN 47907.

A CREEPING LEMMA

R. M. F. Moss and G. T. Roberts, Hull University, England

This article is about a lemma which facilitates the proof of a number of results usually proved in a first course in rigorous analysis. These results include the intermediate value theorem, the mean value theorem and the boundedness of the image of a closed interval under a continuous mapping. They are all essentially dependent on the completeness of the real field and are often proved by

a "creeping" method (see method 5 of [2]). The type of proof we have in mind is one in which a set K is defined by some property, and then proved to be the whole of an interval by assuming it is not and deducing a contradiction on considering $\sup K$. The use of this method often leads to the repetition of very similar reasoning in several proofs all of which are rather sophisticated by the standards of students at this stage in their education. By using the creeping lemma this repetition can be avoided and the logic of such proofs can be seen separately from the details of the particular result that is being proved.

To understand the lemma and its proof the student needs to know what is meant by a transitive relation and to be aware of the completeness of the real field. We state and prove the lemma using neighbourhoods and set theoretic notation, but it is quite possible to avoid this.

THE CREEPING LEMMA. *Let ρ be a transitive relation on the interval $[a, b]$. If each $x\in[a, b]$ has a neighbourhood N_x such that $u\rho v$ whenever $u\in[a, x]\cap N_x$ and $v\in N_x\cap[x, b]$, then $a\rho b$.*

Proof. Let $K=\{x: a\rho x\}$. It is easily proved that K is a nonempty bounded set and so, by the completeness of the real field, has a supremum k. Let N_k contain $[k-\delta, k+\delta]$ where $\delta>0$. Then $k-\delta$ is not an upper bound of K and so there is a number $u\in K$ such that $u>k-\delta$. Clearly this implies that $u\in N_k\cap[a, k]$.

Now suppose that $b\notin N_k$. Put $v=k+\delta$, so that $v\in N_k\cap[k, b]$. Hence, by hypothesis, $u\rho v$ and so, since $a\rho u$ and ρ is transitive, $a\rho v$. This implies that $v\in K$, which is impossible since $v>\sup K$. Thus $b\in N_k$ and so $b\in N_k\cap[k, b]$. From this it follows that $u\rho b$ and so, by using the transitivity of ρ, that $a\rho b$.

The proofs of many of the classical theorems in elementary analysis are straightforward applications of the creeping lemma once it has been decided what the relation ρ should be. We now state a number of these results and give indications of their proofs.

THEOREM 1. *The image of a bounded closed interval under a continuous mapping f is bounded.*

Apply the lemma with ρ the relation defined by requiring $u\rho v$ to be true if and only if f is bounded on $[u, v]$.

THEOREM 2. (The intermediate value theorem.) *If f is continuous on $[a, b]$ and $t\in\,]f(a), f(b)[$, then there is a number $s\in\,]a, b[$ such that $f(s)=t$.*

Assume that $f(x)-t$ is zero for no $s\in[a, b]$. Then prove that $f(a)-t$ and $f(b)-t$ are of the same sign by applying the creeping lemma with ρ the relation defined by requiring $u\rho v$ to be true if and only if $f(u)-t$ and $f(v)-t$ are of the same sign.

THEOREM 3. (The mean value theorem.) *If f is differentiable on $[a, b]$ and $m<f'(x)<M$ for all $x\in[a, b]$, then*

$$m(b-a)<f(b)-f(a)<M(b-a).$$

To prove this take ρ to be the relation defined by requiring $u\rho v$ to mean that $u=v$ or

$$m(v-u) < f(v) - f(u) < M(v-u).$$

If it is ever required, the classical mean value theorem can be obtained from Theorem 3 without much difficulty (see [1]).

THEOREM 4. *If f is continuous on $[a, b]$, f is Riemann integrable over $[a, b]$.*

The crucial step in the proof is to find a dissection (or partition) of $[a, b]$ for which the upper and lower Darboux sums differ by less than $\epsilon(b-a)$. (It is here that uniform continuity is usually used.) This is easily done by taking $u\rho v$ to mean that there is a dissection $\{x_0, x_1, \cdots, x_n\}$ of $[u, v]$ such that, for all $r=1, 2, \cdots, n$,

$$\sup(f[x_{r-1}, x_r]) - \inf(f[x_{r-1}, x_r]) < \epsilon.$$

It is easy to deduce from this proof that the function f must be uniformly continuous. (We are grateful to R. L. Hutchings for this proof.)

THEOREM 5. (Dini's lemma.) *If (f_n) is a nonincreasing sequence of continuous functions that is pointwise convergent to zero on $[a, b]$, then it is uniformly convergent to zero on $[a, b]$.*

Let $\epsilon > 0$. Define ρ by requiring $u\rho v$ to be true if and only if there is a number n such that $f_n(s) < \epsilon$ for all $s \in [u, v]$. It is easy to verify that this relation has the required properties. The conclusion is an immediate consequence of $a\rho b$.

THEOREM 6. (Heine-Borel theorem.) *Every open covering of $[a, b]$ has a finite subcovering.*

This is proved by taking $u\rho v$ to mean that $[u, v]$ can be covered by a finite number of sets of the given covering.

THEOREM 7. (Bolzano-Weierstrass theorem.) *If S is a subset of $[a, b]$ with no points of accumulation, then S is finite.*

This is proved by taking $u\rho v$ to mean that $[u, v] \cap S$ is finite.

THEOREM 8. (Cantor's intersection theorem.) *If (F_n) is a decreasing sequence of closed subsets of the interval $[a, b]$ whose intersection is empty, then one of the sets F_n must be empty.*

This is proved by taking $u\rho v$ to mean that there is a number n such that $[u, v] \cap F_n$ is empty.

It is only in the mean value theorem that it is necessary to use the full force of the creeping lemma, and for the other theorems it is sufficient to use the following weaker form.

THE WEAK CREEPING LEMMA. *Let ρ be a transitive relation on $[a, b]$. If each $x \in [a, b]$ has a neighbourhood N_x such that $u\rho v$ whenever $u, v \in N_x \cap [a, b]$, then $a\rho b$.*

If we define a universal relation on a set to be one in which every pair of elements are related, then the weak creeping lemma can be paraphrased by the statement that any locally universal transitive relation on $[a, b]$ is universal. Since the weak creeping lemma implies the intermediate value theorem, it follows that any totally ordered field in which the weak creeping lemma holds must be complete; however, this result does not seem to be of any importance in undergraduate teaching.

References

1. L. Bers, On avoiding the mean value theorem, this MONTHLY, 74 (1967) 583.

2. H. Petard, A contribution to the mathematical theory of big game hunting, this MONTHLY, 45 (1938) 446.

A MODIFIED NEWTON-RAPHSON ITERATION

WALTER JENNINGS, Naval Postgraduate School

The use of the iteration equation

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}$$

for finding zeros of $f(z)$ has been variously referred to as Newton's Rule [1], the Newton-Fourier [2] and Newton-Raphson Method [3]. Sufficient conditions for its convergence which are dependent on initial value have been found for both real and complex functions [4]. The method applied to entire functions of a complex variable does not seem to be adequately appreciated. We show that it is, in fact, a steepest descent method and hence with a few minor modifications will of necessity converge independently of initial value.

The modified iteration is

$$z_{k+1} = z_k - \xi^{M_k}\frac{f(z_k)}{f'(z_k)}, \qquad 0 < \xi < 1,$$

where M_k *is the smallest nonnegative integer for which* $|f(z_{k+1})| < |f(z_k)|$, *and* $|f'(z_k)|$ *is bounded away from zero.*

We will suppose $f(z)$ to be analytic throughout the complex plane, that the zeros of $f(z)$ are isolated and simple. We set $z = x+iy$ and $f(z) = U(x, y) + iV(x, y)$. Let $F(x, y) = U^2 + V^2$. Then $f(z) = 0$ if and only if $F(x, y) = 0$. Note that $F = |f|^2$. As a consequence of the maximum modulus theorem the only minima of $F(x, y)$ are the zeros of $f(z)$ [5, 6]. Thus the problems of finding zeros of $f(z)$ and of finding the minima of $F(x, y)$ are precisely equivalent.

We now show that the classical Newton-Raphson iteration applied to $f(z)$ is precisely a steepest descent method for finding minima of $F(x, y)$. The iteration can be thought of as replacing z by $z+\Delta z$ where

$$\Delta z = \Delta x + i\Delta y = -\frac{f(z)}{f'(z)} \cdot$$

A simple calculation involving the Cauchy-Riemann partial differential equations gives

$$\Delta x = \frac{-UU_x - VV_x}{U_x^2 + V_x^2}, \qquad \Delta y = \frac{UV_x - VU_x}{U_x^2 + V_x^2},$$

$$F_x = 2UU_x + 2VV_x, \qquad F_y = -2UV_x + 2VU_x.$$

Hence $\Delta x = -hF_x$, $\Delta y = -hF_y$, where $h = 1/2(U_x^2 + V_x^2) > 0$. Now

$$dF = F_x\Delta x + F_y\Delta y = -h(F_x^2 + F_y^2) \leqq 0,$$

with the equality holding only at the zeros of $f(z)$ and at the saddle points of the surface $t = F(x, y)$, i.e., at the zeros of $f'(z)$. Except at those points the direction of the vector from (x, y) to $(x+\Delta x, y+\Delta y)$ is in the direction of decrease of $F(x, y)$. Since this vector is parallel to the gradient vector of $F(x, y)$, it is in the direction of maximum decrease of $F(x, y)$. Hence, if care is taken to choose h so as not to overshoot, then the iteration must converge. At each stage of the calculation one should compare the old and new values of $|f(z)|$ or $F(x, y)$. If there is not a decrease, then we have overshooting and the value of Δz should be reduced by a factor ξ^M, where $0<\xi<1$ and M is smallest positive integer for which $|f(z+\Delta z)| < |f(z)|$. In the unlikely event that the iteration converges on a saddle point (a zero of $f'(z)$), replace last value of Δz by $i\Delta z$ and then proceed with the original iteration. In the case of polynomials

$$f(z) = \sum_{j=0}^{n} (a_j + ib_j)z^j, \qquad a_n + ib_n \neq 0,$$

where a_j and b_j are real, $f(z)$ is an entire function of the complex variable z. The Euclid algorithm can be used to find highest common factor of $f(z)$ and $f'(z)$. Removal of this factor from $f(z)$ will result in a polynomial with simple zeros [7]. Values of $f(z)$ and $f'(z)$ can be calculated using a complex arithmetic package and an algorithm based on nested multiplication. But if one prefers real arithmetic the Šiljak procedure can be recommended [8].

References

1. National Physical Laboratory, Modern Computing Methods, Her Majesty's Stationery Office, London, 1961, p. 55.

2. A. Cayley, The Newton-Fourier imaginary problem, Amer. J. Math., 2 (1879) 97.

3. F. Cajori, Historical note on the Newton-Raphson method of approximation, this MONTHLY, 18 (1911) 29–32.

4. A. M. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966, pp. 59–66.

5. E. C. Titchmarsh, The Theory of Functions, Oxford University Press, London, 1950, p. 167.

6. A. D. Aleksandrov, A. N. Kolmogorov and M. A. Lavrent'ev, Mathematics—its Content, Methods, and Meaning, The M.I.T. Press, Cambridge, 1963, v. I, p. 290.

7. C. C. MacDuffee, Theory of Equations, Wiley, New York, 1960, p. 48.

8. J. B. Moore, A convergent algorithm for solving polynomial equations, J. Assoc. Comput. Mach., 14 (1967) 311–315.

AN ELEMENTARY VERSION OF THE MACKEY-ARENS THEOREM

ALBERT WILANSKY, Lehigh University

A proof of this theorem uses much heavy machinery; for some purposes the part of the result that is most important is the existence of a maximum and a minimum topology, without identifying the maximum (Mackey) topology. The purpose of this note is to give an easy proof of the existence of the maximum topology and even to give a formula for it.

Let X be a linear space and Y a linear space of linear functionals on X. The Mackey-Arens theorem states that Y is the set of continuous linear functionals on (i.e., the dual of) X, where X is given a locally convex topology t, if and only if t lies between two particular topologies; moreover the theorem specifies these two topologies, the larger being called $\tau(X, Y)$, the Mackey topology. We give an elementary proof of the first half as follows.

THEOREM 1. *Y is the dual of (X, t), where t is a locally convex topology for X, if and only if $\sigma(X, Y)\subset t\subset\tau(X, Y)$ where $\sigma(X, Y)$ is the weak topology by Y, and $\tau(X, Y)$ is the supremum of all locally convex topologies T such that Y is the dual of (X, T).*

The only nontrivial part of this result is the assertion that $\tau(X, Y)$, as defined in the statement, is itself compatible with Y; i.e., yields Y as the dual of X. This will follow immediately from the following lemma.

LEMMA 1. *Let $\{T_\alpha: \alpha\in A\}$ be a family of locally convex topologies for a linear space X and let a linear functional f be continuous with $T\equiv\sup\{T_\alpha: \alpha\in A\}$. Then there exists a finite subset $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ of A and linear functionals $g_1, g_2, \cdots, g_n$ such that g_i is T_{α_i} continuous for $i=1, 2, \cdots, n$, and $f=\sum g_i$.*

We may assume that each T_α is given by a family F_α of seminorms, [3], Section 12.1, Theorem 1. Then T is given by $F\equiv\bigcup\{F_\alpha: \alpha\in A\}$, [3], Section 12.1, Fact iii. There are finitely many seminorms $p_1, p_2, \cdots, p_m\in F$ with $|f|\leqq K\sum_{i=1}^m p_i$ for some K; [3], Section 12.1, Fact ix. Adding together those p_i which belong to the same F_α, we have seminorms $q_1, q_2, \cdots, q_n$ with q_i being T_{α_i} continuous for some $\alpha_i\in A$, $i=1, 2, \cdots, n$, and $|f|\leqq K(q_1+q_2+\cdots+q_n)$. The result is now immediate from [3], Section 4.4, Problem 30, or [1], p. 120, Theorem 14.6.

Theorem 1 follows because in this case all T_α allow the same dual.

It is interesting that Lemma 1 is false without the assumption of local convexity; indeed there exists a space X with two topologies T_1, T_2 such that

$(X, T_1)' = (X, T_2)' = \{0\} \neq (X, T_1 \vee T_2)'$; [2], p. 243. Thus Theorem 1 cannot be extended in its given form to nonlocally convex spaces. Moreover, the cited results of [1], [2] show that the diagonal of $(X, T_1) \times (X, T_2)$ allows nonzero continuous linear functionals while the product itself does not.

References

1. J. L. Kelley and I. Namioka, Linear Topological Spaces, Van Nostrand, Princeton, N. J., 1963.

2. Victor Klee, Exotic topologies for linear spaces, *General topology and its relations to modern analysis*, Prague Symposium 1961; J. Novak, Editor.

3. A. Wilansky, Functional Analysis, Blaisdell, New York, 1964.

ON JULIA'S COROLLARY TO PICARD'S GREAT THEOREM

J. W. MACKI, University of Alberta

The purpose of this note is to indicate a rather subtle error in the proof of Julia's Corollary to Picard's Great Theorem, which occurs in two of the few English-language books which prove this corollary (Hille [1] pp. 258–259; Saks and Zygmund [2] pp. 350–352). This error has apparently escaped the attention of a large number of mathematicians and their students. In order to understand its nature, let us recall that the Theorem and Corollary can be stated as follows:

THEOREM (E. Picard, 1879). *If $f(z)$ is holomorphic in $\Delta = \{z \mid 0 < |z| < R\}$ and has an essential singularity at $z = 0$, then there is at most one complex number w such that the equation $f(z) = w$ has only a finite number of roots in Δ.*

COROLLARY (G. Julia, 1919). *Under the assumptions in Picard's Theorem, there exists at least one ray, $\arg z = \theta$, with the property that in any sector of the form $\theta - \delta < \arg z < \theta + \delta$ there is at most one complex number w such that the equation $f(z) = w$ has only a finite number of roots.*

In order to indicate the error, it is necessary to briefly review the "normal families" proof of Picard's Theorem. Recall ([2], p. 50) that a family of functions, each of which is holomorphic in a given domain G, is called *normal* on G if every sequence of functions from the family either

(a) Contains a subsequence which converges, uniformly on compact subsets of G, to a function holomorphic in G, or

(b) Contains a subsequence which diverges to ∞, uniformly on compact subsets of G.

The hard part of Picard's Theorem, then, is to show that any family F of holomorphic functions which omit two fixed values on a domain G is a normal family on G. Once this has been shown, Picard's Theorem follows quite easily by contradiction. That is, we assume $f(z) \neq a$ and $f(z) \neq b$ in Δ, and then define a normal family $F = \{f_n(z)\}$ on the annulus $A = \{z \mid R/4 < |z| < 3R/4\}$ by $f_n(z) = f(z/2^n)$, $z \in A$. Since F is normal, either (a) or (b) above holds. If (a) holds, then the convergent subsequence $\{f_{n(k)}\}$ is bounded on the compact set $|z|$

$=R/2$, i.e., $f(z)$ is bounded on the sequence of circles $(R/2)2^{-n(k)}$. Applying the maximum principle in the annuli between these circles, we see that $f(z)$ is bounded in Δ, in contradiction to the Casorati-Weierstrass Theorem.

Now we come to the critical part of the proof. Suppose we can only find subsequences of $\{f_n\}$ which converge, uniformly on compact subsets of G, to ∞. We choose such a subsequence, and consider

$$g_{n(k)}(z) = 1/(f_{n(k)}(z) - a).$$

Because $f(z) \neq a$ for $z \in \Delta$, $\{g_{n(k)}(z)\}$ is a sequence of holomorphic functions in A, convergent to 0, uniformly on the compact subset $|z| = R/2$. Thus $1/(f(z)-a)$ converges to zero uniformly on the circles $|z| = (R/2)2^{-n(k)}$. Because we can apply the maximum principle to $1/(f(z)-a)$ in the annuli between these circles, we see that $1/(f(z)-a)$ is bounded in a punctured neighborhood of 0, in contradiction to the Casorati-Weierstrass Theorem. Note that in this half of the proof it is essential that $f(z)$ omit some value a.

Now the authors cited earlier assert that the corollary of Julia follows from the fact that the above proof actually shows the family $\{f_n(z)\}$ cannot be normal in A. This, quite clearly, is not true, since the proof required that $f(z)$ omit some value a.

There are several correct ways to prove the Corollary—one of the nicest appears in Montel's book ([3], pp. 79–80). All of these essentially investigate the concept of normal family more closely to show that the family $\{f_n(z)\}$, as defined above, cannot be normal.

The interesting thing about the error described in this note is that it can be immediately carried over to meromorphic functions, to prove that every meromorphic function has a ray of Julia. This is of course a patently false result. A full discussion of the question of rays of Julia for meromorphic functions, including a description of some remarkable recent results by Lehto and Virtanen, can be found in Dinghas' book ([4], pp. 321–323).

References

1. E. Hille, Analytic Function Theory, Vol. II, Ginn, Boston, 1962.
2. S. Saks and A. Zygmund, Analytic Functions, 2nd ed., Polish Sci. Publ., Warsaw, 1965.
3. P. Montel, Leçons sur les Familles Normales, Gauthier-Villars, Paris, 1927.
4. A. Dinghas, Vorlesungen über Funktionentheorie, Springer, Berlin, 1961.

AN APPLICATION OF THE SCHWARZ INEQUALITY

V. G. SIGILLITO, The Johns Hopkins University

In textbooks on linear algebra a complex $n \times n$ matrix U is said to be *unitary* if

$$U^*U = I_n$$

where U^* is the transposed conjugate of U. The following theorem is then given:

THEOREM. *For the matrix U to be unitary it is necessary and sufficient that $\|Ux\| = \|x\|$, for all x in unitary n-space, $\mathfrak{U}_n$.*

Although the necessity is trivial to prove, the sufficiency part is a bit more involved. A variety of proofs, for the most part being variations of those in [1] and [2], are in the literature. None of the proofs which we have seen make use of the Schwarz inequality. In this note we show how the Schwarz inequality can be employed to construct a brief proof of the sufficiency part of the above theorem. This new proof can be used as an application of the Schwarz inequality which usually has been presented earlier in the book where its wide applicability has been duly noted but rarely if ever illustrated.

Thus we assume that

$$\|Ux\|^2 = (Ux,\ Ux) = \|x\|^2 = (x,\ x), \qquad \text{for all } x \in \mathfrak{U}_n, \tag{1}$$

and we prove that U is a unitary matrix. We first note that (1) implies that U, hence also U^*, is nonsingular. Now suppose that x is an arbitrary nonzero vector from $\mathfrak{U}_n$. Then

$$\begin{aligned}(U^*x,\ U^*x)^2 &= (x,\ UU^*x)^2 \leqq (x,\ x)(UU^*x,\ UU^*x)\\ &= (x,\ x)(U^*x,\ U^*x),\end{aligned}$$

where we have used the Schwarz inequality and (1). Then since $(U^*x,\ U^*x) > 0$,

$$(U^*x,\ U^*x) \leqq (x,\ x). \tag{2}$$

Also

$$\begin{aligned}(x,\ x)^2 &= (Ux,\ Ux)^2 = (x,\ U^*Ux)^2 \leqq (x,\ x)(U^*Ux,\ U^*Ux)\\ &\leqq (x,\ x)(Ux,\ Ux)\\ &= (x,\ x)^2\end{aligned} \tag{3}$$

where we have used first the Schwarz inequality and then (2). Thus equality must hold in the Schwarz inequality, which means that $U^*Ux = \alpha x$ for some scalar α. But

$$\alpha = \frac{(U^*Ux,\ x)}{(x,\ x)} = \frac{(Ux,\ Ux)}{(x,\ x)} = 1$$

so that $U^*Ux = x$ for all nonzero $x \in \mathfrak{U}_n$ (equality is trivially true for $x = 0$), or

$$U^*U = I_n.$$

As an alternative way of finishing the proof we note that (3) implies equality in (2) and this with (1) implies that

$$(x - U^*Ux,\ x - U^*Ux) = 0$$

or $x - U^*Ux = 0$ for all $x \in \mathfrak{U}_n$.

References

1. F. E. Hohn, Elementary Matrix Algebra, 2nd ed., Macmillan, New York, 1964.
2. P. R. Halmos, Finite Dimensional Vector Spaces, Van Nostrand, Princeton, N. J., 1958.

THE NEWTON-KANTOROVICH THEOREM

J. M. Ortega, University of Maryland

One of the basic results in numerical analysis is *The Newton-Kantorovich Theorem:* Let X and Y be Banach spaces and $F: D\subset X\to Y$. Suppose that on an open convex set $D_0\subset D$, F is Frechet differentiable and

$$\|F'(x) - F'(y)\| \leqq K\|x - y\|,\ x, y \in D_0.$$

For some $x_0\in D_0$, assume that $\Gamma_0\equiv [F'(x_0)]^{-1}$ is defined on all of Y and that $h\equiv\beta K\eta\leqq\frac{1}{2}$ where $\|\Gamma_0\|\leqq\beta$ and $\|\Gamma_0 Fx_0\|\leqq\eta$. Set

$$(1)\qquad t^* = \frac{1}{\beta K}(1 - \sqrt{1 - 2h}),\qquad t^{**} = \frac{1}{\beta K}(1 + \sqrt{1 - 2h})$$

and suppose that $S\equiv\{x \mid \|x-x_0\|\leqq t^*\}\subset D_0$. Then the Newton iterates $x_{k+1} = x_k - [F'(x_k)]^{-1}Fx_k$, $k=0, 1, \cdots$, are well defined, lie in S and converge to a solution x^* of $Fx=0$ which is unique in $D_0\cap\{x \mid \|x_0-x\|<t^{**}\}$. Moreover, if $h<\frac{1}{2}$ the order of convergence is at least quadratic.

Kantorovich has given two basically different proofs of this result using recurrence relations [1] or majorant functions [2]. It is the purpose of this note to give a proof which is a modification of the second approach and is, we believe, easier to understand and present. Moreover, the concept of a majorizing sequence and estimates of the type (5) have been extended [3] to give a convergence theory for a large class of iterative processes. The proof will be an easy consequence of the following lemmas which serve to isolate the essential points.

Lemma 1. *Let $\{y_k\}$ be a sequence in X and $\{t_k\}$ a sequence of nonnegative real numbers such that*

$$(2)\qquad \|y_{k+1} - y_k\| \leqq t_{k+1} - t_k,\qquad k = 0, 1, \cdots$$

and $t_k\to t^* < \infty$. Then there exists a $y^*\in X$ such that $y_k\to y^*$ and

$$(3)\qquad \|y^* - y_k\| \leqq t^* - t_k,\qquad k = 0, 1, \cdots.$$

The proof is immediate from

$$\|y_{k+p} - y_k\| \leqq \sum_{i=1}^{p} \|y_{k+i} - y_{k+i-1}\| \leqq t_{k+p} - t_k \leqq t^* - t_k,$$

which shows that $\{y_k\}$ is a Cauchy sequence. We shall say that $\{t_k\}$ *majorizes* $\{y_k\}$ if (2) holds.

In the following two lemmas the relevant assumptions of the theorem are assumed to hold.

LEMMA 2. *For all* $x\in Q\equiv\{x|\ \|x-x^0\|<1/\beta K\}\cap D_0$, $[F'(x)]^{-1}$ *is defined on all of* Y *and*

$$\|[F'(x)]^{-1}\| \leqq \beta/(1-\beta K\|x-x_0\|). \tag{4}$$

If x *and* $Nx\equiv x-[F'(x)]^{-1}Fx$ *are in* Q, *then*

$$\|N(Nx)-Nx\| \leqq \frac{1}{2}\,\frac{\beta K\ \|x-Nx\|^2}{1-\beta K\|x_0-Nx\|}. \tag{5}$$

Proof: The first statement follows from the well-known Banach lemma (see, e.g., [4, p. 164]). To prove (5) we note that, since $Fx+F'(x)(Nx-x)=0$,

$$\begin{aligned}\|N(Nx)-Nx)\| &= \|[F'(Nx)]^{-1}F(Nx)\| \\ &\leqq \frac{\beta}{1-\beta K\|x_0-Nx\|}\|F(Nx)-Fx-F'(x)(Nx-x)\|\end{aligned}$$

and the result follows by use of the mean value theorem (see, e.g., [2]):

$$\begin{aligned}\|Fy-Fx-F'(x)(y-x)\| &= \left\|\int_0^1[F'(\theta y+(1-\theta)x)-F'(x)](y-x)d\theta\right\| \\ &\leqq \frac{K}{2}\|y-x\|^2.\end{aligned}$$

LEMMA 3. *The Newton sequence* $\{x_k\}$ *is well-defined and is majorized by the sequence defined by*

$$t_{k+1}=t_k-\frac{(\beta K/2)t_k^2-t_k+\eta}{\beta K t_k-1},\qquad k=0,1,\cdots,t_0=0. \tag{6}$$

Moreover, $t_k\uparrow t^*$, *where* t^* *is defined by* (1).

Proof: We note first that the t_k are simply the Newton iterates for the polynomial $(\beta K/2)t^2-t+\eta$ with roots t^* and t^{**} and it follows immediately that $t_k\uparrow t^*$. Now assume that $x_1,\cdots,x_k$ exist and $\|x_i-x_{i-1}\|\leqq t_i-t_{i-1}$, $i=1,\cdots,k$; this holds by assumption for $k=1$. Then $\|x_k-x_0\|\leqq t_k-t_0\leqq t^*$ so that $x_k\in S$. Hence by Lemma 2, x_{k+1} is defined and

$$\begin{aligned}\|x_{k+1}-x_k\| &= \|N(Nx_{k-1})-Nx_{k-1}\| \leqq \frac{\frac{1}{2}\beta K\|x_k-x_{k-1}\|^2}{1-\beta K\|x_k-x_0\|} \\ &\leqq \frac{\frac{1}{2}\beta K(t_k-t_{k-1})^2}{1-\beta K t_k}=t_{k+1}-t_k,\end{aligned} \tag{7}$$

where the last equality is the result of a simple calculation using the definition of t_k.

The proof of the theorem is now immediate. Lemmas 1 and 3 show that there exists an $x^* \in S$ such that $x_k \to x^*$. That x^* is a solution follows in the usual way from

$$\begin{aligned}\|Fx_k\| &= \|F'(x_k)(x_{k+1}-x_k)\| \leqq [\|F'(x_0)\| + \|F'(x_0) - F'(x_k)\|]\|x_k - x_{k+1}\| \\ &\leqq [\|F'(x_0)\| + Kt^*]\|x_k - x_{k+1}\| \to 0\end{aligned}$$

and the continuity of F in S. If $h < \frac{1}{2}$, the roots t^* and t^{**} are distinct and the order of convergence of t_k to t^* is at least quadratic; hence, by (3) the order of convergence of x_k to x^* is at least quadratic. Finally, the uniqueness statement follows as in [2] by consideration of the simplified Newton iteration $x_{k+1} = x_k - [F'(x_0)]^{-1}Fx_k$.

References

1. L. V. Kantorovich, Functional analysis and applied mathematics, Uspehi Mat. Nauk, 3 (1948) 89–185.
2. L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces, Moscow, 1959. Translated by D. Brown, Pergamon, New York, 1964.
3. W. C. Rheinboldt, A unified convergence theory for a class of iterative processes, SIAM J. Numer. Anal., 5 (1968) 42–63.
4. A. E. Taylor, Introduction to Functional Analysis, Wiley, New York, 1957.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John G. Harvey, Dept. of Math., University of Wisconsin, Madison, WI 53706.

'PRIME' PEDAGOGICAL SCHEMES

S. I. BROWN, Graduate School of Education, Harvard University

Recently I. A. Barnett made a plea for requiring a course in introductory number theory, not only of all mathematics majors, but of prospective elementary and secondary school teachers as well [1]. After considering the beauty of such concepts as quadratic residue properties, the Prime Number Theorem, Euclid's proof of the infinitude of primes and Dirichlet's result on arithmetic progressions, he asserts that though not all of these results need be *proved* in such a course, they should be *introduced*. In addition to justification on aesthetic grounds, he feels that concepts of divisibility and related notions will help clarify much of algebra and arithmetic.

While (as Professor Barnett implies) it would be foolhardy to prove most of these results in their "natural habitat" to an audience of prospective elementary school teachers, there is an appropriate scheme which would encourage such a group to come to grips with some of these concepts in more of a deductive spirit than would otherwise be the case. The scheme requires only a change of environment. That is, we may consider these questions in the context not of the set of natural numbers, but of other domains, which enable us to by-pass some of the deep problems that would otherwise be encountered.

Though the notion of shifting to a new domain is well known with regard to the Fundamental Theorem of Arithmetic [2, 3, 4], there has not been much attention paid to the idea for other concepts. The object here is to enumerate several questions related to number theoretic notions that are more easily answerable in appropriately chosen domains.

Let N represent the natural numbers. For purposes of illustrating the triviality when the domain is changed, of many otherwise deep theorems, we shall choose the set

$$E = \{1\} \cup \{x : x = 2n \text{ for } n \in N\},$$

i.e. 1 together with all the even elements of N. Let us define a prime just as we do in N; i.e. $x \in E$ is *prime* if it has exactly 2 different factors. Though one may wish to argue with such a literal extension of the concept of prime to a new domain, recall that the intention is to enable the mathematically unsophisticated to tackle in a deductive spirit problems that would otherwise be beyond his reach, rather than to generalize the concept of prime.

We shall list several questions, all easily answerable in E, and shall then discuss briefly their deeper counterparts in N as well as the pedagogical relevance of such an approach.

I On Generation, Determination and Distribution of Primes in E.

(A) How many primes are there?

(B) Is there a simple formula which generates (i) only primes (allowing for possible omission), (ii) all the primes?

(C) What is the nth prime?

(D) Given any number, is there a simple test for primality?

(E) For what values of a and b will $a+b\cdot n$ (where n varies in E and a and b are fixed elements of E) generate an arithmetic progression possessing an infinitude of primes?

(F) How many twin primes are there?

(G) Is there a simple formula to determine (or approximate if need be) the number of primes $\leq x$?

(H) If one lists the elements of E in counter-clockwise spiral (below) how do the primes of E cluster?

32	30	28	26	24
34	8	6	4	22
36	10	1	2	20
	12	14	16	18
⋮				

II On the Fundamental Theorem of Arithmetic and Related Properties in E.

(A) Can any number be factored into a product of primes in essentially two different ways? If so, how can all such numbers be characterized?

(B) Can any number be factored into a product of primes in more than two different ways? If so, how can all such numbers be characterized?

(C) Can all fractions be reduced to lowest terms?

(D) Are the number theoretic functions ϕ, τ, σ multiplicative?

(E) Do all divisors of any two numbers always divide their greatest common divisor?

III On Evenness and Oddness in E.

(A) Are there any even primes?

(B) Can every even number greater than 2 be expressed as the sum of two primes?

The reader will recognize most of these as either very difficult or unanswered questions in N. In E, however, many of them are analyzable by bright junior high school students [**5**, **6**].

Since twice an odd element of N always yields a prime in E, we have a simple formula

$$[2(2\cdot n - 1) \text{ for } n \in N]$$

that enables one to both generate and easily characterize the primes in E. (What simple formula works if we restrict the variable to E?) We thus have very immediate answers (in domain E) to questions I(A)–(E). In N, however, these questions suggest Euclid's proof of the infinitude of primes [I(A)]; abortive efforts of Mersenne and Fermat, as well as a brilliant yet tantalizing existential theorem of Mills [I(Bi)], the work of Wilson, Proth and others who have produced theoretically interesting (though in many cases impractical) tests for primality [I(D)]; the results of Dirichlet and more recently Selberg (who produced an "elementary" proof) on characterization of those arithmetic progressions which generate an infinitude of primes [I(E)].

I(F) (an unsolved problem in N) and III(A) are easily answered once we determine what is meant by "even" and "odd" in domain E—an exercise which helps clarify the meaning in N. Once this is done, Goldbach's gnawing conjecture [III(B)] in domain E is reduced to a one line identity:

$$4n = 2\cdot(2n - 1) + 2,$$

wherein we see the power of searching for equivalent expressions.

I(G) in domain N is the famous *Prime Number Theorem* which captured the imagination of Gauss and Legendre in the 18th century, and was finally proven independently in 1896 by Vallée-Poussin and T. Hadamard. In N, we get an approximation to the number of primes $\leqq x$. In the domain E, not only do we get an exact answer in the form of $[(x+2)/4]$, (where $[x]$ is the greatest integer $\leqq x$) but we have an opportunity to introduce a function which can be used in the elementary school to teach estimation as well as some intuitive ideas on the meaning of homorphism (see [**7**]).

I(H) shifts to domain E a hunch of Ulam [**8**] on the clustering of primes as displayed on the cover of an issue of *Scientific American* [**9**]. A pattern barely suggested in N is fully realized in E. In the latter case, *all* elements along *alternating* diagonals (inclined 45° and 135° to the horizontal) of the counterclockwise spiral are primes.

As indicated earlier some aspects of II have been examined in different domains in the literature. The fact that unique factorization fails in E though it succeeds in N suggests that the theorem in N is not as trivial as one might be led to conclude on the basis of many positive instances. The fact that in E, there exist numbers with as many different prime factorizations as one wishes, suggests exercises that may be devised to help further clarify the meaning of prime, factor, and divisibility. See [**10**]. The student will find it worthwhile to verify (and attempt to account for) the fact that [II(C)–(E)] are all answered in the affirmative for N and in the negative for E.

A unit designed around the types of questions we have posed enables the instructor not only to raise and put in historical perspective deep number theory questions, but, even with a mathematically unsophisticated audience, he can deal deductively with most of them. The students should begin to appreciate that simple proofs in E do not work in N, and those capable of doing so may wish to analyze the abstract properties that distinguish the two sets. Students may be encouraged to suggest other domains of their own choosing and to generate additional conjectures therein.

In addition to the virtues enumerated and those that Professor Barnett has extolled for teaching number theory in general, such a unit is useful in demonstrating how both the questions one chooses and the answers he receives are highly sensitive to even "minor" modifications in the phenomenon he investigates.

References

1. I. A. Barnett, The theory of numbers as a required course in the curriculum for majors, this MONTHLY, 73 (1966) 1002–1004.

2. Newcomb Greenleaf and Robert J. Wisner, The unique factorization theorem, Math. Teacher, 52 (1959) 600–603.

3. Bernard Jacobson and Robert J. Wisner, Matrix number theory: An example of non-unique factorization, this MONTHLY, 72 (1965) 399–402.

4. Calvin T. Long, Elementary Introduction to Number Theory, Heath, Boston, Mass., 1965.

5. Stephen I. Brown, Of 'Prime' concern: What domain?, Math. Teacher, 58 (1965) 402–407.

6. Edward J. Stone, New domains, Math. Teacher, 58 (1965) 514–517.

7. David Page, Do something about estimation! Updating Mathematics, 2 (April 1960).
8. M. L. Stein, S. M. Ulam and M. B. Wells, A visual display of some properties of the distribution of primes, this MONTHLY, 71 (1964) 515–520.
9. *Scientific American*, March 1964, Cover (and in that issue, Martin Gardner, "Mathematical Games," 120–128).
10. Stephen I. Brown, 'Prime', 'Elementary,' and 'Fundamental' Comparisons, The Pentagon, 26 (1967) 95–105.

UNIVERSITY MATHEMATICS IN VENEZUELA

P. R. MONTGOMERY, University of Kansas

1. Introduction. The writer has just returned from a two-year stay at the University of Oriente (UDO) in Cumaná, Venezuela, which is a new university dedicated to educational experimentation and innovation. My stay was partially supported by the Ford Foundation, in a cooperative science project between UDO and the University of Kansas. In addition to my normal duties of teaching and serving as advisor to the mathematics department of UDO, I had opportunities to make several short visits to other academic institutions in the country. These visits were generally of an "official" nature and hence may not have been as fruitful as desired. This report, like the visits, is rather short and cursory, but, like the visits, I hope it accurately reflects the general pattern of teaching mathematics at the post-high school level in Venezuela.

There are four different types of such institutions in Venezuela. The most common are the autonomous universities such as Central (UC), Zulia, Los Andes, and Carabobo. At the same level are the two institutions, UDO and the Experimental Center for Higher Education, which are operated directly under the Ministry of Education (ME) through a rector and university council. (ME recently announced plans to start another university similar to UDO in Caracas.) The ME also operates two normal colleges or Pedagogical Institutes (IP and IPE) for the purpose of training high school teachers. These Institutes are not considered equivalent to universities and the degrees they offer are not accepted as university degrees. Finally, there are several small private universities of which Andrés Bello Catholic University is the most important. These private universities do not play a very large role in the educational system. The schools vary in size from tens of thousands of students at UC to a few hundred at IPE. At all of these places, most of the mathematics taught is of the algebra-calculus variety.

2. Students. Most of the university students come from five-year schools or *liceos*. However, there are two other types of secondary schools: normal schools, which prepare primary teachers, and technical schools, which teach such diverse subjects as commercial fishing, agronomy, and industrial arts. These students are not accepted into any of the autonomous schools and may only enter UDO or the IP's.

The mathematical content of the secondary (and primary) programs is very old-fashioned and does not contain any of the newer approaches. It is tightly

controlled by ME and has not had any revisions for many years, although there is presently at work a committee of the Ministry studying possible changes. Algebra and Geometry are combined throughout the curriculum rather than being divided into separate years as is the case in the U.S. In the past, teachers have tended to omit or treat very lightly certain subjects, geometry being the principal victim. Because of the rigidity of the curriculum, then, the student would never be exposed to this material. No more than thirty per cent of the people teaching mathematics in the *liceos* have had any advanced (i.e., college) training in mathematics; the remainder being high school graduates only.

Because of the nature of the high school testing program, the students try to memorize everything, even entire pages. In general they have extreme difficulty applying known material to new problems. Their studying is made further difficult by various other factors, such as classes for 25–30 hours per week; studying usually done in their crowded *pensiones* or under streetlights; a sizable number of students lack textbooks. As would be expected, the failure rate is quite high with the largest contributing factor being poor background.

3. Faculty. In general, the faculty of these institutions is composed of part-time instructors who teach for varying numbers of hours per week, and have sole control over the courses. Two exceptions to this pattern are UDO and IPE, who employ, almost exclusively, full-time professors. This part-time faculty consists of A.B.-level mathematics majors or engineers. However, both UC and UDO have a high percentage of professors with graduate training in mathematics. The faculties have a quite high ratio of non-Venezuelans and this ratio is higher among those professors with advanced training. Rather substantial efforts are being made to overcome this dependence on foreign professors by scholarship programs which have been initiated in several of the universities. Most of these fellowships are for study in the United States, but other students are sent to France, Germany, Italy and Puerto Rico.

The average teaching load is difficult to compute with the part-time help, but at UDO, for example, most professors teach about twelve hours per week. (This represents two courses, since most courses meet six hours weekly.)

There is very little professional interchange between the universities (sometimes not even within the university). The only scientific society with open membership is the Venezuelan Society for the Advancement of Science, which has a small and not very active mathematics section. At least two of the mathematics departments publish quarterlies along the lines of *Mathematics Magazine* which are directed at high school students and teachers.

4. Classes. The size of the classes varies from the large sections (70–100) of the introductory courses to quite small sections (2–10) for the advanced mathematics majors. As stated above, most meet six hours per week. This time includes some hours of "practice" which correspond roughly to laboratory in chemistry or physics. Very little homework is assigned and little studying is done outside of class except for preparation for exams. The practice classes are,

in theory, intended to help overcome this lack of outside work. There are widely differing uses of these hours. Some professors use them strictly for drill, others incorporate the drill in the lectures, spreading it over several class periods, and other professors ignore the drills altogether.

Discipline in classes and exams is far different from what is normally encountered in the U.S., the students being less attentive and less orderly in the classroom. The full-time professors are much stricter, however, than the part-time faculty and those with higher training tend to be even more so.

Texts are not as widespread as in the United States. Many courses have an official text, but not all students buy it. Many professors also use and sell their own notes. Good up-to-date books in Spanish are rare and expensive.

5. Programs. There are two different types of mathematics degrees offered. The most popular is a four-year program for prospective high school teachers and is the only program available at the IP's. Several of the universities also offer this program. The other, which is considered to be a mathematics major, is a five-year program and is offered at most universities. The five-year program is not quite the equivalent of a Master's degree although a thesis is usually a partial requirement.

Both programs are considerably heavier than the corresponding programs in the U.S. This is caused, in part, by the lack of graduate schools in Venezuela and the lack of opportunity for further training elsewhere. At UDO, for example, students take more than one hundred credits in mathematics in the five-year program. At the same time, much of the learning and teaching is superficial.

Usually the students do not have a choice of courses. Instead, the department offers a *pensum*, which is a listing of the courses a student must take during each semester at the university. He then passes the semester as a unit, not the individual courses, although if a previous course is a prerequisite, it must have been passed. (There are several ways of passing a failed course, not all of which involve retaking it.) Some of the *pensums* are quite modern and include such subjects as differential geometry and algebraic topology. Others are very outdated and include such courses as spherical trigonometry and theory of equations. Most of the courses a mathematics student takes are mathematics; very little work is taken outside the department. Minors, or double majors, are unknown. UDO has adopted a credit system similar to that used in the U.S. and although still influenced by the *pensums*, it promises to give more flexibility and adaptability to both the department and the students.

In a given semester, under these *pensums*, a student takes an average of 18–20 credits. The actual number of class hours is somewhat higher due to the practice hours. This represents about six courses and may include one or two courses outside mathematics, in, for example, English, Humanities, or Social Science.

6. Libraries. With the exceptions of the libraries at UC and UDO, these facilities in Venezuela are almost non-existent. About all that is available to a

professor or a student are those books he buys himself, (professors even have to purchase their personal texts) and these are only available through the Caracas bookstores. UC's library is more established and has a better collection of journals. UDO's library has a wider choice of books at all levels, due chiefly to the fact that the mathematics department received $10,000 in 1966 from the International Development Bank to purchase books and journals.

7. Conclusion. Judged overall, mathematics is not very popular in Venezuela. More attention is gradually being paid to it, and several of the universities are trying to build better departments that are beginning to shed the service role which has been their primary function. Still there are little prospects for a "market" outside of the universities.

The most immediate need at the present is qualified teachers at all levels—primary, secondary, and university. For the universities, the scarcity of texts is of crucial importance, and in the secondary schools a curriculum revision is long overdue. As in everything, the problems of the departments and universities have a special character, sometimes referred to as "Latin," which makes them appear almost unresolvable. The University of Oriente, with its experimental nature and approach, could be a strong modernizing factor if its influence could be spread to the other universities.

TEACHING GROUP THEORY TO COLLEGE FRESHMEN

G. E. CARUSO, Lea College on Lake Chapeau, Minnesota

An experimental study was conducted at Nassau Community College during 1965 and 1966 to determine the relative effectiveness of the abstract approach and the concrete approach of teaching the theory of groups, rings, and fields to freshmen. In the abstract approach, with student participation kept to a minimum, rigorous definitions were given first, followed by examples illustrating the properties of the systems. Proofs of ten theorems on groups were then presented deductively. In the concrete approach, specific examples of the systems were presented first. The students were then encouraged to participate in discussions aimed at discovering generalizations from these concrete examples. The proofs of theorems followed particular examples and problems exemplifying each theorem.

The basic hypotheses were: 1. The experimental (abstract) group will show a higher achievement in learning the theory of groups, rings, and fields than the control (concrete) group. 2. The experimental group will show a higher achievement in delayed recall of these systems than the control group.

Two sequences of lesson plans were developed in detail, one for the abstract approach and the other for the concrete approach. Each consisted of eleven 50-minute lessons and extended over a four-week period. From a pilot study preceding the experiment an item analysis was performed on the results of a preliminary test. Two tests were constructed from these results: one to be

administered at the conclusion of the experiment, the other to be administered nine weeks later as a test of recall.

The experiment consisted of five experimental classes and five control classes taught by five instructors. Each teacher used the abstract approach with an experimental class and the concrete approach with a control class. To test the significance of the difference between means, a one-tailed t-test was used between the means of the experimental and control groups. The power of the test was .90 with $\alpha = .05$.

With 184 df no significant difference was found between the two groups on the first test. However, at the .06 level of significance the experimental group performed significantly better than the control group. On the test of delayed recall it was found that the experimental group had a higher achievement than the control group. With 171 df this was statistically significant beyond the .01 level. These results suggest that the abstract approach is a superior method of teaching the basic theory of groups to college freshmen with limited mathematical backgrounds.

Reference

1. George E. Caruso, A Comparison of Two Methods of Teaching the Mathematical Theory of Groups, Rings, and Fields to College Freshmen, Ph.D. thesis, New York University, 1966. For additional information write to the author.

THE SANTA CLARA HIGH SCHOOL MATHEMATICS CONTEST

A. P. Hillman, G. L. Alexanderson, and L. F. Klosinski,
University of New Mexico and University of Santa Clara

Robert E. Maas of the University of Santa Clara, by winning the 1966 Putnam Intercollegiate Mathematics Competition, has provided fresh evidence of the effectiveness of the University of Santa Clara High School Mathematics Contest, initiated in 1959. Three in the top 100 of the 1966 Putnam Contest were first place winners in this local contest; they are Maas, James Hewlett of Harvard and Alan Siegel of Stanford. All have home addresses in Santa Clara County, California, which, as a result, had as many residents in the top 100 as the combined New York City boroughs of Brooklyn, Manhattan, and the Bronx. The winners of 6 successive Santa Clara Contests have gone on to win distinction in the Putnam. The top 100 was achieved twice by each of two of them; all the others achieved honorable mention or better.

The Santa Clara Contest is a 3-hour subjective test composed and graded by a committee of the University of Santa Clara Mathematics Department. It is open to students from the 7th through the 12th grade. The aim of the problems is to test ingenuity and mathematical potential rather than knowledge of courses. Immediately after the contest solutions are sent to the teachers of all contestants. The results indicate to the sponsors that such a contest can be used to discover and encourage talented students in mathematics.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Mich. 47121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before October 31, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2043 [1967, 1262]. **Corrected.** *Proposed by P. Baxandall, University of Keele, Staffordshire, England*

Let S be a semigroup with a right identity e (so $xe=x$ for all $x\in S$) such that given any $y\in S$ there exists $\bar{y}\in S$ with $\bar{y}y=e$. It is well known that S may not be a group. Is this still true if S has only one right identity?

The proposer calls attention to a misprint in the original statement and a number of readers have already submitted solutions interpreting the problem as intended.

E 2095. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Let G be a finite group written multiplicatively, H a subgroup of G, and k a positive integer. Show that if $KH\neq HK$ for every subset K of G containing k elements, then $k<[G:H]$.

E 2096. *Proposed by Seymour Geisser and Paul Schillo, State University of New York at Buffalo.*

Let m and n be positive integers such that $m<n$; and, for each positive integer $t\leqq n$, let x_t be a real number. Can the equations

$$m^2\left(\sum_{t=1}^{m} x_t^2 - 1\right) = m\left(\sum_{t=1}^{m} x_t\right)^2 + 1, \tag{1}$$

$$n^2\left(\sum_{t=1}^{n} x_t^2 - 1\right) = n\left(\sum_{t=1}^{n} x_t\right)^2 + 1 \tag{2}$$

both be true?

E 2097. *Proposed by Harley Flanders, Purdue University*

Let $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$, where $1\geqq a_1\geqq a_2\geqq\cdots\geqq a_n\geqq 0$. Let λ be a complex root of f such that $|\lambda|\geqq 1$. Prove λ is a root of unity.

E 2098. *Proposed by J. O. Kiltinen and T. J. Grilliot, Duke University*

Suppose that A is a commutative ring with identity which has the property that every nonzero polynomial with coefficients in A has only finitely many roots in A. Prove that A is either an integral domain or is finite.

E 2099. *Proposed by R. S. Underwood, Texas Technological College*

Let all numbers involved be real. Given

$$\frac{x^2}{a^2}-\sum_{i=1}^{n}\frac{y_i^2}{b_i^2}=1, \tag{1}$$

let $\sum_{i=1}^{n}A_i^2b_i^2=k^2a^2$, with the A_i arbitrary except that not all are zero. Find the sole common real solution (x, y_i) of (1) and

$$2kx-\sqrt{3}\sum_{i=1}^{n}A_iy_i=ka \qquad (k>0,\ a>0). \tag{2}$$

Show that the solution is unique and that there is no solution if the right side of (2) is replaced by K, with $|K|<ka$.

E 2100. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

Show that any five of the relations

(1) $\dfrac{x-a_1}{a_1-a_2}=\dfrac{a-b}{b-c}$, (2) $\dfrac{x-b_1}{b_1-b_2}=\dfrac{b-c}{c-a}$, (3) $\dfrac{x-c_1}{c_1-c_2}=\dfrac{c-a}{a-b}$,

(4) $x+a=b_2+c_1$, (5) $x+b=c_2+a_1$, (6) $x+c=a_2+b_1$

imply the sixth. Interpret this set of consistent relations geometrically letting a, b, c be the affixes, in the complex plane, of a triangle of reference ABC and other numbers be those of other points.

E 2101. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

ABC is a triangle. Let P_a denote the parabola tangent to the sides AB, AC at B, C respectively. The parabolas P_b and P_c are similarly defined. Let these parabolas intersect to the points A', B', C' inside ABC. Denote the areas of the (curvilinear) triangular regions ABC, $A'B'C'$, $AB'C'$, $BC'A'$, $CA'B'$, $A'BC$, $B'CA$, $C'AB$, by $\triangle$, $\triangle_0$, $\triangle_a'$, $\triangle_b'$, $\triangle_c'$, $\triangle_a''$, $\triangle_b''$, $\triangle_c''$. Then prove

(1) $\triangle_a'=\triangle_b'=\triangle_c'=(\triangle_1)$, $\triangle_a''=\triangle_b''=\triangle_c''=(\triangle_2)$,

(2) $\triangle_0:\triangle_1:\triangle_2:\triangle=15:17:5:81$.

E 2102. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Given an equilateral triangle of side one. Show how, by a straight cut, to get two pieces which can be rearranged so as to form a figure with maximal diameter (a) if the figure must be convex; (b) otherwise.

E 2103. *Proposed by Simeon Reich, The Technion, Haifa, Israel*

Find the seven smallest numbers $a_k (k=1, 2, \cdots, 7)$ with the following property: If a point P is inside a unit cube $A_1A_2 \cdots A_8$ at most k of the eight distances $PA_j (j=1, 2, \cdots, 8)$ are greater than a_k (thus at most one of these distances will be greater than a_1, at most two will be greater than a_2, etc.).

E 2104. *Proposed by F. Dapkus, Seton Hall University*

If a line segment AB of fixed length is moving in such a way that A and B are sliding along two perpendicular, non-intersecting lines, determine the volume bounded by the surface swept out by AB.

SOLUTIONS OF ELEMENTARY PROBLEMS

Special Quadrics

E 1886 [1966, 538; 1967, 1263]. *Proposed by W. M. Sanders, Lawrence University, Appleton, Wis.*

Show that if $ax^2+by^2+cz^2+2fxy+2gzx+2hyz+2rx+2sy+2tz+d=0$ represent two distinct planes, then the matrix

$$E = \begin{vmatrix} a & f & g & r \\ f & b & h & s \\ g & h & c & t \\ r & s & t & d \end{vmatrix}$$

is not the adjoint of any matrix.

III. *Editorial Note.* F. G. Gustavson, of the IBM Watson Research Center, notes that Solution II (Dec. 1967) is not a real generalization of the problem since the solver takes the two planes to be parallel. However, Gustavson shows that the result goes through for two general planes.

Ellipses Passing through a Prescribed Number of Lattice Points

E 1948 [1967, 76]. *Proposed by Gregory Wulczyn, Bucknell University*

(a) Find the equation of a 2-parameter family of ellipses each of which passes through exactly n points of the integral coordinate lattice.

(b) Find the equation of a family of ellipsoids each of which passes through exactly n points having integral Cartesian coordinates.

Solution by the proposer. (a) First, let n be odd, $n=2k+1$. Let a and b be distinct positive integers with $(a, 3)=1$, $b>3$ and b a prime of the form $4\alpha+3$.

Then the ellipse

$$(1) \qquad \frac{(x-\frac{1}{3}a)^2}{a^2}+\frac{y^2}{b^2}=\frac{5^{2k}}{9}, \qquad k=0,1,2,\cdots$$

will have exactly $2k+1$ lattice points upon it.

If an integer value t of y is chosen with $(t, b)=1$, then $5^{2k}b^2-9t^2$ is not a perfect square ($5^{2k}b^2=9t^2+m^2$ is impossible since all decompositions of $5^{2k}b^2$ into the sum of two squares must have b^2 as a common factor). Hence x will be irrational. If $y=0$, $x-a/3=\pm a5^k/3$ will produce one integer solution for x and therefore one lattice point. Let $5^{2k}=9r_i^2+s_i^2$, $i=1, 2, \cdots, k$ be the remaining k decompositions of 5^{2k} into a sum of two squares. Then $y=\pm br_i$ will yield $x-a/3=\pm as_i/3$ or x will have one integral value and there will be two lattice points on the curve for each $y=\pm br_i$; for a total of $2k+1$ lattice points. There can be no more due to the fact that there are only $k+1$ decompositions of 5^{2k} as a square or as the sum of two squares.

Suppose next that n is even, $n=2k$. Let a and b be distinct positive integers with $(a, 2)=1$ and b a prime of the form $4n+3$. Then the ellipse

$$(2) \qquad \frac{(x-a/2)^2}{a^2}+\frac{y^2}{b^2}=\frac{5^{k-1}}{4}, \qquad k=1,2,3,\cdots$$

will have exactly $2k$ lattice points on it. As in the first case, for x to be rational y must be a multiple of b. If $k=2u+1$, then $y=0$, $x-a/2=\pm a5^u/2$ will give two integer values for x and hence 2 lattice points on the ellipse. The remaining u decompositions of 5^{2u} into the sum of two squares will be of the form $5^{2u}=4r_i^2+s_i^2$, $i=1, 2, \cdots, u$ so that $y=\pm br_i$, $x-a/2=\pm as_i/2$ will yield 4 lattice points for each i, and a total of $4u+2$ lattice points on the ellipse, $u=0, 1, 2, \cdots$.

If $k=2u$, then $5^{2u-1}=4r_i^2+s_i^2$, $i=1, 2, \cdots, u$. Then $y=\pm br_i$, $x-a/2=\pm as_i/2$ will yield 4 lattice points on the ellipse for each i and therefore $4u$ lattice points, $u=1, 2, \cdots$. Hence for all k there will be $2k=n$ lattice points on the ellipse.

Note that in both (1) and (2) the integer 5 could be replaced by any prime of the form $4\beta+1$.

(b) If instead of (1) and (2) above we consider the ellipsoids

$$(1') \qquad \frac{(x-a/3)^2}{a^2}+\frac{y^2}{b^2}+\frac{(z-ce)^2}{c^2}=\frac{5^{2k}}{9}+e^2,$$

$$(2') \qquad \frac{(x-a/2)^2}{a^2}+\frac{y^2}{b^2}+\frac{(z-ce)^2}{c^2}=\frac{5^{k-1}}{4}+e^2,$$

where a and b are chosen as above, c is a positive integer and e is the base of natural logarithms or any irrational number which will render it impossible to have simultaneously z a nonzero integer, y an integer and $x-a/3$ (or $x-a/2$)

rational. Since z must therefore be zero for any point with integral coordinates, we have just the cases considered in (a) and the proof is complete.

A Minimum Area Quadrilateral

E 1949 [1967, 76]. *Proposed by Michael Goldberg, Washington, D.C.*

Two circles of radii a and b are tangent externally. What is the area of the smallest quadrilateral enclosing the two circles?

Solution by the proposer. We make use of the lemma: A necessary (but not sufficient) condition for an extremal is that a side which touches only one circle must touch the circle at the mid-point of the side. (Let $A'B'$, $A''B''$ be two neighboring positions of the concerned side of the quadrilateral, chosen on opposite sides of the extremal position AB and such that the determined quadrilateral areas are equal. Then, if $A'B'$, $A''B''$ intersect in P, triangles $A'PA''$ and $B'PB''$ have equal areas, whence $(PA')(PA'')=(PB')(PB'')$. Passing to the limit, where $A'B'$ and $A''B''$ coincide with the required extremal position AB, we find $(PA)^2=(PB)^2$, where P is now the point of contact of AB with the circle. It follows that AB is bisected by its point of contact P with the circle.)

Take $a \geqq b$. If $b \leqq (3-2\sqrt{2})a$, then K, the area of the quadrilateral, is given by $K=4a^2$, since the smallest quadrilateral is the circumscribed square of the larger circle. See Fig. 1.

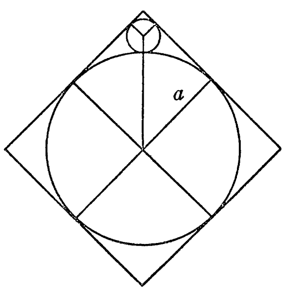


FIG. 1

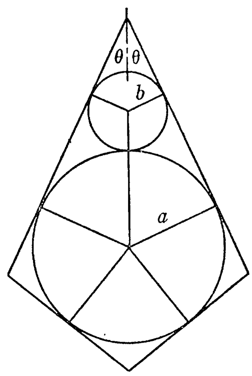


FIG. 2

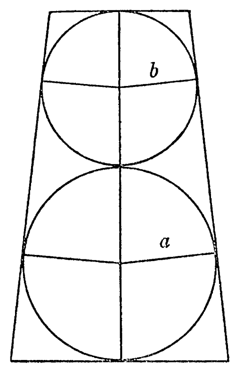


FIG. 3

For larger b, (see Fig. 2), we obtain by elementary trigonometry,

$$\begin{aligned} K &= 3a^2\tan(30° + \theta/3) + b^2\cot\theta + 2(a+b)\sqrt{ab} \\ &= a^2[\cot\theta + 3\tan(30° + \theta/3)], \end{aligned}$$

where $\sin\theta = (a-b)/(a+b)$.

For still larger b, (see Fig. 3),

$$K = 2a^2\cot(45° - \theta/2) + 2b^2\cot(45° + \theta/2) + 2(a+b)\sqrt{ab}$$
$$= 2(a+b)(a^2+b^2)/\sqrt{ab}.$$

The transition occurs at the equivocal case in which

$$a^2[\cot\theta + 3\tan(30° + \theta/3)] = 2(a+b)(a^2+b^2)/\sqrt{ab}.$$

This equation holds when (approximately) $b = 0.54\ a$.

An Extension of the Pythagorean Theorem

E 1965 [1967, 317]. *Proposed by Hyman Gabai and Steven Szabo, University of Illinois and UICSM*

Given any triangle ABC, let P and Q be the midpoints of sides AB and AC, respectively. Let G and G' be points on the perpendicular bisector of AB such that $GG' = AC$, P is the midpoint of GG', and segment GP contains no points of the interior of triangle ABC. Let H and H' be points on the perpendicular bisector of AC such that $HH' = AB$, Q is the midpoint of HH', and segment QH contains no points of the interior of triangle ABC. Prove (a) $GH^2 + G'H'^2 = AB^2 + AC^2$ and (b) $G'H^2 + GH'^2 = BC^2$.

I. *Solution by Jordi Dou, Barcelona, Spain.* Holding side AB fixed in position and letting angle A vary, it is seen that G, G', H, H' all lie on a common fixed circle centered at A. It follows that GH' and $G'H$ intersect at a constant angle (measured by half the sum of the constant arcs GG' and HH'). By taking $A = 90°$, it is found that the constant angle between GH' and $G'H$ is also $90°$. It then follows that $GH^2 + G'H'^2 = GG'^2 + HH'^2 = AB^2 + AC^2$. If H_1 is the reflection of H' in point P, then $H_1H = 2PQ = BC$ and H_1G' is equal and parallel to GH', whence $G'H^2 + GH'^2 = G'H^2 + H_1G'^2 = HH_1^2 = BC^2$.

II. *Solution by Stephen Hoffman, State University of New York at Cortland.* Let

$$\overrightarrow{AB} = 2\mathbf{a}, \qquad \overrightarrow{AC} = 2\mathbf{b}, \qquad \overrightarrow{PG} = \mathbf{c}, \qquad \overrightarrow{QH} = \mathbf{d}.$$

Then

$$|\mathbf{a}| = |\mathbf{d}|, \qquad |\mathbf{b}| = |\mathbf{c}|, \qquad \mathbf{b}\cdot\mathbf{d} = \mathbf{a}\cdot\mathbf{c} = 0, \qquad \mathbf{c}\cdot\mathbf{d} = -\mathbf{a}\cdot\mathbf{b},$$

and

$$\text{(a)}\ GH^2 + G'H'^2 = |\mathbf{b}+\mathbf{d}-\mathbf{a}-\mathbf{c}|^2 + |\mathbf{b}-\mathbf{d}-\mathbf{a}+\mathbf{c}|^2$$
$$= 4(|\mathbf{a}|^2 + |\mathbf{b}|^2) = AB^2 + AC^2;$$
$$\text{(b)}\ G'H^2 + GH'^2 = |\mathbf{b}-\mathbf{d}-\mathbf{a}-\mathbf{c}|^2 + |\mathbf{b}+\mathbf{d}-\mathbf{a}+\mathbf{c}|^2$$
$$= 4|\mathbf{a}-\mathbf{b}|^2 = BC^2.$$

Also solved by Anders Bager (Denmark), J. W. Baldwin, Leon Bankoff, Jhuda Falk (Israel), C. B. Ferree & H. E. Speece, R. L. Fox, M. G. Greening (Australia), B. W. King, E. Kosko,

D. C. B. Marsh, B. D. McLemone, D. G. Merrifield, Norman Miller, Bohuslav Mišek (Czechoslovakia), C. B. A. Peck, J. M. Quoniam (France), Frank Ranhofer, Simeon Reich (Israel), Perry Scheinok, Stephen Spindler, Philip Trauber, Richard Vazzana, C. S. Venkataraman (India), Brother T. C. Wesselkamper (Nigeria), Gregory Wulczyn, Qazi Zameeruddin (India), and the proposers.

Construction of a Regular Tetrahedron

E 1966 [1967, 318]. *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

Show how to construct a regular tetrahedron if the vertices lie on four given parallel planes.

Solution by the proposer. Let the distances between successive planes be a' b and c. Starting with any regular tetrahedron, locate points D, E on edge OA such that $OD:DE:EA=a:b:c$. On edge OB locate F such that $OF:FB=a:b$. Now draw a plane through O, a plane through B and E and a plane through A all parallel to the plane through F, D and C.

This gives us a configuration similar to the one we wish to construct, which can now be done by similar figures.

Also solved by Jordi Dou (Spain), Stephen Clodman, Michael Goldberg, V. F. Ivanoff, Bohuslav Mišek (Czechoslovakia), and J. M. Quoniam (France).

Ivanoff notes that it is possible to construct a tetrahedron under the given conditions similar to any given tetrahedron.

Representable Integers

E 1967 [1967, 318]. *Proposed by W. L. Blanchard, University of Rochester*

Prove that for any positive integers, a, b the set of all $ma+nb$ (m, n positive integers) includes all multiples of (a, b) larger than $[a, b]$. Show also that this is the best possible statement. Note: (a, b) is the greatest common divisor and $[a, b]$ is the least common multiple.

Solution by E. J. F. Primrose, University of Leicester, England. Suppose $(a, b)=h$, so that $a=hc$, $b=hd$. Then we have to show that $h(mc+nd)$ gives all multiples of h larger than hcd. Removing the factor h, we see that this is equivalent to the following: if $(c, d)=1$, $mc+nd$ gives all positive integers larger than cd.

We first show that $mc+nd$ does not give cd. The general solution of $mc+nd=cd$ is $m=d(1-\lambda)$, $n=\lambda c$ (λ integral), and λ cannot be so chosen that both m and n are positive.

Now we show that if $mc+nd=cd+f$ $(f>0)$, then there are positive solutions for m and n. By Euclid's algorithm the equation $m_0c+n_0d=f$ has integral solutions for m_0 and n_0. If m_0 and n_0 can both be positive there is nothing to prove, so suppose $m_0>0$ and $n_0<0$. Other solutions can be obtained by subtracting λd from m_0 and adding λc to n_0, so we may suppose that $n_0>-c$. Now there is a solution of $mc+nd=cd+f$ of the form $m=m_0$, $n=n_0+c$, so m and n are both positive.

An equivalent result is proved in Semple and Kneebone, *Algebraic Curves*, p. 81.

Also solved by Anders Bager (Denmark), R. B. Eggleton (Australia), Neal Felsinger, Bengt Fornberg (Sweden), M. G. Greening (Australia), Donald Jeffords, Lew Kowarski, L. Kuipers (Netherlands), D. A. Marcus, D. C. B. Marsh, Bohuslav Mišek (Czechoslovakia), P. L. Montgomery, Bob Prielipp, Simeon Reich (Israel), David Sibley, Stephen Spindler, H. E. Thomas, Jr., Steven Weintraub, C. S. Venkataraman (India), and the proposer.

Editorial Note. Eggleton and Kuipers note that the problem is contained in I. Niven and H. Zuckerman, *Introduction to the Theory of Numbers*, 1960, 96–97. Bager notes that a fuller statement is given by W. J. LeVeque, *Topics in Number Theory*, 1956, v. I, p. 22, Problem 4. For completeness we reprint the problem:

*4. Let a and b be positive relatively prime integers. Then for certain non-negative integers n (which we shall refer to briefly as the *representable* integers), the equation $ax+by=n$ has a solution with $x\geqq 0$, $y\geqq 0$, while for other n it may not have. For example, if $n=0$, 3, 5, or 6, or if $n\geqq 8$, then $3x+5y=n$ has such a solution. Show that this example is typical, in the following sense:

(a) There is always a number $N(a, b)$ such that for all $n\geqq N(a, b)$, n is representable. (It may be helpful to combine the theory of the present section with the elementary analytic geometry of the line $ax+by=c$, interpreting x and y in the latter case as real variables. Note that so far it is only the existence of $N(a, b)$ that is in question, and not its size.)

(b) The minimal value of $N(a, b)$ is always $(a-1)\ (b-1)$.

(c) Exactly half the integers up to $(a-1)\ (b-1)$ are representable.

On Composite Consecutive Values of a Polynomial

E 1968 [1967, 318]. *Proposed by Erwin Just, Bronx Community College, N. Y.*

(Corrected) Prove that, for any nonconstant integral polynomial $p(x)$ and any integer k, there exist k consecutive integral values of x such that $p(x)$ is composite.

Solution by M. G. Greening, University of New South Wales, Australia. We have $p(x+r)=x\cdot g(x, r)+p(r)$. For fixed r there will be an x_r such that $g(x, r)$ is of constant sign for all $x>x_r$. Let x' be the maximum of the finite set x_r $(r=1, 2, \cdots, k)$. Put $x^*=tp(1)p(2)\cdots p(k)$ and select t so that $x^*>x'$. Then $f(x^*+r)$ is divisible by $f(r)$, $(r=1, \cdots, k)$ so that x^*+r is a set of k consecutive values of x with the required properties.

Also solved by Anders Bager (Denmark), James Baumbach, Mickey Dargitz, R. B. Eggleton (Australia), Neal Felsinger, Bengt Fornberg (Sweden), Jerry Goodman, Donald Jeffords, Douglas Lind, D. A. Marcus, Steven Minsker, Stanley Rabinowitz, and the proposer.

C. B. A. Peck notes that the problem has been solved previously. See E 1320 [1959, 65].

Collinearity of the Centers of Three Squares

E 1969 [1967, 318]. *Proposed by J. M. Quoniam, Saint-Etienne, France*

The centers of the internal squares constructed on the sides BC, CA, AB of a triangle are collinear if $\cot\omega=2$, where ω is the Brocard angle of ABC.

Solution by A. W. Walker, Toronto, Canada. This result is included in problem 3990 [1948, 166], proposed by V. Thébault, who also [1] listed many properties of triangles with $\cot\omega=2$, and [2] noted the association of these

triangles with the arbelos, giving a simple method for their construction.

The solution of 3990 may be generalized as follows. If similar isosceles triangles BCD, CAE, ABF with base angle θ are constructed inwardly on the sides of triangle ABC as bases, and if triangle ABC is acute-angled and its circumcenter O lies inside triangle DEF, then

$$OE = \tfrac{1}{2}b(\cot B - \tan\theta), \qquad OF = \tfrac{1}{2}c(\cot C - \tan\theta),$$

$$\begin{aligned}\triangle OEF &= \tfrac{1}{2}(OE\cdot OF)\sin A\\ &= \tfrac{1}{4}(\triangle ABC)\{\cot B\cot C - (\cot B + \cot C)\tan\theta + \tan^2\theta\},\end{aligned}$$

and, adding this to similar expressions for $\triangle OFD$ and $\triangle ODE$,

$$\begin{aligned}\frac{\triangle DEF}{\triangle ABC} &= \tfrac{1}{4}\{1 - 2\cot\omega\cdot\tan\theta + 3\tan^2\theta\}\\ &= \tfrac{1}{4}\sec^2\theta\cdot\csc\omega\{2\sin\omega - \sin(2\theta+\omega)\}.\end{aligned}$$

Examining the various possibilities, this result is seen to be valid without restriction, having a negative sign when DEF and ABC are in opposite cyclic order. (In an alternative method, areal coordinates are used, noting that $\triangle DEF = 3\{\triangle GEF\}$, where G is the common centroid [3] of triangles DEF and ABC.) It follows that the points D, E, F are collinear if and only if $\sin(2\theta+\omega) = 2\sin\omega$, a result obtained by Brocard [4]. If $\theta = \frac{1}{4}\pi$, this gives $\cot\omega = 2$.

As an associated result, we note that if three circles in mutual external contact all touch the same straight line, their mutual contact triangle is such that $\cot\omega = 2$. For, given three circles touching externally at points A, B, C, then the radius of the larger of the two Soddy circles that touch all three given circles is

$$\rho = \frac{rs}{(4R + r) - 2s} = \frac{r}{\cot\omega - 2}.$$

In the first of these expressions [5, 6], R, r and s are the circumradius, inradius and semiperimeter of the triangle with vertices at the centers of the given circles; in the second, readily obtained from the first (and also from a more general result established by Taylor [7]), r and ω are the circumradius and Brocard angle of the contact triangle ABC.

References

1. Bull. Société Royale des Sciences de Liège, 19 (1950) 575.
2. Scripta Mathematica, 15 (1949) 87.
3. R. A. Johnson, Modern Geometry, (1929) 223.
4. Nouvelle Correspondance Math., 5 (1879) 427.
5. Nouvelles Annales de Math., (2), 15 (1876) 321.
6. H. S. M. Coxeter, Introduction to Geometry, (1961) 16.
7. Proc. London Math. Soc., 20 (1889) 404.

Also solved by Leon Bankoff, D. C. B. Marsh, Simeon Reich (Israel), Sister Stephanie Sloyan,

C. S. Venkataraman (India), Gregory Wulczyn, and the proposer.

Bankoff also found the problem in reference [2] above, where it, along with other properties, is stated without proof.

A Function Which Is a Derivative

E 1970 [1967, 318]. *Proposed by R. E. Dowds, State University College of New York, Fredonia*

Define $f(x)=\sin(1/x)$, $x\neq 0$, and $f(0)=0$. Is the function f a derivative? In other words, does there exist a function F such that $F'(x)=f(x)$ for all x, including $x=0$?

Solution by R. J. Driscoll, Loyola University. If

$$F(0) = 0, \qquad F(x) = x^2\cos(1/x) - 2\int_0^x t\cos(1/t)dt, \qquad x \neq 0,$$

then $F'(x)=f(x)$ for all x. Incidentally, one can show that $F(x)=\int_0^x \sin(1/t)dt$ by using integration by parts in this integral with $u=t^2$, $v=\cos(1/t)$.

Also solved by R. L. Browning, W. G. Dotson, Jr., Neal Felsinger, Bengt Fornberg (Sweden), J. A. Goldstein, G. A. Heuer, Stephen Hoffman, R. E. Johnson, B. G. Klein, M. E. Muldoon, Jürg Rätz (Switzerland), and the proposer.

Editorial Note. Muldoon generalized the problem by showing that $g(x)=|x^\alpha|\sin(1/x)$, $x\neq 0$, $\alpha>-1$, $g(0)=0$, is a derivative using a mean value theorem. The same is true if $|x^\alpha|$ is replaced by x^α; for then $G'(x)=g(x)$ where $G(0)=0$ and

$$G(x) = x^{\alpha+2}\cos(1/x) - (2+\alpha)\int_0^x x^{\alpha+1}\cos(1/x)dx.$$

On the Number of Real and Complex Roots of a Class of Polynomials

E 1971 [1967, 318]. *Proposed by R. A. Whiteman, Illinois Institute of Technology*

Let $g(x)=a_mx^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0$, where the coefficients are real numbers and $a_m>0$. Define a polynomial $G(x)$ by $G(x)=[g(x)]^2-g'(x)$. Prove that, if $g(x)$ has m distinct real roots, then

(a) for m equal to an odd positive integer, the polynomial $G(x)$ has $m+1$ real roots and $m-1$ complex roots, and

(b) for m equal to an even positive integer, $G(x)$ has m real roots and m complex roots.

Solution by L. Carlitz, Duke University. Given

$$g(x_1) = g(x_2) = \cdots = g(x_m) = 0, \qquad x_1 < x_2 < \cdots < x_m,$$

it is evident that

$$g'(x_m) > 0, \qquad g'(x_{m-1}) < 0, \qquad g'(x_{m-2}) > 0, \cdots.$$

It follows that $G(\infty)>0$, $G(x_m)<0$, $G(x_{m-1})>0$, $\cdots$. In particular, $G(x_1)<0$, $G(-\infty)>0$ (m odd), while $G(x_1)>0$ (m even).

Therefore, by Rolle's theorem, $G(x)$ has at least $m+1$ real zeros for m odd and at least m real zeros for m even.

To complete the proof of the stated result it suffices to show that if y_1, y_2 are two consecutive zeros of $G(x)$, there is a root of $g(x)$ between y_1 and y_2. Since the roots of $g(x)$ are real and distinct it follows that $(g'(x))^2 > g(x)g''(x)$ for all real x. (Indeed, since

$$g(x) = a_m(x - x_1)(x - x_2) \cdots (x - x_m), \qquad \frac{g'(x)}{g(x)} = \sum_{j=1}^{m} \frac{1}{x - x_j},$$

$$g(x)g''(x) - (g'(x))^2 = - g^2(x) \sum_{j=1}^{m} \frac{1}{(x - x_j)^2} < 0.)$$

Then

$$\begin{aligned} g(x)G'(x) &= g(x)[2g(x)g'(x) - g''(x)] \\ &= 2g'(x)[g^2(x) - g'(x)] + 2(g'(x))^2 - g(x)g''(x), \end{aligned}$$

so that

$$g(x)G'(x) > 2g'(x)G(x) \tag{1}$$

for all real x. It follows first from (1) that the real roots of $G(x)$ are distinct. Next, if y_1, y_2 are consecutive roots of $G(x)$ then the signs of $G'(y_1)$, $G'(y_2)$ are different; therefore by (1) the signs of $g(y_1)$, $g(y_2)$ are different so that $g(x)$ has a root between y_1 and y_2. Thus for m odd, if $G(x)$ has more than $m+1$ real roots it would follow that $g(x)$ has more than m roots. For m even, if $G(x)$ has more than m real roots then it must have at least $m+2$ real roots and $g(x)$ has more than m roots.

REMARK. Exactly the same argument applies to the more general

$$G_c(x) = g^2(x) - cg'(x) \qquad (0 < c \leqq 2).$$

Also solved by W. G. Dotson, Jr., Neal Felsinger, M. G. Greening (Australia), Norman Miller, and the proposer.

Partition of n Consecutive Squares into Two Equal Sum Sets

E 1972 [1967, 318]. *Proposed by E. P. Starke, Plainfield, N. J.*

There are n consecutive positive integers whose squares can be separated into two sets having the same sum, if and only if there exist two positive integers r, s, $r > s$, such that $n = r^2 - s^2$.

Solution by Jack C. Abad and Paul R. Abad, San Francisco. It is easily shown that $n = r^2 - s^2$ is equivalent to the condition $n \not\equiv 2 \pmod 4$, $n \neq 1$, $n \neq 4$: $4k = (k+1)^2 - (k-1)^2$, $4k+1 = (2k+1)^2 - (2k)^2$, and $4k+3 = (2k+2)^2 - (2k+1)^2$, while $4k+2$ is not the difference of two squares since for all i, $i^2 \equiv 0$ or $1 \pmod 4$.

For these n, we designate n consecutive integers by naming the first (smallest) one and then picking out a subset of the n integers whose squares add up to the squares of the remaining ones.

n	*First*	*Subset*
$8k$	1	Integers congruent to 1, 4, 6, 7 (mod 8)
$8k+4$	1	6, 8, and integers greater than 8 congruent to 1, 4, 6, 7 (mod 8)
$4k+1$	$2k$	Smallest k odd integers and largest k even integers
$4k+3$	$2k+3$	Smallest $k+1$ odd integers and largest $k+1$ even integers

Proving the other half of the theorem indirectly, we note that if n consecutive integers have the given property, the sum of their squares must be even. But if $n \equiv 2 \pmod 4$, half of these $4k+2$ squares are congruent to 0 and the other half to 1, making the sum of the form $2k+1$, an odd number. It remains only to show that such an n is not 1 or 4, which is easily done.

Also solved by Joseph Arkin, Anders Bager (Denmark), L. Carlitz, Thomas Elsner & Dale Meinhold, M. G. Greening (Australia), R. M. Krause, D. C. B. Marsh, T. M. Young III, and the proposer.

A related, more difficult question is: For each n for which solutions exist, how many distinct solutions are there? For $n=3, 5, 7, 8$, there are one, two, four, infinitely many, respectively.

A Property of the Morley Configuration

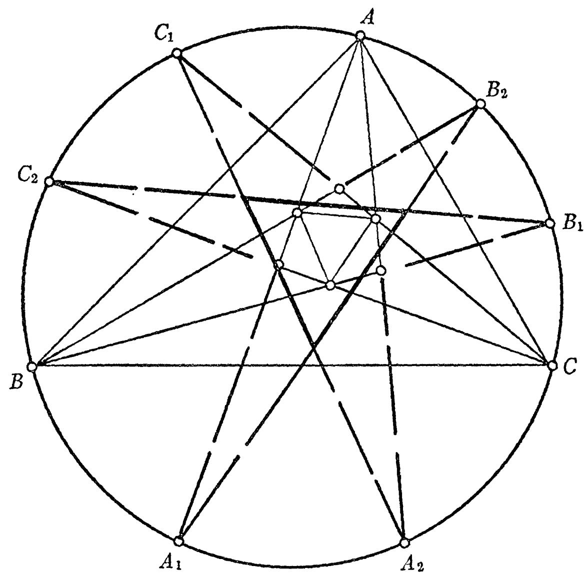
E 1973 (1967, 319]. *Proposed by H. P. Bieri, Spiez, Switzerland*

Consider triangle ABC and its circumcircle. Let A_1, A_2; B_1, B_2; C_1, C_2 be the trisection points of arcs BC, CA, AB of the circumcircle, arranged so that B, A_1, A_2, C, B_1, B_2, A, C_1, C_2, B appear in this cyclic order on the circumference. Prove that A_1B_2, B_1C_2, C_1A_2 determine the side lines of a triangle homothetic to the Morley triangle of ABC.

Solution by A. W. Walker, Toronto, Canada. With reference to triangle ABC, each of the following four associated results is implied by the other three:

1, 2. The lines B_1C_2, C_1A_2, A_1B_2 are parallel to the sides of the Morley triangle (E 1973, E 1304), and are also parallel to the sides of the equilateral "mean triangle" LMN defined in this MONTHLY, 45 (1938), p. 435 and 47 (1940), p. 140 (L, M, N being also the midpoints of arcs B_1AC_2, C_1BA_2, A_1CB_2). This second result is easily proved by arcs and subtended angles on circle ABC.

3, 4. The cusp tangents of the Simson line envelope (Steiner deltoid) are perpendicular to the sides of the Morley triangle (H. F. Baker, *An Introduction to Plane Geometry*, 1943, p. 346), and also to the sides of the mean triangle LMN (this MONTHLY, 53 (1946), p. 200).



Also solved by Anders Bager (Denmark), Leon Bankoff, Michael Goldberg, F. Leuenberger (Switzerland), J. M. Quoniam (France), Jürg Rätz (Switzerland), Simeon Reich (Israel), C. B. A. Peck, Gregory Wulczyn, A. Vandeghen, and the proposer.

Bager and Vandeghen note that the result follows immediately from A. Vandeghen, *A note on Morley's theorem*, this MONTHLY, 1965, p. 638. Rätz refers to H. Dorrie's proof of Morley's theorem in K. Strubecker, *Einführung in die höhere Mathematik*, I, Oldenbourg, München, 1956, p. 595. Goldberg, Leuenberger, Quoniam and Wulczyn refer to A. Coxeter, *Introduction to Geometry*, p. 24. Bankoff, Peck and Walker observe that the present problem is an improved version of E 1304 [1958, 630].

An n-th Order Determinant

E 1974 [1967, 319]. *Proposed by Richard Harter, Brookline, Mass.*

Let the n variables $x_1, \cdots, x_n$ satisfy a relationship $f(x_1, \cdots, x_n)=0$. Define the nth order determinant A as follows: $A_{ii}=1$, $A_{ij}=\partial x_i/\partial x_j$, $i\neq j$. Prove $A=-(n-2)2^{n-1}$.

Solution by E. J. F. Primrose, University of Leicester, England. We denote $\partial f/\partial x_i$ by f_i. Then, if $i\neq j$, $A_{ij}=\partial x_i/\partial x_j=-f_j/f_i$. We now multiply the ith row of the determinant by f_i $(i=1, 2, \cdots, n)$ and divide the jth column by f_j $(j=1, 2, \cdots, n)$. This does not affect the value of the determinant, which now has $A_{ii}=1$, $A_{ij}=-1$ $(i\neq j)$. We now add all the rows to form the new first row, take out the factor $-(n-2)$, and add the new first row to each row in turn. This gives the required value $-(n-2)2^{n-1}$.

Also solved by J. C. Agrawal, L. Carlitz, P. L. Claypool, R. W. Feldmann, Stephen Hoffman, D. C. B. Marsh, Norman Miller, William Parzynski, Edwin A. Power (England), Perry Scheinok, and the proposer.

Editorial Note. The problem is equivalent to evaluating $(-1)^n A(x)$, for $x=-1$, where $A(x) = |a_{ij}|$ with $a_{ii}=x$, $a_{ij}=1\,(i\neq j)$. This is well known. Since by setting $x=1$, n rows become identical, $(x-1)^{n-1}$ is a factor. Also, by adding all the rows, $(x+n-1)$ is another factor. Thus we have $A(x) = A_0(x-1)^{n-1}(x+n-1)$. Since the coefficient of x^n is 1, $A_0 = 1$. Finally $(-1)^n A(-1) = 2^{n-1}(2-n)$.

Two Binomial Identities

E 1975 [1967, 437]. *Proposed by Maurice Machover, Fairleigh Dickinson University and H. W. Gould, West Virginia University*

Prove the following summations for all real z:

$$\sum_{k=0}^{n}\binom{z}{2k}\binom{z-2k}{n-k}2^{2k} = \binom{2z}{2n}, \tag{1}$$

$$\sum_{k=0}^{n}\binom{z+1}{2k+1}\binom{z-2k}{n-k}2^{2k+1} = \binom{2z+2}{2n+1}. \tag{2}$$

Solution by M. T. L. Bizley, London, England. (1). The left side is clearly the coefficient of x^{2n} in

$$\sum\binom{z}{j}2^j x^j(1+x^2)^{z-j}, \qquad |x| < 1, \tag{i}$$

bearing in mind that only even values of j in the sum can give terms in x^{2n} since $(1+x^2)^{z-j}$ yields only even powers. (Thus we can take $j=2k$ and require the term in x^{2n-2k} in $(1+x^2)^{z-2k}$.) But (i) is the binomial expansion of $(2x+(1+x^2))^z = (1+x)^{2z}$, so the coefficient of x^{2n} is $\binom{2z}{2n}$. This completes the proof.

(2). The left side is the coefficient of x^{2n+1} in

$$\sum\binom{z+1}{j}2^j x^j(1+x^2)^{z+1-j}, \qquad |x| < 1. \tag{ii}$$

Here only odd values of j can yield terms in x^{2n+1}; we take $j=2k+1$ and require the term in $x^{2n+1-(2k+1)}$ (i.e. in x^{2n-2k}) in $(1+x^2)^{z+1-(2k+1)}$ (i.e. in $(1+x^2)^{z-2k}$). But (ii) is the binomial expansion of $(2x+(1+x^2))^{z+1} = (1+x)^{2z+2}$, so the coefficient of x^{2n+1} is

$$\binom{2z+2}{2n+1},$$

which completes the proof.

Also solved by L. J. Burton, L. Carlitz, C. A. Church, Jr., Michael Goldberg, Eldon Hansen, D. P. Roselle, Charles Wexler, and the proposers.

Editorial Note. Hansen notes that the sums are special cases of the known result for hypergeometric functions,

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

corresponding to

$$F(-n, n-z; \tfrac{1}{2}; 1) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+z)}{\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n+z)},$$

$$F(-n, n-z; \tfrac{3}{2}; 1) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}+z)}{\Gamma(\frac{3}{2}+n)\Gamma(\frac{3}{2}-n+z)}.$$

He also notes that the sums can be obtained from the convolution

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}.$$

On Perfect Square Triangular Numbers

E 1976 [1967, 438]. *Proposed by J. M. Katri, Baroda, India*

Find all sets of four natural numbers x, y, z, w in arithmetic progression such that $\triangle_x+\triangle_y+\triangle_z=\triangle_w$, where $\triangle_a=\frac{1}{2}a(a+1)$ is a triangular number.

Solution by D. R. Byrkit, University of West Florida. Letting x, $x+d$, $x+2d$, $x+3d$ represent the numbers and multiplying through by 2, we have

$$x(x+1)+(x+d)(x+d+1)+(x+2d)(x+2d+1) = (x+3d)(x+3d+1)$$

which reduces to $d^2=\triangle_x$ or $(2x+1)^2-2(2d)^2=1$. The Pell's equation $p^2-2q^2=1$ has the primitive solution (3, 2) and all other solutions are found from the formula $p_n+q_n\sqrt{2}=(3+2\sqrt{2})^n$. The binomial expansion then gives

$$p_n = \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{2k} 3^{n-2k}2^{3k}, \qquad q_n = \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{2k+1} 3^{n-2k-1}2^{3k+1}.$$

If we set $x_n=\frac{1}{2}(p_n-1)$, $d_n=\frac{1}{2}q_n$, then any value of n will yield values for x, y, z, w fulfilling the stated conditions. The following table gives the first few solutions:

n	p_n	q_n	x	d	y	z	w
1	3	2	1	1	2	3	4
2	17	12	8	6	14	20	26
3	99	70	49	35	84	119	154
4	577	408	288	204	492	696	900
5	3363	2378	1681	1189	2870	4059	5248
6	19601	13860	9800	6930	16730	23660	30590

Also solved by Bernard August, Anders Bager (Denmark), W. T. Bailey, J. W. Baldwin, Leon Bankoff, D. A. Blaeuer, W. J. Blundon, Martin J. Brown, Peter Bundschuh (Germany). Neil

Cameron (New Zealand), P. L. Chessin, William Cutler, E. S. Eby, T. E. Elsner, Jerry Fischer, Bengt Fornberg (Sweden), R. D. Gee, H. M. Gehman, Marc Glucksman, Michael Goldberg, Cornelius Groenewoud, B. A. Hausmann, R. N. Higgins, H. K. Hilton, J. E. Homer, Jr., John Ivie, Margret F. Kothmann, Lew Kowarski, Kenneth Kramer, G. M. Lee, H. J. Ludwig, D. C. B. Marsh, Charles McCracken, A. J. McTernan (Scotland), Norman Miller, Bohuslav Mišek (Czechoslovakia), H. V. Monks, P. L. Montgomery, P. Nagaraju, C. B. A. Peck, Walter Penney, Stanley Rabinowitz, Simeon Reich (Israel), Ira Rosenholtz, Steven Russ, Leo Schneider, A. P. Shah (India), D. R. Stark, D. P. Sumner, H. E. Thomas, Jr., Philip Trauber, Alan Tschetter, Julius Vogel, R. L. Vogt, T. C. Wesselkamper (Nigeria), Charles Wexler, C. P. Wood, Jr., Gregory Wulczyn, and K. L. Yocom.

Higgins, Ivie and Vogel refer to A. Beiler, *Recreation in the Theory of Numbers*, Dover, N. Y., 1964, p. 197 for all the solutions of $d^2=\frac{1}{2}x(x+1)$, i.e., $d=u_n$ where $u_n=6u_{n-1}-u_{n-2}$, $u_1=1$, $u_2=6$. Bankoff, Goldberg and Summer refer to L. Dickson, *History of the Theory of Numbers*, v. II, pp. 10, 36. Gehman refers to E 25 [1933, 426]. Chessin refers to E 954 [1951, 568]. Wesselkamper refers to the Mathematics Student (Indian Math. Soc.) Jan.–Apr. 1959, pp. 55–56.

A Sum of Cubes

E 1977 [1967, 438]. *Proposed by D. R. Rao, Secunderabad, India*

Find all sets of four natural numbers x, y, z, w in arithmetic progression such that $x^3+y^3+z^3=w^3$.

Solution by E. Rosenthall, McGill University, Montreal. Putting $x=a+3d$, $y=a+4d$, $z=a+5d$, $w=a+6d$, the given equation becomes $a(a^2+9ad+21d^2)=0$. The only integral value of a satisfying this equation is $a=0$. Hence the only solution is $\{3d, 4d, 5d, 6d\}$.

Also solved by M. S. Arora, Bernard August, Anders Bager (Denmark), W. T. Bailey, J. W. Baldwin, Leon Bankoff, Merrill Barnebey, D. A. Blaeuer, W. J. Blundon, M. J. Brown, R. L. Browning, Peter Bundschuh (Germany), Neil Cameron (New Zealand), J. D. Carnegie, William Cutler, G. C. Dodds, Gene Dolnick, Ragnar Dybvik (Norway), E. S. Eby, T. E. Elsner, Jerry Fischer, Bengt Fornberg (Sweden), R. D. Gee, Michael Goldberg, M. G. Greening (Australia), Cornelius Groenewoud, H. K. Hilton, Stephen Hoffman, A. E. A. Hunt, J. A. H. Hunter, Donald Jeffords, Erwin Just, Margret F. Kothmann, Sr. M. Celine Kowalska, Lew Kowarski, H. Ramirez López (Puerto Rico), D. C. B. Marsh, James Martin, Patricia Anne Mathews, Charles McCracken, D. E. McLeod, Michael Menn, Norman Miller, Steven Minsker, Bohuslav Mišek (Czechoslovakia), H. V. Monks, P. L. Montgomery, P. Nagaraju, Barbara W. Nason, C. B. A. Peck, Dale Peterson, Bob Prielipp, Stanley Rabinowitz, Simeon Reich (Israel), Ira Rosenholtz, Steven Russ, Leo Schneider, A. P. Shah & A. M. Vaidya (India), John Shochot (Lesotho, Southern Africa), S. P. Singh, Roland F. Smith, Henry Snyder, Stephen Spindler, D. R. Stark, H. S. Sun, H. E. Thomas, Jr., G. C. Thompson, Philip Trauber, Ronald Van Ostenbridge, Julius Vogel, R. L. Vogt, W. M. Waters, Jr., Charles Wexler, Hazel S. Wilson, C. P. Wood, Jr., Gregory Wulczyn, K. L. Yocom, and the proposer.

Several solvers quoted L. Dickson, *History of the Theory of Numbers*, v. II, pp. 583, 585: "V. A. Lebesgue stated that, if x and r are positive integers, $x^3+(x+r)^3+\cdots+(x+(n-1)r)^3=(x+nr)^3$ is impossible except for $n=3$, $x=3r$," and "J. N. Vischers proved Lebesgue's result when $n=3$."

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before December 31, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5600. *Proposed by L. Carlitz, Duke University*

Ramanujan (Collected Papers, p. 325) has stated the following formula for Euler's constant:

$$C = \log 2 - \frac{2}{3^3 - 3} - 2\left(\frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}\right)$$
$$- 3\left(\frac{2}{15^3 - 15} + \frac{2}{18^3 - 18} + \cdots + \frac{2}{39^3 - 39}\right) - \cdots .$$

Provide a proof.

5601. *Proposed by L. Carlitz, Duke University*

Let k be an integer, $k>1$. Show that Euler's constant satisfies

$$C = \sum_{n=1}^{\infty} \frac{\epsilon_n}{n}\left(\frac{\log n}{\log k}\right), \qquad \epsilon_n = \begin{cases} k-1 & (k \mid n) \\ -1 & (\text{otherwise}). \end{cases}$$

For $k=2$ this result reduces to problem 4353 [1951, 116].

5602. *Proposed by A. Wilansky, Lehigh University*

It is well known that R (the reals) has no 3-point compactification. Does it have a 3-point completion? (This means: is there a complete metric space such that the removal of 3 nonisolated points leaves a subspace homeomorphic with R?)

5603. *Proposed by John Milnor, University of California, Los Angeles*

Define the growth function γ of an infinite but finitely generated group G as follows: Choose generators $x_1, \cdots, x_n$ for G and let $\gamma(s)$ be the number of distinct group elements which can be expressed as words $x_{i_1}^{\pm 1} \cdots x_{i_l}^{\pm 1}$ of length l less than or equal to s. Defining γ to be equivalent to γ' if there exist constants c, c' so that

$$\gamma(s) \leqq \gamma'(cs) \quad \text{and} \quad \gamma'(s) \leqq \gamma(c's)$$

for all positive s, the equivalence class of γ is clearly an invariant of G.

Is the function $\gamma(s)$ necessarily equivalent either to a power of s or to the exponential function 2^s? In particular, is the growth exponent

$$e = \lim_{s \to \infty} \log \gamma(s)/\log s$$

always either a well defined integer or infinity? For which groups is $e<\infty$? (A possible conjecture would be that $e<\infty$ if and only if G contains a nilpotent subgroup of finite index.)

5604. *Proposed by R. A. Struble, North Carolina State University at Raleigh*

If the function f is bounded and continuous on $[0, \infty)$ and for some $a>0$ satisfies

$$0 \leqq f(t) \leqq \frac{a}{3}\int_0^t e^{-a(t-s)}f(s)ds + \frac{a}{3}\int_t^\infty e^{-a(t-s)}f(s)ds + e^{-at/2}$$

for $t \geqq 0$, then $f(t) \to 0$ as $t \to \infty$.

5605. *Proposed by Neal Felsinger, Yale University*

Let f be a continuous, periodic function with least positive period p from R onto $[-1, 1]$. Then the sequence $\{f(n)\}$ is known to be dense in $[-1, 1]$ if and only if p is irrational. Does the result still hold if f is not defined on $\{\alpha+kr \mid \alpha$ and r real constants, k an integer$\}$, but continuous on its domain and having range $(-\infty, \infty)$?

5606. *Proposed by J. M. S. Simões Pereira, Lisbon, Portugal*

Let the series $\sum_{n=0}^\infty u_n = S$ be semi-convergent (i.e. $\sum u_n$ converges but $\sum |u_n|$ diverges). By a reordering of its terms, a new series of pre-assigned value, say $\sum_{n=0}^\infty u_{\alpha_n} = A$, may be obtained. If B is such that $|S-B| < |S-A|$ can we find a reordering where $|n-\beta_n| \leqq |n-\alpha_n|$ for every n and such that $\sum_{n=0}^\infty u_{\beta_n} = B$?

5607. *Proposed by E. A. Power, University College, London, England*

If M_i $(i=1, 2, \cdots, n;\ n>1)$ are distinct orthogonal projectors in Hilbert space show that for no set of positive numbers p_i with $\sum_{i=1}^n p_i = 1$ is $M = \sum_{i=1}^n p_i M_i$ a projector. Show that the result is false if M_i are nonorthogonal projectors.

5608. *Proposed by R. E. Shafer, Lawrence Radiation Laboratory, University of California, Livermore*

Evaluate as a convergent series: $|\arg \alpha| \leqq \frac{1}{2}\pi$,

$$\int_0^\infty e^{-u^2} I_0(u^2) K_0(\alpha u)du.$$

5609. *Proposed by R. E. Maas, University of Santa Clara, California*

For any set S and any mapping T: $S \to \mathrm{Re}_{+0}$ (the additive semigroup of non-negative real numbers) define

$$\mathrm{SUM}(T \mid S) = \operatorname*{lub}_{\substack{S' \subset S \\ S' \text{ finite}}} \left(\sum_{n \in S'} T(n) \right).$$

Prove that if SUM $(T|S)<\infty$, then card $\{s\in S|\,T(s)>0\}\leqq\aleph_0$, without assuming the Axiom of Choice.

SOLUTIONS OF ADVANCED PROBLEMS

Sequentially Complete L^p Spaces

5501 [1967, 727]. *Proposed by Harsh Pittie, Princeton University*

Let S be a σ-finite measure space. Prove that $L^p(S)$ is τ sequentially complete, where τ is any locally convex Hausdorff topology such that the conjugate of $L^p(S)$ under τ is L^q, $(q=p/(p-1))$.

Solution by P. R. Chernoff and W. C. Waterhouse, Harvard University. We claim first that the result is true in τ_0, the weak topology induced by L^q. Indeed, for $p>1$ this is standard, since L^p is the dual of the Banach space L^q; for $p=1$ it requires a special argument which was given by Riesz (cf. Banach, *Théorie des opérations linéaires*, pp. 141–143).

The general case then follows. Indeed, let $\{x_n\}$ be τ-Cauchy. It is *a fortiori* τ_0-Cauchy, so it converges to some x in τ_0. Let V be any symmetric closed convex τ-neighborhood of zero; being convex, it is also τ_0-closed. By hypothesis, eventually all x_{n+p} lie in x_n+V, whence $x\in x_n+V$, so $x_n\in x+V$. Thus $x_n\rightarrow x$ in τ.

Also solved by C. L. Fefferman, and Bertram Walsh.

Absolutely Convergent Cauchy Products

5502, [1967, 727]. *Proposed by Roy O. Davies and Alfred Weinman, The University, Leicester, England*

Is it possible for the Cauchy product $\sum_{n=1}^{\infty}c_n$ of two conditionally convergent series $\sum_{n=1}^{\infty}a_n$, $\sum_{n=1}^{\infty}b_n$ to be absolutely convergent?

$$(c_n = a_1b_n + a_2b_{n-1} + \cdots + a_nb_1).$$

Solution by P. J. Owens, University of Surrey, London, England. The binomial series for $(1+t)^m$ is absolutely convergent when $|t|<1$. If $-1<m<0$ it converges when $t=1$ but diverges when $t=-1$. This shows, since the terms are numerically equal in these two cases, that the series is conditionally convergent when $t=1$. It follows that the expansions

$$f(x) = (1+x)(1+x^3)^{-1/2} = \sum_{n=0}^{\infty} a_nx^n,$$

$$g(x) = (1-x+x^2)(1+x^3)^{-1/2} = \sum_{n=0}^{\infty} b_nx^n$$

are absolutely convergent when $|x|<1$ and conditionally convergent when $x=1$. That is, the series $\sum a_n$ and $\sum b_n$ are conditionally convergent.

Since $f(x)g(x)=1$ for all x, it follows by multiplication of the power series that $c_0=1$, $c_n=0$, where $c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$. Thus $\sum c_n$ is absolutely convergent.

Also solved by P. R. Atwood, J. H. E. Cohn (England), O. P. Lossers (Netherlands), George Piranian, P. Rosenthal, and the proposers.

Cohn's solution uses the series of coefficients in the Maclaurin expansions of $(1-x)(1+x)^{-1/2}$ and $(1+x)(1-x)^{-1/2}$. The procedure follows the outline above. Atwood refers to F. Cajori, Transactions of the Amer. Math. Soc., 2 (1901) 25–36; Bull. of the Amer. Math. Soc., 8 (1902) 231–236; A. Pringsheim, Trans. Amer. Math. Soc., 2 (1901) 404–412.

Piranian's solution uses the square of the univalent function $f(z)-f(1)$, where

$$f(z) = z + \sum j^{-1}e^{\pi i/j}[1 + (1 - ze^{-\pi i/j})^{k_j}],$$

where k_j decreases rapidly to 0. The conditional convergence of the coefficients in the expansion of $f(z)$ follows from an analysis of the image of the unit circle; see Erdös, Herzog and Piranian, *Schlicht Taylor series whose convergence on the unit circle is uniform but not absolute*, Pac. J. Math, 1 (1951) 75–82.

Non-differentiable Functions $|f|$

5504 [1967, 728]. *Proposed by Erwin Just, Bronx Community College, New York*

Let f be a function defined on $(0, 1)$ such that $f'(x)=1$ at each point of an infinite subset of $(0, 1)$. Must there exist a point at which $|f|$ is differentiable?

I. *Solution by Einar Andresen, University of Oslo, Norway.* No. To obtain a counterexample, we observe that the function h defined by

$$h(x) = \begin{cases} x & x \text{ rational} \\ \sin x & x \text{ irrational} \end{cases}$$

is continuous only at $x=0$, that $h'(0)=1$, and that $|h|$ is continuous only at $x=0$, and has no derivative here. Define

$$f(x) = \left\{\begin{matrix} x - 3/2^k & x \text{ rational} \\ \sin(x - 3/2^k) & x \text{ irrational}\end{matrix}\right\} \qquad \text{when } x \in \left[\frac{1}{2^{k-1}}, \frac{1}{2^{k-2}}\right).$$

$f(x)$ is continuous at x if and only if $x=3/2^k$, and has derivative 1 at each of these points. $|f|$ is continuous at the same points but, like $|x|$ at $x=0$, has no derivative there.

II. *Solution by J. G. Mauldon, Corpus Christi College, Oxford, England.* The answer is in the negative even if we stipulate that f must be continuous and that the infinite subset has a limit point within the interval (which is not in the subset). For convenience, we take the interval to be $(-1, 1)$ instead of $(0, 1)$.

Let h be a continuous function, nowhere differentiable and such that $|h(x)|<1$ for all $x\in(-1, 1)$ [The existence of such functions is well known.] Then $g(x)=x+x^3h(x)$ is continuous and such that, whereas $g'(0)=1$, $|g|$ is nowhere differentiable.

Finally the function $f(x)=g(x \sin(\log |x|))$ is such that (i) f is continuous and $|f|$ is nowhere differentiable in $(-1, 1)$, (ii) $f'(x)=1$ when $x=\pm e^{-2n\pi}$ $(n=1, 2, 3, \cdots)$.

III. *Comment by R. J. Driscoll, Loyola University.* If the set of points at which f' has the value 1 is nondenumerable, then there is a point at which $|f|$ is differentiable. Let $S=\{x|0<x<1 \text{ and } f'(x)=1\}$. If $x_0 \in S$ and $f(x_0)\neq 0$, then f is of one sign in a neighborhood of x_0 and therefore $|f|$ is differentiable at x_0. There must be such a point x_0, for suppose to the contrary that $f(x)=0$ for all $x\in S$. Then there is a point c which belongs to S such that every neighborhood of c contains infinitely many points of S. [See Natanson, *Theory of Functions of a Real Variable*, vol. 1, pp. 52–53.] But then $f(c)=0$, and $f(x)=0$ for infinitely many x in any neighborhood of c, which implies that either $f'(c)=0$ or $f'(c)$ does not exist, contrary to the assumption that $f'(c)=1$. We may pursue this to show that $|f|$ is differentiable on a nondenumerable set.

Also solved by Roxanne M. Byrne, Roy O. Davies (England), S. B. Eliason, N. J. Fine, G. J. Foschini, M. F. Friedell, D. S. Lawrence, M. D. Mavinkurve (India), Robert Mitchell, P. J. Owens (England), Dale Peterson, Stanley Rabinowitz, Charles Riley, B. L. Schwartz, Walter Stromquist, John Swetik, J. H. van Lint (Netherlands), Michael von Renteln (Germany), and the proposer.

Additive and Multiplicative Functions

5506 [1967, 728]. *Proposed by J. D. Aczel, University of Waterloo, Canada*

Find all real functions satisfying

$$f(xy) = f(x)f(y) \quad \text{and} \quad f(x+k) = f(x) + f(k)$$

for all real x, y and for one real $k\neq 0$.

Solution by C. D. Ahlbrandt, University of Oklahoma. If a and b are reals with $b\neq 0$, then

$$\begin{aligned} f(a+b) &= f[(ab^{-1}k+k)(k^{-1}b)] = f(ab^{-1}k+k)f(k^{-1}b) \\ &= [f(ab^{-1}k)+f(k)]f(k^{-1}b) = f(a)+f(b). \end{aligned}$$

Thus $f(0)=0$, $f(-x)=-f(x)$, and if n is any integer, $f(n)=nf(1)$.

Suppose that $f(x)\not\equiv 0$. Then there exists a real number c such that $0\neq f(c)$ $=f(1)f(c)$ and thus $f(1)=1$. If $x\neq 0$, then $1=f(x)f(x^{-1})$ and $0\neq f(x)=[f(x^{-1})]^{-1}$. If p and q are positive integers,

$$f(pq^{-1}) = f(p)f(q^{-1}) = f(p)[f(q)]^{-1} = pq^{-1}.$$

If $x>0$, then $f(x)=[f(\sqrt{x})]^2>0$. If $y-x>0$, then $f(y-x)=f(y)-(fx)>0$. Since $f(r)=r$, if r is rational, $f(x)=x$ for all real x due to the monotonicity of f. Thus the only solutions are $f(x)\equiv 0$ and $f(x)=x$.

Also solved by I. K. Abruob, Robert Breusch, F. W. Carroll, R. T. Cassel, P. R. Chernoff & W. C. Waterhouse, C. R. Combrink, T. J. Cullen, R. O. Davies (England), D. Ž. Djoković, Mary R. Embry, D. L. Farnsworth, N. J. Fine, M. F. Friedell, Hyman Gabai, R. Goldstein (England), D. A. Hejhal, E. C. Hook, David Huestis, Kenneth Kramer, L. Losonczi (Hungary), O. P. Lossers (Netherlands), Dan Marcus, M. D. Mavinkurve (India), Renate McLaughlin, Steven Minskers, P. L. Montgomery, M. S. Osborne, Charles Riley, S. L. Segal, J. E. Sommese, John Swetik, R. J. Weinacht, Lawrence Zalcman, and the proposer.

Editorial Note. For a recent review of theorems on additive functions see the essay of Albert Wilansky in *Lectures in Calculus*, K. O. May, editor, 1967 (Holden Day), pp. 98, 104, 113, 115.

Inverting Order in a Double Summation

5507 [1967, 728]. *Proposed by B. S. Yadav, Sardar Vallabhbhai Vidyapeeth, India*

If $\{\delta_n\}$ is a sequence of real numbers such that

$$\sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu^2} \left\{ \sum_{k=1}^{\nu} k^2 \delta_k^2 \right\}^{1/2} + \sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu} \left\{ \sum_{k=\nu+1}^{\infty} \delta_k^2 \right\}^{1/2} < \infty.$$

Prove or disprove that $\sum_{n=2}^{\infty} \delta_n^2 \log n < \infty$.

Solution by R. O. Davies, the University, Leicester, England. The hypothesis implies that

$$\sigma_1 \equiv \sum_{\nu=1}^{\infty} \nu^{-2} (\log \nu)^{1/2} \left\{ \sum_{k=1}^{\nu} k^2 \delta_k^2 \right\}^{1/2} < \infty$$

and

$$\sigma_2 \equiv \sum_{\nu=1}^{\infty} \nu^{-1} \left\{ \sum_{k=\nu+1}^{\infty} \delta_k^2 \right\} < \infty;$$

and either of these implies $\sum_{n=2}^{\infty} \delta_n^2 \log n < \infty$, in fact the second is equivalent to it.

(A) If $S_\nu = \sum_1^\nu k^2 \delta_k^2$ and $\sigma_1 < \infty$, then

$$S_\nu^{1/2} = o(\nu/(\log \nu)^{1/2}) \qquad \nu \to \infty,$$

because if $S_N^{1/2} \geqq \epsilon N/(\log N)^{1/2}$ for infinitely many N, where $\epsilon > 0$, then

$$\sum_{N+1}^{2N} \nu^{-2} (\log \nu)^{1/2} S_\nu^{1/2} \geqq \{\epsilon N/(\log N)^{1/2}\} \cdot \sum_{N+1}^{2N} \nu^{-2} (\log \nu)^{1/2} > \tfrac{1}{4}\epsilon,$$

and this is impossible for sufficiently large N. Hence, by comparison with σ_1, $\sigma_3 \equiv \sum_1^\infty \nu^{-3} (\log \nu) S_\nu < \infty$. Now $n^2 \delta_n^2 = S_n - S_{n-1}$, and we find

$$\begin{aligned} \sum_2^N \delta_n^2 \log n &= \sum_1^{N-1} \{n^{-2} \log n - (n+1)^{-2} \log(n+1)\} S_n + N^{-2} (\log N) S_N \\ &= \sum_1^{N-1} \{n^{-2} \log n - (n+1)^{-2} \log(n+1)\} S_n \\ &\quad + \sum_N^\infty \{n^{-2} \log n - (n+1)^{-2} \log(n+1)\} S_N \\ &\leqq \sum_1^\infty \{n^{-2} \log n - (n+1)^{-2} \log(n+1)\} S_n. \end{aligned}$$

Since $0<\{n^{-2}\log n-(n+1)^{-2}\log(n+1)\}<Kn^{-3}\log n$ for all large n, where K is a constant, our first assertion follows by a comparison with σ_3.

(B) The second assertion may be obtained from an interchange of the order of summation.

Also solved by C. Watari (Japan).

Spaces with One- and Two-Point Compactifications

5508 [1967, 728]. *Proposed by Albert Wilansky, Lehigh University*

Let X be a topological space which has a one-point compactification Y and a two-point compactification Z such that Y and Z are homeomorphic. Show that for every natural number n, X has an n-point compactification homeomorphic with Y; and give an example of a space X with this property, where Y is Hausdorff.

Solution by D. E. Sanderson, Iowa State University. The following is a counterexample in the plane to the assertion of the problem.

Let Y consist of the line segments joining $(1, 0)$ to each of the points $(0, 1/n)$ for $n=1, 2, \cdots$ and to $(0, 0)$ together with the vertical line segments joining $(0, 1/(3n-1))$ to $(0, 1/3n)$ for $n=1, 2, \cdots$. (Thus Y consists of alternating arcs and loops joined at $(1, 0)$ and converging to the unit interval on the x-axis.) Let $X=Y-\{(0, 1), (0, \frac{1}{4})\}$ and note that the only T_2 one- and two-point compactifications of X are both homeomorphic to Y (in the former, e.g., X is homeomorphic to $Y-\{(0, \frac{1}{2})\}$) but, of course, X has no n-point compactification for $n>2$ (see, e.g., Theorem 2.6 of K. D. Magill, Jr., *N-point compactifications*, this MONTHLY, 72 (1965) 1075–1081). Thus no n-point compactification of X is homeomorphic to Y for $n>2$.

The fact that the one-point compactification $h_1: X\to Y$ cannot be chosen so that $Y-h_1(x)$ is either of the two points of $Y-h_2(x)$ where h_2 is a two-point compactification is characteristic. Since, if it could, then $h_3=h_2\circ h_1^{-1}\circ h_2: X\to Y$ gives a homeomorphic 3-point compactification with $Y-h_2(x)\subset Y-h_3(x)$ so that, by induction, $h_{n+1}=h_n\circ h_{n-1}^{-1}\circ h_n: X\to Y$ is a homeomorphic $(n+1)$-point compactification if h_n, h_{n-1} are n- and $(n-1)$-point compactifications, respectively, with $Y-h_{n-1}(x)\subset Y-h_n(x)$.

As an example let Y consist of the line segments joining $(1, 0)$ to $(0, 1)$ and to $(0, 1-1/n)$ for $n=1, 2, \cdots$ together with the graphs of the functions $f_n(x)=(x-x^2)2^{-n}$ for $0\leqq x\leqq 1$ and $n=1, 2, \cdots$. Let X consist of the line segments joining $(1, 0)$ to $(0, 1/n)$ and $(0, 1-1/n)$ for $n=1, 2, \cdots$ with the points $(0, 0)$ and $(0, 1/n)$ deleted for $n=2, 3, \cdots$. Clearly Y is homeomorphic to the n-point compactification of X obtained by "adding back" the points $(0, 1/k)$ for $k=2, 3, \cdots, n$ and taking the one-point compactification of the resulting space. "Adding back" all the points gives a homeomorphic $\aleph_0$-compactification.

We raise the question: Are there examples like this for cardinality greater than $\aleph_0$?

Also solved by L. R. King, J. G. Mauldon (England), and M. G. Mavinkurve (India).

Intersecting Curves in the Plane

5509 [1967, 728]. *Proposed by A. K. Dewdney, University of Michigan*

Prove or disprove the following statement: C is a continuous curve from the point (0, 0) to the point $(n, 0)$ in R^2. Denote by $T:R^2 \rightarrow R^2$ the translation $(x, y) \rightarrow (x+1, y)$ of the plane. Then $C \cap T(C) \neq \emptyset$, i.e., the translated curve intersects the original curve at some point.

Solution by L. F. Meyers, Ohio State University. If n is a nonzero integer, then the result is known; see P. Lévy, *Sur une généralisation du théorème de Rolle*, Comptes Rendus Acad. Sci. Paris, 198 (1934) 424–425. A clear exposition, with generalizations, is found in H. Hopf, *Über die Sehnen ebener Kontinuen und die Schleifen geschlossener Wege*, Commen. Math. Helvetici, 9 (1937) 303–319. If C is the graph of a function, then the (specialized) result is also found in R. P. Boas, Jr., *A primer of real functions*, The Carus Math. Monographs, no. 13 (1961) pp. 77–81, as *the universal chord theorem.*

If n is not a nonzero integer, then the result is false. In fact, Hopf's paper contains the following result. *Let C be a continuum containing the points* (0, 0) and $(\alpha, 0)$, *where* $\alpha > 0$. *Then the set of all positive real numbers t such that the translate of C by the vector* $(t, 0)$ *does not intersect C is an open set which is closed under addition. Conversely, for every such nonempty open additive set of positive real numbers, there is a continuum to which the set "belongs."*

Also solved by L. J. Burton, Roxanne M. Byrne, R. O. Davies (England), D. A. Hejhal, B. S. Lalli & R. Manohar & R. Singh, W. A. J. Luxemburg, Stanley Rabinowitz, A. J. Sommese, J. E. Sommese, and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto, 5, Canada. Correspondence about reviews is welcomed.

The Six-Cornered Snowflake. By Johannes Kepler.
Edited and Translated by Colin Hardie, with essays by L. L. Whyte and B. F. J. Mason. The Clarendon Press, Oxford; Oxford University Press, New York, 1966. xvi+76 pp. $3.40.

Kepler's "New Year's Gift" is of great historical interest because it reveals the intimate thoughts of a mathematical philosopher belonging to the generation just before Descartes.

In the present edition, the even-numbered pages contain the original Latin text, made more accessible to modern readers by the expansion of abbreviations and other improvements of spelling. The facing odd-numbered pages contain Colin Hardie's new English translation, which is usually excellent. The mathematics begins on page 10. Kepler considers the enumeration of convex polyhedra whose faces are congruent rhombi. Apart from the rhombohedra, which are merely distorted cubes, he finds two such

solids, having 12 and 30 faces, respectively. (In fact, there are only two others, one with 20 faces, discovered in 1885 by E. S. Fedorov, and another with 12 faces, discovered in 1960 by Stanko Bilinski.) Kepler notices that the first of these solids can be repeated to fill and cover the whole space. In his translation of lines 14 and 15 (. . . *sic rhombici primi, obtusi seu trilateri anguli quaterni idem praestent, et quadrilateri anguli seni similater*), Hardie incorrectly assumed the angles of the rhomb to be 60° and 120° instead of arc sec (± 3). Perhaps a better translation of these words would be: "So with the first kind of rhombic solid [rhombic dodecahedron], four obtuse or trivalent corners do the same [fit together round one point] and similarly six [acute or] quadrivalent corners."

This passage opens up the famous problem of the close-packing of equal spheres, with the marginal note (on p. 13) "Into what shape peas are squeezed." Cubic close-packing is nicely described and illustrated on pp. 14 and 16. (The words *inter ternos inferioris* on p. 14, line 29, should have been translated "between three of the lower" and not "between every three of the lower." Similar words on p. 16, line 5, are translated correctly.) The figures on p. 16 resemble the scheme on p. 149 of W. W. R. Ball, *Mathematical Recreations and Essays* (London, 1956). They show successive hexagonal layers C, D, E, of the packing, and indicate that these layers are to be stacked in the cyclic order CDECDE. . . . If Kepler had gone on to remark that the alternating order DEDEDE. . . . would serve just as well, he would have anticipated by 272 years William Barlow's discovery of the distinction between cubic close-packing and hexagonal close-packing (Nature, **29**, 1883, pp., 186–188). That Kepler did not take this crucial step is almost proved by his remark on lines 14 to 16, *non potest esse ordo triangularis sine quadrangulari* ("the triangular pattern cannot exist without the square"). On the other hand, the first paragraph on page 28 suggests the intriguing possibility that he caught a glimpse of the hybrid patterns of Sidney Melmore (Nature, **159**, 1947, p. 817; see also the review in Math. Rev. **9**, 1948, p. 53).

Another surprise appears on page 10, line 7, where Kepler writes *Archimedeis quattuordecim*. As it was he who gave the thirteen Archimedean solids the names by which they are still known, his casual use of the number 14 indicates that he may have anticipated J. C. P. Miller's discovery of the pseudo-rhombicuboctahedron (see Ball, *op. cit.*, p. 137).

A good example of Kepler's whimsical approach to geometry is the following passage about the five Platonic solids (p. 37): ". . . without doubt the authentic type of these figures exists in the mind of God the Creator and shares His eternity . . . Now among the regular solids, the first, the firstborn and father of all the rest, is the cube, and his wife, so to speak, is the octahedron, which has as many corners as the cube has sides . . ." (Instead of "sides", a better word for *plana* would have been "planes" or "faces".) Again, on p. 43 we read: "But the formative faculty of Earth (*formatrix telluris facultas*) does not take to her heart only one shape; she knows and is practised in the whole of geometry."

Another passage of interest to mathematicians is on p. 21, where a paragraph about the pentagon, the divine proportion (golden section), and the Fibonacci sequence is followed by one about flowers, suggesting an anticipation of phyllotaxis.

In addition to Kepler's text, the book includes a Foreword by L. L. Whyte (author of *Essay on Atomism*), a Synopsis, a facsimile of the title page, a well illustrated commentary ("On the shapes of Snow Crystals") by Professor B. J. Mason, F. R. S. (author of *The Physics of Clouds*), another commentary by L. L. Whyte, ten pages of notes explaining particular words and allusions, and a Bibliography. Whyte (p.5) makes the interesting remark that some of Kepler's ideas were anticipated by an Englishman, Thomas Harriot, who studied the close-packing of spherical atoms before 1600. Mason (p. 52) returns to this subject but does not provide convincing evidence that Harriot was any more successful than Kepler in distinguishing between the two symmetrical types of close-packing (in both of which each sphere is surrounded by 12 others). Whyte's explanation (expressed in a letter) for this lack of evidence is that, as Harriot was a keen pro-atomism

thinker, his relevant manuscript was probably destroyed by a literary executor who disapproved of atomism (as Kepler himself apparently did).

On p. 55, Mason gives a remarkable table of seven ranges of temperature, each having a characteristic shape for snowflakes (for instance, hollow prismatic columns between $-50°C$ and $-25°C$, and again between $-8°C$ and $-5°C$). This provides a striking justification for Kepler's theory that the shape of snowflakes is influenced by variations of temperature.

Whyte remarks (p. 57) that, "in the *New Year's Gift* we observe a mind capable both of defining a mathematical problem in physics [why snowflakes are hexagonal] centuries before it became ripe for solution, and of experiencing a divine magic in two words, *facultas formatrix.* There is no contradiction here. Kepler's attitude is fundamentally natural and truly fertile. He displays that empirical mysticism which is indispensable to science . . . "

H. S. M. COXETER, University of Toronto

Linear Algebra. By W. H. Greub. Third edition. Springer-Verlag, New York, 1967. xii+434 pp. $9.80. (Telegraphic Review, Jan. 1968.)

This tersely written text is a presentation of linear algebra " . . . based on an axiomatic treatment of linear spaces." The new material in the third edition affords a better balance between modern algebraic interests and traditional linear algebra. Working with vector spaces over arbitrary fields of characteristic zero whenever possible and assuming finite dimensionality only when necessary, the author gives comprehensive treatment to many topics including non-associative and graded algebras, inner product spaces, unitary spaces, and polynomial algebras. Masterful use of determinant functions and their duals makes the computational treatment palatable and even delightful in the case of volumes, cross products in oriented three space, and the relationship between rotations in three space and the quaternion algebra. In addition to the usual material on cyclic decompositions and canonical forms, the last chapter on linear transformations treats semi-simple transformations and commutative sets of semi-simple transformations. Problem material at the end of each section nicely supports the text, but for the most part is not suitably graded, i.e., many sections include too few *moderately* difficult problems.

While based on an axiomatic treatment of vector spaces the text (after the three introductory chapters) is not a formal presentation of linear algebra. A format of short crisp sections broken down into subsections devoted to a single idea makes the exposition quite readable. But with only a few exceptions derived statements have not been dignified by the name "theorem . . . " or "proposition . . . ", nor have they been separated from their proofs or derivations by any typographical device. Consequently the novice will have some difficulty focusing upon and sorting the wealth of information presented. Schur's lemma (problem 8, p. 54) is missing the essential hypothesis that the spaces E and F are irreducible with respect to S_E and S_F.

D. J. STERLING, Bowdoin College

Exercises in Probability and Statistics for Mathematics Undergraduates. By N. A. Rahman (University of Leicester). Hafner, New York, 1967. x+307 pp. $10.50. (Telegraphic Review, Feb. 1968.)

With textbooks becoming more abstract and less oriented toward problems and applications, there is a growing need for workbooks and problem collections. This one contains over 400 problems related to a full year undergraduate course at the post calculus level. The coverage is broader than most courses and includes the use of simple difference equations and generating functions, joint and simple derived distributions, tests, inference, bivariate correlation and regression, characteristic functions and their use in deriving sampling distributions explicitly.

At the back of the book there are answers and hints or sketches of solutions to all the

problems. The author includes a list of 17 journals and 11 books from which he has drawn the exercises or ideas for the exercises. In addition he has included explicit references in the answer section for most of the exercises. Presumably, the other problems are either due to the author or their origin is not known. This explicit citation of the exercises is highly commended and long overdue from most other authors. Not only does the citation give credit to the creator of the exercise, it also provides the reader with a starting reference source for deeper investigation.

It would have been nice if the author had even included some other references which he had not used but might be useful to the reader. As an example (for Chap. 1), W. A. Whitworth, *Choice and Chance* (with 1000 exercises), Hafner, New York, 1948, and the corresponding solutions manual with hints or sketches of solutions to all the exercises.

MURRAY KLAMKIN, Ford Scientific Laboratory

The Number Systems and Operations of Arithmetic. By Orval M. Klose. Pergamon Press, Oxford, London, 1966. xi+265 pp. $3.95 (Teleg. Rev. May 1967)

According to the author, "this book was written for the single purpose of explaining to elementary school teachers (both in-service and in-training) the nature of those basic principles of mathematics which form the foundations and structural framework of arithmetic, and how the familiar formal algorithms . . . stem from these principles." The designation "arithmetic" here and in the title covers ordinary algebra as well.

The author asserts that he has used "a frankly intuitive approach" as he considers this a more effective contribution to "common-sense understanding" and the teaching of arithmetic than "premature emphasis on axiomatics".

Part I, entitled "The Number Systems," starts from sets and natural numbers and treats successively the systems of integers, rational numbers, real numbers and complex numbers. Typical procedure is to use lack of closure for specified inverse operations as motivation for introducing new numbers (or number symbols), incorporating or merging these with previous numbers under the assumption that previous laws (commutative, associative, etc.) remain valid, and giving some proofs leading to definitions or derived rules. A separate chapter is included on abstract algebraic systems classified postulationally.

Part II, entitled "Computational Algorithms," includes more detailed application of laws or rules of Part I to standard procedures in arithmetic and algebra, with about half of the space devoted to natural numbers, and with some pedagogical comments. Exercises of various types, from routine or repetitive to more imaginative, are included in both parts. There is an index but there is no bibliography and there are no references to other contemporary books.

Elementary teachers will find some useful and clarifying material in this book, including for example the repeated emphasis on closure, commutative, associative and distributive properties, the treatment of cardinal and ordinal numbers, the discussion (in connection with integers) of the question of division by zero.

On the other hand, some of the explanations or proofs could be clearer or more meaningful; and from time to time statements occur in the book without sufficient qualification, although previous or subsequent discussion may suggest that restrictions are intended. An example is the formula derived for the quotient purportedly of *any* two complex numbers—without restricting the denominator to be nonzero—leading to the flat statement that the complex numbers are closed under division.

Finally, it should be mentioned that there seems to be a potential danger to understanding inherent in the type of approach the author has adopted to promote understanding—at least in the absence of sufficiently strong countermeasures, e.g. putting some emphasis on the need of a more rigorous approach. Specifically, there is the risk of suggesting that logical consistency of properties and symbols of each new number system can be guaranteed by decree, so to speak. The danger results in a full-fledged fallacy

at the complex number stage. Here the assumption is made that $(-1)^{1/2}$, or i, under the defining relation $i^2=-1$, "obeys all applicable laws with respect to all the natural operations and their inverses that are valid in the real number system and its predecessors." One of these applicable laws is $(ab)^c=a^cb^c$, which has previously been identified as stating that exponentiation on the right is distributive over multiplication—a law indicated in a summary table as holding for all number systems and as including the case of root extraction. Clearly it is logically impossible for this law to hold along with all previous agreements when $a=b=i^2$ and $c=\frac{1}{2}$, as it then says:

$$[(-1)(-1)]^{1/2} = (-1)^{1/2}(-1)^{1/2} \quad \text{or} \quad 1 = -1.$$

The same illustration of course shows that $(i^4)^{1/2} \neq i^2$, violating the standard law for taking powers by multiplication of exponents.

E. R. STABLER, Hofstra University

An Introduction to Linear Analysis. By Donald L. Kreider, Robert G. Kuller, Donald R. Ostberg and Fred W. Perkins. Addison-Wesley, Reading, Mass., 1966. xi+772 pp. $13.50. (Teleg. Rev. February 1967)

This is an introduction to the mathematics dealing with linear relations. It is designed for a course of the type usually called "engineering mathematics" and is an above average text for this use. It assumes only a knowledge of elementary calculus and analytic geometry. The book has a nicely printed format and is generous with illustrations. The examples are very good and should be thought provoking. There is a wide range of exercises varying in difficulty, with answers to the odd numbered problems. The presentation is generally formal, with theorems stated but not always proved. The language is not always precise, but the motivational text and introductions are excellent.

The first two chapters provide the fundamentals of linear algebra. Chapters 3, 4, 5 and 6 contain the material of a first introductory course on ordinary differential equations with some more advanced topics in Chapter 6. Chapters 7 and 8 deal with the concept and properties of Euclidean spaces and concepts of convergence in infinite dimensional Euclidean spaces. Chapter 9 follows with Fourier series, and Chapter 10 deals with the convergence properties of the Fourier series and ends with the Fejer theorem and the Weierstrass approximation theorem. Chapter 11 deals with orthogonal systems of polynomials. Chapter 12 deals with the elementary two point boundary value problems using the concepts of eigenvalues and eigenvectors, introducing the theoretical aspects through Green's function at the end of the chapter. Chapters 13, 14 and 15 are concerned with elements of partial differential equations using the previous material (through separation of variables) for problems involving the classical elliptic, hyperbolic and parabolic equations. The book also contains four appendices which provide some background material.

MURRAY WACHMAN, University of Connecticut

Measure and the Integral. By Henri Lebesgue. Edited with a biographical essay by Kenneth O. May. Holden-Day, San Francisco, 1966. xii+194 pp. $6.95.

This book contains translations of two essays by Henri Lebesgue. The first, which takes up most of the book, is entitled "Measure of Magnitudes" and was published serially in *l'Enseignement Mathématique* during the years 1931–1935. With the French school system primarily in mind, Lebesgue discusses possible ways to present the subject on the elementary, secondary and collegiate levels "in as simple and specific a manner as possible but without sacrificing logical rigor." The first four chapters deal with the notions of whole numbers, length and real numbers, area and volume. Much of what Lebesgue suggests could be used profitably in a high school geometry course. Chapter V deals with

curve length and surface area and contains some interesting remarks relating to Lebesgue's own work on the problem of defining the area of a surface. In the final two chapters the notion of a set function is introduced and used as the context for an elegant treatment of differentiation and integration along the lines found in Buck's *Advanced Calculus*. The essay concludes with a lucid discussion of oriented integrals and volumes.

The reader who expects to find Lebesgue's opinions on how to present the concepts of Lebesgue measure and integration will be disappointed by "Measure and Magnitudes." It is perhaps to compensate for this omission that the editor has appended a fifteen page expository article, "The Development of the Integral Concept," published by Lebesgue in 1926. Here we find a clear, non-technical introduction to the Lebesgue integral and its relation to the definitions of Cauchy and Riemann. This is followed by a brief discussion of the manner in which Lebesgue's definition was generalized by Radon and Denjoy.

A short biographical sketch has been supplied by the editor together with a selected bibliography for those who would like to learn more about Lebesgue's life and work.

THOMAS HAWKINS, Swarthmore College

Plateau's Problem; an Invitation to Varifold Geometry. By Frederick J. Almgren, Jr. Benjamin, New York, 1966, xii+74 pp. \$3.95 (paper), \$8.00 (cloth).

With his introduction of the measure theoretic surfaces called *varifolds*, the author has provided a setting in which he can obtain and discuss solutions to a very general form of Plateau's problem, which is roughly that of finding a surface of least area having a given boundary. As with the recent contributions of E. DeGiorgi, H. Federer, W. Fleming, and E. R. Reifenberg, Almgren's work has grown out of the following point of view introduced by L. C. Young in 1942 in his discussion of *generalized surfaces:* An ordinary surface is regarded as acting on a suitable space D of test functions by means of integration over the surface. The concept of "surface" is then extended to include other functions on D which are in some sense continuous. One is now able to use results from geometry and functional analysis as he attempts to find a solution for the problem in the new space of "surfaces" and then show that, because it is extremal, the solution is a surface in the usual sense (or nearly one). This approach is analogous to the one used in distribution theory for discussing differential and integral equations, in which the role of the distribution is similar to that of the generalized "surface."

Since results on Plateau's problem are necessarily important to the calculus of variations, the study of these modern techniques based on measure theoretic geometry is desirable for the student of variational problems. The book is intended to be a survey of these methods and includes a well presented motivation for their development. It is appropriate for use as a supplement for undergraduate courses in the calculus of variations, and will be useful to the more advanced reader as a first exposure.

The material of the book should be accessible to anyone who has taken a course in advanced calculus. The entire discussion takes place in three dimensional euclidean space, but is presented so that the extension to k-dimensional objects in n-space will be clear to a reader used to thinking along these lines. There is just enough of the more sophisticated material of geometric measure theory present to tantalize the student to further study in one of the references. Exercises with hints for solution are provided, permitting the student to develop some nontrivial details together with illuminating examples. There are also thirty-five well-drawn illustrations of examples, many of them in color. These include drawings of a number of soap films, some of them unexpected, which challenge the reader to attempt experimental verification and thereby gain some understanding of the physical basis for Plateau's problem.

J. E. BROTHERS, Indiana University

Approximation of Functions. By G. G. Lorentz (Syracuse University). Holt, Rinehart and Winston, New York, 1966. ix+188 pp. $5.95.

This book is strongly recommended for every college library because of the first eight chapters. Here the author goes out of his way to keep the discussion elementary (not easy!). Chapters nine, ten, and eleven concern entropy and Kolmogoroff's solution of Hilbert's thirteenth problem. Fascinating, but highly special, and not self-contained for nonspecialists.

Students with good advanced calculus background would profit from reading parts of the book, and college teachers should be aware of the elements of the subject. The book is restricted in essence to real functions and uniform approximation, although complex, L^p, and Stone-Weierstrass excursions occur. (The author's concern to be elementary leads him to give a proof of the uniform boundedness principle without any use of Baire category!) The book abounds with amusing and interesting insights. There are problems and notes which extend the results in the text and give their history. A special feature is the coverage of the Russian literature of this field. Excellent, but why, oh why doesn't the author give an index of symbols?

ALBERT WILANSKY, Lehigh University

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses; T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (Junior level)—18 (second graduate year). A boldface star (★) marks a notable book of general interest.

Algebra

Problèmes d'Algèbre Générale. By A. Bigard, M. Crestey, J. Grappy. Dunod, Paris, 1967. Distributed by Gordon and Breach, New York. viii+226 pp. $5.50. May be useful to teachers looking for new problems. S, P.

Einführung in die Verbandstheorie. By H. Hermes. Second edition. Springer-Verlag, New York, 1967. xii+209 pp. $11.50. Minor changes from the first edition of 1954.

Homology Theory. An Introduction to Algebraic Topology. By P. J. Hilton (Cornell University) and S. Wylie (Cambridge University). Cambridge Univ. Press, New York. 1967. xv+484 pp. $3.95 (paper). Paperback reprint of a text first published in 1960, reviewed enthusiastically by A. Heller in *Mathematical Reviews* (Vol. 22, No. 5963) and there described as an excellent transition between an elementary textbook and a research level treatise. Since the hard cover edition costs $13.50, this is an extraordinary bargain in a quality book. T (17), S, P.

Vorlesungen über Artinsche Ringe. By Andor Kertesz (Univ. of Debrecen). Akademiai Kiado, Budapest, 1968. 281 pp. $7.50. P.

Basic Number Theory. By Andre Weil. Grundlehren der mathematischen Wissenschaften, Vol. 144. Springer-Verlag, New York, 1967. xviii+294 pp. $12.50. Algebraic numbers, using adeles, zeta-functions and L-function, followed by local and global classified theory. Pre-supposes such items as duality theory of locally compact commutative groups and Galois theory. T.

Analysis

Differential Equations. By Martinus Esser (Univ. of Dayton). Saunders, Philadelphia, 1968. ix+249 pp. $8.00. Obvious distinguishing features of this book are a facsimile

signature of the author at the end of the preface and an immediate attack on linear differential equations in chapter one, followed by specialization to equations with constant coefficients in chapter two, postponement of first order equations to chapters five and six and a chapter on algebra of operators. The approach is based on the author's belief that mathematics and engineering students should both be taught theory and applications. T (14–15).

★*Ordinary Differential Equations.* By H. Gask. Translated by J. Friberg. M. I. T. Press, Cambridge, Mass., 1968. 84 pp. $2.45 (paper). The Swedish original, written by a group of mathematicians at the University of Lund, is widely used there as an elementary introduction. The ontology of Gask is somewhat similar to that of Bourbaki. T (14–15), S.

Finite-Difference Equations and Simulations. By Francis B. Hildebrand (M. I. T.). Prentice-Hall, Englewood Cliffs, N.J. 1968. ix+338 pp. $12.75. Difference equations with applications to solutions of ordinary and partial differential equations, intended as a self-contained introduction and stepping stone to more advanced treatises by Henrici, Forsyth and Wasow. T (15–16), P, L.

The Numerical Integration of Ordinary Differential Equations. By T. E. Hull (Univ. of Toronto). CUPM, Berkeley, Calif. 1966. 32 pp. Free. Materials intended for inclusion in a course at the postcalculus level. P.

Ordinary Differential Equations. By A. I. Kiselev, M. L. Krasnov and G. I. Makarenko. Translated by E. J. F. Primrose (Univ. of Leicester). Leicester University Press, 1967. 231 pp. 18/– (paper). A collection of exercises with some general exposition and worked examples. Topics include the usual material of an elementary course but with some additional details and novelties, including problems on isoclines, Euler's methods, stability and the Laplace transformation. S.

Integration. By A. C. Zaanen (Univ. of Leiden, Netherlands). North-Holland and Wiley, 1967. xiii+604 pp. $16.75. Comparing this book with *An Introduction to the Theory of Integration*, first published in 1958 and reprinted in 1961 and 1965, the author writes "Although anyone who wishes to compare the two editions will find major or minor changes everywhere, one thing has remained the same, and that is the attempt to produce an advanced textbook on integration theory, which makes the student familiar not only with the measure theoretic approach and the linear functional approach to the theory, but also with the fact that the integral of a non-negative function has something to do with the measure (in a certain specified sense) of the ordinate set of the function." The last chapter on normed Köthe spaces is new. This is a very comprehensive treatise, though the author mentions in the preface several items that he is not going to include. The solutions to exercises occupy 100 pages, and there is an 8 page bibliography. T (17–18), S, P, L.

Applications

An Introduction to Fluid Dynamics. By G. K. Batchelor. Cambridge Univ. Press, 1967. xviii+615 pp. $13.50. Based on the author's research over the past 50 years the book requires no previous knowledge of fluid dynamics and only vector analysis and tensor notation. T (16–17), S, P, L.

Stability of Motion. By Wolfgang Hahn. Translated by Arne P. Baartz. Die Grundlehren der math. Wiss. Vol. 138. Springer-Verlag, New York, 1967. xi+446 pp. $18.00. A comprehensive treatment including application to non-linear vibrations and automatic controls, this book presupposes some knowledge of analysis and differential equations. Ten page bibliography. P, L.

Sets, Functions, and Probability. By John B. Johnston (General Electric Research and Development Center), G. Baley Price (Univ. of Kansas) and Fred S. Van Vleck (Univ. of Kansas). Addison-Wesley, Reading, Mass., 1968. vii+376 pp. $9.50. Developed from an experimental textbook entitled *Introduction to Mathematics* published in 1963, this is intended for students and practitioners of the behavioral sciences. T (13–14), P.

Mathematical Systems Theory. A journal to be published quarterly by Springer-Verlag, New York, Vol. 1, 1967. Edited by D. Bushaw, A. J. Lohwater, M. D. Mesarovic, and G. P. Szegö. Subscription price: $24.00 for institutions. Lower rates for individuals. The scope of mathematical systems theory is said to include such fields as topological dynamics theory of dynamical polysystems, general systems theory, formal systems theory, mathematical theory of automata and algorithms, linguistics, etc. L.

Cartesian Tensors. The Mathematical Language of Engineering. By Nils O. Myklestad (Univ. of Texas at Arlington). Van Nostrand, Princeton, N.J., 1967. vii+141 pp. $3.95 (paper). Cartesian tensors in two and three dimensions for students of science and engineering. The sub-title seems slightly exaggerated. S.

Game Theory. By Guillermo Owen (Fordham Univ.). W. B. Saunders, Philadelphia, 1968. xii+228 pp. $9.00. Designed for a semester each on two person and n-person games, this book treats game theory as "a mathematical description of certain sociological phenomena." Topics include differential games, the bargaining set, and a continuum of players. Convexity and fixed point theorems are treated in an appendix. T (16–17), S, P, L.

Symposia on Theoretical Physics. Lectures presented at the 1965 Summer School of the Institute of Mathematical Sciences, Madras, India. Edited by Alladi Ramakrishnan. Plenum Press, New York, 1967. xi+218 pp. $12.50. In spite of the title, the content of this volume is mathematics with physical motivation. This is the fifth in a series of volumes now totalling nine. P.

Mathematics for Management Series. By Clifford H. Springer, Robert E. Herlihy, Robert T. Mall, and Robert I. Beggs (Consultants, General Electric Company). I. Basic Mathematics. II. Advanced Methods and Models. III. Statistical Inference. IV. Probabilistic models. Richard D. Irwin, Homewood, Illinois, 1965–1968. xii+225 pp., ix+273 pp., x+352 pp., xi+301 pp. Each volume $3.95 (paper). This material was developed at the General Electric Co. for its own training program for management, but it deserves consideration as a textbook or supplement in courses for business majors. T, S.

Logic and Foundations

Sets, Models and Recursion Theory. Edited by J. N. Crossley. North-Holland, Amsterdam, 1967. Distributed by Humanities Press, New York. 331 pp. $16.25. Fifteen papers including "a proof of the relative consistency of the continuum hypothesis" by Carol Karp. P, L.

Variable, Function, Derivative. A Semantic Study in Mathematics and Economics. By Harald Dickson. Universitetsforlaget, Akademiforlaget, Göteborg, 1967. 176 pp. $7.60. Although written by a mathematical economist, this is a thoughtful study of the semantics of analysis as it appears in our elementary courses. P, L.

Elements of Mathematical Logic. Model Theory. By G. Kreisel and J. L. Krivine. North Holland, Amsterdam, 1967. Distributed by Humanities press, New York. xi+221 pp. $12.75. The axiomatic method formulated in set theoretic terms. After the classical

results on propositional calculus, predicate calculus, elimination of quantifiers, hierarchy of finite types, definability, and principal models there are three appendices on applications to current mathematics of the axiomatic method, philosophical aspects of foundations (a section of very general interest to those with the necessary background), and (one page) the semantic versus syntactic approaches. T (17), S. P. L.

First Order Mathematical Logic. By Angelo Margaris (Ohio State Univ.) Blaisdell, Waltham, Mass., 1967. x+211 pp. $6.75. Designed for a one semester course for advanced undergraduate or beginning graduate students with little or no acquaintance with abstract mathematics, it begins with rules of inference and set theory and ends with Gödel's theorem. T.

Algebraic Methods of Mathematical Logic. By Ladislav Rieger. Academic Press, New York, 1967. 210 pp. $10.00. A translation by Michal Basch of an incomplete manuscript discovered after the author's death in 1963. There is a brief explanatory note by the editor Miroslav Katetov. P.

Probability and Statistics

S. S. Wilks: Collected Papers: Contributions to Mathematical Statistics. Edited by Professor T. W. Anderson (Columbia University). Wiley, New York, 1967. xxxii+693 pp. $12.50. Sponsored by the Institute of Mathematical Statistics as a memorial, this volume contains a 20 page biography, a portrait, 48 scientific papers, and a list of 17 other publications. P, L.

Statistics: An Introduction to Tests of Significance. J. K. Backhouse (Dept. of Education, Oxford Univ.). Longmans, London, 1967. ix+197 pp. $2.50 (paper). It is scandalous to publish in 1967 a book on statistics that takes no account of development since 1930!

Principles of Statistics. By M. G. Bulmer (Univ. of Oxford). M. I. T. Press, Cambridge, Mass., 1967. vii+252 pp. $7.50. The aim is to present theory and practice in as simple and non-technical way as possible on the basis of elementary college mathematics. The references include 8 by R. A. Fisher and none by Egon Pearson or J. Neyman.

Time Series Analysis. By E. J. Hannan (Australian National University, Canberra) Science Paperbacks and Methuen, London, 1967. Distributed in the U.S.A. by Barnes and Noble. viii+152 pp. $2.25 (paper), $3.50 (cloth). A paperback reprint of a corrected edition of the book first published in 1960. T, S, P, L.

Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability Edited by Lucien LeCam and Jerzy Neyman. Vol. I: Statistics. Vol. II, Parts I and II: Contributions to Probability Theory. Vol. III: Physical Sciences. University of California Press, 1967. I: xvii+666 pp. $20.00. II, 1: xvi+447 pp. $14.00. II, 2: xvi+483 pp. $15.00. III: xvi+324 pp. These volumes contain papers on general theory, sequential procedures, probability on algebraic structures, distributions in functional spaces, stochastic processes and prediction, martingales, special problems, Markov processes, ergodic theory. Vol. IV will cover biology and health and vol. V weather modification. Together they constitute a rather overwhelming encyclopedia of current activity. The series as a whole is an extraordinary historical record. P, L.

Elementary Statistical Methods. By G. Barrie Wetherill (Imperial College, London). Methuen, London, 1967. Barnes and Noble, New York. xiii+329 pp. $7.95. An old fashioned presentation, ending with multiple regression and analysis of variance. E. S.

Pearson is noticed only for his discussion of graphical methods of presenting statistical information. Neyman and Wald appear not at all!

Essentials of Probability. By Arthur Yaspan (Polytechnic Institute of Brooklyn). Prindle, Weber & Schmidt, Boston, Mass., 1968. viii+200 pp. $7.95. Neither in the brief foreward nor in the table of contents is there any hint of why the author has decided to add yet another elementary book on probability to the large and qualitatively heterogeneous collection already in existence. T (15, for a rather sketchy short course).

NOTABLE PAPERS

If you know of a recent paper that should be mentioned here, drop us a note with complete information (author in full, title in full, name of serial in full, date of serial, volume, page numbers) and a brief indication of content.

Non-Cantorian Set Theory, by Paul J. Cohen and Hersh Ruben in *Scientific American*, December 1967, 104–116. Sketches the new results and their implications in historical perspective.

The Vibrating String of Pythagoras, by E. E. Helm in *Scientific American*, December 1967, 93–103. Relations between music and mathematics through the ages.

Are "Infinity Machines" Paradoxical? by Adolph Grünbaum in *Science* 159 (26 Jan. 1968), 396–406. Can processes involving an infinite sequence of operations be completed in a finite time?

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. H. Bing, University of Wisconsin, has been appointed a member of the National Science Board for a six year term.

Professors Garrett Birkhoff, Harvard University, and I. M. Singer, M.I.T., are among those elected to membership in the National Academy of Sciences.

Professor E. J. Camp, Macalester College, represented the Association at the inauguration of President R. A. Du Fresne of Winona State College on February 2, 1968.

Dr. Einar Hille, Emeritus Professor at Yale University, has been awarded the John Ericsson Medal for 1968 by the Board of Directors of the American Society of Swedish Engineers.

Professor R. C. Meacham, Florida Presbyterian College, represented the Association at the inauguration of President P. F. Geren of Stetson University on January 26, 1968.

Professor Henry Sharp, Jr., Emory University, represented the Association at the inauguration of President H. M. Gloster of Morehouse College on February 17, 1968.

Washburn University: Associate Professor R. E. Shermoen, North Dakota State University, has been appointed Associate Professor and Chairman; Associate Professor R. H. Thompson, Sterling College, has been appointed Associate Professor.

Washington State University: Associate Professor W. E. Walden, University of Omaha, has been appointed Associate Director of the Computing Center and Associ-

ate Professor of Information Science and Mathematics; Assistant Professor D. C. Kent has been promoted to Associate Professor; Associate Professor P. L. Meyer has been promoted to Professor; Assistant Professor R. L. Irwin has been appointed Assistant Professor at the University of Missouri at St. Louis.

Mr. H. D. Allen, Saguenay Valley Schools, has been named a Fellow of the Canadian College of Teachers.

Mr. W. L. Drezdzon, A. C. Electronics, General Motors, has been appointed Assistant Professor at Chicago City College—Wilson Branch.

Mr. George Grossman, Board of Education, New York, has been appointed Director of Mathematics of the New York City School System.

Assistant Professor R. R. Gutzman, Colorado School of Mines, has been promoted to Associate Professor.

Assistant Professor E. C. Young, Florida State University, has been promoted to Associate Professor.

Mr. Edward McGaughy, IBM, New York, died on September 25, 1967. He was a member of the Association for twenty years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The annual business meeting of the Northern California Section of the MAA was held on January 27, 1968, in conjunction with the annual meeting of the MAA in San Francisco. Professor Charles Hayes of the University of California at Davis presided; 22 members of the Association attended.

Professor Mary Sunseri of Stanford University was elected Vice-Chairman of the Section; Dr. Henry Osner of Modesto Junior College succeeds Professor Hayes as chairman. Professor Harold Bacon reported on the Visiting Secondary School Lecturer Program and the members voted that the Executive Board of the Section could appropriate contest funds to support, on a limited basis, a secondary school lecturer program within the Section. It was also decided to award membership in the Association to students within the Section placing high in the Putnam Contest. A report on the activities of the High School Contest Committee was submitted by Mr. William Landis, Contest Committee Chairman.

G. L. ALEXANDERSON, *Secretary-Treasurer*

SUMMER DUES PAYMENTS

Dues in the Association are payable on a calendar year basis. For the past two years, members have been requested to make annual payments for dues during the preceding summer rather than waiting until December or January to make such payments. We have been pleased with the response to this request.

Payment of dues during the summer helps to spread the work load of the office over a longer period and thus contributes to more efficient operation. This in turn helps to keep dues at their present low level.

We, therefore, suggest that all MAA members plan to pay 1969 dues of $8 sometime during the coming summer. Payments should be sent to: Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

CALENDAR OF FUTURE MEETINGS

Forty-Ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

Fifty-Second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
FLORIDA, Florida Atlantic University, Boca Raton, Spring 1969.
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, New Orleans, January 25–27, 1969.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY, Rutgers—The State University, New Brunswick, November 2, 1968.
NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.
NORTHERN CALIFORNIA, University of Santa Clara, Santa Clara, February 8, 1969.
OHIO
OKLAHOMA-ARKANSAS, Arkansas State University, Jonesboro, March 1969.
PACIFIC NORTHWEST
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.
ROCKY MOUNTAIN
SOUTHEASTERN, Winthrop College, Rock Hill, S. C., March 28–29, 1969.
SOUTHERN CALIFORNIA, California State College at Fullerton, March 15, 1969.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, University of Wisconsin, Madison, August 27–30, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, Las Vegas, Nevada, August 27–29, 1968.

ASSOCIATION FOR SYMBOLIC LOGIC, Warsaw, Poland, August 30–31, 1968.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS, University of Wisconsin, Madison, August 27–28, 1968.

MU ALPHA THETA, Trinity University, San Antonio, Texas, August 11–14, 1968.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cedar Rapids, Iowa, August 22–24, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, November 6–9, 1968.

PI MU EPSILON, University of Wisconsin, Madison, August 27–28, 1968.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of Wisconsin, Madison, August 26–30, 1968.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED in 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor-Elect, HARLEY FLANDERS, American Mathematical Monthly, Purdue University, Lafayette, IN 47907.

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

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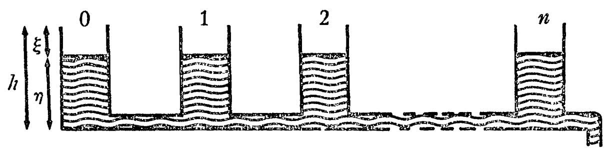
DIFFERENTIAL EQUATIONS FOR FLOW OF A SOLUTION OF VARYING CONCENTRATION

J. C. BURNS, Australian National University, Canberra

1. Introduction. In certain applications of chromatography (gradient elution [1, 2]) it is necessary to produce a flow of a solution through a tube in such a way that the concentration of the solute is a known function of the total volume of solution which has passed through the tube. This varying concentration can be produced in an apparatus shown schematically in its most general form in the diagram.



There are $n+1$ cylinders of cross-sectional areas $A_0, A_1, \cdots, A_n$ connected in series by tubes of negligible volume. Initially the cylinders are filled to a depth h with solution of concentrations $\gamma_0, \gamma_1, \cdots, \gamma_n$ respectively. The fluid is then allowed to run out of the last cylinder so that the levels in all the cylinders fall together; at any stage a depth ξ has run out of each cylinder and a depth η remains, with $\xi+\eta=h$. The solution in each cylinder is kept constantly stirred so that there is a uniform concentration of solute in each at all times. At the stage shown in the diagram we take the concentrations as $c_0(=\gamma_0), c_1, c_2, \cdots, c_n$ and the problem is to find c_n, the concentration of the solution delivered by the apparatus, as a function of the total volume of fluid which has run out or, more conveniently, as a function of ξ.

The differential equation giving the concentration c_k as a function of ξ is obtained by considering the decrease in the total quantity of the solute in cylinder k when the depth of solution which has run out increases from ξ to $\xi+\delta\xi$. This decrease is equal to the difference between the quantity of solute carried out by the volume $\delta\xi \sum_{t=0}^{k} A_t$ of solution which leaves the cylinder and the quantity carried in by the volume $\delta\xi \sum_{t=0}^{k-1} A_t$ of solution which enters the cylinder while the depth $\delta\xi$ runs out of the apparatus. Taking into account the concentrations of these different volumes of solution and denoting the concentration in cylinder k after depth $\xi+\delta\xi$ has run out by $c_k+\delta c_k$, we have for $k=1, 2, \cdots, n$,

$$c_k(h-\xi)A_k - (c_k+\delta c_k)(h-\xi-\delta\xi)A_k = c_k\delta\xi \sum_{t=0}^{k} A_t - c_{k-1}\delta\xi \sum_{t=0}^{k-1} A_t + 0(\delta\xi)^2$$

whence $\delta c_k(h-\xi)A_k + c_k\delta\xi \sum_{t=0}^{k-1} A_t = c_{k-1}\delta\xi \sum_{t=0}^{k-1} A_t + 0(\delta\xi)^2$. Thus, taking the limit as $\delta\xi \to 0$ and defining

$$\alpha_0 = 0; \qquad \alpha_k = A_k^{-1} \sum_{t=0}^{k-1} A_t, \quad k = 1, 2, \cdots;$$

$$x = \xi/h, \quad y = \eta/h, \quad x + y = 1,$$

we obtain the family of differential equations

$$\frac{dc_k}{dx} + \frac{\alpha_k c_k}{1-x} = \frac{\alpha_k c_{k-1}}{1-x}, \qquad k = 1, 2, \cdots, n. \tag{1}$$

By introducing an integrating factor and changing the variable from x to y, we can write (1) more conveniently as

$$\frac{d}{dy}\left\{\frac{c_k}{y^{\alpha_k}}\right\} = -\frac{\alpha_k c_{k-1}}{y^{\alpha_k+1}}, \qquad k = 1, 2, \cdots, n. \tag{2}$$

Equations (2) are to be solved under the initial conditions $c_k = \gamma_k$ when $y = 1$ for $k = 1, 2, \cdots, n$ (together with the condition $c_0 \equiv \gamma_0$).

Equations (2) can be solved for $c_1, c_2, \cdots, c_n$ in succession in any particular case but it is of interest to seek a solution for a general value of n. This can be done in at least two cases. In the first, which corresponds to the usual experimental conditions, all the cross-sectional areas $A_0, A_1, A_2, \cdots, A_n$ are equal. This case can be solved completely and the concentration c_n expressed in several different forms. A similar investigation of the case in which the areas A_k are not all equal can also be carried out provided we assume that all the ratios α_k are unequal. It will be seen that this solution is greatly facilitated by the introduction of two different "generalized binomial coefficients."

2. Cylinders of equal cross-section ($\alpha_k = k$). When all the cylinders have equal cross-sections, $\alpha_k = k$ and equations (2) become

$$\frac{d}{dy}\left\{\frac{c_k}{y^k}\right\} = \frac{-kc_{k-1}}{y^{k+1}}, \qquad k = 1, 2, \cdots, n. \tag{3}$$

We first establish a relation between $c_0, c_1, \cdots, c_n$.

THEOREM 2.1.

$$y^{-n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} c_r = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \gamma_r. \tag{4}$$

The left hand side of (4) is written in the form

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} c_r y^{-r} y^{-(n-r)}. \tag{5}$$

Equations (3) are then used to express the differential coefficient of (5), after some rearrangement, as

$$y^{-n-1}\sum_{r=0}^{n-1}(-1)^r c_r\left\{(r+1)\binom{n}{r+1}-(n-r)\binom{n}{r}\right\}, \tag{6}$$

in which each term is easily seen to vanish.

It follows that the left hand side of (4) is constant, the value of the constant being obtained by using the initial conditions $c_k=\gamma_k$, $k=0, 1, 2, \cdots, n$.

Theorem 2.1 gives

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}c_r = y^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\gamma_r = b_n \text{ (say)} \tag{7}$$

and the problem now is to invert this relation to express c_n in terms of b_r and hence in terms of y.

Riordan [3] gives an orthogonality identity connecting binomial coefficients,

$$\sum_{k=m}^{n}(-1)^{k+m}\binom{n}{k}\binom{k}{m} = \delta_{mn}, \tag{8}$$

which is used to prove that

$$b_n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}c_r \rightarrow c_n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}b_r. \tag{9}$$

(In fact (8) can be used to show that the converse of (9) is also true.)

It is evident that (7) and (9) together give c_n as the polynomial in y given in (10) below which can be arranged in the more compact form (11).

THEOREM 2.2.

$$c_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}y^k\sum_{r=0}^{k}(-1)^r\binom{k}{r}\gamma_r \tag{10}$$

$$= \sum_{r=0}^{n}\binom{n}{r}\gamma_r y^r(1-y)^{n-r}. \tag{11}$$

If, in (11), we replace y by $1-x$ and rearrange the terms appropriately we obtain a very similar expression for c_n involving x rather than y. It is then easily seen how to express c_n as a polynomial in x. These alternative expressions for c_n are given in Theorem 2.3.

THEOREM 2.3.

$$c_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k\sum_{r=0}^{k}(-1)^r\binom{k}{r}\gamma_{n-r}$$

$$= \sum_{r=0}^{n}\binom{n}{r}\gamma_{n-r}x^r(1-x)^{n-r}.$$

It is worth noting that these theorems show that c_n has the expected value in certain special cases. Thus when $x=1$ (and $y=0$), we see that $c_n=\gamma_0$ so that the last drop to run out has the concentration of the solution in the initial cylinder. Similarly, when $x=0$ (and $y=1$), $c_n=\gamma_n$ so that the first drop to run out has concentration γ_n. Also, when all the initial concentrations are equal, say to γ_0, we have, from (10), that

$$c_n = \gamma_0 \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{r=0}^{k} (-1)^r \binom{k}{r} = \gamma_0$$

since

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} = \delta_{k0}.$$

Thus all the solution has the same concentration.

3. Cylinders of unequal cross-section (all α_k unequal). In discussing this more general problem, it will be useful to have two "generalized binomial coefficients," denoted by $\left[{n \atop r}\right]$ and $\left\{{n \atop r}\right\}$ respectively and defined for the set of unequal numbers $\alpha_0=0;\ \alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n$ by the relations

$$\begin{bmatrix} n \\ n \end{bmatrix} = \begin{Bmatrix} n \\ n \end{Bmatrix} = 1 \quad \text{for any } n \geqq 0;$$

for $0 \leqq r \leqq n-1$,

$$\begin{bmatrix} n \\ r \end{bmatrix} = \prod_{k=1}^{n-r} \frac{\alpha_{n-k+1}}{\alpha_n - \alpha_{n-k}}; \quad \begin{Bmatrix} n \\ r \end{Bmatrix} = \prod_{k=1}^{n-r} \frac{\alpha_{n-k+1}}{\alpha_{n-k+1} - \alpha_r}.$$

When $\alpha_k = k$ (as in section 2), each of these coefficients reduces to the ordinary binomial coefficient.

The first step is to prove a theorem analogous to Theorem 2.1.

THEOREM 3.1.

$$y^{-\alpha_n} \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} c_r = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} \gamma_r. \tag{12}$$

The proof follows exactly that outlined for Theorem 2.1, the expression corresponding to (6) being

$$y^{-\alpha_n - 1} \sum_{r=0}^{n-1} (-1)^r c_r \left\{ (\alpha_n - \alpha_r) \begin{bmatrix} n \\ r \end{bmatrix} - \alpha_{r+1} \begin{bmatrix} n \\ r+1 \end{bmatrix} \right\}.$$

Again it is easily verified that each term of this sum vanishes so that the proof of the theorem can be completed as before.

As in section 2, the problem now is to invert relation (12) to obtain c_n. For this purpose we need a result similar to (9) and to prove this we need an orthogonality identity similar to (8). This identity, given in Lemma 3.2, is easily derived from the essentially equivalent result of Lemma 3.1 which will be needed at a later stage of the paper.

LEMMA 3.1.

$$\sum_{k=m}^{n}(-1)^{k+m}\left\{{n \atop k}\right\}\left[{k \atop m}\right]\Big/\left\{{n \atop m}\right\}=\delta_{mn}. \tag{13}$$

When $m=n$, the left hand side of (13) is equal to 1. For $0\leqq m<n$, the first term (for which $k=m$) of the sum is equal to 1 while for $m+1\leqq k\leqq n$, the typical term can be expressed in the form $-p_k(\alpha_m)/p_k(\alpha_k)$ where $p_k(\alpha_m)$ is the polynomial in α_m of degree $n-m-1$ given by

$$p_k(\alpha_m)=\prod_{t=m+1}^{k-1}(\alpha_t-\alpha_m)\prod_{t=k+1}^{n}(\alpha_t-\alpha_m).$$

It can be seen that $p_k(\alpha_m)=0$ when α_m takes any of the values α_{m+1}, α_{m+2}, $\cdots$, α_n except α_k.

It follows that the left hand side of (13) is the polynomial in α_m of degree $n-m-1$ given by

$$1-\sum_{k=m+1}^{n}p_k(\alpha_m)/p_k(\alpha_k)$$

which evidently vanishes for each of $n-m$ values of α_m, namely $\alpha_{m+1},\alpha_{m+2}$, $\cdots$, α_n, and hence vanishes identically.

LEMMA 3.2.

$$\sum_{k=m}^{n}(-1)^{k+m}\left\{{n \atop k}\right\}\left[{k \atop m}\right]=\delta_{mn}. \tag{14}$$

In Lemma 3.1, the factor $\left\{{n \atop m}\right\}$ in the denominator of the left hand side of (13) is the same for each term of the sum and $\left\{{n \atop n}\right\}=1$.

LEMMA 3.3.

$$b_n=\sum_{r=0}^{n}(-1)^r\left[{n \atop r}\right]c_r\rightarrow c_n=\sum_{r=0}^{n}(-1)^r\left\{{n \atop r}\right\}b_r. \tag{15}$$

This result is easily proved by using the orthogonality identity of Lemma 3.2. (The converse of (15), not needed here, can be deduced from an identity which may be described as the dual of (14).)

We can now use Theorem 3.1 and Lemma 3.3 to prove in exactly the same way a theorem analogous to Theorem 2.2.

THEOREM 3.2.

$$c_n = \sum_{k=0}^{n} (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} y^{\alpha k} \sum_{r=0}^{k} (-1)^r \left[\begin{matrix} k \\ r \end{matrix} \right] \gamma_r$$
$$= \sum_{k=0}^{n} \gamma_r \left\{ \begin{matrix} n \\ r \end{matrix} \right\} P_r(y),$$

where $P_r(y) = \sum_{k=r}^{n} (-1)^{k+r} y^{\alpha k} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \left[\begin{smallmatrix} k \\ r \end{smallmatrix} \right] / \left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}$.

From Lemma 3.1 and the definitions of the generalized binomial coefficients, we can see that $P_r(1) = \delta_{rn}$, $P_r(0) = \delta_{0r}$ and hence that $c_n = \gamma_0$ when $y = 0$ and $c_n = \gamma_n$ when $y = 1$ as in the previous case.

Moreover, the result analogous to (12) is easily proved:

$$\sum_{r=0}^{k} (-1)^r \left[\begin{matrix} k \\ r \end{matrix} \right] = \delta_{k0}. \tag{16}$$

(The sum $S_l = \sum_{r=k-l}^{k} (-1)^r \left[\begin{smallmatrix} k \\ r \end{smallmatrix} \right]$ is evaluated by induction on l and it can then be observed that $\left[\begin{smallmatrix} k \\ 0 \end{smallmatrix} \right] + S_{k-1} = 0$.) We also have $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 1$ so it follows as before that when all the initial concentrations γ_k are equal to γ_0, then $c_n = \gamma_0$ for all y.

4. Cylinders of unequal cross-sections (α_k not all unequal). When all the α_k are not unequal, some of the generalized binomial coefficients introduced above may be undefined and c_n will no longer, in general, be so simply expressed in terms of y. For example, in the simple case $n = 2$ with $\alpha_1 = \alpha_2 = \alpha$ say, we obtain

$$c_2 = \gamma_0 + (\gamma_2 - \gamma_0) y^\alpha - \alpha y^\alpha (\gamma_1 - \gamma_0) \log y.$$

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I am greatly indebted to the referee for suggesting the use of the identities given by Riordan in proving the results of section 2.

References

1. R. S. Alm, R. J. P. Williams and A. Tiselius, Acta Chem. Scand., 6 (1952) 826.
2. E. A. Peterson and H. A. Sober, Anal. Chem., 31 (1959) 857.
3. John Riordan, Inverse relations and combinatorial identities, this MONTHLY, 71 (1964) 485–498.

AN IDENTITY AND APPLICATIONS

E. M. WRIGHT, University of Aberdeen, Scotland

1. In what follows all small latin letters denote integers positive, negative or zero. X, Y are any complex numbers such that $|X|<1$, $|Y|<1$. We write

$$\xi_a = 0\ (a < 0),\ \xi_0 = 1,\ \xi_a = \prod_{u=1}^{a} (1 - X^u)^{-1} \qquad (a > 0).$$

We use $\sum_u$ to denote summation over all u and $\sum_u'$ summation over all $u \leqq r+s+t-k$. In every case all but a finite number of the terms vanish, so that the sum is a finite one. We shall prove

THEOREM:

$$\xi_{s+t-k}\xi_{t+r-k}\xi_{r+s-k} \sum_u{}' \frac{X^{u(u-k)}\xi_u\xi_{u-k}\xi_{r-u}\xi_{s-u}\xi_{t-u}}{\xi_{r+s+t-k-u}} = \xi_r\xi_{r-k}\xi_s\xi_{s-k}\xi_t\xi_{t-k}. \tag{1}$$

We first prove two lemmas.

LEMMA 1: *If* $w>0$, *then* $\xi_w \prod_{j=1}^{w}(1+YX^j) = \sum_x X^{x(x+1)/2}\xi_x\xi_{w-x}Y^x$.

The result is trivial for $w=1$. We can establish an induction with respect to w, provided we can show that

$$(1 + YX^{w+1}) \sum_x X^{x(x+1)/2}\xi_x\xi_{w-x}Y^x = (1 - X^{w+1}) \sum_x X^{x(x+1)/2}\xi_x\xi_{w+1-x}Y^x.$$

The coefficient of Y^x on the left-hand side is

$$\begin{aligned} X^{x(x+1)/2}\xi_x\xi_{w-x} &+ X^{w+1+x(x-1)/2}\xi_{x-1}\xi_{w+1-x} \\ &= X^{x(x+1)/2}\xi_x\xi_{w+1-x}\{(1 - X^{w+1-x}) + X^{w+1-x}(1 - X^x)\} \\ &= X^{x(x+1)/2}\xi_x\xi_{w+1-x}(1 - X^{w+1}), \end{aligned}$$

which is the coefficient of Y^x on the right-hand side.

LEMMA 2:

$$\xi_v\xi_w = \xi_{v+w} \sum_x (-1)^x X^{xv+x(x+1)/2}\xi_x\xi_{w-x}. \tag{2}$$

This is trivial if $w\leqq 0$ or if $v+w<0$. We suppose then that $w>0$ and $v+w\geqq 0$. We put $Y=-X^v$ in Lemma 1 and use the result in (2). We find that it is enough to prove that

$$\xi_v = \xi_{v+w} \prod_{j=1}^{w} (1 - X^{j+v}) \qquad (w > 0,\ v + w \geqq 0). \tag{3}$$

If $v<0$, the left-hand side vanishes; also $1\leqq -v \leqq w$ and so one of the factors in the product on the right vanishes. If $v\geqq 0$, (3) is immediate from the definition of ξ_a.

By Lemma 2, we have

$$\xi_{t-u}\xi_{r+s-k} = \xi_{r+s+t-k-u}\sum_{z}(-1)^{z}X^{z(t-u)+z(z+1)/2}\xi_{z}\xi_{r+s-k-z},$$

$$\xi_{t}\xi_{r-k} = \xi_{r+t-k}\sum_{x}(-1)^{x}X^{xt+x(x+1)/2}\xi_{x}\xi_{r-k-x},$$

$$\xi_{t-k}\xi_{s} = \xi_{s+t-k}\sum_{y}(-1)^{y}X^{y(t-k)+y(y+1)/2}\xi_{y}\xi_{s-y}.$$

If we use these in (1), we find that it is enough to prove that

$$\sum_{u}\sum_{z}(-1)^{z}X^{\alpha}\xi_{u}\xi_{u-k}\xi_{r-u}\xi_{s-u}\xi_{z}\xi_{r+s-k-z} = \xi_{r}\xi_{s-k}\sum_{x}\sum_{y}(-1)^{x+y}X^{\beta}\xi_{x}\xi_{r-k-x}\xi_{y}\xi_{s-y},$$

where

$$\alpha = u(u-k) + z(t-u) + z(z+1)/2 = u(u-k-z) + zt + z(z+1)/2$$

and

$$\begin{aligned}\beta &= xt + y(t-k) + x(x+1)/2 + y(y+1)/2\\ &= t(x+y) + y(y-k-x-y) + (x+y)(x+y+1)/2.\end{aligned}$$

Selecting those terms on the right for which $x+y=z$, we see that it is enough to prove that

$$(4)\quad \xi_{z}\xi_{r+s-z-k}\sum_{u}X^{u(u-k-z)}\xi_{u}\xi_{u-k}\xi_{r-u}\xi_{s-u} = \xi_{r}\xi_{s-k}\sum_{y}X^{y(y-k-z)}\xi_{y}\xi_{z-y}\xi_{r-k-z+y}\xi_{s-y}$$

for every z. Again, by Lemma 2,

$$\xi_{u}\xi_{r-u} = \xi_{r}\sum_{v}(-1)^{v}X^{vu+v(v+1)/2}\xi_{v}\xi_{r-u-v},$$

$$\xi_{s-u}\xi_{u-k} = \xi_{s-k}\sum_{w}(-1)^{w}X^{w(s-u)+w(w+1)/2}\xi_{w}\xi_{u-k-w},$$

$$\xi_{y}\xi_{z-y} = \xi_{z}\sum_{v}(-1)^{v}X^{vy+v(v+1)/2}\xi_{v}\xi_{z-y-v},$$

$$\xi_{s-y}\xi_{r-z-k+y} = \xi_{r+s-z-k}\sum_{w}(-1)^{w}X^{w(s-y)+w(w+1)/2}\xi_{w}\xi_{r-z-k+y-w}.$$

We substitute from these in (4). It is then enough to prove that the coefficient of

$$(-1)^{v+w}X^{wz+v(v+1)/2+w(w+1)/2}\xi_{v}\xi_{w}\xi_{z}\xi_{r+s-z-k}\xi_{r}\xi_{s-k}$$

on each side is the same, i.e. to prove that

$$\sum_{u}X^{u(u-k-z+v-w)}\xi_{r-u-v}\xi_{u-k-w} = \sum_{y}X^{y(y-k-z+v-w)}\xi_{z-y-v}\xi_{r-z-k-w+y}.$$

If we put $y=k+z+w-v-u$ in the sum on the right-hand side, it becomes identical with that on the left-hand side. This completes the proof of our theorem.

2. If $a\to\infty$, then $\xi_a\to\xi_\infty=\prod_{u=1}^\infty(1-X^u)^{-1}$, the generating function of the number of partitions of n into any number of parts. If we let $t\to\infty$ in (1) we have the identity

$$\xi_{r+s-k}\sum_u X^{u(u-k)}\xi_u\xi_{u-k}\xi_{r-u}\xi_{s-u} = \xi_r\xi_{r-k}\xi_s\xi_{s-k}. \tag{5}$$

If we let $s\to\infty$ in this, we have the further identity

$$\sum_u X^{u(u-k)}\xi_u\xi_{u-k}\xi_{r-u} = \xi_r\xi_{r-k}. \tag{6}$$

This last identity can be proved independently by induction with respect to r. It has an interpretation in terms of partition theory, but I have not yet found a direct combinatorial proof.

We remark that, if $a\geqq 0$, $b\geqq 0$, then

$$\lim_{X\to 1}\frac{\xi_a\xi_b}{\xi_{a+b}} = \frac{(a+b)!}{a!b!},$$

where $0!=1$. Let us write $c=\max(0,k)$ and $d=\min(r,s,t)$. If $c\leqq d$, we have

$$\sum_{u=c}^{d}\frac{(r+s+t-u-k)!}{u!(u-k)!(r-u)!(s-u)!(t-u)!} = \frac{(s+t-k)!(t+r-k)!(r+s-k)!}{r!(r-k)!\,s!(s-k)!\,t!(t-k)!} \tag{7}$$

if we let $X\to 1$ in (1). Similarly we can deduce from (5) that

$$\sum_{u=\max(0,k)}^{\min(r,s)}\frac{1}{u!(u-k)!(r-u)!(s-u)!} = \frac{(r+s-k)!}{r!(r-k)!s!(s-k)!}.$$

Recently Graham and Riordan [1] have shown that the solution of the recurrence relation

$$\omega_{nm} = \sum_{v=0}^{m}\omega_{mv}\binom{n+v}{2m} \qquad (0\leqq m\leqq n) \tag{8}$$

in terms of the undetermined ω_{vv} is

$$\omega_{nm} = \sum_{v=0}^{m}\frac{2v+1}{m+v+1}\binom{n+v}{m+v}\binom{n-1-v}{m-v}\omega_{vv} \qquad (m<n). \tag{9}$$

An alternative proof to theirs would be to substitute from (9) in (8) and then seek to prove equal the coefficients of ω_{vv} on either side of the result. What is required readily reduces to the identity (7) with $k=1$.

In the same way, the solution of the recurrence relation

$$\Omega_{nm} = \sum_{v=0}^{m} X^{(m-v)(m-v+1)} \xi_{2m}\xi_{n+v-2m}\Omega_{mv}/\xi_{n+v}$$

is

$$\Omega_{nm} = \sum_{v=0}^{m} \frac{(1 - X^{2v+1})\xi_{m+v+1}\xi_{n-m}\xi_{m-v}\xi_{n-1-m}\Omega_{vv}}{\xi_{n+v}\xi_{n-1-v}}$$

and the verification of this reduces to the identity (1) with $k=1$.

Again let us put $r=s=p$ and $k=0$ in (1) and let $X\to 1$. We have

$$\sum_{u=1}^{p} \binom{p}{u}^2 \binom{t+2p-u}{2p} = \binom{t+p}{p}^2,$$

which is, of course, a well-known identity.

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Reference

1. R. L. Graham and J. Riordan, The solution of a certain recurrence, this MONTHLY, 73 (1966) 604–608.

REMARKS ON GROUPS OF ORDER 1

ELVIRA STRASSER RAPAPORT, SUNY at Stony Brook

A group given by generators and relations, $G=(x_1, \cdots; R_1, \cdots)$, may consist of just one element, the identity. The case when there are n generators x_i and n defining relations R_i is particularly interesting to topologists and algebraists. A paper [1] that appeared in this MONTHLY studies these defining relations and offers conjectures regarding their properties. The main purpose of the present remarks is to disprove one of these conjectures and to throw light on the problem posed in that paper. Furthermore, a query of the authors is answered, some of their theorems sharpened and some gaps closed.

Let F_n be the free group of rank n. For this paper n will be 2, but almost every statement remains true for arbitrary n.

The notation and definitions of [1] are retained whenever possible. In addition the following are used.

(s_1, s_2) is a pair of free generators $[s_1(a, b), s_2(a, b)]$ of the free group F of rank two generated by the symbols (a, b).

$\{r, s\}$ is the normal subgroup the elements r and s of F generate in F.

F' is the commutator subgroup of F.

$C, C', C_1, \cdots$ are elements of F'.

$u \sim v$ means that the elements u and v are conjugates in F: $u = xvx^{-1}$ for some element x of F.

An element of F will also be called a word.

It will be necessary to distinguish between two kinds of transformations of words of F into other words. One of these will be denoted by N, the other by Q with suitable subscripts as needed. They will take pairs of words into pairs of words.

N will designate an automorphism of F.

To define Q, let (r, s) be the pair of elements $[r(a, b), s(a, b)]$ to be transformed, ϵ and η a pair of units (that is $+1$ or -1), U, u, V, v any elements of F. Then the transformation that replaces r by $r' = ur^\epsilon u^{-1} \cdot vs^\eta v^{-1}$ and leaves s fixed, will be denoted by Q, and the transformation that replaces s by $s' = Us^\epsilon U^{-1} \cdot Vr^\eta V^{-1}$ and leaves r fixed will be denoted by $\tilde{Q}$. Thus

$$Q(r, s) = (ur^\epsilon u^{-1} \cdot vs^\eta v^{-1}, s),$$
$$\tilde{Q}(r, s) = (r, Us^\epsilon U^{-1} \cdot Vr^\eta V^{-1}).$$

One gets a new Q for a new choice of the words u, v; similarly for $\tilde{Q}$. These and their products, defined below, will be called Q-transformations. They form a group for which the transformations given above are a set of generators if U, V, u, v range over F. It will be assumed that the words r, s are cyclically reduced: they contain no segment of the form AA^{-1} and neither has the form xwx^{-1}. Thus, when cyclically reduced, $babb^{-1}ab^{-1}$ becomes a^2.

To define the product Q_2Q_1, acting on (r, s), let $Q_1(r, s) = (r', s')$. Then $Q_2Q_1(r, s)$ is defined to be $Q_2(Q_1(r, s)) = Q_2(r', s')$. That is, Q_2 acts on the pair (r', s'). For example, let $r = ab$, $s = b^{-1}a^2$, $Q_1(r, s) = (b^{-1}rb \cdot a^6sa^{-6}, s)$ and $Q_2(R, S) = (RS^{-1}, S)$. Then

$$Q_1(r, s) = [(b^{-1}abb)(a^6b^{-1}a^2a^{-6}), b^{-1}a^2] = [b^{-1}ab^2a^6b^{-1}a^{-4}, b^{-1}a^2]$$

and

$$\begin{aligned} Q_2Q_1(r, s) &= Q_2(Q_1(r, s)) = Q_2(b^{-1}ab^2a^6b^{-1}a^{-4}, b^{-1}a^2) \\ &= (b^{-1}ab^2a^6b^{-1}a^{-4} \cdot a^{-2}b, b^{-1}a^2) \sim (b^2a^6b^{-1}a^{-5}, b^{-1}a^2). \end{aligned}$$

In particular, $Q_2Q_1(r) = b^2a^6b^{-1}a^{-5}$, $Q_2Q_1(s) = b^{-1}a^2$. It can be shown with little trouble that the Q-transformations are the "extended Nielsen operations" of [**1**], except for the convenience of lumping together r, s with xrx^{-1}, ysy^{-1} for all x, y in F. The algebraically important difference lies in the way the generators are defined. For example, a single generator can invert r *and* multiply it by a conjugate of s or of s^{-1}. When dealing with arguments of length (of a word), these are the appropriate generators. The length of a word $w(a, b)$ in $F = F(a, b)$ is the number of symbols in it when it is reduced: when it contains no segment AA^{-1} for any symbol A.

THEOREM 1. *For any automorphism N and any pair of elements (R, S) of F, $(R, S)=Q(a, b)$ for some Q if and only if $(NR, NS)=Q^*(a, b)$ for some Q^*.*

The proof uses a combinatorial argument and may be found in [6]. The theorem holds for all finitely generated free groups F_n.

The following special case will be used here.

COROLLARY 1. *Let $R_1=s_1C_1'$, $R_2=s_2C_2'$, the s_i a pair of generators of the free group F, C_i' and C_i elements of F'. Then $Q(R_1, R_2)=(a, b)$ for some Q if and only if $N(R_1, R_2)=(aC_1, bC_2)$ for some N, C_1, C_2 and $Q^*(aC_1, bC_2)=(a, b)$ for some Q^*.*

Proof: Since (s_1, s_2) generate F, there is an N for which $N(s_1)=a$, $N(s_2)=b$. Since F' is invariant under automorphisms of F, the $N(C_i')$ are in F' again. Then, $N(s_1C_1')=N(s_1)N(C_1')=aC_1$. Similarly $N(s_2C_2')=bC_2$. It follows now from Theorem 1 that Q exists if and only if Q^* exists.

THEOREM 2. *If a group G has a presentation on two generators and two defining relations, $(a, b; R_1, R_2)$, then (the commutator subgroup) G' is the whole group G (G is perfect) if and only if $R_1=s_1C_1^{-1}$, $R_2=s_2C_2^{-2}$, for some pair of free generators s_1, s_2, and elements C_1, C_2 of F', in F.*

Proof: If $R_i=s_iC_i$, then $s_1=C_1$ and $s_2=C_2$ in G. As a and b can be expressed in terms of s_1 and s_2, $a=f(s_1, s_2)$, $b=g(s_1, s_2)$, the relations $a=f(C_1, C_2)$, $b=g(C_1, C_2)$ hold in G. Then, since a and b generate G, $G=G'$.

To prove the converse, one must use the well-known fact (see for example Corollary 3.51 in [3]) that for every pair of generators, a^pb^q, a^lb^m, of the free Abelian group F/F' there exist elements C_i' for which $a^pb^qC_1'$, $a^lb^mC_2'$ freely generate F. If $G=G'$ is assumed now, then $G/G'=1$, and it follows that the elements $R_i\equiv a^{x_i}\cdot b^{y_i}$ modulo F' in F together generate F/F'. Therefore $R_i\equiv s_i$ modulo F', as claimed; i: 1, 2.

COROLLARY 2. *If $G=(a, b; R_1, R_2)=1$, then $R_1=s_1C_1$, $R_2=s_2C_2$.*

For then G is perfect.

Both results are true for n generators and n defining relations.

Since $|pm-ql|=1$ in the words above, Proposition 2.2 of [1] follows from the Corollary. Since Q-transformations of the generators of F/F' are simply the automorphisms of F/F', Proposition 3.1 of [1] follows also.

THEOREM 3. *Let s_1, s_2 be any pair of free generators of F and C' any element of F'. Then $(a, b; s_1C', s_2)=1$ with $Q(s_1C', s_2)=(a, b)$.*

Proof: By Corollary 1 it suffices to show this for $(a, b; aC, b)$, with C any element of F'.

Note first that $\{b\}$ contains F' (for $F/\{b\}=F_1$, the free group on the symbol a, is Abelian), and so C is a product $x_1b^{\epsilon}x_1^{-1}\cdots x_kb^{\epsilon_k}x_k^{-1}$ of conjugates of b and b^{-1}. Since multiplication of aC on the right hand side by the conjugate $x_kb^{-\epsilon_k}x_k^{-1}$ of $b^{-\epsilon_k}$ is a Q-transformation, it is clear that $Q(aC, b)=(a, b)$ for some Q.

By noting that a word s_2C_2 of length less than 5 is always a free generator in F, it is seen that Theorem 4.2 of [1] is a special case of this.

THEOREM 4. *Let* (R) *be the set of all pairs of words* (R_1, R_2) *in* F *whose normal closure is* F. *Then* $Q(R_1, R_2)=(a, b)$ *for every pair* (R_1, R_2) *if and only if* $Q(aC_1, bC_2)=(a, b)$ *for each pair* (aC_1, bC_2) *in* R.

This is an obvious application of Corollary 1 to Corollary 2.

Thus, one may restrict attention to presentations of the form $(a, b; aC_1, bC_2)$ of groups of order 1. Similarly for n generators and n defining relations.

THEOREM 5. *For any word* $R_1=s_1C$ *in* F, *there exists a word* R_2 *such that* $Q(R_1, R_2)=(a, b)$ *but* (R_1, R_2) *do not freely generate* F.

Proof: It suffices to take $R_1=aC$, with C an arbitrary element of F'. There are two cases: either R_1 freely generates F with another element s_2^*; or else there is no such s_2^* and so R_1 is not a free generator of F. Suppose first that $R_1=s_1C=s_1^*$ generates F freely with s_2^*. Consider all words of the form $s_2^*\cdot x_1R_1^{\epsilon_1}x_1^{-1}\cdots x_kR_1^{\epsilon_k}x_k^{-1}$, k any natural number. If one of them does not generate F freely with R_1, then by Theorem 3 it is the R_2 required.

Indeed, such a word does exist; for not all pairs of this type are free generators: it follows from known results [4] that the pair (s_1^*, vs_2^*w) are a set of free generators of $F(a, b)=F(s_1^*, s_2^*)$ exactly when v and w are powers of s_1^*. This covers the first case.

The second case is taken care of by Theorem 3.

These results will now be applied to three sets of examples, stated in the form of theorems, because proofs are necessary to show the properties claimed for them.

THEOREM 6. *Let* $R_1=a^{-2}b^{-1}ab$ *and let* $R_2=b^{-1}\cdot b^{-x}a^{-y}b^xa^z$, *with arbitrary integral exponents. With the exception of a finite number of values of* x, y *and* z, *neither* R_i *is a free generator in* F. *For any given values of* x, y, z, *a* Q *can be found such that* $Q(R_1, R_2)=(a, b)$ *for some* Q, *and so* $(a, b; R_1, R_2)=1$.

Proof: If $x=0$ then Theorem 3 applies, since $R_1=a^{-1}C$ and $R_2\sim b^{-1}a^u$, for some integer u, with $a^{-1}=s_1$ and $b^{-1}a^u=s_2$, a pair of free generators of F. If $x\neq 0$ the transformation Q is found in the following steps:

1. Writing $R_1=1$ in the form $a^2=b^{-1}ab$ has the following consequences in F:

$$b^{-2}ab^2 = b^{-1}a^2b = (b^{-1}ab)^2 = (a^2)^2,$$

$$b^{-3}ab^3 = a^{2^3}$$

$$\vdots$$

$$b^{-x}a^{-y}b^x = a^{-y2^x} \quad \text{for} \quad x > 0.$$

As an abbreviation, set the exponent $-y2^x$ equal to v, so $b^{-x}a^{-y}b^x=a^v$. As this equality is the consequence of setting $R_1=1$, the word $b^{-x}a^yb^xa^v$ is a product of conjugates of R_1 and R_1^{-1} in F, namely $w_1R_1^{\epsilon_1}w_1^{-1}\cdots w_tR_1^{\epsilon_t}w_t^{-1}$. Similarly,

$b^x a^z b^{-x} \equiv a^{v^*}$ for some number v^* when $x<0$.

2. Let $R_{21}=b^{-1}\cdot b^{-x}a^{-y}b^x\cdot a^z$ and $R_{22}=b^x a^z b^{-x}\cdot b^{-1}a^{-v}$, so that R_{21} and R_{22} are conjugates of R_2 written as products of conjugates of certain of their segments.

3. If $x>0$, replace the middle segment in R_{21} by a^v. This is the Q-transformation that leaves R_1 fixed and takes R_2 into

$$a^z R_2 a^{-z}\cdot\left[w_1 R_1^{\epsilon_1} w_1^{-1} \cdots w_t R_1^{\epsilon_t} w_t^{-1}\right] = Q^*(R_2) = b^{-1}a^u$$

for some integer u. If $x<0$, replace the first segment in R_{22} by a^{v^*}. This will give a like result: $Q^*(R_2)=b^{-1}a^u$, for some integer u. In either case $Q^*(R_1)=R_1$, and $Q^*(R_2)=b^{-1}a^u$. Set $Q^*(R_1, R_2)=(R_1^*, R_2^*)=(a^{-2}b^{-1}ab, b^{-1}a^u)$.

4. Since $(a^{-1}, b^{-1}a^u)$ is a pair of free generators, all one must do now is to reduce the R_i^* to these. This is quickly done by that Q-transformation which—in effect—replaces the b-symbol in R_1^* with R_2^* and the b^{-1}-symbol in R_1^* by the inverse of R_2^*. By Theorem 1 this gives $Q(R_1, R_2)=(a, b)$.

It remains to show that for almost all values of the exponents neither R_i is a free generator of F.

First let $z=0$. Then $R_2=b^{-1}$ and $(a, b; R_1, R_2)$ is an example for Theorem 3. Letting each exponent equal -1 gives the same result. Finally, let each exponent equal 1. Then $R_2=b^{-2}a^{-1}ba$, and $(a, b; R_1, R_2)$ is an example for Theorem 5. Leaving the cases of combinations of zeros and ± 1 as an exercise, the remaining possibilities are exponents with absolute values greater than 1. It is easy to check (see for example the methods in [**4**]) that none of the R_i involved are free generators of F.

THEOREM 7. *Let* $s_1=a^p b^q C_1$, $s_2=a^l b^m C_2$, *and* $R_1=a^p b^q$, $R_2=a^l b^m$. *Then* $Q(R_1, R_2)=(a, b)$.

This is Theorem 2.1 of [**1**]. Since the proof sketched there does not always go through (try $s_1=a^2b^{1-2x}$, $s_2=a^{-1}b^x$) a proof by induction on the number $t=|p|+|l|$ follows.

If $t=1$ then $p=0$ or $l=0$. Suppose $l=0$. Then $R_2=b$ or b^{-1}, so that Theorem 3 applies. Assume that $Q(R_1, R_2)=(a, b)$ is proven for $t<M$. Then $M>1$ so that for $|p+\epsilon l|=M$, $p\neq 0\neq l$ (because $|pm-ql|=1$). It follows that either $|u| =|p+\epsilon l|<|p|$, or else $|v|=|l+\epsilon p|<|l|$ for $\epsilon=1$ or -1. In the first case $b^m R_2 b^{-m} R_1=b^{\epsilon m}a^{\epsilon l}\cdot a^p b^q \sim a^u b^k$ or else $R_2^{-1}R_1 \sim a^u b^k$, for some integer k. This gives a Q-transformation which reduces the number $|p|+|l|$. In the second case a similar statement holds. By induction hypothesis the conclusion follows.

THEOREM 8. *If* $R_1=s_1C^h$, $R_2=s_2C^k$, C *an element of* F', *then* $Q(R_1, R_2)=(a, b)$.

Proof: One may take a nonnegative integer for k by switching, if necessary, from R_2 to $C^k R_2^{-1} C^{-k}$. Let first $h=1$. If $k=0$, then Theorem 5 applies; else replace R_2 by $R_2 R_1^{-1}=s_2C^k\cdot C^{-1}s_1^{-1}\sim s_1^{-1}s_2C^{k-1}=s_2^* C^{k^*}$. This is a Q-transformation. Since s_1 and s_2^* freely generate F, the new pair satisfies the hypotheses. The new

exponent k^* is smaller than k. This step can be repeated until the exponent k becomes zero. If $h \neq 1$, one may similarly reduce the situation to one where both exponents are nonnegative. Suppose then that $h \leqq k$. Going from (s_1C^h, s_2C^k) to the pair (s_1C^h, s_2C^{k-h}) reduces the second exponent, and so one continues until one exponent is 1 or 0.

This closes the gap in the proof of Theorem 4.3 in [1], which works only if $|p-|q|| < |p|$.

On the pattern of a certain pair (R_1, R_2) encountered above, let $R_1 = a^{-2}b^{-1}ab$, $R_2 = b^{-2}c^{-1}bc$, $R_3 = c^{-2}a^{-1}ca$. It is known [5] that $(a, b, c; R_1, R_2, R_3) = 1$. There is only a finite number of "cyclic conjugates" (that is, conjugates of length 5) [1] of the words R_i, and so it is not too difficult to check that using only these none of the words R_i can be reduced in length by a succession of at most three of the generating Q-transformations given at the outset. It may be remarked that this is true even if one is allowed to use any conjugates of the R_i [6]. The triplet (R_1, R_2, R_3) provides therefore a counterexample to Conjecture 2 in [1], for free groups on three generators. A pair of words $[W(a, b), V(a, b)]$ having the same property in the free group on two generators is given in [6].

They are rather long and will not be given here.

Now if the sum $|R_1| + |R_2| + |R_3|$ [or the sum $|W_1| + |W_2|$] is minimal (that is, cannot be reduced by any Q-transformation), then the triplet (R_1, R_2, R_3) [or the pair (W_1, W_2)] provides a counterexample to Conjecture 1 in [1]; and if not minimal, it provides a counterexample to Conjecture 4 in [1].

These conjectures seem to express the expectation that reductions of length by Q-transformations—if at all feasible—can be effected by a step of the form

$$Q(R_1) = xR_1^{\epsilon_1}x^{-1} \cdot yR_2^{\epsilon_2}y^{-1}, \qquad Q(R_i) = R_i \quad \text{for} \quad i \neq 1;$$

that is, by a generating Q-transformation as defined in the present paper. Past experience does not encourage this expectation, as may be seen, for example, from Problem 2 on page 137 of [3], the example on page 162 of [4] and from Corollary 3.8 of [2].

However, it is true [6] that whenever $|Q(R)| < |R|$ for the Q just given, it can be arranged that the conjugates figuring there have the lengths of the respective R_i. It follows from this that Conjecture 4 would imply Conjecture 3 in [1]. Of course the converse does not follow.

THEOREM 4.1 of [1] states that if the length $|r| + |s|$ is greater than 2 and is minimal under all Q-transformations then neither a nor b is the product of two conjugates of r and s or their inverses. Next the question is asked how to prove this statement if "two" is replaced by the arbitrary natural number n. Now suppose the new statement were true. Then it would follow that $\{r, s\}$ does not contain the element a. For it is part of the new statement that a is not the product of conjugates of r and s and of their inverses. This would contradict the definition of the pair (r, s).—Equivalently, since we know that a and b can be written as products of conjugates of r and s, the assumption that $|r| + |s| > 2$ and is minimal under all Q-transformations contradicts Conjecture 1 in [1].

The same goes for the problem posed in [1]: a proof of Conjecture 1 would provide a solution.

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References

1. J. J. Andrews and M. L. Curtis, Extended Nielsen operations in free groups, this MONTHLY, 73 (1966) 21–28.
2. H. Federer and B. Jónsson, Some properties of free groups, Trans. Amer. Math. Soc., 68 (1950) 1–27.
3. W Magnus, A. Karrass, and D. Solitar, Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, Interscience Publ., New York, 1966.
4. E. S. Rapaport, On free groups and their automorphisms, Acta Math., 99 (1958) 139–163.
5. ———, Groups of order 1, Proc. Amer. Math. Soc., 15 (1964) 828–833.
6. ———, Groups of order 1. Some properties of presentations, Acta Math., 120 (1968) 127–150.

ANOTHER PROOF OF THE FUNDAMENTAL THEOREM OF GALOIS THEORY

FRANK DEMEYER, Purdue University

This note presents a new proof of the Fundamental Theorem of Galois Theory for fields. The proof is based on the techniques contained in [2], and these ideas have been used in a very general setting in [4] and [5].

First we fix notation which will be maintained throughout. Let K be a field and let G be a finite group of automorphisms of K. We denote the subfield of K consisting of all elements left fixed by the automorphisms in G as K^G and call K^G the fixed field of G. If A is a subfield of K containing K^G then we let G_A be the subgroup of G consisting of all automorphisms in G which leave the elements of A fixed. The key to our approach is the following:

PROPOSITION 1. *Let G be a finite group of automorphisms of the field K and let A be a subfield of K containing K^G, then there are elements $a_1, \cdots, a_n \in A$ and $y_1, \cdots, y_n \in K$ such that for all $g \in G$*

$$\sum_{i=1}^{n} a_i g(y_i) = \begin{cases} 1 & \text{if} \quad g \in G_A \\ 0 & \text{if} \quad g \notin G_A \end{cases}$$

Proof: We observe first that if $G = G_A$ then $n=1$ and $a_1 = y_1 = 1$ works. If $G \neq G_A$ let S be the subset of G containing G_A together with one other element $g \in G$. Since $G \neq G_A$ there is an $a \in A$ with $g^{-1}(a) - a \neq 0$, so there is a $b \in K$ with $b(g^{-1}(a) - a) = 1$. Let $a_1 = a$, $a_2 = 1$, $y_1 = -b$, $y_2 = bg^{-1}(a)$. For any $h \in G_A$, since $h(a) = a$ for all $a \in A$,

$$\sum_{i=1}^{2} a_i h(y_i) = h\left(\sum_{i=1}^{2} a_i y_i \right) = h(1) = 1.$$

Also

$$\sum_{i=1}^{2} a_i g(y_i) = g\left(\sum_{i=1}^{2} g^{-1}(a_i)y_i\right) = g(-bg^{-1}(a) + bg^{-1}(a)) = 0.$$

Now let S and S' be any two subsets of G containing G_A for which there are elements $a_1 \cdots a_n \in A$, $a_1' \cdots a_m' \in A$; $y_1 \cdots y_n \in K$, $y_1' \cdots y_m' \in K$ so that for all $g \in S$, $g' \in S'$

$$\sum_{i=1}^{n} a_i g(y_i) = \begin{cases} 1 & g \in G_A \\ 0 & g \notin G_A \end{cases}$$

$$\sum_{j=1}^{m} a_j' g'(y_j') = \begin{cases} 1 & g' \in G_A \\ 0 & g' \notin G_A. \end{cases}$$

Then for any $g \in S \cup S'$

$$\sum_{i,j=1}^{n,m} a_i a_j' g(y_i y_j') = \left(\sum_{i=1}^{n} a_i g(y_i)\right)\left(\sum_{j=1}^{m} a_j' g(y_j')\right) = \begin{cases} 1 & g \in G_A \\ 0 & g \notin G_A. \end{cases}$$

Since $G = \bigcup_{g \in G} G_A \cup \{g\}$ the proposition follows.

COROLLARY 2. *With the hypothesis of Proposition* 1, *let* $g_1, \cdots, g_m$ *be left coset representatives of* G_A *in* G. *If* $k_1, \cdots, k_m$ *are in* K *with*

$$\sum_{j=1}^{m} k_j g_j(b) = 0 \quad \textit{for all} \quad b \in A$$

then $k_j = 0$ $(j = 1, \cdots, m)$.

Proof: Let $a_1 \cdots a_n$, $y_1 \cdots y_n$ be the elements satisfying the conclusion of proposition 1. Then for each j

$$\sum_{i=1}^{n} g_j(a_i)y_i = g_j\left(\sum_{i=1}^{n} a_i g_j^{-1}(y_i)\right) = \begin{cases} 1 & g_j \in G_A \\ 0 & g_j \notin G_A. \end{cases}$$

Let $y_i' = g_p(y_i)$ for some p, $1 \leqq p \leqq m$, then for each j,

$$\sum_{i=1}^{n} g_j(a_i)y_i' = g_j\left(\sum_{i=1}^{n} a_i g_j^{-1}(g_p(y_i)\right) = \begin{cases} 1 & p = j \\ 0 & p \neq j. \end{cases}$$

Thus for each p, $1 \leqq p \leqq m$, if $\sum_{j=1}^{m} k_j g_j(a) = 0$ then

$$0 = \sum_{i=1}^{m} \sum_{j=1}^{m} k_j g_j(a_i)y_i' = \sum_{j=1}^{m} k_j\left(\sum_{i=1}^{n} g_j(a_i)y_i'\right) = k_p$$

which proves the corollary.

REMARK: This corollary is a special case of a theorem of Artin (see for example Theorem 12 of [**1**]).

COROLLARY 3. *Let K be a field and let G be a finite group of automorphisms of K, then there is an element $c \in K$ such that $\sum_{g\in G} g(c)=1$.*

Proof: By Corollary 2, (with $G_A=\{e\}$ and $A=K$) there exists $b\in K$ with $x=\sum_{g\in G} g(b)\neq 0$. It is easy to check that $x\in K^G$ so $x^{-1}\in K^G$ and

$$1 = x^{-1}x = x^{-1}\sum_{g\in G} g(b) = \sum_{g\in G} g(x^{-1}b).$$

We observe that K can be viewed as a vector space over K^G and that A is a subspace. A mapping T from A into K is called a K^G linear transformation in case $T(x+y)=T(x)+T(y)$ and $T(bx)=bT(x)$ for all $x, y\in A$, $b\in K^G$. Observe that every element of G can be viewed via restriction as a K^G linear transformation from A into K.

LEMMA 4. *Let G be a finite group of automorphisms of the field K and let A be a subfield of K containing K^G. Let T be a K^G linear transformation from A into K and let $g_1 \cdots g_m$ be left coset representatives of G_A in G. Then there exist unique elements $k_1 \cdots k_m$ in K so that*

$$T(x) = \sum_{j=1}^{m} k_j g_j(x) \quad \text{for all} \quad x \in A.$$

Proof: By Proposition 1 there exist $a_1 \cdots a_n \in A$, and $y_1 \cdots y_n \in K$ such that for all $g\in G$,

$$\sum_{i=1}^{n} a_i g(y_i) = \begin{cases} 1 & g \in G_A \\ 0 & g \notin G_A. \end{cases}$$

Let $g_1 \cdots g_m$ be a set of left coset representatives of G_A in G; and let $c\in K$ with $\sum_{h\in G_A} h(c)=1$, such an element exists by Corollary 3. We observe that for any $a\in A$

$$\begin{aligned} T(a) &= \sum_{i=1}^{n} \sum_{h\in G_A} T(a_i h(y_i c a)) = \sum_{i=1}^{n} \sum_{g\in G} T(a_i g(y_i c a)) \\ &= \sum_{i=1}^{n} \sum_{g\in G} T(a_i) g(y_i c a) \quad \left(\text{since} \sum_{g\in G} g(x) \in K^G\right) \\ &= \sum_{g\in G} \left(\sum_{i=1}^{n} T(a_i) g(y_i c)\right) g(a). \end{aligned}$$

If we let $k_j=\sum_{g\in g_j G_A} \sum_{i=1}^{n} T(a_i)g(y_i c)$ then $T(a)=\sum_{j=1}^{m} k_j g_j(a)$, for all $a\in A$.

To prove uniqueness of the k_j assume

$$\sum_{j=1}^{m} k_j g_j(x) = \sum_{j=1}^{m} k_j' g_j(x) \quad \text{for all} \quad x \in A.$$

Then $\sum_{j=1}^{m}(k_j-k_j')g_j(x)=0$ for all $x\in A$. Thus $k_j-k_j'=0$, for all j by Corollary 2. This proves the lemma.

We now give information about the automorphisms of a field K.

THEOREM 5. *Let G be a finite group of automorphisms of the field K and let A be a subfield of K containing K^G. If τ is an isomorphism from A into K leaving K^G element-wise fixed, then τ is the restriction of some element $g \in G$.*

Proof. One quickly checks that τ is a K^G linear transformation from A to K so by Lemma 4 there exist $k_1 \cdots k_m \in K$ so that if $g_1 \cdots g_m$ are left coset representatives of G_A in G then for any $a \in A$

$$\tau(a) = \sum_{j=1}^{m} k_j g_j(a).$$

If $a, b \in A$ then $\tau(a)\tau(b) = \tau(a) \sum_{j=1}^{m} k_j g_j(b)$ and $\tau(ab) = \sum_{j=1}^{m} k_j g_j(a) g_j(b)$. So for every $b \in A$

$$\sum_{j=1}^{m} (k_j\tau(a) - k_j g_j(a)) g_j(b) = 0.$$

By Corollary 1 this implies $k_j\tau(a) - k_j g_j(a) = 0$ for all $a \in A$. Since $\tau \neq 0$ some $k_j \neq 0$. For this j we have $\tau(a) = g_j(a)$, proving the theorem.

THEOREM 6 (Fundamental theorem). *Let K be a field and let G be a finite group of automorphisms of K. Then there is a one to one correspondence between the subfields A of K containing K^G and the subgroups H of G given by $H \to K^H$.*

The inverse of the correspondence is $A \to G_A$. There is a finite group of automorphisms of A which leaves K^G fixed if and only if G_A is a normal subgroup of G. In this case the group of automorphisms of A which leaves K^G fixed is isomorphic to the factor group G/G_A.

Proof: Let A be a subfield of K containing K^G. By Proposition 1 there exists $a_1 \cdots a_n \in A$ and $y_1 \cdots y_n \in K$ so that

$$\sum_{i=1}^{n} a_i g(y_i) = \begin{cases} 1 & g \in G_A \\ 0 & g \notin G_A. \end{cases}$$

By Corollary 3 there is a $c \in K$ with $\sum_{h \in G_A} h(c) = 1$. Let x be an element in the fixed field of G_A, then

$$\begin{aligned} x = \sum_{h \in G_A} h(c)x &= \sum_{h \in G_A} h(cx) \\ &= \sum_{h \in G_A} \sum_{i=1}^{n} a_i h(y_i cx) \\ &= \sum_{g \in G} \sum_{i=1}^{n} a_i g(y_i cx) \\ &= \sum_{i=1}^{n} a_i \left(\sum_{g \in G} g(y_i cx) \right) \in A \end{aligned}$$

thus the correspondence $H \to K^H$ is onto.

If H and H' are subgroups of G with $K^H = K^{H'}$ then $H = H'$ by Theorem 5 (with $A = K$, $G = H$). Thus the correspondence $H \to K^H$ is both one to one and onto. Our argument also proves the second statement of the theorem.

For the last assertions assume G_A is a normal subgroup of G. Then $g^{-1}G_A g = G_A$ for all $g \in G$ so if $g|_A$ denotes the restriction of the automorphism $g \in G$ to an isomorphism of A into K it must be the case that $g|_A$ is an automorphism of A. The map $g \to g|_A$ then induces a homomorphism from the group G onto the group G/G_A so G/G_A may be viewed as a group of automorphisms of A whose fixed field is K^G.

Conversely if $\bar{G}$ is a group of automorphisms of A then by Theorem 5 the elements of $\bar{G}$ are restrictions of elements in G. The restriction map from G to $\bar{G}$ is a homomorphism whose kernel must be a normal subgroup of G. One easily checks that the kernel of this homomorphism is G_A. This proves the theorem.

We have observed that K may be viewed as a vector space over K^G. Let $[K: K^G]$ be the dimension of K over K^G. Let $[G: 1]$ be the order of the group G. Using some elementary facts from linear algebra we conclude by showing

COROLLARY 7. *Let G be a finite group of automorphisms of the field K and let K^G be the subfield of K left elementwise fixed by the automorphisms in G, then* $[G: 1] = [K: K^G]$. (See also Theorem 14 of [1].)

Proof: In proposition 1 let $A = K$, one can check that if $x \in K$ then $\sum_{g\in G} g(y_i x)$ is in K^G and

$$x = \sum_{i=1}^{n} a_i \left(\sum_{g\in G} g(y_i x) \right).$$

Thus the elements $a_1, \cdots, a_n$ form a generating set for K as a vector space over K^G and $[K: K^G]$ is finite.

One turns the set of K^G linear transformations on K into a K^G vector space by defining $(T+Q)x = T(x) + Q(x)$ and $(a \cdot T)(x) = aT(x)$ for all $x \in K$, $a \in K^G$ and all K^G linear transformations T and Q. By Lemma 4 (with $A = K$), every K^G linear transformation T on K is of the form $T(x) = \sum_{g\in G} k_g g(x)$ for uniquely determined $k_g \in K$.

Thus one can check that if $k_1, \cdots, k_m$ is a K^G basis for K then the set $\{k_i g\}_{i=1}^{m} (g \in G)$ forms a K^G basis for the space of all linear transformations. The number of elements in this basis is $[G:1][K:K^G]$. On the other hand it is well known that the dimension of the space of all K^G linear transformations of K is $[K:K^G]^2$ (see, for example, page 219 of [3]). We conclude that $[G:1][K \cdot K^G] = [K \cdot K^G]^2$. So $[G \cdot 1] = [K:K^G]$, proving the corollary.

References

1. E. Artin, Galois Theory, Notre Dame Mathematical Lectures, No. 2 (1959).

2. S. Chase, D. Harrison, and A. Roseberg, Galois Theory and Galois Cohomology of Commutative Rings. Mem. of the Am. Math. Soc., No. 52 (1965).

3. I. N. Herstein, Topics in Algebra, Blaisdell, Waltham, Mass., 1964.

4. M. Kriemer, Galois theory for non-commutative rings, Trans. Am. Math. Soc., 127 (1967) 29–41.

5. Merle M. Manis, Valuations on Commutative Rings. Ph.D. Thesis, U. of Oregon, 1966.

THE SMALLEST NUMBER WITH A GIVEN NUMBER OF DIVISORS

M. E. GROST, Michigan State University

1. Introduction. Throughout this paper, all symbols will stand for positive integers, and p_i for the ith prime, while P_i and q_i refer to any primes.

Given $h = q_1 \cdot q_2 \cdots q_n$, with $q_1 \geqq q_2 \geqq \cdots \geqq q_n$, let $A(h)$ be the smallest number with h divisors. In many cases,

$$A(h) = 2^{q_1-1} \cdot 3^{q_2-1} \cdots p_n^{q_n-1}. \tag{1}$$

The primary objective of this paper is to determine the exceptions to (1).

In Section 2 we derive lemmas giving necessary conditions on $A(h)$, and in Section 3 we determine the exceptions to (1) for the cases $1 \leqq n \leqq 6$. Section 4 is devoted to a formula for $A(h)$ when h is a prime power.

A number is minimal when there is no smaller number with the same number of divisors. In Section 5 we test both $N!$ and the least common multiple of the numbers $1, 2, \cdots, N$ for minimality with the surprising result that both are minimal for only finitely many values N.

Ramanujan [2] investigated extensively what he called highly composite numbers, which are related to our minimal numbers in the sense that every highly composite number is minimal, but not vice versa. The concept of highly composite numbers was generalized by Alaoglu and Erdös [1].

2. The problem and some lemmas. Our problem is to find the smallest number x such that $\tau(x) = h$, where h is given and $\tau(y)$ is the number of divisors of y. We will denote this smallest number by $A(h)$. As is well known, if $y = P_1^{e_1} \cdots P_n^{e_n}$ in canonical representation, then $\tau(y) = (e_1+1)(e_2+1) \cdots (e_n+1)$, so $A(h)$ must be of the form

$$P_1^{a_1-1} \cdot P_2^{a_2-1} \cdots P_k^{a_k-1} \quad \text{where} \quad a_1 \cdot a_2 \cdots a_k = h.$$

LEMMA 1. *Let $b_1 \geqq b_2 \geqq \cdots \geqq b_k$, and the c_j's some permutation of the b_i's. Then the smallest number of the form $2^{c_1} \cdot 3^{c_2} \cdots p_k^{c_k}$ is $2^{b_1} 3^{b_2} \cdots p_k^{b_k}$.*

Proof. Assume the minimum is of the form $M = 2^{c_1} \cdot 3^{c_2} \cdots p_k^{c_k}$ where there exist I, J such that $I > J$ and $c_I > c_J$. Then since $p_I^{c_I} p_J^{c_J} > p_I^{c_J} p_J^{c_I}$, $M = 2^{c_1} 3^{c_2} \cdots p_J^{c_J} \cdots p_I^{c_I} \cdots p_k^{c_k} > 2^{c_1} \cdot 3^{c_2} \cdots p_J^{c_I} \cdots p_I^{c_J} \cdots p_k^{c_k}$; hence, M is not minimal and we have a contradiction. Therefore, $2^{b_1} \cdot 3^{b_2} \cdots p_k^{b_k}$ is the minimum.

LEMMA 2. $P_1^{AB-1} > P_1^{A-1} P_2^{B-1}$ *if* $A > 1$, $B > 1$, *and* $P_1^A > P_2$.

Proof. $P_1^{AB-1} > P_1^{A-1} P_2^{B-1}$ if and only if $P_1^{A(B-1)} > P_2^{(B-1)}$, which is true if $B > 1$ and $P_1^A > P_2$.

LEMMA 3. *$A(h)$ is of the form $2^{a_1-1} 3^{a_2-1} \cdots p_k^{a_k-1}$ where $a_1 \geqq a_2 \geqq \cdots \geqq a_k$, $a_1 \cdot a_2 \cdots a_k = h$ and where for all i, and for all proper divisors A of a_i, we have $p_i^A < p_{k+1}$.*

Proof. The canonical form of $A(h)$ stems from the previous remarks and the observation that if $A(h)$ is to be minimal its prime factors must obviously be minimal. Lemma 1 guarantees that $a_1 \geqq a_2 \geqq \cdots \geqq a_k$ in order that $2^{a_1-1}3^{a_2-1} \cdots p_k^{a_k-1}$ be minimal. And, if there exists an i such that $p_i^A > p_{k+1}$, with $A>1$, $B>1$, $AB=a_i$, then by Lemma 2, $x=2^{a_1-1} \cdots p_i^{a_i-1} \cdots p_k^{a_k-1} > 2^{a_1-1} \cdots p_i^{A-1} \cdots p_k^{a_k-1} p_{k+1}^{B-1} = y$, with $\tau(x)=\tau(y)$; hence, x would not be minimal, which is a contradiction.

3. The values of $A(q_1 \cdot q_2 \cdots q_n)$, when $n \leqq 6$. It will be convenient to set $h=q_1q_2 \cdots q_n$, where $q_1 \geqq q_2 \geqq \cdots \geqq q_n$ in order to investigate this problem. We will set $A(h)=2^{a_1-1}3^{a_2-1} \cdots p_k^{a_k-1}$ and draw conclusions on the a_i using Lemma 3, with various values of n. If $A(h)=2^{q_1-1}3^{q_2-1} \cdots p_n^{q_n-1}$, then we will denote h *ordinary*, otherwise h is *exceptional.*

I. *Cases* $n=1$, 2.

The case $n=1$ is trivial and is included here for completeness.

THEOREM 1. *If $n=1$, h is ordinary, that is, $A(h)=2^{q_1-1}$. If $n=2$, h is ordinary, that is, $A(h)=2^{q_1-1}3^{q_2-1}$.*

Proof. If $a_1=q_1 \cdot q_2$, then $2^{q_1} \geqq 2^2 > 3 = p_{k+1}$, which contradicts Lemma 3; hence $a_1=q_1 \geqq q_2=a_2$ and h is ordinary.

II. *Cases* $n=3$, 4.

THEOREM 2. *If $n=3$ or 4, h is ordinary except for the values $h=8$, 16, and* 24 (see table).

Proof. Let $n=4$. If $a_1=AB$, $A \geqq 3$, $B \geqq 2$, then $2^A \geqq 2^3 > 7 \geqq p_{k+1}$, contradicting Lemma 3, so a_1 is either prime or 4. If $i \geqq 2$, $a_i=AB$, $A \geqq 2$, $B \geqq 2$, then $p_i^A \geqq p_i^2 \geqq 3^2 > 7 \geqq p_{k+1}$, again contradicting Lemma 3; hence a_i is prime when $i \geqq 2$. Therefore, $A(h)$ must be one of the following forms:

$$2^{4-1}3^{q_1-1}5^{q_2-1},\ 3 \geqq q_1 \geqq q_2 \geqq 2,\ h = q_1q_2 \cdot 4, \tag{2}$$

$$2^{q_1-1}3^{q_2-1}5^{q_3-1}7^{q_4-1} \tag{3}$$

the latter being the ordinary case.

Since $A(h)$ is of form (2) in only finitely many cases, it can be investigated directly, producing the result given. Hence, all other h are ordinary, which completes the proof.

When $n=3$, $A(h)$ can be investigated similarly, h being ordinary except for the value $h=8$, (see table).

III. *Cases* $n=5$, 6.

THEOREM 3. *If $n=5$ or 6, h is ordinary except for the values listed in the table.*

Proof. Let $n=6$. If $a_1=AB$, $A \geqq 4$, $B \geqq 2$, then $2^A \geqq 2^4 > 13 \geqq p_{k+1}$, contradicting Lemma 3. If $a_2=AB$, $A \geqq 3$, $B \geqq 2$, then $3^A \geqq 3^3 > 23 > p_{k+1}$, again contradict-

TABLE. Exceptional Values for $n \leqq 6$

n	h	$A(h)$
3	2^3	$2^3 \cdot 3$
4	2^4	$2^3 \cdot 3 \cdot 5$
	$3 \cdot 2^3$	$2^3 \cdot 3^2 \cdot 5$
5	2^5	$2^3 \cdot 3 \cdot 5 \cdot 7$
	$3 \cdot 2^4$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
	$3^2 \cdot 2^3$	$2^5 \cdot 3^2 \cdot 5 \cdot 7$
	$3^3 \cdot 2^2$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7$
	$3^4 \cdot 2$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7^2$
	3^5	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^2$
	$q_1 \cdot 2^4$, $q_1 \geqq 5$	$2^{q_1-1} \cdot 3^3 \cdot 5 \cdot 7$
6	2^6	$2^3 \cdot 3^3 \cdot 5 \cdot 7$
	$3 \cdot 2^5$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
	$3^2 \cdot 2^4$	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
	$3^3 \cdot 2^3$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
	$3^4 \cdot 2^2$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11$
	$3^5 \cdot 2$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11$
	3^6	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2$
	$5 \cdot 3^5$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2$
	$q_1 \cdot 2^5$, $q_1 \geqq 5$	$2^{q_1-1} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$

ing Lemma 3. If $i \geqq 3$, $a_i = AB$, $A \geqq 2$, $B \geqq 2$, then $p_i^A \geqq p_i^2 \geqq 5^2 > 23 > p_{k+1}$, contradicting Lemma 3. Therefore, $A(h)$ must be one of the following forms, where the Q_i are the q_j in some order:

(4) $\quad 2^{a_1-1}3^{4-1}5^{Q_1-1}7^{Q_2-1}$, $a_1 = 4, 6$, or 9, $3 \geqq Q_1 \geqq Q_2$,

(5) $\quad 2^{a_1-1}3^{Q_1-1}5^{Q_2-1}7^{Q_3-1}11^{Q_4-1}$, $a_1 = 4, 6$, or 9, $a_1 > Q_1 \geqq Q_2 \geqq Q_3 \geqq Q_4$,

(6) $\quad 2^{q_1-1}3^{4-1}5^{q_2-1}7^{q_3-1}11^{q_4-1}$, $q_1 \geqq 5$, $3 \geqq q_2 \geqq q_3 \geqq q_4 \geqq 2$,

(7) $\quad 2^{q_1-1}3^{q_2-1}5^{q_3-1}7^{q_4-1}11^{q_5-1}13^{q_6-1}$,

the latter being the ordinary case.

Cases (4) and (5) are the form of $A(h)$ for only finitely many values h, and can be investigated directly. In order that in Case (6) h be exceptional, the ratio:

$$\frac{2^{q_1-1}3^{q_2-1}5^{q_3-1}7^{q_4-1}11^{2-1}13^{2-1}}{2^{q_1-1}3^{4-1}5^{q_2-1}7^{q_3-1}11^{q_4-1}} = \frac{13}{3^{4-q_2}5^{q_2-q_3}7^{q_3-q_4}11^{q_4-2}}$$

must be greater than 1, that is, $q_2 = q_3 = q_4 = 2$. These values $h = q_1 \cdot 2^5$ are the infinitely many exceptions listed in the table. Hence, all other h are ordinary, which completes the proof.

The proof for $n = 5$ is similar except that form (4) for $A(h)$ need not be considered because $3^2 > 7 = p_{k+1}$, contradicting Lemma 3.

4. The value of $A(h)$, if h is a prime power. Let q be any fixed prime and $G(i, l) = p_i^{(q-1)q^{(l-1)}}$, $i \geqq 1$, $l \geqq 1$. Set the $G(i, l)$ in increasing order, g_j being the jth $G(i, l)$.

THEOREM 4. $A(q^n) = \prod_{j=1}^{n} g_j$.

Proof. If $\tau(Y) = q^n$ then the exponents in the canonical form of Y are $q^{b_1}-1$, $q^{b_2}-1, \cdots, q^{b_k}-1$, where $b_1+b_2+\cdots+b_k = n$. Since $p^{q^b-1} = p^{(q-1)\cdot 1} \cdot p^{(q-1)q} \cdot p^{(q-1)\cdot q^2} \cdots p^{(q-1)q^{b-1}}$, Y is the product of n distinct $G(i, l)$. It is easily seen that $\tau(\prod_{j=1}^{n} g_j) = q^n$, hence, $A(q^n)$ is the product of the n minimal values $G(i, l)$; that is, $A(q^n) = \prod_{j=1}^{n} g_j$, which completes the proof.

For example, the $\{g_j\}$ when $q=2$ are $\{2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23, 25, \cdots\}$, so $A(2^{10}) = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$.

5. Minimal numbers. A given number X is defined to be *minimal* if and only if $A(\tau(X)) = X$, that is, a number is minimal if and only if there is no smaller number with the same number of divisors. If X is minimal, then it must be of the form given in Lemma 3; hence, Lemma 3 provides a negative criterion while Theorems 1–4 are positive ones. In looking for a general class of minimal numbers, we want numbers that have many divisors for their size.

PROBLEM 1. *For what values N is $N!$ minimal?*

Solution. $N!$ is minimal if and only if $1 \leqq N \leqq 7$.

Proof. We are using in this paper one of Chebyshev's results, namely that if $\epsilon > 1/5$, there exists a prime p such that $x < p \leqq (1+\epsilon)x$ whenever $x \geqq x_\epsilon$; it is well known that x_1 can be chosen as 1. One can choose $x_{1/3}$ as 33/4; see J.-A. Serret, Cours d'Algèbre Supérieure, Seventh ed. (1928), Vol. II, pp. 236–239; in particular (3), p. 238.

Let $N \geqq 33$. Then there exists a prime p_i such that $N/4 < p_i \leqq N/3$. The power of p_i contained in $N!$ is equal to $[N/p_i] + [N/p_i^2] + [N/p_i^3] + \cdots$, and since $4 > N/p_i \geqq 3$ and $p_i^J \geqq p_i^2 > N^2/16 > N$ when $J \geqq 2$, the power of p_i contained in $N!$ is exactly 3. In the notation of Lemma 3, let $p_{k+1} > N \geqq p_k$. Then, by Chebyshev's theorem, $p_i^2 > N^2/16 > 2N \geqq p_{k+1}$, contradicting Lemma 3. Hence, $N!$ is not minimal when $N \geqq 33$; the remaining cases can be investigated directly, producing the result given, which completes the proof.

Let $V(N)$ be the least common multiple of the numbers $1, 2, \cdots, N$. Although $V(N)$ satisfies Lemma 3 it is minimal for only 14 values N.

PROBLEM 2. *For what values N is $V(N)$ minimal?*

Solution. $V(N)$ is minimal only for cases $1 \leqq N \leqq 6$, $8 \leqq N \leqq 12$, $N = 16$, 27 and 28.

Proof. A prime P_i is contained exactly to the kth power in $V(N)$ when $P_i^k \leqq N < P_i^{k+1}$, that is, $N^{1/(k+1)} < P_i \leqq N^{1/k}$. By Chebyshev's theorem, (with $\epsilon = 1/3$, $x_\epsilon = 33/4$), there exist primes P, Q and R such that $N^{1/3} < P \leqq (4/3)N^{1/3}$

$\leqq N^{1/2}$, $N^{1/2} < Q \leqq (4/3)N^{1/2} \leqq N$, $N^{1/2} \leqq (3/4)N < R \leqq N$; in order that N satisfies these inequalities as well as the condition $N^{1/3} \geqq 33/4$, it suffices to assume that $N \geqq (33/4)^3$. It follows that P is contained in $V(N)$ exactly to the second power, and Q and R to the first. Note that $N \geqq (33/4)^3 > (4/3)^{18}$, and therefore that $R > (3/4)N \geqq (4/3)N^{1/2} \cdot (4/3)N^{1/3} \geqq PQ$, and so $P^2 \cdot Q \cdot R > P^3Q^2$, each of these two numbers having twelve divisors. Hence, we can replace $P^2 \cdot Q \cdot R$ by P^3Q^2 in $V(N)$, producing a smaller number with the same number of divisors, so $V(N)$ is not minimal when $N \geqq (33/4)^3$. The remaining cases can be investigated directly, producing the result given, which completes the proof.

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References

1. L. Alaoglu and P. Erdös, On highly composite and similar numbers, Trans. Amer. Math. Soc., 56 (1944) 448–469.

2. S. V. Ramanujan, Highly composite numbers, Proc. London Math. Soc., (2) 14 (1915) 347–409.

SERIES WITH SUMS INVARIANT UNDER REARRANGEMENT

C. W. McARTHUR, Florida State University

1. Introduction. In this note it is shown that the terms of a series with invariant sum in a Hausdorff topological group commute with one another. Generalizations of known sufficient conditions for a series to have an invariant sum are given. The question is raised: "What are necessary and sufficient conditions in order that the sum of a series be invariant under rearrangement?" Throughout the note, except when otherwise stated, E will denote a (not necessarily commutative) Hausdorff topological group and $x \circ y$ will represent the group operation between elements x and y of E. When the group is commutative the group operation will be denoted by $+$. Let $\{x_i\}$ denote a sequence in E and let $\sum_{i=1}^{m} x_i = x_1 \circ x_2 \circ \cdots \circ x_m$. Then the series $\sum_{i=1}^{\infty} x_i$ in E is *convergent* to x in E if and only if corresponding to each open set G containing x there exists an integer N such that $\sum_{i=1}^{n} x_i \in G$ if $n \geqq N$. By a *rearrangement* of the series $\sum_{i=1}^{\infty} x_i$ is meant a series $\sum_{i=1}^{\infty} x_{\tau(i)}$ where τ is a permutation of ω, the set of positive integers. A series is said to have an *invariant sum* x if and only if the series converges to x, and, whenever a rearrangement of the series converges, it too converges to x. A series is *conditionally* convergent if it converges but some rearrangement of it does not converge. A series is *unconditionally convergent* if each of its rearrangements converges. B. Riemann [2, pp. 301, 302] has shown that a series of real numbers is unconditionally convergent, or, equivalently, absolutely convergent, if and only if it has an invariant sum. Hadwiger [1]

showed that in Hilbert space there exist series with invariant sum which do not converge unconditionally. The author [3] has shown that in each infinite dimensional Banach space there exist series which have invariant sums but do not converge unconditionally.

2. Invariant sum and commutativity. We state the following straightforward lemmas without proof.

LEMMA 1. *If $\sum_{i=1}^{\infty} x_i$ is a series which converges to x then for each $n \in \omega$, the series $\sum_{i=n+1} x_i$ converges to $(\sum_{i=1}^{n} x_i)^{-1} \circ x$.*

LEMMA 2. *If $\sum_{i=1}^{\infty} x_i$ is a convergent series and τ is a permutation of ω such that $\tau(i) = i$ for all but a finite number of $i \in \omega$ then $\sum_{i=1}^{\infty} x_{\tau(i)}$ also converges.*

THEOREM 1. *If $\sum_{i=1}^{\infty} x_i$ is a series with invariant sum then $x_i \circ x_j = x_j \circ x_i$ for all $i, j \in \omega$.*

Proof. Let x denote the invariant sum and let $i, j \in \omega$ with $i < j$. Let τ be a permutation of ω such that $\tau(k) = k$ for all $k > j$ and $\tau(1) = i$ and $\tau(2) = j$. Let τ' be a permutation of ω such that $\tau'(1) = j$ and $\tau'(2) = i$ and $\tau'(k) = \tau(k)$ for $k \geqq 3$. Thus, $x = \sum_{k=1}^{\infty} x_{\tau(k)} = \sum_{k=1}^{\infty} x_{\tau'(k)}$ using Lemma 2 and the hypothesis of the theorem. By Lemma 1,

$$\begin{aligned} \sum_{k=3}^{\infty} x_{\tau(k)} &= (x_{\tau(1)} \circ x_{\tau(2)})^{-1} \circ x \\ &= x_j^{-1} \circ x_i^{-1} \circ x. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{k=3}^{\infty} x_{\tau(k)} = \sum_{k=3}^{\infty} x_{\tau'(k)} &= (x_{\tau'(1)} \circ x_{\tau'(2)})^{-1} \circ x \\ &= x_i^{-1} \circ x_j^{-1} \circ x. \end{aligned}$$

Hence, $x_j^{-1} \circ x_i^{-1} \circ x = x_i^{-1} \circ x_j^{-1} \circ x$ from which the conclusion follows.

3. Sufficient conditions for invariant sum. Orlicz [4] has given a proof of Theorem 2 for Banach spaces. Theorem 3 below has been noted for Banach spaces [3].

THEOREM 2. *If $\sum_{i=1}^{\infty} x_i$ is an unconditionally convergent series and $x_i \circ x_j = x_j \circ x_i$ for all $i, j \in \omega$ then $\sum_{i=1}^{\infty} x_i$ has an invariant sum.*

Proof. Let τ be a permutation of ω. We shall show that

$$\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} x_{\tau(i)}. \text{ Let } x = \sum_{i=1}^{\infty} x_i.$$

Let $m_1 = 1$. For positive integer n let $[1, n]$ denote the set of positive integers from 1 to n. Thus $[1, m_1] = \{1\}$. Let m_2 denote the smallest positive integer such

that $[1, m_1]$ is a proper subset of $\tau([1, m_2])$. Let m_3 denote the smallest positive integer such that $\tau([1, m_2])$ is a proper subset of $[1, m_3]$ and m_4 the smallest positive integer such that $[1, m_3]$ is a proper subset of $\tau([1, m_4])$. Proceeding in this manner we obtain a strictly increasing sequence of positive integers $\{m_n\}$ such that $[1, m_{2k-1}]$ is a proper subset of $\tau([1, m_{2k}])$ which is in turn a proper subset of $[1, m_{2k+1}]$, $k=1, 2, \cdots$. Note that the cardinality of $\tau([1, m_{2k}])$ is m_{2k} since τ is 1-1. Let τ' denote a permutation of ω constructed by using the integers in $[1, m_1]$ followed by the integers $\tau([1, m_2])-[1, m_1]$ in some order followed by the integers $[1, m_3]-\tau([1, m_2])$ in some order etc. Thus $\tau'([1, m_{2k-1}])$ consists of the integers $[1, m_{2k-1}]$ in some order so that

$$\sum_{i=1}^{m_{2k-1}} x_{\tau'(i)} = \sum_{i=1}^{m_{2k-1}} x_i, \qquad k = 1, 2, \cdots,$$

because of the commutativity of the elements x_i, x_j. Also, $\tau'([1, m_{2k}])$ consists of the integers $\tau([1, m_{2k}])$ in some order so that

$$\sum_{i=1}^{m_{2k}} x_{\tau'(i)} = \sum_{i=1}^{m_{2k}} x_{\tau(i)}.$$

Now

$$x = \sum_{i=1}^{\infty} x_i = \lim_{k\to\infty} \sum_{i=1}^{m_{2k-1}} x_i = \lim_{k\to\infty} \sum_{i=1}^{m_{2k-1}} x_{\tau'(i)}.$$

By hypothesis $\lim_{n\to\infty} \sum_{i=1}^{n} x_{\tau'(i)}$ exists. Thus,

$$x = \lim_{k\to\infty} \sum_{i=1}^{m_{2k-1}} x_{\tau'(i)} = \lim_{k\to\infty} \sum_{i=1}^{m_{2k}} x_{\tau'(i)} = \lim_{k\to\infty} \sum_{i=1}^{m_{2k}} x_{\tau(i)} = \sum_{i=1}^{\infty} x_{\tau(i)}.$$

THEOREM 3. *If $\sum_{i=1}^{\infty} x_i$ is a convergent series in a locally convex Hausdorff space E such that $\sum_{i=1} |f(x_i)| < +\infty$ for every continuous linear functional f on E, then $\sum_{i=1}^{\infty} x_i$ has an invariant sum.*

A proof of Theorem 3 has been given by the author [3, Lemma 4.1]. It is known that a series in a weakly sequentially complete locally convex space satisfies the hypothesis of Theorem 3 if and only if the series is unconditionally convergent. However, it has been shown [3, Theorem 4.9] that in the space c_0 a conditionally convergent series satisfies the hypothesis of Theorem 3.

References

1. H. Hadwiger, Über die Konvergenzarten unendlicher Reihen in Hilbertschen Raum, Math. Zeit., 47 (1941) 325–329.

2. E. Hille, Analysis, Vol I, Blaisdell, New York, 1964.

3. C. W. McArthur, On relationships amongst certain spaces of sequences in an arbitrary Banach space, Canad. J. Math., 8 (1956) 192–197.

4. W. Orlicz, Beiträge zur Theorie der Orthogonalentwicklungen II, Studia Math., 1 (1929) 241–255.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

J. H. McKAY, Oakland University

The following results of the twenty-eighth William Lowell Putnam Mathematical Competition held on December 2, 1967, have been determined in accordance with the regulations governing the Competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of **Michigan State University,** East Lansing, Michigan. The members of the team were Allen J. Beadle, Steven C. Ferry, and Michael E. Grost; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of **California Institute of Technology,** Pasadena, California. The members of the team were Gregory S. Harkness, Jeffrey S. Leon, and Allen J. Schwenk; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of **Harvard University,** Cambridge, Massachusetts. The members of the team were Marshall W. Buck, Irwin Gaines, and Neal I. Koblitz; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of **Massachusetts Institute of Technology,** Cambridge, Massachusetts. The members of the team were Gerald S. Gras, Richard C. Schroeppel, and Robert S. Winternitz; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of **The University of Michigan,** Ann Arbor, Michigan. The members of the team were Robert F. Brammer, Robert L. Scott, and Paul E. Weiss; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are **David R. Haynor,** Harvard University; **Dennis A. Hejhal,** University of Chicago; **Peter L. Montgomery,** University of California, Berkeley; **Richard C. Schroeppel,** Massachusetts Institute of Technology; and **Don B. Zagier,** Massachusetts Institute of Technology. Each of these have been designated as Putnam Fellows by the Mathematical Association of America and are awarded a prize of seventy-five dollars.

The five persons ranking second highest in the examination, named in alphabetical order, are *Marshall W. Buck*, Harvard University; *Michael E. Grost*, Michigan State University; *Neal I. Koblitz*, Harvard University; *Jonathan D. Melvin*, Yale University; and *William F. Wilson*, Princeton University. To each of these a prize of thirty-five dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: *University of California, Berkeley*, the members of the team were Stephen T. Fisk, Arthur A. Mirin and Peter L. Montgomery; *The Cooper Union*, the mem-

bers of the team were Robert T. Baumel, Curtis N. Browne and Harry Ploss; *Oberlin College*, the members of the team were Gerald R. Butters, Douglas C. Ravenel and Roger A. Smith; *McGill University*, the members of the team were David Berengut, Peter M. Doubilet and Steven L. Tanny; *University of Toronto*, the members of the team were Bruce M. Amos, Edward Bierstone and James M. Kavanagh.

Honorable mention is given to the following thirty individuals, named in alphabetical order: Michael A. Amling, *University of California, Santa Barbara*; Bruce M. Amos, *University of Toronto*; Daniel A. Asimov, *Massachusetts Institute of Technology*; Robert T. Baumel, *The Cooper Union*; Allen J. Beadle, *Michigan State University*; Alan R. Beale, *Rice University*; Stephen P. Chilton, *Brown University*; Frederick V. Favor, *University of Illinois*; Steven C. Ferry, *Michigan State University*; William J. Gordon, *Columbia University*; Gordon E. Gullahorn, *Stanford University*; Gregory S. Harkness, *California Institute of Technology*; Ronald A. Hunsinger, *Humboldt State College*; Jacques LaBelle, *Université de Montréal*; Douglas A. Lind, *University of Virginia*; George Markowsky, *Columbia University*; John C. Mather, *Swarthmore College*; Robert L. Mercer, *University of New Mexico*; John P. Monahan, *Princeton University*; Mason S. Osborne, *University of Washington*; Rodger E. Poore, *Carleton College*; Stanley Rabinowitz, *Polytechnic Institute of Brooklyn*; Gary L. Russell, *University of Rochester*; S. David Shapiro, *Harvard University*; George Sicherman, *Harvard University*; Roger A. Smith, *Oberlin College*; Robert D. Trent, *University of Kentucky*; Adrian R. Wadsworth, *Yale University*; Robert J. Weber, *Princeton University*; and Gregg J. Zuckerman, *Yale University*.

The other 62 individuals who were ranked in the top one hundred, arranged by college, are: Mark A. Peterson, *Amherst College;* William G. Dwyer, *Boston College;* Robert Epp, *University of British Columbia;* Nancy A. Evraets, Michael L. Ruchlis, *Brooklyn College;* Kenneth A. Ribet, *Brown University;* Arthur A. Mirin, *University of California, Berkeley;* Jeffrey S. Leon, James A. Maiorana, Allen J. Schwenk, *California Institute of Technology;* Kenneth Astbury, *Carnegie-Mellon University;* Richard W. Rose, *Chico State College;* Robert M. Ephraim, Louis H. Rowen, *Columbia University;* Donald R. Smith, *Cornell University;* J. Lawrence Carter, *Dartmouth College;* Daniel E. Frohardt, *Grinnell College;* Avner D. Ash, Peter M. Winkler, *Harvard University;* Peter A. Bloniarz, *College of the Holy Cross;* Agnis Kaugers, *Kalamazoo College;* Walter R. Stromquist, *University of Kansas;* Paul R. Dippolito, *Kenyon College;* Joseph D. Horton, *University of Manitoba;* Charles E. Blair, Daniel C. Galehouse, Gerald S. Gras, Mark L. Green, John J. Keary, Jeffery C. Lagaria, Alan S. Pollack, Michael Speciner, *Massachusetts Institute of Technology;* Peter M. Doubilet, *McGill University;* Brian C. Smith, *McMaster University;* Robert F. Brammer, James A. Reeds, Robert L. Scott, *University of Michigan;* Alan C. Stickney, *Michigan State University;* Pierre Bouchard, *Université de Montréal;* Simon I. Aloff, *New York University;* John M. Masley, James H. Mulflur, *University of Notre Dame;* Douglas C. Ravenel, *Oberlin College;* David V. Martin, *Ohio State University;* Richard F. Lary, *Polytechnic Institute of Brooklyn;* Brian M. Scott, John E. Shapard, *Pomona College;* Shelby J. Haberman, Steven H. Weintraub, *Princeton University;* Stephen J. Spindler, *Purdue University;* William G. Doubleday, *Queen's University;* Richard E. Crandall, *Reed College;* John W. Morgan, *Rice University;* Lansing J. Sloan, *South Dakota School of Mines;* Alan R. Siegel, Robert A. Van Wesep, *Stanford University;* Ronald R. Miller, *University of Toronto;* Richard L. Ferch, *University of Waterloo;* Michael G. Wasserman, *Williams College;* William L. Hibbard, *University of Wisconsin;* Gregory L. Cherlin, Dean G. Huffman, *Yale University.*

One thousand five hundred and ninety-two students from two hundred and eighty-six colleges and universities participated in the examination on December 2, 1967.

A listing of the top five hundred contestants may be obtained from the Director. The list, which includes addresses and expected dates of graduation, may be helpful to departments of mathematics in selecting graduate students.

The Questions Committee, consisting of R. E. Greenwood (chairman), N. D. Kazarinoff, and Leo Moser, prepared the problems (listed below) for the competition.

PROBLEMS. PART A

A-1. Let $f(x)=a_1\sin x+a_2\sin 2x+\cdots+a_n\sin nx$, where $a_1, a_2, \cdots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)|\leqq|\sin x|$ for all real x, prove that

$$|a_1+2a_2+\cdots+na_n|\leqq 1.$$

A-2. Define S_0 to be 1. For $n\geqq 1$, let S_n be the number of $n\times n$ matrices whose elements are nonnegative integers with the property that $a_{ij}=a_{ji}$, $(i,j=1, 2, \cdots, n)$ and where $\sum_{i=1}^n a_{ij}=1$, $(j=1, 2, \cdots, n)$. Prove

$$\text{(a) } S_{n+1}=S_n+nS_{n-1},$$

$$\text{(b) } \sum_{n=0}^{\infty} S_n\frac{x^n}{n!}=\exp(x+x^2/2), \qquad \text{where } \exp(x)=e^x.$$

A-3. Consider polynomial forms ax^2-bx+c with integer coefficients which have two distinct zeros in the open interval $0<x<1$. Exhibit with a proof the least positive integer value of a for which such a polynomial exists.

A-4. Show that if $\lambda>\frac{1}{2}$ there does not exist a real-valued function u such that for all x in the closed interval $0\leqq x\leqq 1$, $u(x)=1+\lambda\int_x^1 u(y)\,u(y-x)\,dy$.

A-5. Show that in a convex region in the plane whose boundary contains at most a finite number of straight line segments and whose area is greater than $\pi/4$ there is at least one pair of points a unit distance apart.

A-6. Given real numbers $\{a_i\}$ and $\{b_i\}$, $(i=1, 2, 3, 4)$, such that $a_1b_2-a_2b_1\neq 0$. Consider the set of all solutions (x_1, x_2, x_3, x_4) of the simultaneous equations

$$a_1x_1+a_2x_2+a_3x_3+a_4x_4=0 \text{ and } b_1x_1+b_2x_2+b_3x_3+b_4x_4=0,$$

for which no x_i $(i=1, 2, 3, 4)$ is zero. Each such solution generates a 4-tuple of plus and minus signs (signum x_1, signum x_2, signum x_3, signum x_4).

(a) Determine, with a proof, the maximum number of distinct 4-tuples possible.

(b) Investigate necessary and sufficient conditions on the real numbers $\{a_i\}$ and $\{b_i\}$ such that the above maximum number of 4-tuples is attained.

PART B

B-1. Let $(ABCDEF)$ be a hexagon inscribed in a circle of radius r. Show that if $\overline{AB}=\overline{CD}=\overline{EF}=r$, then the midpoints of $\overline{BC}$, $\overline{DE}$, $\overline{FA}$ are the vertices of an equilateral triangle.

B-2. Let $0\leqq p\leqq 1$ and $0\leqq r\leqq 1$ and consider the identities

$$\text{(a) } (px+(1-p)y)^2=Ax^2+Bxy+Cy^2,$$

$$\text{(b) } (px+(1-p)y)(rx+(1-r)y)=\alpha x^2+\beta xy+\gamma y^2.$$

Show that (with respect to p and r)

(a) $\max\{A, B, C\} \geqq 4/9$,
(b) $\max\{\alpha, \beta, \gamma\} \geqq 4/9$.

B-3. If f and g are continuous and periodic functions with period 1 on the real line, then $\lim_{n\to\infty} \int_0^1 f(x)\, g(nx)\, dx = (\int_0^1 f(x)dx)(\int_0^1 g(x)dx)$.

B-4. (a) A certain locker room contains n lockers numbered $1, 2, 3, \cdots, n$ and all are originally locked. An attendant performs a sequence of operations $T_1, T_2, \cdots, T_n$ whereby with the operation T_k, $1 \leqq k \leqq n$, the condition of being locked or unlocked is changed for all those lockers and only those lockers whose numbers are multiples of k. After all the n operations have been performed it is observed that all lockers whose numbers are perfect squares (and only those lockers) are now open or unlocked. Prove this mathematically.

(b) Investigate in a meaningful mathematical way a procedure or set of operations similar to those above which will produce the set of cubes, or the set of numbers of the form $2m^2$, or the set of numbers of the form m^2+1, or some nontrivial similar set of your own selection.

B-5. Show that the sum of the first n terms in the binomial expansion of $(2-1)^{-n}$ is $\frac{1}{2}$, where n is a positive integer.

B-6. Let f be a real-valued function having partial derivatives and which is defined for $x^2+y^2 \leqq 1$ and is such that $|f(x, y)| \leqq 1$. Show that there exists a point (x_0, y_0) in the interior of the unit circle such that

$$\left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^2 \leqq 16.$$

SOLUTIONS. PART A

The number in parentheses, immediately following the problem number, is the number of participants who received a score of 8, 9 or 10 on the problem. The maximum possible score on a problem is 10.

A-1 (310)

$$\left| a_1 + 2a_2 + \cdots + na_n \right| = |f'(0)| = \lim_{x\to 0}\left|\frac{f(x) - f(0)}{x}\right|$$
$$= \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right|\cdot\left|\frac{\sin x}{x}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \leqq 1.$$

A-2 (35) S_n is the number of symmetric $n\times n$ permutation matrices (a permutation matrix has exactly one 1 in each row and column with 0's elsewhere). Let the 1 in the first row be in the kth column. If $k=1$, then there are S_{n-1} ways to complete the matrix. If $k\neq 1$ then $a_{1k}=a_{k1}=1$ and the deletion of the 1st and kth rows and columns leaves a symmetric $(n-2)\times(n-2)$ permutation matrix. Consequently $S_n = S_{n-1} + (n-1)S_{n-2}$.

For part (b), let

$$F(x) = \sum_{n=0}^{\infty}\left\{S_n \frac{x^n}{n!}\right\}. \quad F'(x) = \sum_{n=1}^{\infty}\left\{S_n \frac{x^{n-1}}{(n-1)!}\right\}$$
$$= \sum_{n=1}^{\infty}\left\{S_{n-1}\frac{x^{n-1}}{(n-1)!} + (n-1)S_{n-2}\frac{x^{n-1}}{(n-1)!}\right\}$$

$$= \sum_{n=0}^{\infty}\left\{S_n\frac{x^n}{n!}\right\} + \sum_{n=2}^{\infty}\left\{S_{n-2}\frac{x^{n-1}}{(n-2)!}\right\} = F(x) + xF(x).$$

Hence $F'(x)/F(x)=1+x$. Integration and use of $F(0)=S_0=1$ yields $F(x) =\exp(x+x^2/2)$. Now the series for $F(x)$ is uniformly convergent for all x, so all the operations are legal.

Comment: S_n is the number of permutations π, with $\pi^2=1$, in the symmetric group on n symbols. Some contestants used this observation together with known formulas for the number of permutations with a given cycle structure to check part (b) by multiplying the series for $\exp(x)$ and $\exp(x^2/2)$.

A-3 (109) Let $f(x)=ax^2-bx+c=a(x-r)(x-s)$. Then $f(0)\cdot f(1)=a^2r(r-1)\cdot s(s-1)$. The graph of $r(r-1)$ shows that $0<r<1$ implies $0<r(r-1)\leqq\frac{1}{4}$, with equality if and only if $r=\frac{1}{2}$. Similarly, $0<s(s-1)\leqq\frac{1}{4}$. Since $r\neq s$, $r(r-1)s(s-1) <1/16$ and $0<f(0)\cdot f(1)<a^2/16$. The coefficients a, b, c are integers and thus $1\leqq f(0)\cdot f(1)$. Consequently $a^2>16$, i.e. $a\geqq 5$.

The discriminant b^2-4ac shows that the minimum possible value for b is 5. Furthermore, $5x^2-5x+1$ has two distinct roots between 0 and 1.

Comment: The above elegant solution was presented only by Don B. Zagier.

A-4 (0) Assuming that there is a solution u, then integrating with respect to x from 0 to 1, one obtains $\int_0^1 u(x)dx=\int_0^1 1\cdot dx+\lambda\int_0^1\{\int_x^1 u(y)u(y-x)dy\}dx$. In the iterated integral, one can interchange the order of integration, and letting $\int_0^1 u(x)dx=\alpha$, one gets $\alpha=1+\lambda\int_0^1\{u(y)\int_0^y u(y-x)dx\}dy$. Now, holding y fixed, let $z=y-x$ to get $\alpha=1+\lambda\int_0^1\{u(y)\int_0^y u(z)dz\}dy$. Set $f(y)=\int_0^y u(z)dz$. Then $\alpha=1 +\lambda\int_0^1 f'(y)f(y)dy=1+\lambda\{\frac{1}{2}f^2(1)-\frac{1}{2}f^2(0)\}=1+\lambda\cdot\frac{1}{2}\alpha^2$, or $\lambda\alpha^2-2\alpha+2=0$. The discriminant of this quadratic shows that if $\lambda>\frac{1}{2}$ then the roots are imaginary.

A-5 (0) Let the maximum diameter be $2d$ and assume $d<\frac{1}{2}$. Take such a diameter as the x-axis with the origin at the mid-point. Since this is a maximum diameter the region is bounded between the lines $x=-d$ and $x=d$. The upper and lower boundaries of the region are functions, because of convexity. Denote them by $f(x)$ and $-g(x)$, where f and g are nonnegative for $-d\leqq x\leqq d$. Calculating the distance between $(x, f(x))$ and $(-x, -g(x))$ shows that $f(x)+g(-x) <\sqrt{(1+4x^2)}$, for $-d\leqq x\leqq d$. Area$=\int_{-d}^{d}\{f(x)+g(-x)\}dx<\int_{-d}^{d}\sqrt{(1+4x^2)}dx <\int_{-1/2}^{1/2}\sqrt{(1+4x^2)}dx=\frac{1}{4}\pi$. This contradiction proves that $d\geqq\frac{1}{2}$ and so there must be at least two points a unit distance apart.

Comment: The requirement that the boundary contain at most a finite number of straight line segments was extraneous.

The above solution is due to Fritz Herzog, whereas the Questions Committee suggested a similar solution in polar coordinates.

A-6 (1) Solving the given equations in terms of x_3 and x_4, leads to the equivalent system: $x_1=A_1x_3+B_1x_4$, $x_2=A_2x_3+B_2x_4$, $x_3=x_3$, $x_4=x_4$, where $A_1=(a_2b_3-a_3b_2)/(a_1b_2-a_2b_1)$, etc.

Each point in the x_3, x_4-plane corresponds uniquely to a solution $(x_1, x_2,$

x_3, x_4). Signum x_1 is positive for (x_3, x_4) on one side of $A_1x_3+B_1x_4=0$ and negative on the other side. Similarly for signum x_2, signum x_3, and signum x_4, using $A_2x_3+B_2x_3=0$, $x_3=0$ and $x_4=0$, respectively. These four lines through the origin, in general, divide the plane into eight regions, each having a different 4-tuple of signum values. Hence the maximum number of distinct 4-tuples is eight.

The maximum number of eight will occur if and only if there are actually four distinct lines. This is equivalent to the conditions $A_1\neq 0$, $A_2\neq 0$, $B_1\neq 0$, $B_2\neq 0$ and $A_1B_2-A_2B_1\neq 0$ or, simply, $a_ib_j-a_jb_i\neq 0$ for $i, j=1, 2, 3, 4$ and $i<j$.

SOLUTIONS. PART B

B-1 (36) Consider the figure in the complex plane with the center of the circle at the origin. We can take A, B, C, D, E, F as complex numbers of absolute value r. Furthermore $B=A\omega$, $D=C\omega$ and $F=E\omega$, where $\omega=\cos(\pi/3)+i\sin(\pi/3)$. Since $\omega^3=1$ and $\omega\neq 1$, $\omega^2-\omega+1=0$. The mid-points of BC, DE and FA are $P=\frac{1}{2}(A\omega+C)$, $Q=\frac{1}{2}(C\omega+E)$ and $R=\frac{1}{2}(E\omega+A)$. If the segment from Q to R is rotated through $\pi/3$ about Q, then R is carried into $Q+\omega(R-Q)$, which equals P. Thus P, Q, R are vertices of an equilateral triangle.

B-2 (55) For part (a) one has immediately that $A=p^2$, $B=2p(1-p)$, and $C=(1-p)^2$. The result follows by examination of the graphs for A, B and C on $0\leqq p\leqq 1$.

For part (b), $\alpha=pr$, $\beta=p(1-r)+r(1-p)$, $\gamma=(1-p)(1-r)$. Consider the region R in the p, r-plane defined by $0\leqq p\leqq 1$ and $0\leqq r\leqq 1$. We will show that there is no point in R with $\alpha<4/9$, $\beta<4/9$ and $\gamma<4/9$. If $\alpha<4/9$ and $\gamma<4/9$ then (p, r) is between the hyperbolas $pr=4/9$ and $(1-p)(1-r)=4/9$. These hyperbolas have vertices in R at $(2/3, 2/3)$ and $(1/3, 1/3)$, respectively. The symmetry about $(\frac{1}{2}, \frac{1}{2})$ suggests setting $p'=p-\frac{1}{2}$ and $r'=r-\frac{1}{2}$. Then $\beta=\frac{1}{2}-2p'r'$ and thus $\beta<4/9$ if and only if $p'r'>1/36$. Note that the vertices for the hyperbola $p'r'=1/36$ are at $(p, r)=(1/3, 1/3)$ and $(2/3, 2/3)$. By looking at asymptotes, we see graphically that the region $\beta<4/9$ does not overlap the region in R where $\alpha<4/9$ and $\gamma<4/9$.

Comment: The above solution for part (b) is adapted from the solution presented by Robert Baumel.

B-3 (17) Since f is uniformly continuous, for any $\epsilon>0$ there is an integer n such that $|x-y|<1/n$ implies $|f(x)-f(y)|<\epsilon$.

$$\int_0^1 f(x)g(nx)dx = \sum_{m=0}^{n-1}\int_{m/n}^{m+1/n} f(x)g(nx)dx = \sum_{m=0}^{n-1}\int_{m/n}^{m+1/n} f(m/n)g(nx)dx$$
$$+\sum_{m=0}^{n-1}\int_{m/n}^{m+1/n}(f(x)-f(m/n))g(nx)dx.$$

The first term equals $\sum_{m=0}^{n-1}(1/n)f(m/n)\int_0^1 g(t)dt$ and becomes $(\int_0^1 f(x)dx)(\int_0^1 g(x)dx)$

as $n \to \infty$. Furthermore,

$$\left| \int_{m/n}^{m+1/n} \{f(x) - f(m/n)\} g(nx)dx \right| < \int_{m/n}^{m+1/n} | f(x) - f(m/n) | \cdot | g(nx) | \, dx$$
$$< \int_{m/n}^{m+1/n} \epsilon | g(nx) | \, dx < \epsilon/n \int_0^1 | g(t) | \, dt.$$

Thus the absolute value of the second term is less than or equal to $\epsilon \int_0^1 | g(t) | dt$ which becomes 0 as $\epsilon \to 0$.

B-4 (145) Locker m, $1 \leqq m \leqq n$, will be unlocked after the n operations are performed if and only if m has an odd number of positive divisors. If $m = p^\alpha q^\beta \cdots r^\gamma$, then the number of divisors of m is $(\alpha+1)(\beta+1) \cdots (\gamma+1)$, which is odd if and only if $\alpha, \beta, \cdots, \gamma$ are all even. This is equivalent to the condition that m is a perfect square.

For part (b), the set of numbers of the form $2m^2$ are obtained by having T_k change lockers whose numbers are multiples of $2k$. The set m^2+1 results from T_k changing locker i if $i-1$ is a multiple of k, with the stipulation that locker number 1 is changed only by T_1.

B-5 (20) Let A_n be the sum of the first n terms in the binomial expansion of $(2-1)^{-n}$.

$$A_n = \sum_{i=0}^{n-1} \binom{n+i-1}{i} 2^{-n-i} = 2^{-n} + \sum_{i=1}^{n-1} \left\{ \binom{n+i-2}{i} + \binom{n+i-2}{i-1} \right\} 2^{-n-i}$$
$$= 2^{-n} + \left\{ \sum_{i=0}^{n-2} \binom{n+i-2}{i} 2^{-n-i} + \binom{2n-3}{n-1} 2^{-2n+1} - 2^{-n} \right\} + \sum_{j=0}^{n-2} \binom{n+j-1}{j} 2^{-n-j-1}$$
$$= 2^{-n} + \tfrac{1}{2} A_{n-1} + \binom{2n-3}{n-1} 2^{-2n+1} - 2^{-n} + \tfrac{1}{2} A_n - \binom{2n-2}{n-1} 2^{-2n}$$
$$= \tfrac{1}{2} A_{n-1} + \tfrac{1}{2} A_n + 2^{-2n} \left\{ 2\binom{2n-3}{n-1} - \binom{2n-3}{n-1} - \binom{2n-3}{n-2} \right\} = \tfrac{1}{2} A_{n-1} + \tfrac{1}{2} A_n.$$

Thus $A_n = A_{n-1}$, but $A_1 = 2^{-1} = \frac{1}{2}$ and so $A_n = \frac{1}{2}$ for all positive integers n.

Alternate solution: Consider a random walk starting at (0, 0), such that if one is at (x, y) the probability of moving to $(x+1, y)$ is $\frac{1}{2}$ and the probability of moving to $(x, y+1)$ is $\frac{1}{2}$. Let S_n be the square with vertices at (0, 0), $(n, 0)$, (n, n), $(0, n)$. By symmetry, the probability $R_i(n)$ of first touching S_n at (n, i), $0 \leqq i < n$, equals the probability $T_i(n)$ of first touching S_n at (i, n), $0 \leqq i < n$, and hence $\sum_{i=0}^{n-1} R_i(n) = \sum_{i=0}^{n-1} T_i(n) = \frac{1}{2}$. Furthermore, $R_i(n) = \binom{n+i-1}{i} (\frac{1}{2})^{n+i}$.

B-6 (8) Consider the function g whose values are defined by $g(x, y) = f(x, y) + 2(x^2+y^2)$. On the circumference of the unit circle, $g(x, y) \geqq 1$, and at the origin, $g(0, 0) \leqq 1$. Hence, either $g(x, y) =$ constant and $f(x, y) =$ constant $-2(x^2+y^2)$, or there is a minimum value for $g(x, y)$ at some interior point. In the case

$g(x, y) = \text{constant}$, the result is immediate. Otherwise, let (x_0, y_0) be the coordinates of a point where $g(x, y)$ has a minimum. Then

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0 \text{ at } (x_0, y_0)$$

$$\text{and} \quad \left|\frac{\partial f}{\partial x}(x_0, y_0)\right| \leqq 4|x_0|, \quad \left|\frac{\partial f}{\partial y}(x_0, y_0)\right| \leqq 4|y_0|.$$

Thus the conclusion follows.

Comment: This problem is a special case of problem E 1986 [1967, 589] in this MONTHLY. Solutions to problem E 1986 will show that the inequality can be sharpened.

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MATHEMATICAL NOTES

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

TOPOLOGY ON FINITE SETS

D. STEPHEN, Madurai University (S. India)

The purpose of this article is to prove two results connected with topologies on a finite set of elements.

Let X be a set of $n(>1)$ elements.

THEOREM 1. *The only topology on X, having more than $3\times 2^{n-2}$ open sets is the discrete topology.*

Proof. Let T be a topology on X. For each x in X, let O_x be the intersection of all T-open sets containing x. Since X is finite, there are only a finite number of open sets containing x and hence their intersection O_x is T-open. If (o) be the collection of all such O_x, for each x in X, then (o) is the unique minimal base for T. Clearly, the number of sets in the family (o) is at the most n. Suppose (o) contains k sets. If $k=1$, then $(o)=X$ and hence T is the trivial topology on X. We assume that $k>1$. Since (o) is the base for T, each T-open set can be expressed as the union of members of (o).

Case (i). There does not exist in (o) a pair (O_i, O_j) of distinct members such

that $O_i \subset O_j$. In this case $N(T)$, the number of open sets in T, including the empty set $\varnothing$ is $1+{}^kC_1+{}^kC_2+\cdots+{}^kC_k=2^k$. Thus $N(T)=2^n$ when $k=n$ and consequently T is discrete. Again, when $k\leqq n-1$ $N(T)\leqq 2^{n-1}\leqq 3\times 2^{n-2}$.

Case (*ii*). There exists just one pair (O_i, O_j) of distinct members in (o) such that $O_i\subset O_j$. In this case,

$$N(T) = 1 + C_{k,1} + (C_{k,2} - 1) + (C_{k,3} - C_{k-2,1}) + \cdots + (C_{k,k} - C_{k-2,k-2})$$
$$= 2^k - 2^{k-2} = 3 \times 2^{k-2}.$$

Therefore $N(T)=3\times 2^{n-2}$ when k has the maximum value n.

Case (*iii*). There exists at least one pair (O_i, O_j) of distinct sets in (o) such that $O_i\subset O_j$. In this case, $N(T)\leqq N(T)$ of case (ii) $\leqq 3\times 2^{n-2}$.

Hence we find that in any case the maximum number of open sets a topology T on X can have is exactly $3\times 2^{n-2}$ unless T is discrete. The proof will be complete if the existence of a topology on X with exactly $3\times 2^{n-2}$ open sets is established. Consider the topology T', generated by the family S of subsets of X, consisting of $(n-1)$ singletons say (a_1), (a_2), $\cdots$, (a_{n-1}) and one pair which contains a_n, say (a_1a_n). Clearly all the 2^{n-1} subsets of X which do not contain a_n are T'-open and all the 2^{n-2} supersets of (a_1a_n) in X are T'-open. Thus T' contains $2^{n-1}+2^{n-2}=3\times 2^{n-2}$ open sets and clearly not more than that. It is easy to verify that S is the minimal base for T'.

Let $f(n)$ be the number of chain topologies—topologies whose open sets are completely ordered by inclusion—on X.

THEOREM 2. $f(n)=1+C_{n,1}f(1)+\cdots+C_{n,n-1}f(n-1)$, $(n>0)$.

Proof. The proof depends upon the fact that any chain of sets is a topology for their union.

Let T be a chain topology for X, defined as follows:

$$\varnothing \subset O_p \subset O_q \subset \cdots \subset O_r \subset X.$$

Let O_r contain r elements of X. Now the chain of sets $\varnothing\subset O_p\subset O_q\subset\cdots\subset O_r$ constitute a chain topology for the set O_r. Conversely, every chain topology for O_r gives rise to a chain topology for X. The number of chain topologies on O_r is $f(r)$. There are $C_{n,r}$ sets of r elements and hence there are $C_{n,r}f(r)$ chain topologies on X having a set of r elements as the second largest open set. The indiscrete topology being a chain topology, we get

$$f(n) = 1 + C_{n,1}f(1) + \cdots + C_{n,n-1}f(n-1).$$

THEOREM 3. *If* $S_{mn}=m^n-C_{m,1}(m-1)^n+C_{m,2}(m-2)^n+\cdots+(-1)^{m-1}C_{m,m-1}$, *then*

$$f(n) = S_{1n} + S_{2n} + \cdots + S_{nn} \qquad (n > 0).$$

Proof. We observe that S_{kn} is the coefficient of $(x^n)/n!$ in the expansion of $(e^x-1)^k$. As a consequence of this it can be easily verified that

(a) $S_{kn} = 0$ if $k > n$ (b) $S_{kk} = k!$

(c) $S_{(k+1)n} = C_{n,1}S_{k(n-1)} + C_{n,2}S_{k(n-2)} + \cdots + C_{n,n-k}S_{kk}$

By the order of a chain topology T, we shall mean the number of nonempty open sets of T. With n elements we can form a chain of, at the most, n nonempty sets. So the order of a chain topology on X cannot exceed n. We shall first prove by induction that $N(r, n)$, the number of chain topologies of order r on a set of n elements is equal to S_{rn}.

The indiscrete topology on X, is a chain topology and it is the only topology of order one on X. Therefore, $N(1, n) = 1 = S_{1n}$. Any nonempty proper subset of X together with the empty set and X will form a chain topology of order 2. Therefore $N(2, n) = 2^n - 2 = S_{2n}$. Now, assume that $N(m, r) = S_{mr}$. We shall compute $N(m+1, n)$ as follows.

Let X' represent a proper nonempty subset of X having at least m elements. Now each chain topology of order $m+1$ on X can be thought of as the result of adding X to some chain topology of order m on X'. If X' contains $p(\geqq m)$ elements then X' itself can be chosen in ${}^nC_p$ ways. Hence

$$N(m + 1, n) = C_{n,1}S_{m(n-1)} + C_{n,2}S_{m(n-2)} + \cdots + C_{n,n-m}S_{mm} = S_{(m+1)n} \quad \text{by (c)}.$$

Hence $N(m, n) = S_{mn}$ for $m = 1$ to n. Therefore, $f(n) = N(1, n) + N(2, n) + \cdots + N(n, n) = S_{1n} + S_{2n} + \cdots + S_{nn}$. It is easy to verify that the expression for $f(n)$ given above satisfies the recurrence relation given in the statement of Theorem 2.

NUMERICAL INTEGRATION OVER ANY NUMBER OF EQUAL INTERVALS

G. O. Peters and C. E. Maley, General Electric Company, Philadelphia

Improvement of the Simpson rules [1] for numerical integration might, at this late date, seem impossible. However, examination of their utilization shows two difficulties: first, the number n of intervals must be a multiple of the degree m of polynomial fit used; second, when increasing the number of intervals during a computation to increase accuracy, the data have to be regrouped each time.

This paper develops a set of rules for numerical integration which allows any number of intervals (not less than the degree of fit), using simple addition of ordinates and correction terms requiring the first and last $m+1$ ordinates. They are similar to the trapezoidal rule, which is a special case of this set of rules for $m = 1$.

Some useful formulas. Consider the derivation of the original Simpson's rule ($m = 2$). A function $f(x)$ is approximated by a quadratic one,

$$\phi(x) = a + bx + cx^2,$$

through three of its points (x_i, f_i), $(x_i + h, f_{i+1})$, $(x_i + 2h, f_{i+2})$. Since translating the origin of coordinates to $(x_i, 0)$ does not affect the area that is the geometric

interpretation of the desired integral, the algebra—solving three simultaneous linear equations for the three unknowns—is quite simple:

$$a = f_i$$
$$b = (-3f_i + 4f_{i+1} - f_{i+2})/(2h)$$
$$c = (f_i - 2f_{i+1} + f_{i+2})/(2h^2).$$

Then

$$\int_{x_i}^{x_{i+2}} f(x)dx \doteq \int_0^{2h} \phi(x)dx,$$

or

$$\text{(S)} \qquad \int_{x_i}^{x_{i+2}} f(x)dx \doteq (f_i + 4f_{i+1} + f_{i+2})h/3.$$

As a check, the sum of the f coefficients in all such integration rules must be nh, where h is the common length of the intervals.

The usual method for applying rule (S) is in the "abutting" fashion

$$\text{(A)} \quad \text{total area} = (P_0P_2)+(P_2P_4)+(P_4P_6)+(P_6P_8)+ \cdots +(P_{n-4}P_{n-2})+(P_{n-2}P_n),$$

where (P_2P_4), for example, represents the area bounded by the x axis, the curve and two ordinates to points P_2 and P_4 on the curve. The resultant formula,

$$\int_{x_0}^{x_{2k}} f(x)dx \doteq \left[(f_0 + f_{2k}) + 4\sum_{i=1}^{k} f_{2i-1} + 2\sum_{i=1}^{k-1} f_{2i}\right]h/3, \qquad (2k = n),$$

shows the two difficulties mentioned.

There seems no reason, though, why (S) cannot be applied in the "overlapping" fashion

$$\text{(02)} \quad \text{total area} = \tfrac{1}{2}[(P_0P_1) + (P_0P_2) + (P_1P_3) + (P_2P_4) + \cdots + (P_{n-2}P_n) + (P_{n-1}P_n)],$$

except that (S) will not calculate either (P_0P_1) or $(P_{n-1}P_n)$. But,

$$\text{(LS)} \qquad \int_{x_0}^{x_1} f(x)dx \doteq \int_0^h \phi(x)dx = (5f_0 + 8f_1 - f_2)h/12$$

and

$$\text{(SR)} \qquad \int_{x_{n-1}}^{x_n} f(x)dx \doteq \int_{(n-1)h}^{nh} \phi(x)dx = (-f_{n-2} + 8f_{n-1} + 5f_n)h/12.$$

Neglecting the common factor $h/12$ in (S), (LS) and (SR), the central coefficients are 4, 16, 4, the lefthand coefficients are 5, 8, -1 and the righthand

coefficients are -1, 8, 5. By (02) we may construct a "Simpson's rule" at will for any number of intervals, say $n=3$:

	f_0	f_1	f_2	f_3
(P_0P_1)	5	8	-1	
(P_0P_2)	4	16	4	
(P_1P_3)		4	16	4
(P_2P_3)		-1	8	5
$2(P_0P_3)$	9	27	27	9

Multiplying the results by the omitted common factor and also dividing by 2 to counter the overlap, the resultant formula is

$$\int_{x_0}^{x_3} f(x)dx \doteq 3[(f_0+f_3)+3(f_1+f_2)]h/8,$$

(which "happens" to be Simpson's rule for $m=3$!).

As a matter of fact, it is not more difficult to extend the above table to f_n, producing the general quadratic form valid for n equal to or greater than 2:

$$\int_{x_0}^{x_n} f(x)dx \doteq \left[\sum_{i=0}^{n} f_i - 5/8(f_0+f_n) + 1/6(f_1+f_{n-1}) - 1/24(f_2+f_{n-2})\right]h.$$

Its similarity to the linear form, the trapezoidal rule valid for n equal to or greater than 1,

$$\int_{x_0}^{x_n} f(x)dx \doteq \left[\sum_{i=0}^{n} f_i - 1/2(f_0+f_n)\right]h,$$

is apparent. Clearly, this overlapping may be extended to cubic approximations,

$$\text{(03)}\quad \text{total area} = 1/3[(P_0P_1)+(P_0P_2)+(P_0P_3)+(P_1P_4)+\cdots + (P_{n-3}P_n)+(P_{n-2}P_n)+(P_{n-1}P_n)],$$

and beyond.

(03) will produce the cubic approximation, valid for all n equal to or greater than 3,

$$\int_{x_0}^{x_n} f(x)dx \doteq \left[\sum_{i=0}^{n} f_i - 23/36(f_0+f_n) + 5/24(f_1+f_{n-1}) - 1/12(f_2+f_{n-2}) + 1/72(f_3+f_{n-3})\right]h,$$

and its next extension will produce the quartic approximation, valid for all n equal to or greater than 4,

$$\int_{x_0}^{x_n} f(x)dx \doteq \Big[\sum_{i=0}^{n} f_i - 193/288(f_0 + f_n) + 77/240(f_1 + f_{n-1})$$
$$- 7/30(f_2 + f_{n-2}) + 73/720(f_3 + f_{n-3}) - 3/160(f_4 + f_{n-4}) \Big] h.$$

The general formula. One may derive the general formula either by generalizing the above method or by solving directly for A_j^m in the desired relation

$$\int_{x_0}^{x_n} f(x)dx \doteq \left[\sum_{j=0}^{n} f_j - \sum_{j=0}^{m} A_j^m (f_j + f_{n-j}) \right] h.$$

For this purpose any polynomial of the mth degree may be substituted (we used x^m).

Making use of the fact ([2], [3]) that the (ij)-th term, a_{ij}, of the inverse of the Vandermonde matrix is

$$a_{ij} = \sum_{k=0}^{m} (-1)^{j+k} S_k^i C_j^k / k!,$$

where the S and C entries are Stirling numbers of the first kind and binomial coefficients respectively, then

$$A_j^m = 1 - 1/m \left[\sum_{i=0}^{m} 1/(i+1) \left\{ a_{ij} \sum_{k=0}^{m-1} k^{i+1} + m^{i+1} \sum_{k=0}^{j} a_{ik} \right\} \right].$$

Comparison of degrees of fit. It is easy to find the relations between approximations of varying degree m, I_n^m, applied to a constant number n of intervals, whenever n is equal to or greater than m:

$$I_n^1 - I_n^2 = [3(f_0 + f_n) - 4(f_1 + f_{n-1}) + (f_2 + f_{n-2})]h/24$$
$$I_n^2 - I_n^3 = [(f_0 + f_n) - 3(f_1 + f_{n-1}) + 3(f_2 + f_{n-2}) - (f_3 + f_{n-3})]h/72$$
$$I_n^3 - I_n^4 = [5(f_0 + f_n) - 18(f_1 + f_{n-1}) + 24(f_2 + f_{n-2})$$
$$- 14(f_3 + f_{n-3}) + 3(f_4 + f_{n-4})]h/160$$

. . . .

These provide an estimate of the worth of proceeding to a higher approximation.

References

1. I. S. Sokolnikoff and R. M. Redheffer, Mathematics of Physics and Modern Engineering, McGraw-Hill, New York, 1958, pp. 715–721.

2. N. Macon and A. Spitzbart, Inverses of Vandermonde matrices, this MONTHLY, 65 (1958) 95–100.

3. J. F. Traub, Associated polynomials and uniform methods for the solution of linear problems, SIAM Rev., 8, 3, p. 277.

THE SUPERSET TOPOLOGY

NORMAN LEVINE, Ohio State University

1. Let $\mathfrak{J}$ be a topology for the set X. We will call $\mathfrak{J}$ an *S-topology* for X if and only if for $\emptyset \neq O \subseteq A \subseteq X$ and $O \in \mathfrak{J}$, then $A \in \mathfrak{J}$. For example, let $\mathfrak{J}_1 = \{\emptyset, X\}$, $\mathfrak{J}_2 = \{A \mid A \subseteq X\}$, $\mathfrak{J}_3 = \{A \mid A = \emptyset$ or $\mathcal{C}A$ is finite$\}$ where $\mathcal{C}$ is the complement operator, and $\mathfrak{J}_4 = \{A \mid A = \emptyset$ or $A^* \subseteq A \subseteq X\}$, A^* a fixed subset of X. It is clear that $\mathfrak{J}$ is an S-topology for X if and only if the neighborhood system $\mathfrak{N}(x) \subseteq \mathfrak{J}$ for all $x \in X$. ($N \in \mathfrak{N}(x)$ if and only if $x \in O \subseteq N$ for some $O \in \mathfrak{J}$.)

2. Let $\Delta \neq \emptyset$ be an arbitrary index set and for each $\alpha \in \Delta$, let $\mathfrak{J}_\alpha$ be an S-topology for the set X. Let $\mathfrak{J}_1 = \inf\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$ and $\mathfrak{J}_2 = \sup\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$. The reader will recall that $O \in \mathfrak{J}_1$ if and only if $O \in \mathfrak{J}_\alpha$ for all $\alpha \in \Delta$, and that $O \in \mathfrak{J}_2$ if and only if either $O = \emptyset$, or for $x \in O$ there exist $O_i \in \bigcup\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$ such that $x \in O_1 \cap \cdots \cap O_n \subseteq O$. Then $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are each S-topologies for X.

Proof. We will show only that $\mathfrak{J}_2$ is an S-topology for X. Let $\emptyset \neq O \subseteq A \subseteq X$ with $O \in \mathfrak{J}_2$. To show that $A \in \mathfrak{J}_2$, let $y \in A$ and take $x \in O$. Then $x \in O_1 \cap \cdots \cap O_n \subseteq O$ for some $O_i \in \bigcup\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$. Let $O_i^* = O_i \cup \{y\}$ for $i = 1, \cdots, n$. Then $O_i^* \in \bigcup\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$ and $y \in O_1^* \cap \cdots \cap O_n^* \subseteq A$. Thus $A \in \mathfrak{J}_2$.

3. Let $(X, \mathfrak{J})$ be a topological space. There exists a topology $\mathfrak{J}^*$ for X such that (1) $\mathfrak{J}^* \subseteq \mathfrak{J}$, (2) $\mathfrak{J}^*$ is an S-topology and (3) $\mathfrak{J}^*$ is the largest topology for X with properties (1) and (2).

Proof. Let $\mathfrak{J}^* = \sup\{\mathfrak{J}' : \mathfrak{J}' \subseteq \mathfrak{J}$ and $\mathfrak{J}'$ is an S-topology for $X\}$.

4. Let $\mathfrak{J}$ be an S-topology for X. Suppose $\emptyset \neq O_i \in \mathfrak{J}$ for $i = 1, 2$ and $O_1 \cap O_2 = \emptyset$. Then $(X, \mathfrak{J})$ is discrete.

Proof. Let $x \in X$.
Case 1. $x \notin O_1 \cup O_2$. Then $\{x\} = \{O_1 \cup \{x\}\} \cap \{O_2 \cup \{x\}\} \in \mathfrak{J}$.
Case 2. $x \in O_1$. Then $\{x\} = O_1 \cap \{O_2 \cup \{x\}\} \in \mathfrak{J}$.

5. Let $\mathfrak{J}$ be an S-topology for X. Then $(X, \mathfrak{J})$ is connected or discrete (or both if X is a singleton set).

This follows immediately from section 4.

6. Let $(Y, \mathfrak{U}) = (X, \mathfrak{J}) \times (X, \mathfrak{J})$. Then $\mathfrak{U}$ is generally not an S-topology for Y when $\mathfrak{J}$ is an S-topology for X. To see this, let $X = \{a, b\}$ and $\mathfrak{J} = \{\emptyset, \{a\}, X\}$. Then $\{(a, a)\} \in \mathfrak{U}$, but $\{(a, a), (b, b)\} \notin \mathfrak{U}$.

7. Let $(Y, \mathfrak{U})$ be a subspace of $(X, \mathfrak{J})$. If $\mathfrak{J}$ is an S-topology for X, then $\mathfrak{U}$ is an S-topology for Y.

8. $\mathfrak{J}$ is an S-topology for the set X if and only if $\emptyset \neq O \in \mathfrak{J}$ implies that $(\mathcal{C}O)' = \emptyset$, $(\mathcal{C}O)'$ denoting the derived set of $\mathcal{C}O$.

The proof is left for the reader.

9. Let $\mathfrak{J}$ be an S-topology for the set X and let $f: X \to Y$ be a single valued

function. If $\mathfrak{U}=\{U \mid U\subseteq Y \text{ and } f^{-1}U\in\mathfrak{J}\}$, then $\mathfrak{U}$ is an S-topology for Y.

Proof. Let $\emptyset\neq U\subseteq V\subseteq Y$ and $U\in\mathfrak{U}$. Then $f^{-1}V\supseteq f^{-1}U\in\mathfrak{J}$ and hence $f^{-1}V\in\mathfrak{J}$. Thus $V\in\mathfrak{U}$.

10. Let $\mathfrak{U}$ be an S-topology for the set Y and let $f: X\to Y$ be a single valued, one-to-one function. If $\mathfrak{J}=\{f^{-1}U \mid U\in\mathfrak{U}\}$, then $\mathfrak{J}$ is an S-topology for X.

Proof. Let $\emptyset\neq f^{-1}U\subseteq V\subseteq X$ where $U\in\mathfrak{U}$. Then $V=f^{-1}fV\cup f^{-1}U=f^{-1}\{fV\cup U\}\in\mathfrak{J}$ since $fV\cup U\in\mathfrak{U}$.

11. In this section we will characterize some of the well-known topologies for a set X in terms of S-topologies. Let $\mathfrak{J}$ be a topology for X. Then

(a) $\mathfrak{J}$ is the cofinite topology for X if and only if $(X, \mathfrak{J})$ is compact and $\mathfrak{J}$ is an S-topology for X.

(b) $\mathfrak{J}$ is the cocountable topology for X if and only if $(X, \mathfrak{J})$ is Lindelöf (every open cover of X has a cocountable subcover) and $\mathfrak{J}$ is an S-topology for X.

(c) If X has two or more elements, then $(X, \mathfrak{J})$ is discrete if and only if $(X, \mathfrak{J})$ is disconnected and $\mathfrak{J}$ is an S-topology for X.

(d) $\mathfrak{J}=\{A \mid A=\emptyset \textit{ or } A^*\subseteq A\subseteq X\}$ for some fixed set A^* in X if and only if $\mathfrak{J}$ is an S-topology for X and $\bigcap\{O \mid \emptyset\neq O\in\mathfrak{J}\}\in\mathfrak{J}$.

The proofs are left as exercises for the reader.

DIRECT AND SUBDIRECT SUMS OF SIMPLE RINGS WITH UNIT

E. P. Armendariz, University of Texas, Austin

In what follows R denotes a ring with unit 1. Call R an *S-ring* if R is a direct sum of a finite number of simple rings with unit, while R is an *SD-ring* if R is a subdirect sum of simple rings with unit. We consider S- and SD-rings in connection with the following properties for a ring R:

$D1$: *Every maximal ideal of R is a direct summand of R.*
$D2$: *Every proper ideal of R is a direct summand of R.*
$D3$: *Every proper ideal of R is contained in a proper direct summand of R.*
$D4$: *Every nonzero ideal of R contains a nonzero direct summand of R.*

Call R a *Dk-ring* if R satisfies condition Dk, $k=1, 2, 3, 4$. We remark that similar conditions for groups were considered by T. Head in [1], and our results are patterned after these.

Since R has a unit, it contains maximal ideals so that any $D2$- or $D3$-ring is a $D1$-ring. It is easy to see that every S-ring is a Dk-ring for $k=1, 2, 3, 4$.

Proposition 1. *A ring R is an S-ring if and only if R is a Dk-ring for any one of* $k=1, 2, 3$.

Proof. As noted above, R an S-ring$\Rightarrow D2\Rightarrow D3\Rightarrow D1$, so it remains to show that every $D1$-ring is an S-ring. Let A be a maximal ideal of R and write $R=A\oplus I$. Then $I\cong R/M$ is a simple ring with unit so I is a minimal ideal of R.

We now follow a standard argument, as for example in [2, p. 14], taking the collection of all independent families of minimal ideals of R, and applying Zorn's Lemma to get a maximal family $\{A_j: j\in J\}=\mathfrak{A}$. Let $K=\sum_{j\in J}A_j$; if $K\neq R$ then $K\leqq A$, a maximal ideal of R and so $R=A\oplus I$, where I is a minimal ideal of R. Then $\mathfrak{A}\cup\{I\}$ contradicts the maximality of $\mathfrak{A}$. Write $1=\sum_{j\in J}e_j$, where $e_j\in A_j$ and all but a finite number of the e_j are zero. We can assume these are labelled so that $e_1, \cdots, e_n$ are nonzero and $e_j=0$ for $j\neq 1, \cdots, n$. Then $R=\sum_{j=1}^{n}A_j$ and the sum is direct since $\mathfrak{A}$ is independent. This completes the proof.

Concerning subdirect sums we have

PROPOSITION 2. *A ring R is an SD-ring if and only if for every ideal $I\neq 0$ of R there exists an ideal $A\neq R$ such that $I+A=R$.*

Proof. Suppose $I\neq 0$ is an ideal of R, where R is an SD-ring. Then R contains maximal ideals $\{M_j: j\in J\}$ with $\bigcap_{j\in J}M_j=0$. Hence $I\not\subseteq M_j$ for some j, and for any such j, $I+M_j=R$. Conversely, assume R satisfies the condition. We show that every nonzero element is excluded by some maximal ideal of R, so that every nonzero element is excluded by some maximal ideal of R, so that the intersection of the maximal ideals of R is zero. Let $0\neq x\in R$; then $RXR\neq 0$ and we may assume $RXR\neq R$. Then $RXR+A=R$ for some ideal $A\neq R$. Choose M maximal relative to $A\subseteq M$ and $x\notin M$. Then M is a maximal ideal of R, since if $M\subset B$ and $M\neq B$ then $x\in B$ implies $R=RXR+A\subseteq B$. It follows that R is an SD-ring.

COROLLARY. *If R is a $D4$-ring then R is an SD-ring.*

Proof. If $I\neq 0$ is an ideal of R then $N\subseteq I$, where N is a nonzero direct summand of R, so $R=N\oplus A\subseteq I+A$ with $A\neq R$. Hence R is an SD-ring by the above proposition.

The ring Z of integers is an example of an SD-ring which is not a $D4$-ring. With respect to the ring Z we note that if x, $y\in Z$, $x\neq 0$, then $Z=Zx+A$ for some $A\neq Z$ so that $y=gx+r$, where $r\in A$, $g\in Z$; in fact this holds for any commutative SD-ring.

PROPOSITION 3. *A ring R with minimum condition on (two-sided) ideals is an S-ring if and only if R is a $D4$-ring.*

Proof. By the corollary R is an SD-ring if R is a $D4$-ring, so the intersection of the maximal ideals of R is zero. Among the collection of finite intersections of maximal ideals of R, we obtain a minimal element K. For any maximal ideal M, $K\cap M\subseteq K$ implies $K\cap M=0$, in which case $K=0$ by minimality, or else $K\cap M=K$ so that $K\subseteq M$. If the first case occurs we obtain a finite collection of maximal ideals with zero intersection. By [3, Theorem 3.18, p. 59] R is an S-ring. If, on the other hand, $K\cap M=K$ for all maximal ideals M of R then $K=0$ since R is an SD-ring. Thus again R is an S-ring.

Combining this last result with Proposition 1, we have

COROLLARY. *For a ring R with minimum condition on ideals the conditions Dk, $k=1, 2, 3, 4$ are equivalent.*

We note that if $Lk(Rk)$ denotes the corresponding statement for Dk obtained by replacing "ideal" by "left ideal" ("right ideal"), the proofs above, with suitable modifications yield.

PROPOSITION 1′: *A ring R is semisimple (with left minimum condition) if and only if R satisfies any one of Lk or Rk, $k=1, 2, 3$.*

PROPOSITION 2′: *A ring R is Jacobson semisimple and hence a subdirect sum of primitive rings if and only if for each left ideal $I\neq 0$ there exists a left ideal $A\neq R$ such that $I+A=R$. Thus if R satisfies $L4(R4)$, then R is a subdirect sum of primitive rings.*

PROPOSITION 3′: *If R is a ring with left (right) minimum condition then R satisfies $L4(R4)$ if and only if R is semisimple.*

Finally, the equivalence of $L2$ and R being semisimple with left minimum condition can be found in [4, p. 61].

References

1. T. J. Head, Note on the occurrence of direct factors in groups, Proc. Amer. Math. Soc., 15 (1964) 193–195.
2. J. P. Jans, Rings and Homology, Holt, Rinehart and Winston, New York, 1964.
3. N. H. McCoy, Theory of Rings, Macmillan, New York, 1964.
4. J. Lambek, Lectures on Rings and Modules, Blaisdell, Waltham, Mass., 1966.

ON ASSOCIATORS IN JORDAN ALGEBRAS

C. E. TSAI, Michigan State University

Jacobson proved in [1] that if A is an (associative) algebra over a field of characteristic zero and a, b are elements in A such that $[a, [a, b]]=0$, then $[a, b]$ is a nilpotent element in A where $[a, b]=ab-ba$. Kuźmen [3] has generalized this result to certain nonassociative algebras.

In this note, we shall look at this problem from another angle and prove the following: If J is a finite dimensional Jordan algebra and if $(x, (x, a, y), y)=0$, then (x, a, y) is a nilpotent, where $(x, a, y)=(xa)y-x(ay)$ is the associator.

LEMMA 1. *If J be a power associative algebra, D be a derivation of J and $D^2a=0$ for some element a in J; then $D^m a^n=0$ if m, n are positive integers and $m>n$.*

Proof (by induction). Let $n=1$ and m be any integer greater than 1. Then $D^m a=0$ is given. If the lemma holds for all $n'<n$, then

$$D^{n+1}a^n = D^{n+1}(a\cdot a^{n-1})$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} D^i(a) D^{n+1-i}(a^{n-1})$$
$$= aD^{n+1}(a^{n-1}) + (n+1) D(a) D^n(a^{n-1}) = 0.$$

Suppose $D^{n+k'}(a^n)=0$ for every positive integer $k'<k$ and all n; then $D^{n+k}a^n = D[D^{n+(k-1)}a^n]=0$.

LEMMA 2. *If J is a power associative algebra with derivation D and if a is an element of J such that $D^2a=0$, then $D^na^n=n!(Da)^n$ for all positive integer n.*

Proof (by induction). The lemma is obviously true if $n=1$. If we assume that $D^{n-1}(a^{n-1})=(n-1)!(Da)^{n-1}$, then

$$D^n(a^n) = D^n(a\cdot a^{n-1}) = \sum_{i=0}^{n} \binom{n}{i} D^i(a) D^{n-i}(a^{n-1}) = aD^n(a^{n-1})$$
$$+ nD(a)D^{n-1}(a^{n-1}) = nD(a)\cdot(n-1)!(Da)^{n-1} = n![D(a)]^n.$$

THEOREM. *Let J be a finite dimensional power associative algebra over a field K and D be a derivation on J. If a is an element of J and if the characteristic of K is either 0 or greater than the degree of the minimal polynomial of a, then $D^2a=0$ implies $D(a)$ is nilpotent.*

Proof. Let $\phi(\lambda)=\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2+\cdots+\alpha_t\lambda^t$ be the minimal polynomial of a. Then

$$0 = \phi(a) = \alpha_0 e + \alpha_1 a + \alpha_2 a^2 + \cdots + \alpha_t a^t.$$

Hence

$$0 = D^t(\phi(a)) = D^t[\alpha_0 e + \alpha_1 a + \alpha_2 a^2 + \cdots + \alpha_t a^t]$$
$$= \alpha_0 D^t(e) + \alpha_1 D^t(a) + \cdots + \alpha_t D^t(a^t)$$
$$= a_t D^t(a^t) = t! a_t [D(a)]^t.$$

Thus $D(a)$ is nilpotent by our assumption on the characteristics of the field K.

COROLLARY. *Let a, x, y be elements of a finite dimensional Jordan algebra over a field K. If the characteristic of K is either 0 or greater than the degree of the minimal polynomial of a, then $(x, (x, a, y), y)=0$ implies $(x, a, y)=0$.*

Proof. It is well known that the mapping $a \rightarrow (x, a, y)=(xa)y-x(ay)$ is a derivation on a Jordan algebra (see, for example, [4]).

References

1. N. Jacobson, Rational methods in the theory of Lie algebras, Ann. of Math., 36(1935) 875–881.

2. D. Kleinecke, On operator commutators, Proc. Amer. Math. Soc., 8(1957) 535–536.

3. E. N. Kuźmin, On commutators in flexible rings, Math. Reviews, 23-A 1682 (1962).

4. R. Schafer, An Introduction to Non-Associative Algebras, Academic Press, New York, 1966.

A SHORT PROOF OF THE NEST CHARACTERIZATION OF COMPACTNESS

D. E. Catlin, University of Massachusetts, Amherst

It is well known that the compactness of a topological space X can be characterized by either of the following equivalent conditions: (i) every family of closed sets in X having the finite intersection property (f.i.p.) has a nonempty intersection, (ii) every nest of closed sets has a nonempty intersection [1]. It is obvious that (i) implies (ii), however, the proof that (ii) implies (i) is somewhat more difficult, generally involving two applications of the axiom of choice. In this note we give a short proof that (ii) implies (i) and we use the axiom of choice only once.

Assume (ii) and let $\mathcal{C}$ be a collection of closed sets in X having f.i.p. Define

$$\mathcal{B} = \{B\colon B \text{ is closed, } \mathcal{C} \cup \{B\} \text{ has f.i.p.}\}$$

and let τ be any maximal nest in $\mathcal{B}$. By hypothesis, $T_0 = \bigcap\tau \neq \emptyset$. Note that for any $A \in \mathcal{C}$, $\{A \cap T\colon T \in \tau\}$ is a nest of nonempty closed sets and so $A \cap T_0 = \bigcap\{A \cap T\colon T \in \tau\} \neq \emptyset$. We claim that $T_0 \subset \bigcap\mathcal{C}$. If not, there exists $A_0 \in \mathcal{C}$ such that $A_0 \cap T_0$ is a nonempty closed set properly contained in T_0. But for any $A_1, \cdots, A_n \in \mathcal{C}$, $\{\bigcap_{i=1}^n \{A_i\} \cap A_0 \cap T\colon T \in \tau\}$ is a nest of nonempty closed sets and hence

$$\bigcap_{i=1}^{n} \{A_i\} \cap A_0 \cap T_0 = \bigcap \left\{ \bigcap_{i=1}^{n} \{A_i\} \cap A_0 \cap T\colon T \in \tau \right\} \neq \emptyset.$$

Thus $A_0 \cap T_0 \in \mathcal{B}$, contradicting the maximality of τ.

Reference

1. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955, p. 163.

RANDOM SIMPLE CLOSED PATHS IN A LINEAR GRAPH

S. K. Stein, University of California, Davis

Let G be a finite connected linear graph with a_1 edges. By a closed path based at a vertex V we mean a sequence of edges $V_0 V_1$, $V_1 V_2, \cdots, V_{n-1} V_n$, such that $V_0 = V = V_n$ and no $V_i = V$, $0 < i < n$. The length of such a path is n. The degree of a vertex V is the number of edges incident to V; we denote it $d(V)$. The following theorem is a special case of a result on mean recurrence times in a stationary process. (See W. Feller, *An introduction to probability theory*, Wiley, 1957, vol. 1, p. 396.) However, a short geometrically oriented proof of it may be of interest.

We will consider random closed paths based at a vertex V. The expected length of such paths we define as the sum of the expectations of the directed edges of G. Letting $E(V'V'')$ denote the expectation of the directed edge $V'V''$, we make two assumptions:

(1) If $V_1, V_2, \cdots, V_j$ are the vertices adjacent to a vertex V', then

$$E(V'V_1) = E(V'V_2) = \cdots = E(V'V_j).$$

This reflects the assumption that when a random path has reached a vertex, then it continues to any of the adjacent vertices with equal likelihood.

(2) If V' and V'' are adjacent vertices, then

$$E(V'V'') = E(V''V').$$

This reflects the assumption that the reverse of a closed path at V is just as likely to occur as the path itself.

THEOREM. *The expected length of a random closed path based at V is $2a_1/d(V)$.*

Proof. The expected length is the sum of the expectations of the $2a_1$ directed edges. By assumptions (1) and (2) these expectations are all equal. Since $E(VV_k)=1/d(V)$ for any vertex adjacent to V, it follows that all directed edges have the expectation $1/d(V)$. Thus the expected length of a random closed path based at V is $2a_1(1/d(V))$. The theorem is proved.

THE JORDAN CANONICAL FORM OF A PARTICULAR MATRIX

A. ARCESE, The MITRE Corporation, Bedford, Massachusetts

This note gives the Jordan canonical form of an upper diagonal equiprobability transition matrix

$$P = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \cdots & \cdots & \frac{1}{n} \\ 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \cdots & \frac{1}{n-1} \\ 0 & 0 & \frac{1}{n-2} & \cdots & \cdots & \frac{1}{n-2} \\ 0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

where the last state is an absorbing state. The canonical form is $P=H\lambda H^{-1}$ where λ is the diagonal matrix of eigenvalues $\lambda_1=1/n, \lambda_2=1/(n-1), \cdots, \lambda_n=1$, and H and H^{-1} are upper diagonal matrices of binomial coefficients

$$H = \left\{\binom{n-i}{j-i}\right\}_{i\le j} \qquad H^{-1} = \left\{(-1)^{i+j}\binom{n-i}{j-i}\right\}_{i\le j}.$$

For example, if there are five states

$$P=\begin{bmatrix}1&4&6&4&1\\0&1&3&3&1\\0&0&1&2&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{5}&0&0&0&0\\0&\frac{1}{4}&0&0&0\\0&0&\frac{1}{3}&0&0\\0&0&0&\frac{1}{2}&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&-4&6&-4&1\\0&1&-3&3&-1\\0&0&1&-2&1\\0&0&0&1&-1\\0&0&0&0&1\end{bmatrix}$$

It is interesting to note that the inverse of H with binomial coefficients is simply the binomial coefficients with alternating signs.

This transition matrix might be used for a number of problems of which some simple examples are a ball bouncing down a flight of stairs or communication in a forward oriented complete graph.

SOME PROPERTIES OF THE FAREY SERIES

J. D. MAITLAND WRIGHT, Christ Church, University of Oxford

For each natural number n greater than 1 the Farey series F_n is the strictly increasing sequence of irreducible rational fractions in the closed interval $[0, 1]$ whose denominators do not exceed n. Thus F_n is the sequence of those h/k for which $1 \leqq k \leqq n$, $0 \leqq h \leqq k$ and $(h, k) = 1$.

In an interesting recent paper [1] Blake proved two theorems on Farey series. These appear here as Theorems 1 and 3. We give alternative shorter proofs of Blake's results. We also obtain, very easily, a formula for the sum of the Farey series in terms of Euler's function ϕ.

LEMMA 1. *If h/k is a term of the sequence F_n then $(k-h)/k$ is also a term of the sequence.*

For, if $(h, k) = 1$ and $0 \leqq (h/k) \leqq 1$, then $(k-h, k) = 1$ and $0 \leqq 1-(h/k) \leqq 1$.

In what follows we use $\sum$ with no suffixes to denote summation over all the terms h/k of the sequence F_n.

THEOREM 1. *The sum of the numerators of the fractions of a Farey series F_n is half the sum of the denominators.*

By Lemma 1 $\sum h = \sum (k-h)$, so that $2\sum h = \sum k$. This is Theorem 1.

THEOREM 2. *The sum of the fractions of the Farey series is half the number of terms.*

For, by Lemma 1, $\sum (h/k) = \sum \{1-(h/k)\}$ and so $2\sum (h/k) = \sum 1$.

COROLLARY: $2\sum (h/k) = 2 + \sum_{s=2}^{n} \phi(s)$.

For h/k is a term in F_n if $1 \leqq k \leqq n$, $0 \leqq h \leqq k$ and $(h, k) = 1$. For any fixed $k > 1$, the number of terms of the form h/k is $\phi(k)$. So the number of terms of F_n is

$$2+\sum_{s=2}^{s=n}\phi(s).$$

Blake quotes the well-known results:

LEMMA 2. *If h/k and h'/k' are successive terms of F then $kh'-k'h=1$ and $k+k'>n$.*

This lemma is Theorems 28 and 30 of [2].

THEOREM 3. *In F_n the denominator of the immediate predecessor and of the immediate successor of $1/2$ is equal to the greatest odd integer $\leqq n$.*

Let h/k be the immediate predecessor of $1/2$ in F_n. Then by Lemma 2 we have

$$k-2h=1, \tag{1}$$

$$k+2>n. \tag{2}$$

It follows from (1) that k is odd and from (2) that $k\geqq n-1$. But from the definition of F_n we have $k\leqq n$. So k is the greatest odd integer less than or equal to n. The other part of the theorem follows from the observation that $1-(h/k)$ is the immediate successor of $1/2$.

References

1. J. A. Blake, Some characteristic properties of the Farey series, this MONTHLY, 73 (1966), 50–52.

2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed. Oxford, 1960, pp. 23 *et seq.*

ON THE ORDER OF INTEGRAL FUNCTIONS DEFINED BY DIRICHLET SERIES

J. S. GUPTA, Kurukshetra University, India

1. Consider the Dirichlet series $f(s)=\sum_{n=1}^{\infty}a_ne^{s\lambda_n}$ where $\lambda_{n+1}>\lambda_n$, $\lambda_1\geqq 0$, $\lim_{n\to\infty}\lambda_n=\infty$, $s=\sigma+it$, and

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=D<\infty.$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$.

When $D=0$, then [1, p. 4] $\sigma_c=\sigma_a$. If $\sigma_c=\sigma_a=\infty$, $f(s)$ represents an integral function. It will be supposed throughout that for all functions $D=0$ and $\sigma_c=\sigma_a=\infty$.

If ρ be the Ritt order [2, p. 78] of $f(s)$, then [2, p. 79]

$$\limsup_{n\to\infty}\frac{\lambda_n\log\lambda_n}{\log|a_n|^{-1}}=\rho=\limsup_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\sigma},$$

where $M(\sigma)$ is the l.u.b. of $|f(\sigma+it)|$, $-\infty<t<\infty$.

A similar result, for the lower order λ of $f(s)$, namely

$$\liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \lambda = \liminf_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma}$$

holds when [3, Th. A] (i) $\log \lambda_n \sim \log \lambda_{n+1}$ and (ii) $[\log|a_n/a_{n+1}|]/[\lambda_{n+1}-\lambda_n]$ forms a nondecreasing function of n for $n>n_0$.

2. We have the following

THEOREM 1. *If* $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an integral function of order* ρ *and lower order* $\lambda (0\leqq\lambda\leqq\infty)$ *such that*

(i) $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=h$, *where* h *is a positive constant* (*the condition* (i) *implies that* $\log \lambda_n\sim\log \lambda_{n+1}$),

(ii) $\log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ *is a nondecreasing function of* n *for* $n\geqq N$, *then*

$$\frac{\rho}{\lambda} = \lim_{n\to\infty} \frac{\sup}{\inf} \frac{\lambda_n \log\lambda_n}{\log |a_n|^{-1}} = h \lim_{n\to\infty} \frac{\sup}{\inf} \frac{\log \lambda_n}{\log |a_n/a_{n+1}|}. \tag{2.1}$$

Proof. Let

$$\lim_{n\to\infty} \frac{\sup}{\inf} \frac{\log |a_n/a_{n+1}|}{\log \lambda_n} = \frac{\beta}{\alpha} \tag{2.2}$$

and

$$\limsup_{n\to\infty} \frac{\log |a_n|^{-1}}{\lambda_n \log\lambda_n} = \frac{1}{\lambda} = B \text{ say.} \tag{2.3}$$

Therefore it is obvious from (2.2) and from (2.3) that $B\leqq\beta/h$. Let

$$\phi(n) = \frac{\log |a_n/a_{n+1}|}{\lambda_{n+1}-\lambda_n},$$

and suppose first that $0<\beta<\infty$, then $\phi(n)>(\log \lambda_n^{\beta-\epsilon})/(\lambda_{n+1}-\lambda_n)$ for $n=N_1, N_2, \cdots, N_p, \cdots$. Now, for $n\geqq n_0$,

$$h-\epsilon<\lambda_{n+1}-\lambda_n<h+\epsilon. \tag{2.4}$$

Hence, for $N_1>\max(n_0, N)$,

$$\begin{aligned}\log|1/a_n| &> (h-\epsilon)(\phi(N_1)+\phi(N_1+1)+\cdots+\phi(n-1)) - \log|a_{N_1}| \\ &> (h-\epsilon)(n-N_1)\phi(N_1) - \log|a_{N_1}|.\end{aligned}$$

Therefore

$$\frac{\log|1/a_n|}{\lambda_n\log\lambda_n} > \frac{(h-\epsilon)(n-N_p)\phi(N_p)}{\lambda_n\log\lambda_n} - o(1).$$

Now using (2.4) for $N_p, N_p+1, \cdots, n-1$, we get, on adding the $n-N_p$ inequal-

ities thus obtained, $\lambda_n - \lambda_{N_p} < (n - N_p)(h+\epsilon)$. Hence

$$\frac{\log|1/a_n|}{\lambda_n \log \lambda_n} > \frac{(h-\epsilon)(\lambda_n - \lambda_{N_p})\phi(N_p)}{(h+\epsilon)(\lambda_n \log \lambda_n)} - o(1).$$

Let $\lambda_n = \lambda_{N_p} \log \lambda_{N_p} + O(\lambda_{N_p})$, then

$$\frac{\log|1/a_n|}{\lambda_n \log \lambda_n}$$

$$> \frac{(h-\epsilon)(\lambda_{N_p} \log \lambda_{N_p} + O(\lambda_{N_p}) - \lambda_{N_p}) \log \lambda_{N_p}^{\beta-\epsilon}}{(h+\epsilon)(\lambda_{N_p} \log \lambda_{N_p} + O(\lambda_{N_p})) \log (\lambda_{N_p} \log \lambda_{N_p} + O(\lambda_{N_p}))(\lambda_{N_{p+1}} - \lambda_{N_p})} - o(1)$$

$$> \frac{(h-\epsilon)(\lambda_{N_p} \log \lambda_{N_p} + O(\lambda_{N_p}) - \lambda_{N_p})^{(\beta-\epsilon)} \log \lambda_{N_p}}{(h+\epsilon)^2(\lambda_{N_p} \log \lambda_{N_p} + O(\lambda_{N_p})) \log (\lambda_{N_p} \log \lambda_{N_p} + O(\lambda_{N_p}))} - o(1).$$

Hence $B \geqq \beta/h$ which holds also when $\beta = 0$. If β be infinite the above argument, with an arbitrary large number instead of $\beta - \epsilon$, gives that $B = \infty$. Hence we get $B = \beta/h$ i.e. $\beta/h = 1/\lambda$. Similarly it can be shown that $\alpha/h = 1/\rho$ and hence the theorem.

We note the hypotheses of Theorem 1 do not imply that $f(s)$ is of regular growth. In fact we have the following:

THEOREM 2. *There exists an integral function* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *for which* (i) $a_n > 0$, (ii) $\limsup_{n\to\infty}(\log n/\lambda_n) = 0$,

(iii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = h$ *where h is a positive constant,*

(iv) $\log|a_n/a_{n+1}|/(\lambda_{n+1} - \lambda_n)$ *is a steadily increasing function of n, and* (v)$\rho > \lambda$.

Proof. Let $n_1 = 2$, $n_{k+1} = n_k^4$, $(k = 1, 2, \cdots)$, $r_1 = 1$, $r_m = m$ for $n_k \leqq m < n_k^2$

$$r_m = n_{k+1} - \frac{n_{k+1} - m}{((n_{k+1})!)^{(n_{k+1})!}} \quad \text{for } n_k^2 \leqq m < n_{k+1},$$

and let $f(s) = \sum_{n=1}^{\infty} (e^{ns}/r_1 r_2 \cdots r_n)$.

Then clearly all the conditions (i) to (iv) are satisfied. Also

$$\psi(n) = \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} = \frac{\log r_1 + \cdots + \log r_n}{n \log n}.$$

Hence

$$\psi(n_{k+1}) \sim \frac{(n_k^4 - n_k^2) \log n_k^4}{n_k^4 \log n_k^4} \sim 1,$$

$$\psi([n_k^2 \log n_k]) \sim \frac{(n_k^2 \log n_k - n_k^2) \log n_k^4 + O(n_k^2 \log n_k)}{n_k^2 \log n_k \log (n_k^2 \log n_k)} \sim 2.$$

Thus it is easily seen that $\limsup_{n\to\infty} \psi(n) = 2$, $\liminf_{n\to\infty} \psi(n) = 1$. Hence $f(s)$ is an integral function of order 1 and lower order 1/2.

References

1. V. Bernstein, Leçons sur les progrès récents de la théorie des séries de Dirichlet, Gauthier-Villars, Paris, 1933.
2. J. F. Ritt, On certain points in the theory of Dirichlet series, Amer. J. Math., 50 (1928) 73–86.
3. R. S. L. Srivastava, On the order of integral functions defined by Dirichlet series, Proc. Amer. Math. Soc., 12 (1961) 702–708.

FINITE DIMENSIONAL TRANSLATION INVARIANT SPACES

K. O. LELAND, Illinois Institute of Technology

In [1] Anselone and Korevaar give several simple proofs of the following.

THEOREM. *Every finite dimensional translation invariant linear space F of complex valued continuous functions on $R=(-\infty,+\infty)$ is the span of a set of exponential monomials*

$$x^{\mu-1}e^{\rho x}, \qquad \mu=1,\cdots,m(\rho), \qquad \rho=\rho_1,\cdots,\rho_k.$$

It is the purpose of this note to give an even simpler proof of this theorem.

Proof. Let G be the linear space of all continuously differentiable functions g on R such that $g'\in F$. Let $g_0\in G$, and set for $n=1,2,\cdots,x\in R$,

$$g_n(x)=n[g_0(x+1/n)-g_0(x)].$$

Since F is translation invariant, $g_n\in G$ for $n=1,2,\cdots$. Clearly the sequence $g_1,g_2,\cdots$ converges to g_0' uniformly on bounded intervals. Set $I=[-a,a]$, $a>0$, and let $\bar{h}$ be the restriction of h to I for $h\in F$, G. Set $\bar{G}=\{\bar{g};g\in G\}$, and set $\|h\|=\sup\{|h(t)|;t\in I\}$ for $h\in\bar{G}$. Then $\bar{G}$ is a normed linear space. Since $\bar{G}$ is finite dimensional, $\bar{G}$ is a Banach space, and thus the limit $\bar{g}_0'$ of the sequence $\bar{g}_1,\bar{g}_2,\cdots$ must lie in $\bar{G}$.

Let $h\in\bar{G}$. Then $h=\bar{g}$ for some $g\in G$, and $h'=\bar{g}'\in\bar{G}$. For $f\in F, f=g'$ for some $g\in G$, and $\bar{f}=\bar{g}'\in\bar{G}$. Thus the elements of $\bar{G}$ and hence those of F are infinitely differentiable.

Although not needed for the proof, it is easy to prove analyticity at this stage. For $h\in\bar{G}$, set $T(h)=h'$. Then T is a linear operator from $\bar{G}$ into $\bar{G}$. Since $\bar{G}$ is finite dimensional, T must be bounded. Let $f\in F$. Then $\bar{f}\in\bar{G}$, and for all $x\in I$, $n=1,2,\cdots,$

$$|\bar{f}^{(n)}(x)|\leqq\|f^{(n)}\|\leqq\|T\|^n\|\bar{f}\|.$$

Hence the remainder terms of the Taylor series expansion of $\bar{f}$ converge uniformly to 0 on I, and thus the Taylor series of $\bar{f}$ and hence that of f converges uniformly on I to f.

Resuming the proof, let n be the dimension of F, and let $f\in F$. Then $\{f,f',\cdots,f^{(n)}\}$ is a linearly dependent subset of F. Hence there exist $\rho_0,\cdots,\rho_p$, $p\leqq n$, such that

$$\sum_0^p \rho_i f^{(i)} = 0, \tag{1}$$

and $\{f, f', \cdots, f^{(p-1)}\}$ is linearly independent and $\rho_p \neq 0$. Let F_0 be the linear space generated by $\{f, f', \cdots, f^{(p-1)}\}$; differentiation of (1) shows that every element of F_0 satisfies this differential equation.

By direct computation [2] one shows that the set of p monomials

$$x^{k-1}e^{rx}, \qquad k = 1, \cdots, m(r), \qquad r = r_1, \cdots, r_q,$$

where $r_1, \cdots, r_q$ are the roots of $\sum_0^p \rho_i r^i = 0$ of multiplicity $m(r_1), \cdots, m(r_q)$, are a linearly independent set of solutions of (1) and thus generate the p dimensional solution space F_1. Comparing dimensions one concludes that $F_1 = F_0$.

Thus every $f \in F$ belongs to a span of exponential monomials $F_0 = F_1$ contained in F. It follows that F is the span of all exponential monomials which it contains.

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References

1. P. M. Anselone and J. Korevaar, Translation invariant subspaces of finite dimension, Proc. Amer. Math. Soc., 15 (1964) 747–752.

2. T. M. Apostol, Calculus, Volume II, Blaisdell, New York, 1962, 342–350.

A COMPACTNESS CONDITION WEAKER THAN T_2

P. R. MEYER, Hunter College of C.U.N.Y.

The purpose of this note is to show the independence of the condition

(I) *The intersection of two compact sets is compact* from some standard separation axioms weaker than T_2. (The condition (I) arose from [2].)

Following [4], we say that a topological space is a KC space if every compact set is closed, and a US space if no sequence converges to two distinct points. Then $T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$, but no converse implication holds.

THEOREM. *In a KC space* (I) *holds, but a US space in which* (I) *holds need not be a KC space. Furthermore, a US space need not satisfy* (I), *and a T_1 space satisfying* (I) *need not be a US space.*

Proof. That KC implies (I) is trivial; Example 1 is a US space which satisfies (I) but is not KC. Example 2 provides a US space which fails to satisfy (I). Example 1 of [4] is a T_1 space which satisfies (I) but is not a US space, as was pointed out by the referee.

Example 1. This example was used by Cullen [1] to show that $US \not\Rightarrow KC$. Let $X = R \cup \{p\}$, where R is the set of all real numbers and p is any element not in R. The open sets in X are as follows: (i) a set not containing p is open iff it is open in the usual topology for R,(ii) a set containing p is open iff its complement

is the union of the ranges of a finite number of convergent sequences in R together with their limits. The set $[0, 1]$ is compact but not closed, since it has p as a limit point. But p is the only posssible missing limit point for a compact set which is not closed. (If A is compact with limit point x, $x \neq p$, then we can find a sequence of distinct points (x_n) which converges to x. The infinite set $\{x_n\}$ must have a limit point in A, but x is its only limit point in X. Therefore x is in A.) Thus for any compact set A we have $A \cap R$ is closed in R. Conversely, if B is closed in R, then $B \cup \{p\}$ is closed in the compact space X and hence is compact. It now follows by cases that the intersection of two compact sets is compact.

Example 2. Let W^* be the set of all ordinals $\leqq \Omega$ with the order topology (Ω denotes the first uncountable ordinal). In $W^* \times \{0, 1\}$ identify $(\alpha, 0)$ and $(\alpha, 1)$ for all $\alpha \neq \Omega$; let α denote the resulting equivalence class and X the quotient space. In X sequential limits are unique, because $(\Omega, 0)$ and $(\Omega, 1)$ are the only two points which cannot be separated by open sets. (No sequence in W^* converges to Ω.) Let $A_i = \{\alpha: \alpha < \Omega\} \cup \{(\Omega, i)\}$, $i = 0, 1$. Each A_i is compact (it is homeomorphic to W^*), but $A_0 \cap A_1 = \{\alpha: \alpha < \Omega\}$ is not compact. That X is locally compact follows directly from the corresponding property of W^*. The relevant facts about W^* are proved for example in [**3**].

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References

1. Helen F. Cullen, Unique sequential limits, Boll. Un. Mat. Ital., (3) 20 (1965) 123–124.
2. T. S. Frank and W. A. Lopez, Problem 5444, this MONTHLY, 73 (1966) 1124.
3. L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand, Princeton, N. J., 1960, pp. 72–74.
4. Albert Wilansky, Between T_1 and T_2, this MONTHLY, 74 (1967) 261–266.

SOLUTIONS OF $A^k + B^k = C^k$ IN $n \times n$ INTEGRAL MATRICES

E. D. BOLKER, Bryn Mawr College

In this note we generalize an observation of R. Z. Domiaty [**2**], who discussed the Diophantine equation in the title for $k = 4$ and $n = 2$. A subsequent Elementary Problem proposed by J. L. Brenner and B. Jacobson [**1**] considered the cases $n = 2$, arbitrary $k > 0$ and the case $k = n$. Their problem appeared after a preliminary version of this note was accepted for publication.

Let $\mathrm{Z}(n)$ be the ring of $n \times n$ matrices with integer entries. $\mathrm{Z}(1) = \mathrm{Z}$.

THEOREM. *There are nonsingular elements A, B and C of $\mathrm{Z}(n)$ such that*

$$A^{2n} + B^{2n} = C^{2n}. \tag{1}$$

Proof. Let P be the representation of the symmetric group on n letters, S_n, by permutation matrices. That is, for $\pi \in S_n$, $P_\pi(i, j) = \delta_{\pi(i),j}$. Then P is an

antihomomorphism of S_n into the group of units in $Z(n)$. Suppose $a \in Z^n$; think of a as an n-tuple and as a function from $\{1, \cdots, n\}$ to Z. Write $[a]$ for the diagonal matrix with diagonal $a(1), \cdots, a(n)$. Thus if $a = \langle \alpha, \beta, \gamma \rangle$ then

$$[a]P_{(123)} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}.$$

Note that for $\pi \in S_n$, $P_\pi[a]P_\pi^{-1} = [a \circ \pi]$. Now choose a cyclic $\pi \in S_n$. Then

$$\begin{aligned} ([a]P_\pi)^n &= [a]P_\pi[a]P_\pi \cdots [a]P_\pi \\ &= [a](P_\pi[a]P_\pi^{-1})(P_\pi^2[a]P_\pi^{-2}) \cdots (P_\pi^{n-1}[a]P_\pi) \\ &= [a][a \circ \pi][a \circ \pi^2] \cdots [a \circ \pi^{n-1}]. \end{aligned}$$

This product of diagonal matrices has

$$a(j)a(\pi(j)) \cdots a(\pi^{n-1}(j))$$

for its j, jth entry. Since π is cyclic we have shown

$$([a]P_\pi)^n = a(1)a(2) \cdots a(n)I.$$

Now let α, β, γ be a Pythagorean triple: $\alpha^2 + \beta^2 = \gamma^2$. Let $a = \langle \alpha, 1, \cdots, 1 \rangle$ and $A = [a]P_\pi$; similarly define B and C. Then A, B and C satisfy (1).

REMARK 1. We have clearly proved that $A^{pn} + B^{pn} = C^{pn}$ has nonsingular solutions in $Z(n)$ whenever $\alpha^p + \beta^p = \gamma^p$ has a nonzero solution in Z. When $p = 1$ this remark settles the second problem in [1].

REMARK 2. If we demand only that A^k, B^k and C^k be nonzero then the equation in the title has trivial solutions for $n \geqq 2$ and arbitrary positive k. Simply set the solution

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^k + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^k$$

into the upper left hand corners of $n \times n$ matrices which are zero elsewhere. This settles the first problem in [1]. Of course nonsingularity is sacrificed.

REMARK 3. The first interesting case not covered by Remark 1 is $k = 3$, $n = 2$. I suspect that no nonsingular solution exists. Note that a proof must either use or reprove the impossibility of $\alpha^3 + \beta^3 = \gamma^3$ in Z.

REMARK 4. $Z(n)$ resembles the ring of integers in an algebraic number field of degree n since each of its elements satisfies a monic integral polynomial of degree n. The equation in the title is just one Diophantine equation, albeit a famous one, in that noncommutative number theory.

References

1. J. L. Brenner and Bernard Jacobson, Problem E2030, this MONTHLY, 74 (1967) 1133.

2. R. Z. Domiaty, Solutions of $x^4 + y^4 = z^4$ in 2×2 Integral Matrices, this MONTHLY, 73 (1966) 631.

THE MAXIMAL T_0 (RESPECTIVELY, T_1) SUBSPACE LEMMA IS EQUIVALENT TO THE AXIOM OF CHOICE

PAUL S. SCHNARE, University of Florida

The following problem is considered. Given some topological property P, is this statement true: (MP) every topological space (X, τ) has a subspace $(Y, \tau | Y)$ (where $\tau | Y$ is the relativization of τ to Y) with property P which is maximal (with respect to inclusion)? ([2] is a blanket topological reference.) In a recent paper [1], Gaifman proved that the axiom of choice (AC) implies MT_0 and noted that AC implies MT_1. One may give a quick proof of these facts and by proving the converses establish the apparently new [3]:

THEOREM A. *AC iff MT_0 iff MT_1.*

Proof. ($AC \rightarrow MT_0$ (resp. MT_1)). Let $\mathcal{A} = \{A \subset X: (A, \tau | A) \text{ is } T_0 \text{ (resp., } T_1)\}$. Then, $\mathcal{A}$ is a family of finite character so by Tukey's Lemma has a maximal element. (See [2], pp. 32–3.)

(MT_0 (resp., MT_1)$\rightarrow AC$). Given $\{X_a: a \in A\}$ a disjoint family of nonempty sets. Give $X = \bigcup\{X_a: a \in A\}$ the topology $\tau = \{U \subset X: U \cap X_a \neq \emptyset \text{ implies } X_a \subset U\}$. Let T be a maximal T_0 (resp., T_1) subspace of X. By maximality T intersects each X_a but, clearly, since T is T_0 (resp., T_1) in exactly one point.

A source of counterexamples is provided by the easy:

THEOREM B. *Let P be possessed by every finite discrete space but not by a countably infinite minimal T_1 space. Then, MP is false.*

COROLLARY. *MP is false for $P = T_2$, regular, completely regular, normal, T_3, T_4, paracompact, metrizable, totally disconnected or discrete.*

REMARKS. MP is true for P = connected, as is well known, the maximal connected subspaces being connected components. By considering discrete spaces one readily sees that MP is false for P = compact, countably compact, Lindelöf, separable, or second countable. The reader may wish to prove that the Cartesian product of the real numbers an infinite number of times with the box topology (see [2], pp. 107–108) has no maximal first countable subspace.

Also, it can be shown that MP is false for P = local connectedness or P = local compactness. A totally disconnected T_1 space with no isolated points (e.g., the Cantor discontinuum) never has a maximal locally connected subspace. The space of rational numbers in the usual topology has no maximal locally compact subspace.

This research was supported in part by an NSF Science Faculty Fellowship.

References

1. H. Gaifman, Remarks on complementation in the lattice of all topologies, Canad. J. Math., 18 (1966) 83–88.

2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N.J., 1955.

3. H. Rubin and J. Rubin, Equivalents of the axiom of choice, Studies in Logic, North Holland, Amsterdam, 1963.

ON VON NEUMANN'S AXIOM SYSTEM FOR SET THEORY

AZRIEL LÉVY, The Hebrew University of Jerusalem

In [2] and [3] von Neumann gave, for the first time, an axiom system for what we call now set theory with classes. Here we shall prove that one of his axioms is redundant. Von Neumann used functions instead of classes, but this aspect of his system is rather inessential and has no bearing on our discussion. Therefore, we shall deal with a set theory with sets and classes like that of Gödel [1]. We shall assume as axioms the axioms of groups A, B and axiom 3 of group C of [1] (these are, essentially, the axioms of extensionality, pairing, predicative comprehension and the power-set axiom), as well as the following axiom of von Neumann (Axiom IV2 of [2]) which, formulated in our terminology, reads

N A class A is not a set if and only if there is a function F which maps A onto the class V of all sets.

As von Neumann proved in [3] (see Footnote 8 of [3]), N, together with the other axioms mentioned above, implies the axioms of replacement and choice (Axioms C4 and E of [1]). We shall prove here that N implies also the axiom of union (Axiom C2 of [1]). Accordingly, the axiom of union is redundant in von Neumann's system (of [2]).

Let us first outline the idea of the proof. Given a set a we want to prove that its union-class $\cup a$ is a set. Von Neumann used the Burali-Forti paradox to show that the class On of all ordinals is not a set and hence, as a consequence of N, it is equinumerous to the class V of all sets. Because of this fact it is easily seen that it is enough to verify the axiom of union for the case where a is a set of sets of ordinals. For this case the axiom can be seen to follow rather directly from the assumption that every set of ordinals is bounded by some ordinal. It is the latter assumption which we want to verify now. Let us notice that this is just the assumption that the "ordinal" On is regular. Suppose On is singular, then, informally, $On=\sum_{\lambda<\beta}c_\lambda$ where $\{c_\lambda\}_{\lambda<\beta}$ is an increasing sequence of cardinal numbers (which are sets) and β is a limit number $<On$. By the Zermelo-König inequality we have $On<\Pi_{\lambda<\beta}c_\lambda\leqq\overline{\overline{V}}$, which is a contradiction, since by N and the Burali-Forti paradox $\overline{\overline{On}}=\overline{\overline{V}}$.

The rigorous proof of the axiom of union will be even simpler than the outline above in the sense that it will avoid all the advanced technical notions of set theory such as cardinal numbers and regular ordinals.

The class existence theorem (see Theorem M1 in [1], p. 8) uses only the axioms of groups A and B. Therefore we can now introduce, by means of explicit definitions which do not involve quantifying over all classes, classes and functions, where by functions we mean classes of ordered pairs which satisfy the obvious requirement ([1], Def. 4.61). As von Neumann proved in [3], the axiom of replacement, and the axiom of subsets which follows from it, are now provable.

One defines a set x to be an *ordinal* if x is *transitive* (i.e., if $z \in y \in x$ then $z \in x$) and if the membership relation ϵ well-orders x. Then one develops the theory of ordinal numbers (as in [1], Ch. III, with obvious modifications owing to the different definition of the notion of ordinals which allows us to dispense with Axiom D). In this development the axiom of union is not used; to prove that for every ordinal x its successor $x \cup \{x\}$ is a set we use the power-set axiom (incidentally, this is our only use of the power-set axiom). The class On of all ordinals is easily seen to be transitive and well-ordered by the ϵ-relation, hence if On were a set it would be an ordinal number and we would have $On \in On$, and thus On would have a member $x(=On)$ such that $x \in x$, contradicting what we assumed that On is well-ordered by ϵ. Thus On is a proper class and therefore, by N, there is a function F mapping On on the class V of all sets. Let G be the inverse of F defined on V by $G(x) =$ the least ordinal y such that $F(y) = x$.

Lemma. *Every set of ordinals is bounded.*

Proof. Let b be an unbounded set of ordinals. Every function defined on b is a set, by the axiom of replacement. Let us define a function h on b as follows. For $x \in b$ let $h(x)$ be the least ordinal which is not in $\{F(y)(x) \mid y \in x\}$ (where F is the function mentioned above which maps On on V; if $F(y)$ is a function and x belongs to its domain then $F(y)(x)$ is the value of that function for x, and $F(y)(x)$ is 0 otherwise). $\{F(y)(x) \mid y \in x\}$ is a set, by the axiom of replacement, hence it does not contain all the ordinals. Let $\beta = G(h)$ then $h = F(\beta)$. Since b is unbounded we have that for some $\gamma \in b$, $\beta < \gamma$. By definition of $hh(\gamma)$ $\notin \{F(x)(\gamma) \mid x \in \gamma\}$ hence $h(\gamma) \neq F(\beta)(\gamma) = h(\gamma)$, which is a contradiction.

Theorem. *The axiom of union holds.*

Proof. Let a be a set, we want to prove that the class $\cup a$ is a set too. For every $b \in a$ the class $G[b]$ (the set of all images of members of b under G) is a set, by the axiom of replacement, hence, by the lemma, it is bounded by some ordinal. For $b \in a$ let $H(b)$ be the least strict upper bound of $G[b]$. Again by the axiom of replacement and the lemma the class $H[a]$ is bounded by some ordinal α. We have now, obviously, $F[\alpha] \supseteq \cup a$ and hence, by the axiom of replacement $\cup a$ is a set.

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References

1. K. Gödel, The consistency of the continuum hypothesis, Princeton, 1940.

2. J. von Neumann, Eine Axiomatisierung der Mengenlehre, J. Reine Angew. Math., 154 (1925) 219–240. Also in [4], 35–56.

3. ———, Die Axiomatisierung der Mengenlehre, Math. Z., 27 (1928) 669–752. Also in [4], 339–422.

4. ———, Collected Works, Vol. I, Oxford 1961.

ON THE DISTRIBUTION OF PRIMES

ABRAHAM WAKSMAN, Stanford Research Institute

Let Q denote the set of primes (including 1) and let Z_+ denote the nonnegative integers. We would like to find all integers K such that $Q \subset S_K$ where

$$S_K = \{m: m = Kn + p,\ p \in Q,\ p \nmid K,\ p < K,\ n \in Z_+\} \cup \{p: p \in Q,\ p \mid K\}.$$

This is equivalent to requiring that for such an integer K, if we generate an array of the form

$$\begin{array}{cccc} 1 & 2 & 3 & \cdots K \\ (K+1) & (K+2) & (K+3) & \cdots 2K \\ (2K+1) & (2K+2) & (2K+3) & \cdots 3K \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{array}$$

etc.,

all the columns beginning with a prime p such that $p \mid K$ will contain no other prime numbers, and all the columns beginning with composite numbers will never contain a prime number.

THEOREM. *For a positive integer K, $Q \subset S_K$ if and only if $p_j^2 \geqq K$, where $p_j = \min\{p: p \in Q,\ p < K,\ p \nmid K\}$.*

Proof. $p_j \nmid K \Rightarrow (p_j, K) = 1 \Rightarrow (p_j^2, K) = 1 \Rightarrow \{m: m = Kn + p_j^2,\ n \in Z_+\}$ contains infinitely many primes by Dirichlet's theorem.

Thus if $p_j^2 < K$, the column under p_j^2 will contain infinitely many primes, contradicting our requirement, since p_j^2 is a composite number.

Conversely, if $p_j^2 \geqq K$, for all $a < K$, $a \in (Z_+ - Q)$, we have $(a, K) > 1$, since the prime factorization of a contains at least one factor of K. Hence $\{m: m = Kn + a,\ n \in Z_+\}$ never contains a prime number.

In search of such K's we attempt to maximize p_j^2 by requiring that

(a) the prime factors of K will include *all* the primes from 1 to some p_r, i.e., $p_1 \times p_2 \times \cdots \times p_r \mid K$ and

(b) observing that $p_{r+1} \leqq 2p_r - 1$ and $p_r - 2p_{r-1} - 1$ since there is always a prime between n and $2n$.

Thus for $p_j^2 \geqq K$ we require that

$$1 \leqq \left[\frac{p_j^2}{K}\right] \leqq \left[\frac{p_{r+1}^2}{p_1 \times \cdots \times p_r}\right] \leqq \left[\frac{4p_r}{p_1 \times \cdots \times p_{r-1}}\right]$$
$$\leqq \left[\frac{8}{p_1 \times \cdots \times p_{r-2}}\right].$$

Since $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $p_4 = 5$, $p_5 = 7$, etc., and

$$1 \times 2 \times 3 \times 5 = 30 > 8, \qquad 1 \times 2 \times 3 = 6 < 8,$$

the upper bound on $p_1 \times p_2 \times p_3 \times \cdots \times p_{r-2}$ is $1\times 2\times 3=6$ and the upper bound on $K=p_1\times p_2\times p_3\times \cdots \times p_r$ is $1\times 2\times 3\times 5\times 7=210$. This proves that there is no K larger than 210 of the form required. A computer search for K among the integers smaller than 210 reveals the existence of only 9 such numbers, namely 2, 3, 4, 6, 8, 12, 18, 24, 30. The first few rows of the array for $K=30$ are given.

EXAMPLE FOR $K=30$

1 *represents a prime*

0 *represents a composite number*

1	1	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0

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ACKNOWLEDGMENT OF PRIORITY

My attention has recently been directed to the paper "Eine Erweiterung eines Theorems von Steinhaus-Rademacher" by H. Hadwiger, Comment. Math. Helv., 19 (1946) 236–239, which contains a theorem essentially the same as Theorem 2 of my note "Similar Configurations in Measurable Sets," in this MONTHLY, 75 (1968), 31–34. Although the proofs are substantially different, Professor Hadwiger clearly has prior claim to the result.

T. A. BICK

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the MONTHLY. *Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.*

STRAIGHT SETS

DUANE M. BROLINE, South Dakota School of Mines and Technology

The purpose of this paper is to extend the notion of collinearity from Euclidean k-space (E^k) to arbitrary metric spaces. This is done by observing that for every three points of a straight line, one of the three possible triangle inequalities involving these points is an equality.

DEFINITION. *A subset M of a metric space (X, ρ) is a straight set provided for every three points x, y, and z in M at least one of the following holds:*

(i) $\rho(x, y)+\rho(y, z)=\rho(x, z)$,
(ii) $\rho(x, z)+\rho(z, y)=\rho(x, y)$,
(iii) $\rho(y, x)+\rho(x, z)=\rho(y, z)$.

A straight set is of TYPE I provided it has only four members, i.e. $M=\{a, b, c, d\}$, and $\rho(a, b)=\rho(c, d)=m$, $\rho(a, d)=\rho(b, c)=n$, and $\rho(a, c)=\rho(b, d)=m+n$, where m and n are positive real numbers. A straight set is of TYPE II if and only if it is not of Type I.

In a straightforward manner it is possible to prove the following

THEOREM 1. *A straight set of Type I is not the proper subset of any straight set.*

THEOREM 2. *A straight set is of Type II if and only if it is isometric with a subset of a line in E^k.*

Thus we are able to characterize straight sets. Those of Type I are explicitly defined and are not isometric with a subset of E^1. Those of Type II are metrically equivalent to subsets of the real line.

The author is presently doing graduate work at the University of Wisconsin, Madison.
Mail Address: Dept. of Math., University of Wisconsin, Madison, Wis. 53706.

SPHERES UNIFORMLY WEDGED BETWEEN BALLS ARE TAME IN E^3

H. C. GRIFFITH, Florida State University

A 2-sphere S in E^3 is said to be wedged between tangent balls (of radius δ) at $p \in S$ if there are two round balls (of radius δ) which are tangent to each other at p and which, except for p, lie in opposite components of $E^3 \backslash S$. Bing [1] has asked whether or not a sphere which is wedged between tangent balls at each of its points must be tame, i.e., must be the image of a polyhedron under a homeomorphism of E^3 onto itself.

THEOREM. *If there is a $\delta > 0$ such that the 2-sphere S in E^3 is wedged between tangent balls of radius δ at each of its points, then S is tame.*

The proof is based on a result due to Burgess [2]. An example of a sphere which is wedged between tangent balls at each of its points, but for which the δ of the theorem does not exist, is given. Since this sphere is tame, Bing's question remains unanswered.

Mail Address: Mathematics Department, Florida State University, Tallahassee, Fla. 32306.

References

1. R. H. Bing, Spheres in E^3, this MONTHLY, 71 (1964) 353–363.
2. C. E. Burgess, Characterizations of tame surfaces in E^3, Trans. Amer. Math. Soc., 114 (1965) 80–97.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

ON THE EVALUATION OF THE WRONSKIAN DETERMINANT

HARRY HOCHSTADT, Polytechnic Institute of Brooklyn

We consider a linear homogeneous system of n first order linear homogeneous differential equations.

$$x_i' = \sum_{j=1}^{n} a_{ij}(t)x_j, \qquad i = 1, 2, \cdots, n. \tag{1}$$

It will be convenient to use the more compact vector notation

$$X' = A(t)X, \tag{2}$$

where X is a vector with components x_i and $A(t)$ is the matrix $(a_{ij}(t))$. Let $X_1, X_2, \cdots, X_n$ be n solutions of (2). The matrix

$$\Phi = (X_1 X_2 \cdots X_n), \tag{3}$$

whose columns are these n solutions, is known as a fundamental matrix if it is nonsingular. To determine whether it is or is not singular one customarily considers its determinant

$$W(t) = \det \Phi. \tag{4}$$

The latter is known as the Wronskian Determinant or the Wronskian. It is well known that

$$W(t) = W(t_0) \exp \int_{t_0}^{t} \sum_{k=1}^{n} a_{kk}(t)\,dt, \tag{5}$$

where $\sum_{k=1}^{n} a_{kk}(t) \equiv \operatorname{tr} A(t)$ is known as the trace of $A(t)$. From (5) we observe immediately that if $W(t_0)$ does not vanish then $W(t)$ will not vanish in the whole domain of existence of the n solutions X_i. If, however, it vanishes at a single point it must vanish identically.

It is the purpose of this article to discuss a number of different derivations of (5). The first to be treated is standard and relatively elementary, but requires a number of rather clumsy manipulations. The second is a variant on the first. The last two are believed to be new and require fewer manipulations, at the added expense of more conceptual reasoning. The third is an induction scheme and the fourth depends on vital and general properties of the Wronskian. For general background to this material the books by Hochstadt [1], John [2], Pontryagin [3], may be consulted.

Method 1. Upon differentiating $W(t) = \det \Phi$ we find $W'(t) = \sum_{i=1}^{n} W_i(t)$. $W_i(t)$ has the same rows as $W(t)$, except for the ith row. Its ith row is the derivative of the ith row of $W(t)$. Upon replacing these differentiated terms by suitable linear combinations of undifferentiated terms, as given by (1), and using standard row manipulations we obtain

$$W'(t) = \left(\sum_{k=1}^{n} a_{kk}(t) \right) W(t) = \operatorname{tr} A(t) W(t), \tag{6}$$

from which (5) follows immediately. For details we refer to [1], [3].

Method 2. This method is a variant on the above found in [2]. We note that the matrix Φ satisfies the differential equation

$$\Phi' = A(t)\Phi.$$

Then, assuming that $A(t)$ is continuous, we see that

$$\Phi(t + h) = \Phi(t) + h\Phi'(t) + o(h)$$

by a suitable mean value theorem. It follows that

$$\Phi(t + h) = (I + hA)\Phi(t) + o(h).$$

Next we take determinants of both sides, and use the fact that the determinant of a product of matrices is the product of their determinants.

$$W(t+h) = \det(I+hA)W(t) + o(h).$$

For small values of h we find that

$$\det(I+hA) = 1 + h\sum_{1}^{n} a_{kk}(t) + o(h).$$

Finally

$$\frac{W(t+h)-W(t)}{h} = \sum_{k=1}^{n} a_{kk}(t)W(t) + o(1). \tag{7}$$

Letting $h \rightarrow 0$ we again obtain (6) from which (5) follows.

We note that in both methods we are forced to deal with specific and clumsy determinant manipulations.

Method 3. This method is based on a mathematical induction scheme. For $n=1$ we have $x'=a(t)x$ and

$$W(t) = x(t) = x(t_0)\exp\int_{t_0}^{t} a(t)dt = W(t_0)\exp\int_{t_0}^{t} a(t)dt.$$

For general values of n we return to (2). Suppose $\overline{X}$ is some solution, and, without loss of generality, we assume that $\bar{x}_n(t_0) \neq 0$. Then $\bar{x}_n(t)$, being continuous, will not vanish in a neighborhood of t_0. We now introduce the matrix

$$\Psi(t) = \begin{bmatrix} 1 & 0\cdots 0 & \bar{x}_1 \\ 0 & 1\cdots 0 & \bar{x}_2 \\ 0 & 0\cdots 0 & \bar{x}_3 \\ \cdot & \cdots\cdots & \cdot \\ 0 & 0\cdots 1 & \bar{x}_{n-1} \\ 0 & 0\cdots 0 & \bar{x}_n \end{bmatrix}$$

whose inverse is given by

$$\Psi^{-1}(t) = \begin{bmatrix} 1 & 0\cdots 0 & -\bar{x}_1/\bar{x}_n \\ 0 & 1\cdots 0 & -\bar{x}_2/\bar{x}_n \\ 0 & 0\cdots 0 & -\bar{x}_3/\bar{x}_n \\ \cdot & \cdots\cdots & \cdot \\ 0 & 0\cdots 1 & -\bar{x}_{n-1}/\bar{x}_n \\ 0 & 0\cdots 0 & 1/\bar{x}_n \end{bmatrix}$$

and let $X=\Psi(t)Y$. Substitution in (2) leads to an equation for Y.

$$Y' = [\Psi^{-1}(t)A(t)\Psi(t) - \Psi^{-1}(t)\Psi'(t)]Y \equiv B(t)Y \tag{8}$$

and $B(t)$ is defined by (8). The method being used is one that is a standard method of reduction of order of the type discussed in [1]. An inspection of $\Psi(t)$ makes it evident that the solution $\overline{X}$ for (2) is equivalent to $\overline{Y}$ in (8) where

$$\overline{Y} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}. \tag{9}$$

Insertion of $\overline{Y}$ in (8) shows that the last column of $B(t)$ consists of zero elements. Now $\Psi^{-1}A\Psi$ is a matrix similar to A and it is well known that the trace tr A is invariant under similarity transformations. Then

$$\operatorname{tr} \Psi^{-1}(t)A(t)\Psi(t) = \operatorname{tr} A(t).$$

It is immediate that $\operatorname{tr} \Psi^{-1}(t)\Psi'(t) = \bar{x}_n'/\bar{x}_n$ so that

$$\operatorname{tr} B(t) = \operatorname{tr} A(t) - \frac{\bar{x}_n'}{\bar{x}_n} \cdot \tag{10}$$

If $\Phi_x(t)$ is given by (3) and $\Phi_y(t)$ is defined by $\Phi_x(t)=\Psi(t)\Phi_y(t)$ then

$$W_x(t) = \det \Psi(t)W_y(t) = \bar{x}_n W_y(t), \tag{11}$$

where $W_x(t)$ and $W_y(t)$ denote the Wronskians corresponding to (2) and (8) respectively.

The first $n-1$ scalar equations in (8) can be written as follows

$$y_i' = \sum_{j=1}^{n-1} b_{ij}(t)y_j, \qquad i = 1, 2, \cdots, n-1. \tag{12}$$

Let $W_B(t)$ be the Wronskian corresponding to (12). We assume that for systems of order $n-1$ the basic formula (5) holds. Then

$$W_B(t) = W_B(t_0) \exp \int_{t_0}^{t} \sum_{k=1}^{n-1} b_{kk}(t)dt. \tag{13}$$

But since $b_{nn}=0$ (in fact $b_{in}=0$ for all i)

$$W_B(t) = W_B(t_0) \exp \int_{t_0}^{t} \sum_{k=1}^{n} b_{kk}(t)dt = W_B(t_0) \exp \int_{t_0}^{t} \operatorname{tr} B(t)dt.$$

Using (10) in the above

$$W_B(t) = \frac{W_B(t_0)}{\bar{x}_n} \exp \int_{t_0}^{t} \operatorname{tr} A(t)dt. \tag{14}$$

Inspection of (8), (9) and (12) shows that

$$W_y(t) = W_B(t) \tag{15}$$

since the last column of $W_y(t)$ is given by (9). Combining (11), (14) and (15) finally shows that

$$W_x(t) = W_x(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(t)dt$$

showing that if (5) holds for $n=1$ and $n-1$ it must hold for n.

Finally we turn to the last method which entails the least amount of calculation.

Method 4. We consider two matrices Φ_1 and Φ_2 of type (3) where Φ_2 is nonsingular. We will show that a constant matrix C exists such that

$$\Phi_1 = \Phi_2 C. \tag{16}$$

To see this, we differentiate both sides of (16).

$$\Phi_1' = A\Phi_1 = \Phi_2' C + \Phi_2 C' = A\Phi_2 C + \Phi_2 C'$$

so that $\Phi_2 C' = 0$. Since Φ_2 is nonsingular C must be constant. Taking determinants in (16) we have $W_1(t) = kW_2(t)$, where $k = \det C$. If at some point t_0, $W_1(t_0)$ and $W_2(t_0)$ do not vanish we have by differentiation and elimination of the constant k

$$\frac{W_1'(t)}{W_1(t)} = \frac{W_2'(t)}{W_2(t)}\,. \tag{17}$$

From (17) we can conclude that

$$W_1'(t)/W_1(t) = f(t), \tag{18}$$

where $f(t)$ is a function that is independent of the choice of the Wronskian $W_1(t)$, in view of the fact $W_2(t)$ on the right of (17) can be replaced by any other nonvanishing Wronskian.

To determine $f(t)$ in (18) we proceed as follows. Select Φ in such a way that at $t=\tau$

$$\Phi(\tau) = I, \tag{19}$$

so that $W(\tau) = 1$. Differentiating $W(t)$ by columns we note that

$$W'(t) = \sum_{k=1}^{n} W^{(k)}(t), \tag{20}$$

where the columns in $W^{(k)}(t)$ are those in $W(t)$, except for the kth which is replaced by its derivative. Then if $\Phi(t) = (X_1 X_2 \cdots X_k \cdots X_n)$

$$
\begin{aligned}
W^{(k)}(t) &= \left| X_1 X_2 \cdots X_k' \cdots X_n \right| \\
&= \left| X_1 X_2 \cdots A X_k \cdots X_n \right| .
\end{aligned}
\tag{21}
$$

From (19) we observe that the columns $X_k(t)$ are unit vectors such that

$$
X_j(\tau) = \begin{bmatrix} \delta_{1j} \\ \delta_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \delta_{nj} \end{bmatrix} \qquad \text{where } \delta_{ij} = 0, \quad i \neq j \\ \qquad\qquad = 1, \quad i = j.
$$

Then

$$
A(\tau) X_k(\tau) = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{nk} \end{bmatrix}
\tag{22}
$$

Then (21) and (22) combined show that $W^{(k)}(\tau) = a_{kk}(\tau)$. The latter coupled with (20) and (18) shows that

$$
f(\tau) = \frac{W'(\tau)}{W(\tau)} = \sum_{k=1}^{n} a_{kk}(\tau) = \operatorname{tr} A(\tau).
$$

Since τ is an arbitrary point $W'(t) = f(t) W(t) = \operatorname{tr} A(t) W(t)$ which leads us back to (5).

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References

1. H. Hochstadt, Differential Equations—A Modern Approach, Holt, Rinehart and Winston, New York, 1964.

2. F. John, Ordinary Differential Equations, Lecture Notes: New York University, 1965.

3. L. S. Pontryagin, Ordinary Differential Equations, Addison-Wesley, Reading, Mass, 1962.

A SIMPLE IRRATIONALITY PROOF FOR QUADRATIC SURDS

M. V. SUBBARAO, University of Missouri

Let N be a positive integer which is not a square of another integer. If $\sqrt{N}$ is rational, we will obtain contradictions in three ways, thus providing three different proofs for the irrationality of $\sqrt{N}$.

Write $\sqrt{N} = a/b$, where the fraction on the right is chosen so that:

1. The numerator a is the smallest possible positive integer (for the first proof);

2. The denominator b is the smallest possible positive integer (for the second proof);

3. The sum of the numerator and denominator is the smallest possible (for the third proof).

Since $a^2=Nb^2$ we have $a^2-Aab=Nb^2-Aab$, where A is the unique positive integer given by $A<\sqrt{N}<A+1$. Hence, $a(a-Ab)=b(Nb-Aa)$ giving $\sqrt{N}=a/b=(Nb-Aa)/(a-Ab)$. But, in this new expression for $\sqrt{N}$, the numerator $Nb-Aa$ is less than a, the denominator $a-Ab$ is less than b, and the sum of the numerator and denominator is less than $a+b$—three contradictions to complete the three proofs!

Whether this kind of reasoning can be extended to establish the irrationality of the kth root of a non-kth power integer for $k>2$ is an open question; the writer's attempts in this direction ran into difficulties.

LEAST SQUARES LINE BY GRAPHICAL METHOD

J. B. WILSON, North Carolina State University

1. Introduction. To find the best fitting line $y=ax+b$ for a set of points (x_k, y_k) of equal weight, $k=0, 1, \cdots, n$, according to the theory of least squares, one determines the coefficients a and b from the normal equations

$$\begin{aligned} \sum_{k=0}^{n} y_k &= a \sum_{k=0}^{n} x_k + (n+1)b \\ \sum_{k=0}^{n} x_k y_k &= a \sum_{k=0}^{n} x_k^2 + b \sum_{k=0}^{n} x_k. \end{aligned} \tag{1}$$

The graphical procedure discussed below for determining this line has been in use for some time, but apparently is not widely known. The procedure and its justification are felt to be of classroom interest.

2. Procedure. For convenience, the abscissa x_0 is taken to be zero, there being no loss of generality. The difference x_k-x_{k-1}, assumed to be constant, is denoted by h, and the given points (x_k, y_k) are denoted by P_k.

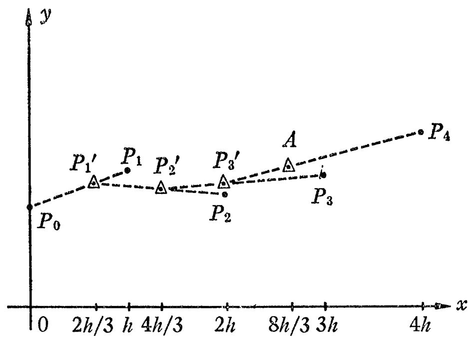
Let P_1' denote the point on the segment P_0P_1 with abscissa $2h/3$, P_2' the point on $P_1'P_2$ with abscissa $4h/3$, and, in general, P_k' the point on $P_{k-1}'P_k$ with abscissa $2kh/3$. The final such point is the point A (or P_n') on $P_{n-1}'P_n$ with abscissa $2nh/3$. A point B is located by a similar "two-thirds" procedure, but beginning at P_n, instead of P_0, and proceeding to the left.

The line drawn on points A and B is the required least squares line. The adjoining figure illustrates the procedure for finding point A in the case of five points. For this case we would have

$$A(8h/3, (y_0 + 2y_1 + 3y_2 + 4y_3 + 5y_4)/15)$$

and

$$B(4h/3,\ (5y_0 + 4y_1 + 3y_2 + 2y_3 + y_4)/15).$$



3. Justification. The ordinate of A is given by $\sum_{k=0}^{n} c_k y_k$, where

$$c_k = 2(k + 1)/(n + 1)(n + 2), \qquad k = 0, 1, \cdots, n. \tag{2}$$

This result is established by induction, considering, for a given value of k, the coefficient of y_k in the expression for the ordinate y_i' of P_i', $i=k, k+1, \cdots, n$. The coefficient of y_k in the ordinate y_k' is found by direct calculation to be $2/(k+2)$, the value given by (2) with $n=k$. The assumption that the coefficient of y_k in y_N', $N \geqq k$, is given by (2) with $n=N$ leads, by calculation of the ordinate of the point P_{N+1}' on the segment $P_N'P_{N+1}$, to the conclusion that the coefficient of y_k in y_{N+1}' is given by (2) with $n=N+1$. Hence, (2) is valid for all integers n greater than or equal to k.

The coefficient c_k of y_k in the ordinate of A is also the coefficient of y_{n-k} in the ordinate of B. The graphically determined points are, therefore,

$$\begin{aligned} &A\left(2nh/3,\ \sum_{k=0}^{n} 2(k+1)y_k/(n+1)(n+2)\right), \\ &B\left(nh/3,\ \sum_{k=0}^{n} 2(k+1)y_{n-k}/(n+1)(n+2)\right). \end{aligned} \tag{3}$$

With $x_k = kh$, so that, for example, $\sum_{k=0}^{n} x_k = hn(n+1)/2$ and

$$\sum_{k=0}^{n} x_k^2 = h^2 n(n + 1)(2n + 1)/6,$$

the normal equations (1) yield the values

$$\begin{aligned} a &= 6 \sum_{k=0}^{n} (2k - n)y_k/hn(n + 1)(n + 2) \\ b &= 2 \sum_{k=0}^{n} (2n - 3k + 1)y_k/(n + 1)(n + 2). \end{aligned} \tag{4}$$

That A and B are indeed on the line $y=ax+b$ may be verified now by substitution of their respective coordinates into the equation. The results are identities in the ordinates y_k.

4. Uniqueness of the ratio involved. That a ratio other than 2/3 cannot be used is readily shown. The ordinate, call it Y, of the point on the line $y=ax+b$ corresponding to $x=nrh$ is given, using (4), by

$$Y = \sum_{k=0}^{n} [6r(2k-n) + 2(2n-3k+1)]y_k/(n+1)(n+2).$$

The ordinate of the point (nrh, y_n') determined by the procedure described herein, with r in the place of the ratio 2/3, may be written in the form

$$y_n' = ry_n/[n-(n-1)r] + n(1-r)y_{n-1}'/[n-(n-1)r],$$

in which it is noted that y_{n-1}' is a function of y_k, $k=0, 1, \cdots, n-1$. These ordinates Y and y_n' can be equal for arbitrary y_n only if the coefficients of y_n are equal, which condition determines the equation

$$6r^2 - 7r + 2 = 0$$

with roots 2/3 and 1/2, provided the trivial cases $n=0$ and $n=1$ are removed. It can be shown that $r=1/2$ leads to identical points A and B coincident with the centroid of the points (x_k, y_k), $k=0, 1, \cdots, n$, which lies on the least squares line. Thus, while $r=1/2$ provides a point for checking the work, $r=2/3$ is the only ratio which determines the least squares line by this procedure.

A REMARK ON THE CONVERSE OF BANACH'S CONTRACTION THEOREM

SHOURO KASAHARA, Kobe University

Let M be a subspace of a metrizable topological space S. Let $0<\alpha<1$; a mapping f of M into itself is called an *α-contraction* on M, if there exists a metric d on M generating the topology of M such that

$$d(f(x), f(y)) \leqq \alpha d(x, y)$$

for every $x, y \in M$. Denote by $\mathcal{K}_M$ the set of all α-contractions on M with $0<\alpha<1$, and by $\mathcal{I}_M$ the set of all continuous mappings f of M into itself such that $\bigcap_{n=1}^{\infty} f^n(M)$ are singletons.

In a recent paper [2], Janos has shown that if S is compact, then for every α with $0<\alpha<1$, each $f\in\mathcal{I}_S$ is an α-contraction on S. Consequently, Banach's contraction theorem yields that if S is compact, then $\mathcal{I}_M=\mathcal{K}_M$ for every nonempty closed subset M of S. The purpose of this note is to establish the converse of this proposition, that is, to prove the "if part" of the following theorem.

THEOREM. *A metrizable space S is compact if and only if $\mathcal{I}_M=\mathcal{K}_M$ for every nonempty closed subset M of S.*

To prove the "if part," it suffices to show that S is precompact; for since $\mathcal{K}_M \subset \mathcal{G}_M$ implies that each $f \in \mathcal{K}_M$ has a fixed point, it follows from a theorem due to Hu [1] that S is complete. Assume now that S is not precompact, and let d' be a metric on S generating the topology of S. Then there exist an $\epsilon_0 > 0$ and an infinite sequence $\{x_n\}$ of points of S such that $d'(x_m, x_n) > \epsilon_0$ whenever $m \neq n$. It is clear that the nonempty set $M = \{x_n\}$ is closed. The mapping f defined by $f(x_0) = x_0$ and $f(x_n) = x_{n+1}$ for every positive integer n, does belong obviously to $\mathcal{G}_M$, and so by the assumption, it is an α-contraction on M for some α with $0 < \alpha < 1$. Let d be a metric on M associated with f. We shall show that $\{x_n\}$ is a Cauchy sequence with respect to the metric d; this yields a contradiction since then $\{x_n\}$ is also a Cauchy sequence with respect to d'. Now let $\epsilon > 0$. We can find then a positive integer m such that $(1-\alpha)^{-1}\alpha^{m-1}d(x_1, x_2) < \epsilon$. Thus if $p > q > 0$, we have

$$\begin{aligned} d(x_{m+p}, x_{m+q}) &\leqq \alpha^{m+q-1} d(x_1, x_{p-q+1}) \\ &\leqq \alpha^{m+q-1}[d(x_1, x_2) + \cdots + d(x_{p-q}, x_{p-q+1})] \\ &\leqq \alpha^{m+q-1}(1 + \alpha + \cdots + \alpha^{p-q-1}) d(x_1, x_2) \\ &= \alpha^{m+q-1} \frac{1 - \alpha^{p-q}}{1 - \alpha} d(x_1, x_2) \\ &< (1 - \alpha)^{-1}\alpha^{m-1} d(x_1, x_2) < \epsilon. \end{aligned}$$

This completes the proof.

This theorem may be sharpened, as the theorem of Hu is sharpened, by replacing "nonempty closed subset" by "countably infinite closed subset."

References

1. T. K. Hu, On a fixed-point theorem for metric spaces, this MONTHLY, 74 (1967) 436–437.
2. Ludvik Janos, A converse of Banach's contraction theorem, Proc. Amer. Math. Soc., 18 (1967) 287–289.

HARDING'S REPRESENTATION IS THE REGULAR REPRESENTATION

BARRY SIMON, Princeton University

In this MONTHLY [1], C. F. Harding points out that one can obtain a faithful representation of a finite group directly from its multiplication table. He observes the following:

THEOREM. *If, for each α_i in the finite group $G = \{\alpha_1, \cdots, \alpha_n\}$, we construct an $n \times n$ matrix $M(\alpha_i)$ by placing a 1 where α_i appears in the multiplication table of G and a 0 otherwise, then the mapping $\alpha_i \rightarrow R(\alpha_i) = M(\epsilon)M(\alpha_i)$ is a faithful representation of G when ϵ is the neutral element.*

This representation is just the right regular representation in disguise. The

regular representation is defined on the vector space of functions from G to the complex numbers by: [2]

$$(U(\alpha_i)f)(\alpha_j) = f(\alpha_j\alpha_i).$$

To show this we use the symbol $\delta(\alpha;\beta)$ which is 1 when $\alpha=\beta$, and 0 otherwise. For $\alpha\in G$, $M(\alpha)_{ij}=1$ if $\alpha=\alpha_i\alpha_j$ and 0 otherwise. Thus $M(\alpha)_{ij}=\delta(\alpha;\alpha_i\alpha_j)$ and $M(\epsilon)_{ij}=\delta(\epsilon;\alpha_i\alpha_j)=\delta(\alpha_i^{-1};\alpha_j)$, so that

$$\begin{aligned} R(\alpha)_{ij} &= \sum_{r=1}^{n} \delta(\alpha_i^{-1};\alpha_r)\delta(\alpha;\alpha_r\alpha_j) \\ &= \delta(\alpha;\alpha_i^{-1}\alpha_j). \end{aligned}$$

If f is a function from G into the complex numbers, we can identify f with the vector $[f(\alpha_j)\,|\,j=1,\cdots,n]$. Then

$$\begin{aligned} (R(\alpha_i)f)(\alpha_j) &= \sum_{r=1}^{n} \delta(\alpha_i;\alpha_j^{-1}\alpha_r)f(\alpha_r) \\ &= \sum_{r=1}^{n} \delta(\alpha_j\alpha_i;\alpha_r)f(\alpha_r) \\ &= f(\alpha_j\alpha_i) = (U(\alpha_i)f)(\alpha_j). \end{aligned}$$

Thus Harding's representation is the right regular representation. It has many additional properties besides being faithful; for example, it contains every irreducible representation.

The author would like to thank the referee for a suggestion that simplified the notation and the National Science Foundation for the support of a Pre-Doctoral Fellowship at the time this note was prepared.

References

1. C. F. Harding, A natural matrix representation of finite groups, this MONTHLY, 73 (1966) 880.

2. M. Burrow, Representation Theory of Finite Groups, Academic Press, New York, 1965, p. 60.

ANOTHER PROOF OF THE EGYPTIAN FRACTION THEOREM

J. C. OWINGS, JR., University of Maryland

A well-known proposition of elementary number theory states that any positive rational is the sum of a finite number of distinct Egyptian fractions (an Egyptian fraction is a positive fraction with numerator 1). The proof we present here has an air of magic; one feels that the lemma is almost accidental.

Let $S(0)=\{2\}$, $S(n+1)=\{c\,|\,(\exists a)(a\in S(n)$ and $(c=a+1 \vee c=a(a+1)))\}$.

LEMMA. *$S(n)$ has cardinality 2^n.*

Proof. Suppose not and let k be the least integer n such that $\mathrm{Card}(S(n)) \neq 2^n$. Then obviously $\mathrm{Card}(S(k)) < 2^k$ and there exist integers a, $b \in S(k-1)$ such that $a(a+1) = b+1$. It follows from the definition of $S(n)$ that its least member is $n+2$. So $a \geqq k+1$.

Let $t(n)$ $(0 \leqq n \leqq k)$ be a finite sequence such that $t(0) = b+1$, $t(1) = b$, and, for $n \geqq 1$, $t(n+1) \in S(k-(n+1))$ and $t(n) = t(n+1)+1$ or $t(n) = t(n+1) \cdot (t(n+1)+1)$. Obviously $t(k) = 2$. Since $b+1 = a(a+1)$, $b+1 > k+2$. Now if $t(n) = t(n+1)+1$ for all $n < k$, then $b+1 = t(0) = k+2$, contradicting the above. So let d be the least $n < k$ such that $t(n) = t(n+1)(t(n+1)+1)$ and set $c = t(d+1)$. Then $b+1 = c(c+1)+d$, so that $a(a+1) = c(c+1)+d$ where $a > k$ and $1 \leqq d < k$. So $c \leqq a-1$ and $d \geqq a(a+1) - a(a-1) = 2a > k$, a contradiction. Thus the lemma is proved.

Since, for any integer m, $1/m = (1/(m+1)) + (1/m(m+1))$, it follows immediately from the lemma that $\sum 1/a (a \in S(n))$ is equal to $1/2$ for every n. Since $\min S(n) = n+2$ and $\max S(n) = \max S(n-1)(\max S(n-1)+1)$, it is easy to find an increasing sequence $k(0)$, $k(1)$, $\cdots$ of integers such that $S(k(i)) \cap S(k(j)) = \varnothing$ if $i < j$. For example we could set $k(0) = 0$, $k(n+1) = \max S(k(n))$. Consequently, there are infinitely many mutually disjoint finite sets of Egyptian fractions whose sum is $1/2$. The first four generated by our algorithm are

$$\{1/2\},\ \{1/3,\ 1/6\},\ \{1/4,\ 1/12,\ 1/7,\ 1/42\},$$

and

$$\{1/5,\ 1/20,\ 1/13,\ 1/156,\ 1/8,\ 1/56,\ 1/43,\ 1/1806\}.$$

Since any positive rational is equal to $r(1/2)/s$ for some pair of integers r, s, the original proposition follows. Another easy consequence is the divergence of the harmonic series $\sum 1/n$ $(n > 0)$.

We remark that if one sets $T(0) = \{1\}$, $T(1) = \{2, 3, 6\} = S(0) \cup S(1)$ and for $n > 1$ defines

$$\begin{aligned} T(n+1) &= \{c \mid (\exists a)(a \in T(n) \quad \text{and} \quad (c = a+1 \vee c = a(a+1)))\} \\ &= S(n) \cup S(n+1), \end{aligned}$$

one can show $\sum 1/a (a \in T(n))$ is equal to 1 for all $n \geqq 0$. In other words, $S(n) \cap S(n+1) = \varnothing$ for all $n \geqq 0$. In fact, one can show, given any $m > 0$, that there exists a $k \geqq 0$ such that $S(n) \cap S(n+q) = \varnothing$ for all $n \geqq k$ and all q, $1 \leqq q \leqq m$.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before January 31, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards. **Note new address for mailing Elementary Solutions.**

E 2093 [1968, 543]. **Correction.** *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York*

Delete "for each positive integer n."

E 2105. *Proposed by H. L. Nelson, Livermore, California*

Characterize those real valued functions f, continuous on the positive real axis for which $f^{-1}=f'$. (Cf. E 1894 [1967, 1268].)

E 2106. *Proposed by Bernt Lindström, University of Stockholm, Sweden*

Let S be a set with n elements and $M_1, M_2, \cdots, M_{n+1}$ be nonempty subsets of S. Prove that one can find r, s and $r+s$ distinct indices $i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_s$ such that

$$M_{i_1} \cup \cdots \cup M_{i_r} = M_{j_1} \cup \cdots \cup M_{j_s}.$$

E 2107. *Proposed by Bernt Lindström, University of Stockholm, Sweden*

Counterfeit coins weigh a and genuine coins weigh b, $a \neq b$. One is given two samples of three coins each and knows that each sample has one counterfeit coin. How many weighings are needed to isolate the two counterfeit coins by the aid of an accurate scale (not a balance)?

E 2108. *Proposed by L. J. Lander and T. R. Parkin, Aerospace Corporation, Los Angeles*

A polyomino is a finite rookwise-connected set of squares chosen from an

infinite plane chessboard. A square in the polyomino is called removable if after removing it the remainder of the set is still a polyomino. Prove that every polyomino having at least two squares contains at least two removable squares.

E 2109. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let ABC be a triangle and A' be any fixed point on the side BC. Construct the inscribed triangle $A'B'C'$ which is directly similar to a given triangle XYZ.

E 2110. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If, in a plane, the triangles AUV, VBU, UVC are directly similar to a given triangle, then so is ABC.

E 2111. *Proposed by R. D. Jenks, Brookhaven National Laboratory, New York*

Is it true that for any odd number $2n+1$ $(n\geqq 1)$ of positive numbers $x_1, x_2, \cdots, x_{2n+1}$,

$$\frac{x_1x_2}{x_3}+\frac{x_2x_3}{x_4}+\cdots+\frac{x_{2n}x_{2n+1}}{x_1}+\frac{x_{2n+1}x_1}{x_2}\geqq x_1+x_2+\cdots+x_{2n+1}$$

with equality only if all are equal?

E 2112. *Proposed by D. E. Daykin and D. G. Neal, University of Malaya, Kuala Lumpur*

The n squares of sides $1, 3, 5, \cdots, 2n-1$ respectively are closed on two edges and open on two edges. They are to be arranged without overlapping on the xy-plane so that their edges are parallel to the x- and y-axes and so that no line parallel to an axis is to pass through an odd number of the squares.

(i) For which integers n does such an arrangement exist?

(ii) With n as small as you can, find an arrangement which also satisfies the line condition for lines equally inclined to the axes.

E 2113. *Proposed by Francis Sand, Mathematica, Princeton, N.J.*

Given an arbitrary finite set of n pairs of positive numbers $\{(a_i, b_i): i=1, \cdots, n\}$, show that

$$\prod_{i=1}^{n}[xa_i+(1-x)b_i]\leqq \max\left[\prod_{i=1}^{n}a_i,\ \prod_{i=1}^{n}b_i\right]\quad \text{for all } x\in[0,1],$$

with equality attained only at $x=0$ or $x=1$, if and only if

$$\left(\sum_{i=1}^{n}\frac{a_i-b_i}{a_i}\right)\left(\sum_{i=1}^{n}\frac{a_i-b_i}{b_i}\right)\geqq 0.$$

E 2114. *Proposed by P. Richman and P. Rosenthal, Stanford University*

For each integer $n\geqq 1$, does there exist a fixed, finite set of $s\times s$ squares (s an integer, $s\geqq n$) such that, for all integers $k\geqq n$, any $k\times k$ square can be

covered without overlapping by some or all of the $s \times s$ squares (allowing repetitions)? (In other words, is there a finite $s \times s$ basis for each n?) For example, if $n=2$, the $s \times s$ squares for $s=2$, 3, 5 and 7 will do.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Geometric Max-Min Problem

E 1958 [1967, 198]. *Proposed by S. R. Petrick, Air Force Cambridge Research Laboratories*

Find a point P in the plane of the triangle ABC such that

$$\min\{\,|PA - PB|,\ |PB - PC|,\ |PC - PA|\,\}$$

is maximized and determine the value of this maximum.

Solution by Oswald Wyler, Carnegie-Mellon University. Let a, b, c be the three sides of the triangle ABC, and put

$$u = |PB - PC|, \quad v = |PC - PA|, \quad w = |PA - PB|.$$

Then $w \leqq c$, with $w=c$ if and only if P is on the line AB, but not strictly between A and B. Similarly, $u \leqq a$ and $v \leqq b$.

Let us suppose (without loss of generality) that $a \leqq b \leqq c$. The greatest of the three differences u, v, w is the sum of the other two, so that

$$\min\{u, v, w\} \leqq \tfrac{1}{2}\max\{u, v, w\} \leqq \tfrac{1}{2}c, \tag{1}$$

for every point P. Also $\min\{u, v, w\} \leqq u \leqq a$ for every point P.

If $a \leqq \frac{1}{2}c$, then for $P=B$, $u=a \leqq c-a=v$, $w=c$, so that the maximum $\min\{u, v, w\}=a$ is attained at $P=B$.

If $\frac{1}{2}c < a < b$, let M be the midpoint of AB, and construct R on the line AB equidistant from M and C. Then B is between R and M, and

$$RC = RM = RA - \tfrac{1}{2}c = RB + \tfrac{1}{2}c.$$

Thus, for $P=R$, $u=v=\frac{1}{2}c$, $w=c$, and $\min\{u, v, w\}=\frac{1}{2}c$ attains its maximum.

If $\frac{1}{2}c < a=b$, and if $\max\{u, v, w\} < c$, then $\min\{u, v, w\} < \frac{1}{2}c$ by (1). If, on the other hand, $w=c$, then $\min\{u, v, w\}=PM+\frac{1}{2}c-PC < \frac{1}{2}c$, with M as above, and $\frac{1}{2}c - \min\{u, v, w\}$ becomes arbitrarily small for PM large. By the same argument, $\min\{u, v, w\} < \frac{1}{2}c$ if $a=b=c$ and $u=a$ or $v=b$. Thus the problem has no solution in this case; $\min\{u, v, w\}$ does not attain its least upper bound $\frac{1}{2}c$.

The solution constructed above is the only one if $\frac{1}{2}c \leqq a < b < c$, and there is a second solution if $\frac{1}{2}c \leqq a < b = c$. If $a < \frac{1}{2}c$, then the solutions form one or several, bounded or unbounded, intervals on the line BC.

Also solved by Michael Goldberg, Bohnslav Mišek (Czechoslovakia), and the proposer.

A Minimum Number of Intersections

E 1978 [1967, 438]. *Proposed by T. L. Saaty, U. S. Arms Control and Disarmament Agency, Washington, D.C.*

Given n collinear points in a plane, what is the minimum possible number of

intersections of semicircular arcs in the plane joining all nonadjacent pairs of points? What is the minimum number of intersections if simple curves are used instead of semicircles?

Solution by E. M. Holroyd, Bracknell, Berkshire, England. Let the n collinear points be $P_1, P_2, \cdots, P_n$ in order. Then (Construction I) a semicircle on P_iP_j as diameter ($|i-j| \geqq 2$) is drawn on one side of the line if either

$$2 \leqq i+j \leqq \left\lceil \frac{n+3}{2} \right\rceil \quad \text{or} \quad n+2 \leqq i+j \leqq \left\lceil \frac{3n+3}{2} \right\rceil,$$

and on the other side of the line if not.

This construction is an adaptation of one (Construction II) given by J. Blazek and M. Koman (*A minimal problem concerning complete plane graphs*, Theory of Graphs and its Applications, Smolenice Symposium, 1963, Prague, 1964, pp. 113–117.) In this a regular n-gon with vertices $Q_1, Q_2, \cdots, Q_n$ has all the diagonals drawn which are parallel to the lines Q_iQ_{i+1} for $i=1, 2, \cdots, [\frac{1}{4}(n+1)]$ and the lines $Q_{i-1}Q_{i+1}$ for $i=1, 2, \cdots, [\frac{1}{4}(n+3)]$ (taking Q_0 to be Q_n). The remaining pairs of nonadjacent vertices are linked by arcs outside the n-gon so that no pair of arcs crosses more than once. (The sides of the n-gon are also drawn, but as they make no intersections they are not relevant here.) Blazek and Koman show that the number of intersections of the graph so formed is

$$c(n) = \begin{cases} (n-1)^2(n-3)^2/64 & (n \text{ odd}), \\ n(n-2)^2(n-4)/64 & (n \text{ even}). \end{cases}$$

It is easy to see that Constructions I and II are equivalent in the sense that P_gP_h intersects P_iP_j if and only if Q_gQ_h intersects Q_iQ_j, and it follows that the number of intersections given by Construction I must also be $c(n)$.

Suppose there were a construction of the type required giving fewer than $c(n)$ intersections. By adding the straight lines $P_1P_2, P_2P_3, \cdots, P_{n-1}P_n$ we should obtain a complete graph on n vertices and should not introduce any further intersections. We should thus have disproved the conjecture (R. K. Guy, *A combinatorial problem*, Bull. Malayan Math. Soc., 1960, 7, 58–72) that the minimum number of intersections of such a graph is $c(n)$. If we assume the truth of the conjecture, therefore, it follows that $c(n)$ is the minimum number of intersections for the type of construction required.

Editorial Note. Several questions remain unanswered in connection with this (*not* elementary) problem. Further results (though not the complete solution) and references may be found in The University of Calgary, Department of Mathematics, Research Papers, 1967, Nos. 8 and 18, by R. K. Guy, *et al.*

Professor Umbugio Wrong Again

E 1979 [1967, 438]. *Proposed by R. T. Hood, Franklin College, Indiana*

Professor Euclides Paracelso Bombasto Umbuggio, who will be remembered by readers of this Department, has emerged from retirement. Having heard

some of the recent discussion about the independence of the continuum hypothesis, he now appears in order to demolish it forever. To reach the widest possible audience, he has chosen the most elementary means of doing so: obviously, $1<\aleph_0$. Therefore, $\aleph_0<\aleph_0^{\aleph_0}$. Moreover, $\aleph_0<2^{\aleph_0}$. Therefore $\aleph_0^{\aleph_0}<(2^{\aleph_0})^{\aleph_0}<2^{\aleph_0 \cdot \aleph_0}=2^{\aleph_0}=c$. Therefore, $\aleph_0<\aleph_0^{\aleph_0}<c$. Deflate the professor, demolish his demolishment.

Solution by Eric Rosenthal, Student, West Orange Mountain High School, New Jersey. The fallacy arises from attributing the properties of finite numbers to cardinals. Specifically, $a<b$ does not imply $a^e<b^e$ for $e\neq 0$. A counterexample may be obtained starting from $2<c$. If $a<b$ implies $a^e<b^e$, then

$$2^{\aleph_0} < c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$$

so that $2^{\aleph_0}<2^{\aleph_0}$, which is not true.

Also solved by Anders Bager (Denmark), W. T. Bailey, D. Bollman, J. D. Carnegie, T. E. Elsner, Jerry Fischer, R. S. Fishman, Michael Goldberg, W. S. Hatcher, G. A. Heuer, John Ivie, Lew Kowarski, David Kriegman, Steven Minsker, D. E. Penney, Dale Peterson, Stanley Rabinowitz, Alfred Raws III, Simeon Reich (Israel), G. F. Schumm, D. L. Silverman, Francis Siwiec, R. C. Steinlage, Ronald Stern, Alan Tschetter, Steven Weintraub, and the proposer.

Identifying a Certain Subset

E 1981 [1967, 438]. *Proposed by M. V. Subbarao, University of Alberta*

Let b and k be given positive numbers, $k\leqq 1$, and let B be the union of the open intervals $|x-(4n+k)b|<b$ $(n=1, 2, \cdots)$. If A is a subset of B such that $x\in A$ implies $2x\in A$, show that $A=\{4b, 8b, 12b, \cdots\}$ if $k<1$ and A is null if $k=1$.

Solution by M. Perisastri, M. R. College, Vizianagram, India. Let $0<k<1$ and let P be the set of all positive integers. By hypothesis $x\in A\Rightarrow 2^r x\in A$ for every $r\in P$. Since $A\subset B$, it follows that

(i) $|x-(4n+k)b|<b$, for some $n\in P$, and

(ii) $|2^r x-(4n_r+k)b|<b$, for every $r\in P$ and some $n_r\in P$.

Now (ii) implies $|2x-(4n_1+k)b|<b$, which along with (i) implies $n_1=2n$. Hence, by induction we have $n_r=2^r n$. Therefore from (ii) it follows that $|x-4nb|=|x-4bn_r/2^r|<b(k+1)/2^r$, for every $r\in P$. Thus it follows that $x=4nb$. Hence $A=\{4nb\mid n\in P\}$.

Let $k=1$ and $x\in A$. Then arguing as above, it follows that $x=4nb$, $\Rightarrow 4nb\in A$, $\Rightarrow 4nb\in B$, which is false. Hence A is the empty set.

Also solved by Donald Batman, Neal Felsinger, Jerry Fischer, Toyomasa Fujinawa (Japan), S. M. Gagola, Jr., Donald Jeffords, Dan Marcus, F. H. Munck (Denmark), Stanley Rabinowitz, Simeon Reich (Israel), John Swetik, Philip Trauber, and the proposer.

Rational Polygons

E 1982 [1967, 439]. *Proposed by J. S. Biggerstaff, Portland, Ore.*

A rational polygon is one whose sides, diagonals and area are all rational

numbers. (1) Find sets of distinct rational triangles (not right-angled or isosceles) which have the same area; and (2) Find a pair of rational quadrilaterals (not parallelograms or trapezoids) which have the same area.

Solution by E. P. Starke, Plainfield, N.J. Daykin proved that all triangles formed by producing the sides and diagonals of a rational polygon are rational triangles (*Mathematics Magazine*, 39(1966), 299–301, Theorem 1). Similarly the triangles into which the altitude of a rational triangle divides it are rational triangles. ($h_a=2\Delta/a$, $\cos B=(a^2+c^2-b^2)/2ac$, thus the sides of the subtriangle h_a, c, $\cos B$ are rational, as are h_a, b, $b\cos C$. Being right triangles, they have rational areas. Δ stands for the area.)

Rational triangles are then easily formed by putting together two Pythagorean triangles having a common leg: if the two desired legs are not equal, use appropriate similar triangles. Further, all rational triangles can be obtained in this way.

A different procedure is the following: $\Delta^2=s(s-a)(s-b)(s-c)$, $\Delta=rs$ imply $r^2s^2=s(s-a)(s-b)(s-c)=s\alpha\beta(s-\alpha-\beta)$, where $\alpha=s-a$, $\beta=s-b$, $\alpha+\beta=c$. So

$$s = \alpha\beta(\alpha+\beta)/(\alpha\beta - r^2).$$

Take arbitrary α, β, $r(r^2<\alpha\beta)$, compute s. Then $a=s-\alpha$, $b=s-\beta$, $c=\alpha+\beta$, $\Delta=rs$.

There are three trivial types of rational quadrilaterals. The kite and parallelogram are formed by placing two identical rational triangles on opposite sides of a common side, with reversed or similar orientation. The isosceles trapezoid (*trapezium*—to all but American readers!) is formed from two identical rational triangles placed on the same side of a common side but with opposite orientations.

Using the notation $(a, b, c):\Delta$, we give some instances of (1). (13, 14, 15): 84, (10, 17, 21): 84, (8, 29, 35): 84, (12, 35/2, 53/2): 84, (15/2, 26, 61/2): 84, etc. Also (3, 148, 149): 210, (7, 65, 68): 210, (15, 91/3, 116/3): 210, (17, 28, 39): 210, (35/2, 26, 73/2): 210, (21/2, 41, 89/2): 210, (21/2, 89/2, 50): 210, (17, 25, 28): 210, (25/2, 42, 101/2): 210, (89/6, 170/6, 189/6): 210, (15/2, 56, 113/2): 210, etc.

For an illustration of (2): (i) $AB=AD=21$, $BC=DC=72$, $AC=75$, $BD=1008/25$, area 1512, a kite. (ii) $AB=20$, $BC=34$, $CD=56$, $DA=70$, $AC=42$, $BD=78$, area$=1512$.

Also solved by Anders Bager (Denmark), Merrill Barnebey, L. J. Burton, Michael Goldberg, and the proposer.

Goldberg quotes a result, due to Kummer: If the four sides of a quadrilateral which is inscriptable in a circle have the values

$$a = (u^2+v^2)(x^2-y^2), \qquad b = (u^2-v^2)(x^2+y^2),$$
$$c = 2xy(u^2+v^2), \qquad d = 2uv(x^2+y^2),$$

where u, v, x, y are rational, then both diagonals, the segments of them, the area of the quadrilateral and the diameter of the circumscribed circle are all rational. (See Dickson, *History of the Theory of Numbers*, vol. 2, p. 217.) For example, with $u, v, x, y=5, 4, 3, 2$, the sides are 205, 117, 492 and 520. Three different cyclic quadrilaterals of the same area 80850 are formed, depending on the order of arrangement of the sides. The three possible diagonals are 308, 525 and 533; each quadrilateral uses two of these diagonals.

Impossibility of Covering a Rectangle with L-Hexominos

E 1983 [1967, 439]. *Proposed by Robert Spira, University of Tennessee*

Show that replicas of the 3×4 L-hexomino cannot be fitted to form a rectangle. (An L-hexomino is formed when straight lines are drawn joining the following points in order: (0, 0), (3, 0), (3, 1), (1, 1), (1, 4), (0, 4), (0, 0).)

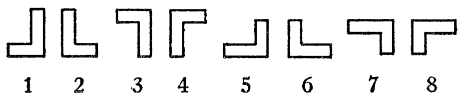


FIGURE 1

Solution by Dennis Gannon (submitted by the proposer). In giving an outline of the proof we set up the following notation. Designate positions of the 3×4 L-hexomino by numbers 1 to 8 as in Figure 1. We set up an xy-coordinate system and suppose the positive x-axis and the negative y-axis are edges of a rectangle we are trying to construct, whose corner lies at the origin. Label the unit squares in the fourth quadrant by a row and column number, e.g., square 25 has corners $(4, -1)$, $(4, -2)$, $(5, -2)$ and $(5, -1)$. In the proof it is a convenient accident that the specification of a square to be covered and the position number of the hexomino completely determine how the hexomino is to be placed, so that we do not have to number the individual squares of the hexomino.

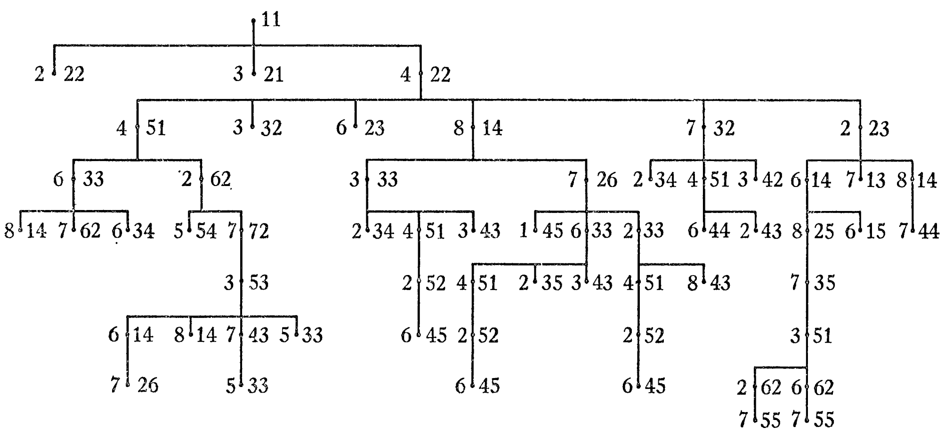


FIGURE 2

Figure 2 gives the tree of possibilities of such a construction, showing that each possibility ends with a square which cannot be filled. Each node of the tree has a number on the left indicating the position number of the hexomino which is placed on the unit square indicated by the pair of numbers to the *right* of the node *above*. Where the tree terminates in a node the number pair on the right indicates a square which is impossible to fill or, if it can be filled, the

choice leads quickly to another obvious square which cannot be filled.

For example, we may cover square 11 with a hexomino in position 2. Then, although we can cover square 22 with position 3 or 7 or 8, it is not hard to see that we cannot proceed much further without leaving some neighboring square surrounded and uncovered. Again, if we cover 11 with 4, then 22 with 8, 14 with 7, 26 with 2, 33 with 4, 51 with 2, 52 with 6, the tree indicates that 45 cannot be covered (without leaving 44 surrounded).

This, and a more general study of polyominoes, was a project of Study Group Four of the NSF Summer Institute in Mathematics for High School Students at Berkeley, 1964.

Editorial Note. C. B. A. Peck cites the following reference in which is a table of such theorems, given without proof: S. W. Golomb, *Tiling with Polyominoes*, The Journal of Combinatorial Theory, Vol. 1, No. 2, September 1966, pp. 280–296.

A Number Theoretic Inequality

E 1984 [1967, 439]. *Proposed by Erwin Just, Bronx Community College*

If the divisors of n are $d_1, d_2, \cdots, d_k$, prove that

$$\prod_{i=1}^{k} \phi(d_i) \leqq (n/k)^k.$$

Solution by C. S. Venkataraman, Trichur, India. We establish a general result of which the given inequality is a special case.

Let $f(n)$ be a function defined on the integers, $f(n)>0$ for $n\geqq 1$, and $\sum_{i=1}^{k} f(d_i)=F(n)$, where $d_1=1, d_2, \cdots, d_k=n$ are the divisors of n. Then

$$\prod_{i=1}^{k} f(d_i) \leqq \{F(n)/k\}^k.$$

Using the well-known arithmetic-geometric means inequality we have at once

$$\left\{\prod_{i=1}^{k} f(d_i)\right\}^{1/k} \leqq \frac{1}{k}\left\{\sum_{i=1}^{k} f(d_i)\right\} = F(n)/k.$$

That is, $\prod_{i=1}^{k} f(d_i) \leqq \{F(n)/k\}^k$.

Since $\phi(n)>0$ for $n\geqq 1$ and $\sum_{i=1}^{k}\phi(d_i)=n$, the desired inequality follows, with $f(n)=\phi(n)$, $F(n)=n$. It may also be remarked that equality arises in the given result for $n=1$ and $n=2$ only. For, then, $\phi(d_1)=\phi(d_2)=\cdots=\phi(d_k)$, that is, $\phi(n)=\phi(1)=1$. Hence $n=1$, or $n=2$.

Also solved by A. N. Aheart, Peter Ash, Anders Bager (Denmark), W. T. Bailey, M. G. Beumer (Netherlands), Peter Bundschuh (Germany), Lindley Burton, G. C. Dodds, Neal Felsinger, Jerry Fischer, Michael Goldberg, Jerry Goodman, M. G. Greening (Australia), Donald Jeffords, Kenneth Kramer, James Martin, B. Mišek (Czechoslovakia), P. Nagaraju (India), Barbara W. Nason, C. B. A. Peck, Dale Peterson, Bob Prielipp, Stanley Rabinowitz, Simeon Reich (Israel), Ira Rosenholtz, D. B. Shapiro, Stephen Spindler, Hugo Sun, C. H. Toll, Philip Trauber, A. M. Vaidya (India), K. L. Yocom, and the proposer.

Upper Bound for a Gradient

E 1986 [1967, 589]. *Proposed by H. S. Shapiro, New York University*

Let B denote the set $\sum_{i=1}^{n} x_i^2 \leqq 1$ in Euclidean n-space. Suppose f is a differentiable function on an open set containing B, and that $|f(x)| \leqq 1$ for $x=(x_1, \cdots, x_n) \in B$. Then show that there is a point interior to B at which grad f has a magnitude less than 4.

Solution by the proposer. Let $g(x)=\sum_{i=1}^{n} x_i^2$ and $h(x)=f(x)+2g(x)$. Then $h(0)=f(0)\leqq 1$, and for x on the boundary of B, $f(x)+2g(x)=f(x)+2\geqq 1$. Therefore there is at least one interior point y of B such that

$$h(y) = \min_{x\in B} h(x).$$

At this point y, grad h vanishes. Thus at this point

$$\frac{\partial f}{\partial x_i} = -2\frac{\partial g}{\partial x_i} = -4y_i \quad\text{and}\quad \sum\left(\frac{\partial f}{\partial x_i}\right)^2 = 16\sum_{i=1}^{n} y_i^2 < 16,$$

completing the proof.

We remark that the function $f(x)=x$ shows that 4 cannot be replaced by 1 in the above result. It would be of interest to find the best value of the constant.

Also solved by E. D. Gingerich, and H. A. Smith.

A Diophantine Equation

E 1987 [1967, 589]. *Proposed by Joseph Arkin, Suffern, N.Y.*

Show that there are infinitely many solutions of

$$(a/b)^2 + (b/a)^2 - (x/y)^2 - (y/x)^2 = L^2,$$

where $L(\neq 0)$ is a rational number and each of a, b, x, y is the sum of squares of two integers.

I. *Solution by Steven Russ, Brandeis University.* Choose a, b and x arbitrarily, each a sum of two squares, and take $y=x$. Then the given equation is evidently satisfied and $L=(a/b)-(b/a)$. Although a very special case, this does provide infinitely many solutions.

II. *Solution by the proposer.* Let $L=L_1/(abxy)$ so that the given equation can be put in the form

$$(1) \qquad (ax+by)(ax-by)(ay+bx)(ay-bx) = L_1^2.$$

If now we put $ax+by=m^2$, $ax-by=u^2$, $ay+bx=z^2$, $ay-bx=n^2$, and note $(ax+by)^2+(ay-bx)^2=(ay+bx)^2+(ax-by)^2$, our problem reduces to

$$(2) \qquad m^4+n^4 = z^4+u^4, \qquad mnuz = L_1.$$

Now let $m=A+B$, $n=C-D$, $z=C+D$, $u=A-B$ to further reduce (2) to

$$(3) \qquad AB(A^2+B^2) = CD(C^2+D^2), \quad (A^2-B^2)(C^2-D^2) = L_1.$$

Using the known solution of (3) (Euler's: see R. D. Carmichael, *Diophantine Analysis*, Dover, p. 82), and substituting back, we have (after much computation and contriving to have a, b, x, y each a sum of two squares):

$$x = (36(p^2+q^2)^4-1)^2+(216(p^2+q^2)^6)^2,$$
$$y = (1296(p^2+q^2)^8-36(p^2+q^2)^4)^2+(6(p^2+q^2)^2)^2,$$
$$a = (216(p^2+q^2)^6+12(p^2+q^2)^2)^2+1^2,$$
$$b = (36p^6-36q^6-540p^4q^2+540p^2q^4)^2+(720p^3q^3-216pq^5-216p^5q)^2.$$

Also solved by J. W. Baldwin

Editorial Note. It is not necessary that each of the individual factors in (1) be made a perfect square. Consider, for example, $a=26=5^2+1^2$, $b=8=4^2+4^2=y$, $x=10=3^2+1^2$. These satisfy the original equation, giving $L=189/65$. The smallest solution resulting from the proposer's formulas (with $p=1$, $q=0$) is $x=47881$, $y=1587636$, $a=51985$, $b=1296$.

Integer Solution for a Surd Equation

E 1988 [1967, 589]. *Proposed by R. S. Luthar, Colby College, Waterville, Me.*

Given any integers a, k, $a\geqq k\geqq 0$, show that, for each positive integer n there exists an integer b such that

$$[\sqrt{a}-\sqrt{(a-k)}]^n=\sqrt{b}-\sqrt{(b-k^n)}.$$

Solution by Mannis Charosh, New Utrecht High School, Brooklyn, N.Y. We have $b-(b-k^n)=k^n$ identically, and

$$\sqrt{b}-\sqrt{(b-k)^n}=[\sqrt{a}-\sqrt{(a-k)}]^n.$$

For $a=k=0$, any value of b will satisfy the given equation. Otherwise, by division

$$\sqrt{b}+\sqrt{(b-k^n)}=[\sqrt{a}+\sqrt{(a-k)}]^n.$$

Thus $\sqrt{b}=\frac{1}{2}[\sqrt{a}+\sqrt{(a-k)}]^n+\frac{1}{2}[\sqrt{a}-\sqrt{(a-k)}]^n$. If n is even, all surds cancel in the right member, so that $\sqrt{b}$ and therefore b is integral. If n is odd, the right member contains only like surds, which combine into a single radical. Squaring both sides yields an integral value for b.

Also solved by Jerome Cherniack, T. P. Dence, D. Ž. Djoković, E. S. Eby, Neal Felsinger, N. J. Fine, M. G. Greening (Australia), Stephen Hoffman, Donald Jeffords, Kenneth Kramer, Beatriz Margolis (Argentina), D. C. B. Marsh, Helen Marston, Norman Miller, Steven Minsker, Barbara W. Nason, Stanley Rabinowitz, Simeon Reich (Israel), P. A. Scheinok, R. Singh, B. S. Lalli, R. Manchar, Charles Wexler, and the proposer.

Coloring a Generalized Map

E 1989 [1967, 589]. *Proposed by W. A. McWorter, University of British Columbia*

Define a generalized map on the plane to be a partition of the plane into a

finite family of pairwise disjoint, nonempty, connected sets. These sets are called countries. We say two countries are adjacent if their union is connected. Prove or disprove that such a map on the plane can be colored in five colors so that no two adjacent countries have the same color. What happens if we replace "connected" with "arcwise connected"?

Solution by Stanley Rabinowitz, Far Rockaway, N.Y. Finitely many colors will not suffice to color a generalized map. We show how to construct a generalized map with k countries each adjacent to each other for any k.

Let $A_1, A_2, \cdots, A_k$ be any k distinct points in the plane. Country E_i originally owns point A_i, $i=1, \cdots, k$. In step 1 of an expansion program, country E_1 extends its territory by claiming land along a continuous curve of finite length which starts at A_1 and does not intersect itself or A_j $(j\neq 1)$ and passes within a distance ϵ of each point A_j $(j\neq 1)$. In no way has this country enclosed any land, so all unoccupied points are accessible to any country by extensions along continuous curves. In step n $(n=1, 2, \cdots)$, country E_n (subscript reduced modulo k) extends its previously owned territory by claiming land along a continuous curve of finite length which starts at A_n or any other point it already owns and does not intersect itself or the property of any other country and which passes within $\epsilon/2^{n-1}$ of each point A_j ($j\neq n$ modulo k). This is possible because in no previous step has any unoccupied land been made inaccessible. If step n is completed in a time of $1/2^n$ minutes, after one minute each country will be disjoint, connected, and adjacent to each other country. We must merely verify that countries E_i and E_j are adjacent $(i\neq j)$. The expansion process assures that A_j is a limit point of country E_i. Hence $E_i\subset E_i\cup\{A_j\}\subset\overline{E}_i$, and so $E_i\cup\{A_j\}$ is connected. But $E_i\cup E_j$ is the union of the two intersecting connected sets E_j and $E_i\cup\{A_j\}$ and is therefore connected.

If we replace "connected" with "arcwise connected" in the definition, then five colors will suffice to color any generalized map. For pick a point in each country and connect those points in adjacent countries by a continuous simple curve lying wholly in their union, and not meeting any other such curve. These points will then be the vertices of a planar network and can thus be colored with five colors (so that endpoints of each arc are colored differently) by the dual of the usual five-color map theorem. Hence the generalized map can be colored with these five colors in an analogous way.

Also solved by L. J. Burton, and by the proposer.

Properties of General Pentagons

E 1990 [1967, 590]. *Proposed by V. F. Ivanoff, San Carlos, California*

Given a pentagon with the sides 1, 2, $\cdots$, 5 in that order. Denoting by $p(q:r)$ the segment on the side p intercepted by the sides q and r, prove that

$$\text{(a) } \prod [1(3:5)] = \prod [5(1:3)], \qquad \text{(b) } \prod [1(4:5)] = \prod [5(1:2)],$$

where each factor is obtained by increasing the numbers of its predecessor by 1,

the resulting numbers to be taken modulo 5. (Note: Both statements have been proved for cyclic pentagons. See W. B. Carver, *Cyclic Polygons*, this MONTHLY, 68 (1961) 537, with two illustrations.)

Solution by Stanley Rabinowitz, Far Rockaway, N.Y. The stated results, and others similar to them, are easily established by use of the law of sines. Let B be the intersection of sides 1 and 2 ($B=12$), $G=13$, $J=14$, $A=15$, $C=23$, $H=24$, $F=25$, $D=34$, $I=35$, $E=45$. Then

$$\text{(a)}\ \frac{\prod[1(3:5)]}{\prod[5(1:3)]}=\frac{AG}{AI}\cdot\frac{BH}{BJ}\cdot\frac{CI}{CF}\cdot\frac{DJ}{DG}\cdot\frac{EF}{EH}$$

$$=\frac{\sin\measuredangle I}{\sin\measuredangle G}\cdot\frac{\sin\measuredangle J}{\sin\measuredangle H}\cdot\frac{\sin\measuredangle F}{\sin\measuredangle I}\cdot\frac{\sin\measuredangle G}{\sin\measuredangle J}\cdot\frac{\sin\measuredangle H}{\sin\measuredangle F}=1.$$

$$\text{(b)}\ \frac{\prod[1(4:5)]}{\prod[5(1:2)]}=\frac{AJ}{EJ}\cdot\frac{BF}{AF}\cdot\frac{CG}{BG}\cdot\frac{DH}{CH}\cdot\frac{EI}{DI}$$

$$=\frac{\sin\measuredangle E}{\sin\measuredangle A}\cdot\frac{\sin\measuredangle A}{\sin\measuredangle B}\cdot\frac{\sin\measuredangle B}{\sin\measuredangle C}\cdot\frac{\sin\measuredangle C}{\sin\measuredangle D}\cdot\frac{\sin\measuredangle D}{\sin\measuredangle E}=1.$$

Also solved by M. G. Greening (Australia), and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate signed sheets and should be mailed before March 31, 1969. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5575 [1968, 305]. **Correction.** *Proposed by Alexandru Lupas, Cluj, Rumania*

Let $Q(0, a)$ be the set of functions which are defined and bounded on $[0, \infty)$ and which are continuous on the interval $[0, a]$, $a>0$. On $Q(0, a)$ we define, for a positive integer n, the linear positive operator L_n by

$$(*)\qquad (L_nf)(x)=\sum_{k=0}^{\infty}\binom{k+n-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right).$$

(1) Prove that $\|L_nf-f\|\to 0$ where $\|f\|$ is the maximum of $|f(x)|$ for $x\in[0, a]$.

(2) We say that a function f is P-convex of order s on the reals if the divided differences of order $s+2$, $[x_1, x_2, \cdots, x_{s+2}; f]$, are positive for every real x_i, $i=1, 2, \cdots, s+2$. Show that if f is convex of order s then L_nf is also a convex function of order s.

(3) The sequence of operators $\{L_n\}$, defined by (*) is decreasing on the class of convex functions of order 1, and satisfies $L_{n+1}-L_n=0$, for sufficiently large n, on the class of linear functions.

5610. *Proposed by K. S. Menger, Jr., Cambridge, Mass.*

(i) Show that any $n-1$ cycles, of lengths $2, 3, \cdots, n$ respectively, generate S_n, the symmetric group of degree n.

(ii) Show that if $m=rs$ then there exist $r+s-2$ cycles, no two of the same length, which do not generate S_m.

(iii) Show that any transposition and any cycle of length n generate S_n if and only if n is a prime.

5611. *Proposed by Leonard Carlitz, Duke University*

Let D denote the polynomial domain $GF[p, x]$, p odd prime. Let $P(x)$ be a monic irreducible polynomial in D of degree n. Show that the congruence $U^2 \equiv P'(x) \pmod{P(x)}$ is solvable with $U \in D$ if and only if $n(n-1)(p-1)/4 \equiv n-1 \pmod 2$.

5612. *Proposed by Franklin C. Smith, Minneapolis, Minn. and H. W. Gould, West Virginia University*

Evaluate the summations:

$$\sum_{k=0}^{n} \binom{2x+i}{2k+j}\binom{x-k}{n-k}, \qquad n \geqq 0,$$

where x is any real number, for $i=0, 1$ and $j=0, 1$ (four sums). Is there a general closed formula valid for all integers i, j?

5613. *Proposed by Robert Goldstein, The Northern Polytechnic, London, England*

Let $\omega(z)$ be an entire function such that there exists a nonconstant function $u(z)$, harmonic for all z, for which $u(z)=u(\omega(z))$ for all z. Show that this implies $\omega(z)=\zeta z+\alpha$, where ζ is a root of unity.

This problem is a generalization of 5329 [1966, 904].

5614. *Proposed by Tung-Po Lin, San Fernando State College, California*

Assume that the earth is a sphere of homogeneous density. An initially stationary object at a point A on the surface of the earth slides under gravitational force through a frictionless tunnel leading to another point B on the surface. Show that the time needed to slide from A to B is a minimum when the tunnel takes the shape of a hypocycloid. Furthermore, show that this minimum time is equal to $(1-b^2/a^2)^{1/2}T$, where a is the radius of the earth, b is the distance from earth center to the nearest point on the tunnel, and T is the time needed to slide from A to B through a frictionless straight-line tunnel.

5615. *Proposed by G. F. Schumm, University of Chicago*

Suppose S is any infinite set of cardinality m, and let $\mathcal{P}(S)$ denote the power of S. Using the generalized continuum hypothesis, E. S. Wolk proves that there exists a chain $\mathcal{C}$ in $\mathcal{P}(S)$ with $\mathrm{card}(\mathcal{C})=2^m$ (*A theorem on power sets*, this MONTHLY, 72 (1965) 397–398). How many such chains are there?

5616. *Proposed by E. R. Gentile, University of Buenos Aires, Argentina*

Let K be a ring with an identity and without zero divisors $\neq 0$. Then, if there is a natural number n such that $(a\cdot b)^k = a^k \cdot b^k$ for $k=n, n+1$, K is a commutative ring.

(Cf. I. N. Herstein, *Topics in Algebra*, 1964, p. 31, nos. 4, 5.)

5617. *Proposed by C. J. Mozzochi, University of Connecticut*

Prove (by elementary methods): If f is continuous on $I=[-\pi, \pi]$, then for every $\epsilon>0$ there exists an integer N such that for all $n \geqq N$ we have $|S_n(x)-f(x)| \leqq (n+1)\epsilon$ for all x in I where $S_n(x)$ is the nth partial sum of the Fourier series for f.

5618. *Proposed by David Boyd, University of Alberta*

Let $0 \leqq r < 1$, and define

$$f(r) = \lim_{n\to\infty} 2^{-n} \sum \left| \pm 1 \pm r \pm \cdots \pm r^{n-1} \right|,$$

where the summation is over all 2^n possible choices of sign. Show that

$$\text{(a) } f[(\sqrt{5}-1)/2] = 4/(6-\sqrt{5}), \qquad \text{(b) } f(2^{-1/2}) = 7/6.$$

5619. *Proposed by G. L. Britton, South Dakota School of Mines and Technology*

Define a finite, undirected graph $G=(V, E)$ to be minimally of diameter 2 if and only if G has diameter 2 and for every $e \in E$ the graph $G'=(V, E-\{e\})$ has diameter greater than 2. Are there any planar graphs which are minimally of diameter 2 and (a) enclose only regions bounded by three edges, (b) enclose only regions bounded by three or four edges?

SOLUTIONS OF ADVANCED PROBLEMS

5493 [1968, 559]. **Correction.** *Solution II was contributed by P. G. Comba* (not Combea as erroneously printed).

Embedding a Finite Group in a Finite Group

5500 [1967, 727]. *Proposed by W. A. McWorter, University of British Columbia*

Let H be a nontrivial finite group. Then there exists a finite group G in which H is properly embedded as its own normalizer in G.

Solution by C. R. Combrink, Texas Christian University. Let p be a prime which does not divide the order of H, and let $W = A\,\mathrm{wr}\,H$ be the standard wreath product of a cyclic group A of order p with H. Let 1 denote the identity element of all groups. Then, if $F=\{f \mid f \text{ is a function from } H \text{ into } A\}$, W is a split extension of F by H. By Lemma 3.1 and Corollary 4.5 of Newmann, Peter M., *On the*

structure of standard wreath products of groups, Math. Z., 84 (1964) 343–373, $N_W(H)=DH$ and $W'=MH'$, where

$$D = \{f \in F \mid f(x) = f(1) \quad \text{for all } x \in H\},$$
$$M = \{f \in F \mid \Uparrow \{f(h) \mid h \in H\} = 1\}.$$

Since $(p, o(H))=1$, $D\cap M=(1)$. Thus if $G=W'H=MH$,

$$N_G(H) = N_W(H) \cap G = (DH) \cap (MH) = (D \cap M)H = H.$$

$H \neq G$ since $[F: M]=p$.

Also solved by P. R. Chernoff & W. C. Waterhouse, Eleanor D. Jones, Joanne Peeples, W. R. Scott, and the proposer.

Nonsimple Finite Groups

5505 [1967, 728]. *Proposed by Erwin Just, Bronx Community College*

If p is a prime and $p>m^{1/2}$, prove that any group of order p^2m is not simple.

Solution by Irving Katz, George Washington University. This problem is a special case of Exercise 3 on page 163 of W. Burnside, *Theory of Groups of Finite Order* (Dover). In fact, the requirement that the group have order p^2m may be relaxed to allow p^nm, $0<n$.

If $1+kp$ is the number of Sylow p-subgroups, then $(1+kp)\mid m$ and so $1+kp\leqq m<p^2$. Then apply the cited result.

Also solved by W. O. Alltop, J. L. Berggren, Laura Conte, M. G. Greening (Australia), R. P. Grimaldi, O. Maners, Joanne Peeples, J. H. Ryshpan, F. J. Samaniego, Z. Z. Uoiea, and the proposer.

On Hypergeometric Functions

5510 [1967, 871]. *Proposed by M. L. Glasser, Battelle Memorial Institute, Columbus, Ohio*

Show that

$$\sum_{k=-\infty}^{\infty} (-1)^k(x+kz)\, \exp(-k^2z^2/x)\cdot {}_1F_1\left[\frac{1}{2}, \frac{3}{2}; (x+kz)^2/x\right] = 0.$$

Does this generalize to other hypergeometric functions?

Solution by the proposer. In the relation

$$\sum_{k=0}^{\infty} (-1)^k \int_c^d g(y) \cosh kydy = \tfrac{1}{2} \int_c^d g(y)dy,$$

set $g(y)=(2z)^{-1}\exp(xy^2/4z^2)$, $c=0$, $d=2z$. The stated result follows on using the formula

$$\int e^{ay^2+by}\,dy = \left(y-\frac{b}{2a}\right)e^{-b^2/4a}\cdot {}_1F_1\left[\frac{1}{2}, \frac{3}{2}; a\left(y-\frac{b}{2a}\right)^2\right] + \text{const.}$$

Groups with Elements of Order 2

5511 [1967, 871]. *Proposed by R. S. Kulkarni, Harvard University*

Characterize groups of order $2n$ which contain exactly n elements of order 2 (i.e., $a^2=1$, $a\neq 1$). (Note that the symmetry group of a polygon with an odd number of sides has this form.)

Solution by A. D. Sands, The University, Dundee, Scotland. Let G be such a group. Elements g with $g\neq g^{-1}$ occur in pairs, namely g, g^{-1}. Thus the total number of elements with $g=g^{-1}$ is even. Omitting the identity 1 it is seen that n must be odd. Let $A=\{a_1, \cdots, a_n\}$ be the set of elements of order 2. It follows easily that if $a_ia_j=a_ja_i$ then $\{1, a_i, a_j, a_ia_j\}$ is a subgroup of G and so $4\mid 2n$, which is false. Further $a_ia_j\neq a_k$, as $a_ia_j=a_k$ would imply $a_ja_i=a_k^{-1}=a_k$. Thus if $B=\{b_1, b_2, \cdots, b_n\}$, where $a_ia_j=b_j$, then B is the set of elements of G not of order 2 including the identity. Now for b_rb_s to be an element of A would imply $(a_1a_ra_1)a_s\in A$. But $a_1a_ra_1\in A$, as it has order 2 and so, from above, $(a_1a_ra_1)a_s\notin A$. Therefore $b_rb_s\in B$. Thus B is a subgroup of G. From $a_ra_1a_s=a_sa_1a_r$ it follows that $b_rb_s=b_sb_r$, whence B is abelian. Further $a_1b_ra_1=a_ra_1=b_r^{-1}$.

Conversely, if B is an abelian group of odd order n and α is the automorphism of B given by $\alpha(b)=b^{-1}$ for each $b\in B$, then the group G obtained by extending B by α, where $\alpha^2=1$, is of the required form.

Also solved by J. L. Berggren, D. M. Bloom, P. R. Chernoff & W. C. Waterhouse, D. Z. Djoković, M. G. Greening (Australia), M. E. Harris, Rudolf Kochendörffer (Germany), Ka Menehune, Hugo Sun, and the proposer.

Djoković reports that the problem appeared in the Canadian Mathematical Bulletin as Problem 129.

Drawing Balls from an Urn

5512 [1967, 871]. *Proposed by M. F. Neuts, Purdue University*

At time 0 an urn contains N balls ($N\geqq 0$). Every time unit later, k balls are added and immediately thereafter (i.e., at times $n+0$), r balls are drawn out at random and discarded ($0<r<k$).

1. What is the probability that a particular ball which is in the urn at time $n+\frac{1}{2}$ will be thrown out later than time $n+\nu+0$? $\nu=1, 2, \cdots$.

2. What is the probability that a particular ball will eventually be thrown out?

3. What is the expected number of drawings required before the given ball is thrown out?

Solution by F. W. Steutel, Technische Hogeschool Twente, Enschede, Netherlands. 1. The number of balls present at time $n+\frac{1}{2}$ is $N+n(k-r)$, so that the probability p_ν that a ball present at time $n+\frac{1}{2}$ is still present at time $n+\nu+\frac{1}{2}$ is given by

$$p_\nu=\prod_{t=1}^{\nu}\left(1-\frac{r}{\alpha+t(k-r)}\right), \tag{a}$$

where $\alpha=N+r+n(k-r)$.

2. As the product in (a) diverges to zero for every α (even if $k=r$), the probability that a ball present at time $n+\frac{1}{2}$ is thrown out eventually equals

$$1-\lim_{\nu\to\infty} p_\nu = 1.$$

3. If by T_n we denote the average throw-out time (number of drawings) for a ball present at time $n+\frac{1}{2}$, we have

$$T_n = 1+\sum_{\nu=1}^{\infty} p_\nu = 1+\sum_{\nu=1}^{\infty}\prod_{t=1}^{\nu}\left(1-\frac{r}{\alpha+t(k-r)}\right), \tag{b}$$

which is finite for $k<2r$ and infinite for $k\geqq 2r$ because, by (a),

$$p_\nu \sim \text{const}\cdot \nu^{-r/(k-r)}, \qquad \nu\to\infty.$$

Also it can easily be deduced from (b) that $T_n=O(n)$ for $n\to\infty$. On the other hand the T_n are uniquely determined by the relations

$$T_n = 1+\frac{N+(n+1)(k-r)}{N+n(k-r)+k}\,T_{n+1}, \qquad T_n=O(n). \tag{c}$$

But (c) is also seen to be satisfied by

$$T_n=\frac{N+n(k-r)+r}{2r-k} \qquad (n=0, 1, 2, \cdots).$$

Also solved by Roy O. Davies (England), D. A. Hejhal, E. S. Langford, Oswald Wyler, and the proposer.

Translating a Set to its Complement

5513 [1967, 871]. *Proposed by L. F. Meyers, Ohio State University*

If a is a real number and S is a subset of the set R of all real numbers, then the translate T_aS of S by a is defined to be $\{x+a: x\in S\}$. For certain subsets S of R it is possible to find a real number a such that T_aS is the complement S' of S in R. For example, if $S=\bigcup_{k=-\infty}^{\infty}[2k, 2k+1)$, then the translate of S by each odd integer is S'. For each set S allowing a translation onto its complement, let

$$f(S) = \inf\{a: a>0 \quad\text{and}\quad T_a S = S'\}.$$

In the above example, $f(S)=1$. Is it possible to find a set such that $f(S)=0$? In other words, is there a subset of the real line which admits arbitrarily small nontrivial translations of itself onto its complement?

I. *Solution by J. C. Morgan, II, University of California at Berkeley.* Such sets S do exist. Let I denote the set of integers and let $D=\{m/3^n: m, n\in I, n\geqq 0\}$. The relation $\equiv$ defined by $x\equiv y$ if and only if $x-y\in D$ is an equivalence relation on R. Select a set E containing one and only one element from each distinct

equivalence class. For each $x \in E$, the equivalence class containing x is divided into the two disjoint sets

$$A_x = \{x + m/3^n: m, n \in I, m \text{ even}, n \geqq 0\},$$

and

$$B_x = \{x + m/3^n,: m, n \in I, m \text{ odd}, n \geqq 0\}.$$

Define $a = 1/3^n$ and observe that $T_a A_x = B_x$ for each $x \in E$ and each nonnegative integer n. Therefore, if $S = \bigcup_{x \in E} A_x$, then $T_a S = \bigcup_{x \in E} T_a A_x = \bigcup_{x \in E} B_x = S'$ for each nonnegative integer n. Hence $f(S) = 0$.

II. *Solution by D. P. Giesy, University of Southern California.* Let G be a subgroup of the real numbers under addition such that G has a subgroup H of index 2 in G. Let A be the other coset of H in G. Let $\{x_t: t \in T\}$ be a complete set of coset representatives of G in the real numbers. Let

$$S = \bigcup \{x_t + H: t \in T\}.$$

Then it is routine to verify that $S' = \bigcup \{x_t + A: t \in T\}$, and that for each $a \in A$, $S + a = S'$. So the problem will have an affirmative answer if we can display a G as above such that A has arbitrarily small positive elements. We offer, for example,

$$G = \{m/2n: n \text{ odd}\} \qquad \text{with } H = \{m/n: n \text{ odd}\},$$

so $A = \{m/2n: m \text{ and } n \text{ odd}\}$.

Also solved by R. D. Berlin, P. R. Chernoff & W. C. Waterhouse, C. R. Combrink, Roy O. Davies (England), N. J. Fine, Jan Hejcman (Czechoslovakia), Dennis Henkel, James S. Johnson, N. K. Krier, O. P. Lossers (Netherlands), Anastassios Nacassis (Greece), L. T. Ollmann, P. J. Owens (England), Charles Riley, J. T. Rosenbaum, J. F. Wirth, Oswald Wyler, and the proposer.

Editorial Note. All of the solutions involved the use of the axiom of choice. This is not surprising in view of the demonstrations by Chernoff & Waterhouse, Davies, Lossers, and Warren Page that all such sets S are nonmeasurable (Lebesgue).

A Functional Inequality

5514 [1967, 872]. *Proposed by J. W. Wyman, Pasadena College, California*

Find all positive valued functions f defined on the positive real numbers such that $ab \leqq \{\frac{1}{2}af(a) + bf^{-1}(b)\}$ for every $a > 0$, $b > 0$.

Solution by Charles Riley, Keene State College, New Hampshire. We will show that to satisfy the given condition, f must be cx, $c > 0$. The hypothesis implies that f is 1-1 and onto the positive reals. Let x and y be any positive numbers, and $z = f(y)$. Then $2xz \leqq xf(x) + zf^{-1}(z) = xf(x) + yf(y)$. That is

$$2xf(y) \leqq xf(x) + yf(y), \tag{1}$$

and

$$2yf(x) \leqq xf(x) + yf(y). \tag{2}$$

Thus $xf(y)+yf(x)\leqq xf(x)+yf(y)$ and $(x-y)(f(x)-f(y))\geqq 0$. This shows that f is increasing. The fact that f is onto now implies the continuity of f. Inequalities (1) and (2) imply

$$\frac{y-x}{y}f(x)\leqq f(y)-f(x)\leqq\frac{y-x}{x}f(y).$$

Thus $(f(y)-f(x))/(y-x)$ is between $f(x)/y$ and $f(y)/x$. Using the continuity of f, $f'(x)=f(x)/x>0$ for each $x>0$. Thus $f(x)=cx$ with $c>0$. It is easily verified that the stated inequality holds for such functions.

Also solved by P. R. Chernoff, Ted Cullen, Roy O. Davies (England), Anastassios Nacassis (Greece), C. B. A. Peck, Alberto Torchinsky, and the proposer.

Davies comments that if f is not one-one, the inequality is to be assumed true for every value of $f^{-1}(b)$. If f is many-to-one and the inequality is understood as being true merely for at least one value of $f^{-1}(b)$, then nothing much can be concluded. Any function such that each value is taken for arbitrarily large x will satisfy this condition, and it is easy to construct such functions: for example, $f(x)=\sin x+2$.

Equivalent (?) Metrics

5515 [1967, 872]. *Proposed by J. W. Wyman, Pasadena College, California*

Let f: $M_1\rightarrow M_2$, where M_1 and M_2 are metric spaces, and define f to be absolutely continuous if for each $\epsilon>0$, there exists $\delta>0$ such that if $\{a_1,\cdots,a_n\}$ is any finite set of points of M_1 such that $\sum_{i=1}^{n-1}d(a_i,a_{i+1})<\delta$, then $\sum_{i=1}^{n-1}d(f(a_i),f(a_{i+1}))<\epsilon$. If f is continuous, do there exist metrics equivalent to the given metrics in M_1 and M_2 respectively such that f is absolutely continuous with respect to those metrics?

I. *Solution by Jan Hejcman, Prague, Czechoslovakia.* Let f: $M_1\rightarrow M_2$ be continuous, M_i be metrized by p_i, $i=1, 2$. Put, for x, $y\in M$, $p'(x,y)=p_1(x,y)+p_2(f(x),f(y))$. Then p' is a metric on M_1; if $p_1(x_n,x)\rightarrow 0$, then $p_2(f(x_n),f(x))\rightarrow 0$ and therefore $p'(x_n,x)\rightarrow 0$. The converse follows from the inequality $p'(x,y)\geqq p_1(x,y)$. Thus p' and p_1 are equivalent. For any x, $y\in M_1$, $p_2(f(x),f(y))\leqq p'(x,y)$, which implies that f: $(M_1,p')\rightarrow(M_2,p_2)$ is absolutely continuous.

II. *Solution by Peter W. Gray, Clarkson College of Technology.* Let f: $(0,1)\rightarrow[0,\infty)$, where $f(x)=1/x$. Let $(0,1)$ and $[0,\infty)$ have the usual metrics. Suppose that there are equivalent metrics so that f is absolutely continuous with respect to these metrics. Then f is uniformly continuous. So we may extend f uniquely to a uniformly continuous function on $[0,1]$ which is a contradiction. So f must not be uniformly continuous and hence is not absolutely continuous in the new metric. (You may replace absolutely continuous by uniformly continuous and the answer still is no.)

III. *Comment by P. R. Chernoff and W. C. Waterhouse, Harvard University.* The answers to this problem are different—depending on what meaning is given to "equivalent." If equivalent means inducing the same topology, see I above.

If equivalent means inducing the same uniform structure, see II above.

Also solved by Anastassios Nacassis (Greece), and the proposer who refers to Norman Levine, *Remarks on Uniform Continuity in Metric Spaces*, this MONTHLY, 67 (1960) 562.

Basis for a K-module

5516 [1967, 872]. *Proposed by D. J. Simanaitis and L. R. Anderson, Case-Western Reserve University*

Let E be a nonzero unitary K-module which is generated by one element. Suppose the annihilator of E in K is $\{0\}$. Prove or disprove: E admits a basis.

Solution by Oswald Wyler, Carnegie-Mellon University. If K is a commutative ring, then the stated conditions imply that E has a one-element basis, consisting of the generator of E. The following example shows that E need not have a basis if K is noncommutative.

Let K be the ring of n by n matrices over a commutative ring R with identity, $n>1$, and let L be the set of all matrices in K with zero first column. Then L is a left ideal of K, and the congruence classes $a+L$, $a\in K$, form a left K-module K/L, generated by the congruence class of the identity matrix. For any matrix $a\in K$, there is a matrix $b\neq 0$ in K such that $ba\in L$, and then $b(a+L)=0$ in K/L. Thus K/L has rank 0, and cannot have a basis. On the other hand, for every matrix $a\neq 0$ in K there is a matrix $c\in K$ such that $ac\notin L$, and then $a(c+L)\neq 0$ in K/L. Thus the K-module $E=K/L$ satisfies all conditions of the problem and does not have a basis.

Also solved by R. A. Howland, John Kieffer, T. M. Viswanathan, W. C. Waterhouse, B. J. Winkel, J. F. Wirth, and the proposers.

Note. The proposers offered this problem as a counterexample for Exercise 27.18. p. 278, of S. Warner, *Modern Algebra*, vol. I, with the additional comment that the converse of the statement of the problem—If E admits a basis, then the annihilator of E is $\{0\}$—is valid.

The Kernel of a Relation

5517 [1967, 872]. *Proposed by W. A. McWorter, University of British Columbia*

Let $\bigcirc$ be a relation on a set X. A subset A of X is said to be a kernel of $\bigcirc$ if (1) $x\varnothing y$ for every x, y in A, and (2) for every x not in A, there is a y in A such that $x\bigcirc y$.

Suppose $\bigcirc$ is such that for every nonempty subset B of X there is an x in B such that $x\varnothing b$ for every b in B. Prove that $\bigcirc$ has a unique kernel.

I. *Solution by P. S. Schnare, University of Florida.* If X is empty, so is the kernel of $\bigcirc$; we consider henceforth that X is not empty. Let B' denote the set of minimal elements of B with respect to the relation $\varnothing$, i.e., $x\in B'$ iff $x\in B$ and $b\in B$ imply $x\varnothing b$. By hypothesis B is not empty and so B' is not empty. Say that two subsets of X: M and N are *nice* iff (i) if $x\in M$ and $y\in X-N$, then $x\varnothing y$; and (ii) if $x\in N$, then there exists $y\in M$ such that $x\bigcirc y$. Let $\mathfrak{F}=\{(M, N)$: M and N are nice disjoint subsets of $X\}$. $\mathfrak{F}$ is not empty since $(X', \phi)\in\mathfrak{F}$. Par-

tially order $\mathfrak{F}$ with the relation $\prec$ defined by $(M_1, N_1)\prec(M_2, N_2)$ iff $M_1\subset M_2$ and $N_1\subset N_2$. By Zorn's lemma $\mathfrak{F}$ has an element (A, B) maximal with respect to $\prec$.

Suppose now that $X\neq A\cup B$. Let $z\in[X-(A\cup B)]'$. By maximality of (A, B), $(A\cup\{z\}, B)\notin\mathfrak{F}$. Therefore $z\bigcirc y$ for some $y\in X-B$. By our choice of z, $y\notin X-(A\cup B)$. Therefore, $y\in A$. But then, $(A, B)\prec(A, B\cup\{z\})\in\mathfrak{F}$, a contradiction. Thus $X=A\cup B$ and A is clearly a kernel of $\bigcirc$, as a result of (i), (ii) above.

Suppose that A_1 is also a kernel of $\bigcirc$. Let $x_0\in[(A-A_1)\cup(A_1-A)]'$. If $x_0\in A-A_1$, then there exists $y\in A_1$ such that $x_0\bigcirc y$. Since x_0 is in A, $y\notin A$, whence $y\in A_1-A$. Hence, $x_0\varnothing y$, a contradiction. Thus $x_0\notin A-A_1$ and by symmetry $x_0\notin A_1-A$. Therefore $(A-A_1)\cup(A_1-A)$ is empty, i.e., $A=A_1$.

II. *Solution by Fred Galvin, St. Paul, Minn.* The result is an interpretation of problem 5332 [1966, 1022]. For any subset A of X, let $f(A)=\{x\in X: x\varnothing y$ for every $y\in A\}$. Then f satisfies the conditions of problem 5332 and there is a unique subset of X which is a fixed point of f, and this subset is the unique kernel of $\bigcirc$.

III. *Solution by D. P. Sumner, University of Massachusetts.* The notation in this solution will conform to that in *The Theory of Graphs* by Claude Berge (translated by A. Doig). For the given set X, we define a graph (X, Γ) by $x\in\Gamma(y)$ iff $y\bigcirc x$. We claim that (X, Γ) is progressively finite (i.e., for each $x_0\in X$, there are no infinite paths originating at x_0). Suppose for some $x_1\in X$, $[x_1, x_2, \cdots, x_n, \cdots]$ is an infinite path in (X, Γ). Then $x_2\in\Gamma(x_1)$, $x_3\in\Gamma(x_2)$, $\cdots$, $x_{n+1}\in\Gamma(x_n)$ for every $n=1, 2, \cdots$. Therefore, $x_n\bigcirc x_{n+1}$ for each $n=1, 2, \cdots$. Let $B=\{x_1, x_2, \cdots\}$; then for every $x_i\in B$, $x_i\bigcirc x_{i+1}$. But this contradicts the condition that there exists an $x\in B$, $x\varnothing b$ for each $b\in B$. Thus (X, Γ) is progressively finite. Using Theorem 4, p. 48, *loc. cit.* (A progressively finite graph possesses a kernel and this kernel is unique), we know that (X, Γ) has a unique kernel in the graph-theoretic sense. That is, there exists a unique set $S\subset X$ such that

$$x\in S\Rightarrow\Gamma(x)\cap S \text{ is empty.} \tag{1}$$

$$x\notin S\Rightarrow\Gamma(x)\cap S \text{ is not empty.} \tag{2}$$

But (1) is equivalent to the condition that for every $x, y\in S$, $x\varnothing y$; and (2) is equivalent to the condition that $x\notin S$ implies that there exists a $y\in S$ such that $x\bigcirc y$. So X contains a unique kernel in the sense of the problem.

Also solved by R. A. Christiansen, Roy O. Davies (England), D. P. Geller, M. G. Greening (Australia), D. A. Hejhal, Peter Kornya, M. J. Lempel, George Mazaitis, R. D. Meredith, Lois E. Minning, Anastassios Nacassis (Greece), Tivis Nelson, C. B. A. Peck, S. B. Rao (India), Simeon Reich (Israel), Henry Ricardo, Charles Riley, A. H. Shuchat, Stephen Tice, Hwai-chiuan Wang (Taiwan), Mary M. Whiteside, J. C. Williams, Oswald Wyler, and the proposer.

The Traces of the Powers of a Matrix

5518 [1967, 872]. *Proposed by P. J. Schweitzer, Institute for Defense Analyses, Arlington, Va.*

Let $a_k=\sum_{i=1}^n(x_i)^k$, $k=1, 2, \cdots$. Show that a_k, $(k=n+1, n+2, \cdots)$ can be expressed as a polynomial in $a_1, a_2, \cdots, a_n$. Stated differently, show that the traces of the higher powers of an $n\times n$ matrix can be expressed as polynomials in the traces of the first n powers.

Solution by John Kieffer, University of Illinois. The a_k are symmetric polynomials in the x_i. Each a_k can be expressed as a polynomial in the elementary symmetric functions $\sigma_1, \sigma_2, \cdots, \sigma_n$ with integral coefficients,

$$\sigma_1=\sum x_i,\ \sigma_2=\sum_{i<j} x_ix_j, \cdots, \sigma_j=\sum_{i_1<i_2<\cdots<i_j} x_{i_1}x_{i_2}\cdots x_{i_j}, \cdots,$$

$\sigma_n=x_1x_2\cdots x_n$, (cf. van der Waerden, *Modern Algebra*, vol. I, p. 80).

We have also the following formula due to Newton:

$$a_j-a_{j-1}\sigma_1+a_{j-2}\sigma_2-\cdots+(-1)^{j-1}a_1\sigma_{j-1}+(-1)^j j\sigma_j=0,$$

$1\leqq j\leqq n$, whence we can express each σ_i as a polynomial in $a_1, a_2, \cdots, a_n$ with rational coefficients. Hence, each a_k $(k=n+1, n+2, \cdots)$ can be expressed as a polynomial in $a_1, a_2, \cdots, a_n$ with rational coefficients.

As an application, let A be an $n\times n$ matrix whose characteristic values are $x_1, x_2, \cdots, x_n$. Then we can interpret a_k to be the trace of A^k, for we know that the trace of A^k can be expressed as the sum of the kth powers of the characteristic values of A. It follows that trace A^k $(k=n+1, n+2, \cdots)$ has been expressed as a polynomial in trace A, trace $A^2, \cdots$, trace A^n with rational coefficients.

Also solved by P. M. Brady, Jr., P. R. Chernoff & W. C. Waterhouse, C. G. Cullen, D. Ž. Djoković, N. J. Fine, D. A. Hejhal, D. A. Klarner, Rudolf Kochendörffer (Germany), Bohuslav Míšek (Czechoslovakia), Tassos Nacassis (Greece), P. V. Subba Rao (India), Simeon Reich (Israel), Steven Weintraub, Oswald Wyler, and the proposer.

On the Order of a Special Integral

5519 [1967, 872]. *Proposed by W. O. Egerland, U.S.A. Nuclear Defense Analyses, Arlington, Va.*

Let $\sum_{n=1}^\infty a_nz^n$ be a power series in $z=x+iy$, $|a_n|\leqq cn^p$, c and p positive constants. Determine, for every $\alpha<1$,

$$\lim_{n\to\infty} n^\alpha\iint_{|z|\leq 1}\log(1+|a_nz^n|)dxdy.$$

Solution by D. A. Hejhal, Student, University of Chicago. Writing I for $\iint_{|z|\leq 1}\ln(1+|a_nz^n|)dxdy$ and transforming to polar coordinates we get

$$0\leqq I\leqq 2\pi\int_0^1 r\ln(1+Ar^n)dr,\ A=cn^p.$$

Using the transformation $t=Ar^n$, the integral becomes

$$\frac{1}{nA^{2/n}}\int_0^A t^{2/n-1}\ln(1+t)dt.$$

The integrand $t^{2/n}\ln(1+t)/t$ is bounded on (0, 1). Thus

$$\frac{n^\alpha}{nA^{2/n}}\int_0^1 t^{2/n-1}\ln(1+t)dt \to 0 \text{ as } n\to\infty.$$

For the remainder of the integration range we have

$$\frac{n^\alpha}{nA^{2/n}}\int_1^A t^{2/n-1}\ln(1+t)dt \leqq \frac{n^{\alpha-1}}{A^{2/n}}A^{2/n}\int_1^A \frac{\ln(2t)}{t}dt$$
$$= \tfrac{1}{2}n^{\alpha-1}(\ln 2A)^2 \leqq n^{\alpha-1}[\ln(2cn^p)]^2 \to 0 \text{ as } n\to\infty.$$

Thus the required limit is zero for all $\alpha<1$.

Also solved by Rudolf Gorenflo (Germany), O. P. Lossers (Netherlands), and the proposer.

Semi-simple Rings

5520 [1967, 1013]. *Proposed by T. J. Head, University of Alaska*

Let Λ be a ring with identity. A (left) Λ-module M is said to have a *projective cover* if there is a projective Λ-module P and a Λ-homomorphism ϕ: $P\to M$ of P onto M for which the only submodule S of P satisfying $S+\ker(\phi)=P$ is $S=P$. Each projective Λ-module has a projective cover, namely the identity mapping of P onto P. For which rings Λ is it true that only the projective Λ-modules have projective covers?

Solution by H. H. Storrer, Technological University, Zürich, Switzerland. We show that the rings in question are the semiprimitive ($\equiv$Jacobson semisimple) rings. Call a submodule K of a Λ-module M small if $K+N=M$ implies $M=N$ for all submodules N of M. We have to characterize those rings for which no projective module P contains a nonzero small submodule.

Since the (Jacobson) radical $N=\text{rad}\,\Lambda$ of a ring Λ is a small submodule of Λ, a ring with the required property must be semiprimitive. This condition is also sufficient. A small submodule of P is obviously contained in rad P, the intersection of all maximal submodules of P. Since $\text{rad}\oplus M_i=\oplus\text{rad}\, M_i$ and $\text{rad}\,\Lambda=0$ by hypothesis, we have rad $F=0$ for any free Λ-module F. If $P\oplus Q=F$, we get $\text{rad}\,P\oplus\text{rad}\,Q=0$ and hence rad $P=0$, which proves that P has no nonzero small submodules.

Also solved by J. S. Golan (Israel), W. W. Leonard, Oswald Wyler and the proposer.

Homomorphism in a Semigroup

5522 [1967, 1014]. *Proposed by T. C. Brown, Kiev State University, U.S.S.R.*

Let S be a periodic semigroup whose idempotents lie in the center. Then the

mapping $x \rightarrow x^{n(x)}$, where $x^{n(x)}$ is idempotent, is a homomorphism.

Solution by E. D. Dixon and Franklin Cheatham, Tennessee Technological University. Since S is periodic there exist integers r and m such that $x^r = x$ and $y^m = y$. Since x^{r-1} and y^{m-1} are idempotent, $x^{n(x)} = x^{r-1}$ and $y^{n(y)} = y^{m-1}$. Using the fact that $x^{n(x)}$, $y^{n(y)}$, $(xy)^{n(xy)}$ are in the center, we get

$$\begin{aligned}(xy) \rightarrow (xy)^{n(xy)} &= (x^r y^m)^{n(xy)} = x^{n(x)} y^{n(y)} (xy)^{n(xy)} = xy^{n(y)} x^{n(x)-1} (xy)^{n(xy)} \\ &= (xy)^{n(xy)+1} y^{n(y)-1} x^{n(x)-1} = (xy) y^{n(y)-1} x^{n(x)-1} = y^{n(y)} x^{n(x)} \\ &= x^{n(x)} y^{n(y)}.\end{aligned}$$

Therefore the mapping $x \rightarrow x^{n(x)}$ is a homomorphism.

Also solved by J. L. Brenner, D. P. Sumner, and the proposer.

Coverage by Sets of Fixed Minimum Measure

5523 [1967, 1014]. *Proposed by J. T. Rosenbaum, University of Pittsburgh*

What pairs, ϵ, θ, with $0 < \epsilon < \theta < 1$, make the following statement true? If $\{E_n\}$ is any sequence of measurable subsets of $[0, 1]$, each with measure at least θ, there will exist a set of measure ϵ which is covered by E_n for infinitely many n. (Note: This is not implied by $|\overline{\lim}\, E_n| \geqq \epsilon$.)

Solution by P. R. Chernoff and W. C. Waterhouse, Harvard University. There are no such pairs ϵ, θ. This is essentially a restatement of problem 5408 [1967, 743] and the solutions and comments given there obtain here too.

Also solved by N. J. Fine, and by the proposer.

Moments of Distances of Uniformly Distributed Points

5524 [1967, 1014]. *Proposed by P. A. Schweitzer, Institute for Defense Analyses, Arlington, Va.*

Let d_n be the nth moment of the distance from a point P outside the unit circle to a randomly chosen point within the circle. Show that

$$d_n = R^n \cdot F\left(-\frac{n}{2}, -\frac{n}{2}, 2; \frac{1}{R^2}\right),$$

where $R > 1$ is the distance from P to the center of the circle, and F is the hypergeometric function.

Let e_n be the nth moment of the distance between two points chosen randomly within the unit circle. Show that

$$e_n = \frac{4}{(n+4)} F\left(-\frac{n}{2}, -\frac{n}{2}, 2; 1\right) = \frac{4(n+1)!}{(n+4)\Gamma(2+\frac{1}{2}n)^2}.$$

I. *Solution by Oswald Wyler, Carnegie-Mellon University.* We shall need the formula

$$\int_0^{2\pi} (1 - 2x\cos\phi + x^2)^p d\phi = 2\pi \sum_{k=0}^{\infty} \binom{p}{k}^2 x^{2k}. \tag{1}$$

In order to obtain (1) we observe that

$$(1 - 2x\cos\phi + x^2)^p = (1 - xe^{i\phi})^p(1 - xe^{-i\phi})^p.$$

The binomial series for the factors at the right converge absolutely and uniformly in x and ϕ for $|x| \leqq c < 1$ and all real ϕ. The same is true for the product series

$$(1 - 2x\cos\phi + x^2)^p = \sum_{k=0}^{\infty} (-1)^k \sum_{h=0}^{k} \binom{p}{h}\binom{p}{k-h} e^{i\phi(2h-k)} x^k.$$

Integrating this series term by term we obtain (1). We note that the sum in (1) breaks off if p is a natural number, converges uniformly and absolutely for $|x| \leqq c < 1$ for any p, and for $|x| \leqq 1$ if $p > \frac{1}{2}$.

Using polar coordinates we have

$$d_{2p} = \frac{1}{\pi}\int_0^1 rdr \int_0^{2\pi} (R^2 - 2rR\cos\phi + r^2)^p d\phi.$$

Putting $r = xR$ and using (1), we get

$$d_{2p} = 2R^{2p+2}\int_0^{1/R} \sum_{k=0}^{\infty} \binom{p}{k}^2 x^{2k+1} dx.$$

Integrating term by term we obtain

$$d_{2p} = R^{2p} \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{p}{k} R^{-2k} = F(-p, -p, 2; R^{-2}). \tag{2}$$

Again using polar coordinates r, ϕ and s, ψ, we have

$$e_{2p} = \frac{1}{\pi^2}\int_0^1 rdr \int_0^1 sds \int_0^{2\pi} d\phi \int_0^{2\pi} (r^2 - 2rs\cos(\phi - \psi) + s^2)^p d\psi.$$

We note that the two inner integrals can be replaced by a single integral

$$2\pi \int_0^{2\pi} (r^2 - 2rs\cos\theta + s^2)^p d\theta,$$

and that the double integral in the (r, s)-plane is the sum of two equal double integrals, one for the domain $0 \leqq r \leqq s \leqq 1$, and one for the domain $0 \leqq s \leqq r \leqq 1$. Thus we have

$$e_{2p} = \frac{4}{\pi}\int_0^1 rdr \int_0^r sds \int_0^{2\pi} (r^2 - 2rs\cos\theta + s^2)^p d\theta.$$

Putting $s = xr$ and using (1) we get

$$e_{2p} = 8\int_0^1 r^{2p+3}dr \int_0^1 \sum_{k=1}^{\infty} \binom{p}{k}^2 x^{2k+1}dx.$$

The integral at the right is $\frac{1}{2}F(-p, -p, 2; 1)$ for all $p>0$. Thus

$$e_{2p} = \frac{2}{p+2}F(-p, -p, 2; 1) = \frac{2(2p+1)!}{(p+2)[\Gamma(2+p)]^2} \tag{3}$$

for all $p>0$, where the last value has been obtained from a well-known formula for $F(a, b, c; 1)$.

We remark that (2) is valid for all real p, and (3) for all real $p \geqq 0$.

II. *Solution by Günter Bach, Braunschweig, Germany.* The results are given essentially in G. N. Watson, *A Quadruple Integral*, Math. Gazette, 43 (1959) pp. 280–283. In Mathematical Gazette, 44 (1960) p. 287, J. M. Hammersley remarks that previously R. Deltheil has shown in his *Probabilités Géométrique* (1926) the following result:

If PQ is the distance between two points P and Q uniformly distributed in the interior of an m-dimensional hypersphere of radius 1, then the mean value of $[PC]^n$ is

$$\frac{m\Gamma(m+1)\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2})2^n}{(m+n)\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(m+\frac{1}{2}n+1)}.$$

Also solved by Michael Skalsky, and by the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. All unsigned material is by the editor. Correspondence about Reviews will be welcome.

Beginning with the January 1969 issue, film reviews will be edited by Seymour Schuster, Carleton College, Northfield, MN 55057. All correspondence concerning films should be sent to him.

Noncommutative Rings. By I. N. Herstein (University of Chicago) Carus Mathematical Monographs, Num. 15. Mathematical Association of America. Distribution by Wiley, New York, 1968. xi+199 pp. (Telegraphic Review, Aug.–Sept. 1968.)

This seems to be an excellent text for a first year graduate course on ring theory. It should follow a sound introductory course in abstract algebra and if such a prerequisite is available at the junior year then this can be used for undergraduate seniors.

It begins with the Jacobson radical, done via modules, obtains the Wedderburn results for rings with descending chain conditions, the Jacobson results on general rings, the

Goldie theorems for rings with ascending chain conditions, and closes with the Golod-Skalarevitch theorem which separates locally nilpotent algebras from nilalgebras and gives a negative answer to the general Burnside problem. In between there are pithy sections on rings with polynomial identities, commutativity theorems (Herstein's personal specialty), representations of finite groups and division algebras.

The book is up to date, easy to read, well motivated and filled with fine insights and appealing open questions.

N. DIVINSKY, University of British Columbia

Initiation to Combinatorial Topology. By Maurice Fréchet and Ky Fan. Translated from the French, with some notes, by Howard W. Eves. Prindle, Weber and Schmidt, Boston, Mass. 1967. xii+124 pp. $2.95 (paper). (Telegraphic Review, May, 1968.)

The authors wrote in a most masterful way, producing a veritable gem of elementary exposition. Anyone having a small acquaintance with high school geometry can read this work with comprehension. There are only three short sections (printed in small type) which assume a somewhat higher mathematical knowledge and can be skipped without harm to the understanding of the main body of the text.

The translator's notes are helpful to those readers who would like to have a little more. These notes supply proofs of some of the unproved statements of the original text, point out some extensions and applications of concepts of the work, and engage in some historical comment.

S. T. HU, University of California, Los Angeles

Unbounded Linear Operators. By Seymour Goldberg. McGraw-Hill, New York, 1966. vii+199 pp. $10.50. (Telegraphic Review, January 1967.)

This review is based on using the book in the second half of a year course in functional analysis for first year graduate students, who had been exposed to the basics in a rather unsophisticated form during the first semester. Instead of following with a second semester of general theory, I preferred to treat a special topic which could lead, in one semester, to significant recent results, and this book was a most rewarding choice.

The book is very clear and elegant with well organized detailed proofs and easy-to-follow references throughout. Although rather advanced, it requires only a basic background in classical and functional analysis sufficient to begin with Chapter II on Linear Operators and their Conjugates. Preliminaries and Chapter I on Introduction to Normed Linear Spaces are suitable only as a catalog of results. It is not easy to cover the entire book in one semester, and some selection of material has to be made. A guide to interdependence of paragraphs would be helpful. But an instructor can do the necessary cuttings, for example going from Chapter II directly to Chapter IV on Operators with Closed Range, and then to Chapter V on Perturbation Theory where the beautiful results of Gokhberg, Krein and Kato are discussed. Some results of Chapter III on Strictly Singular Operators are used, but it is not difficult to go back to what is needed. Chapters VI on Applications to Differential Operators and VII on the Dirichlet Operator could develop into seminars, using the many good references provided.

Summing up, it is a very stimulating book. Besides, it has the special virtue of generating motivation and providing orientation, bibliographical material and background for a good seminar on an advanced, interesting and very active subject.

U. D'AMBROSIO, University of Rhode Island

Mathematics for the Physical Sciences. By Laurent Schwartz (University of Paris). Addison-Wesley, Reading, Mass., 1966. 358 pp. $14.00. (Telegraphic Review, November 1967.)

This book is divided into roughly four parts. The first (pp. 13–70) is a collection of

topics in series and integration theory, including Lebesgue integration. The second (pp. 71–241), and by far the longest, is devoted to distributions and related topics such as convolutions, Fourier series, Fourier and Laplace transforms. The third (pp. 242–310) is a discussion of applications to the solution of the heat and wave equations. The fourth (pp. 311–356) is a development of some properties of gamma, beta, and Bessel functions. As might be expected, the part devoted to distributions is the heart of the book and the best done. Unfortunately the other parts seem peripheral and do not come up to the standards of the sections on distributions. One wonders why the author chose to discuss series over an arbitrary index set when he never makes use of this generalization later in the book. The sections on integration are for the most part too sketchy to be of much use. It is doubtful whether anyone could learn Lebesgue integration from this text. The biggest surprise comes in the chapter on applications, for having developed distribution theory for 170 pages, the author hardly mentions them in the applications. A physicist, who had heard a rumor that distributions are very important, might wonder why all the fuss. The chapters on special functions are conventional and quite routine.

The question of where this text might fit into the curriculum is a difficult one to answer. The material here on real variables and distributions is at the first or second year graduate level in most American universities. However, the level of applications and topics on special functions is mostly undergraduate. One could probably teach a good semester course in distribution theory based on Chapters II–VI, after at least a semester of real variables out of another book. However, there are serious omissions if one considers this as a text for a mathematical physics course. For example, Hilbert spaces, variational methods, Sturm-Liouville problems, and integral equations are either totally missing or just barely mentioned. There are many exercises at the end of each chapter.

J. W. DETTMAN, Oakland University

Matrices and Linear Transformations. By Charles Cullen. Addison-Wesley, Reading, Mass., 1966. 227 pp. $8.95.

Typical of modern mathematical exposition, this book is mathematically masterful, grammatically grim, literarily limp and pedantically pompous. It tells the undergraduate more about matrices than he wants to know, presuming, at the same time, that he knows more than he does.

I found few errors including one erroneous theorem for which, however, proofs will be readily produced, since left to the reader. There is a nice balance between computational and theoretical problems. An author who was more of a writer and less of a mathematician could produce a better book, but until one comes along, this book is as good as any I've seen.

PAUL YEAROUT, Brigham Young University

An Introduction to Fourier Series and Integrals. By Robert T. Seeley. Benjamin, New York, 1966. 104 pp. $5.00.

Series and integral are unified here by the Dirichlet problems on the disc and the half-plane. The first chapter poses the original heat conduction problem of Fourier and deftly exhibits some of the difficulties about the boundary values of the series solution. The last chapter recapitulates the theme in the half-plane, arriving at the Fourier integral. The intervening two chapters are concerned with the possibilities of the separation of variables method in partial differential equations and the application of Abel summability of Fourier series to approximation theory. Riemann integrals are used throughout.

The style, as soon as an idea occurs, is to plunge ahead with calculations and forget about hypotheses. This is excellent for holding the reader's interest and for conveying much the same spirit as original work. Proper conditions are provided later in a theorem. There are asides to the neophyte about the more global setting of a theorem or definition. For a learner these virtues and others, like the rollicking pace of the book, make Seeley's

enterprise far more informative and valuable than a traditional text.

Exercises are frequently useful extensions of the theorems in the text, and some are rudimentary developments of whole fields, like summability or orthogonal polynomials. There are few misprints, though one, on page 28, occurs unfortunately in a definition.

STEPHEN PUCKETTE, University of Kentucky

Integral, Measure and Derivative: a Unified Approach. By G. E. Shilov and B. L. Gurevich. Translated from the Russian by Richard A. Silverman. Prentice-Hall, Englewood Cliffs, N. J., 1966. xiv+233 pp. $10.00.

This book is intended for students of mathematics and physics at the graduate or advanced undergraduate level. The modern theory of the integral is presented by the Daniell scheme which starts from a family of functions for which an elementary integral is defined axiomatically. There is a good introduction where the authors outline their reasons for selecting this approach to the theory, and the first chapter rephrases the Darboux sum and Riemann integral of the typical advanced calculus course and provides motivation and background for the generalization. There are a few problems at the end of each of the chapters. Even though the book is appealing, it is doubtful that undergraduates will be able to make much use of it.

E. R. MULLINS, JR., Grinnell College

The Elements of Integration. By Robert G. Bartle. Wiley, New York and London. 1966. x+129 pp. $6.95.

The aim of this book is to provide an introduction to Lebesgue integration that gets quickly to the principal convergence theorems with a minimum of topological and set theoretic preliminaries, yet without slighting the idea of measure. The theory of the integral over an abstract measure space is developed in the usual manner, but consideration of the techniques for generating and extending measures is deferred to a late chapter. Meanwhile the reader is asked to accept temporarily the existence of linear Borel measure. Thus he learns, for instance, how to apply the dominated convergence theorem to the differentiation of a Laplace integral before he has seen the definition of a Lebesgue measurable set. This arrangement is designed to emphasize the analytical advantages of the integral, and to subordinate considerations of a more technical character. (The user may, of course, take up outer measures sooner if he wishes.) Despite the small size of the book its coverage is fairly complete on the analytical side: L_p spaces, product measure, Radon-Nikodym and Riesz representation theorems. There is no mention of topological or metric measures, or of groups. The exposition is concise but clear, suitable for advanced undergraduate or beginning graduate students. An abundance of exercises is provided.

J. C. OXTOBY, Bryn Mawr College

Calculus of Residues. By D. S. Mitrinovic. P. Noordhoff, The Netherlands, 1966. 87 pp. $1.90 (paper).

This problem book should prove useful to students and instructors as a source for exercises in a first year course on complex analysis. Each chapter treats some application of residue calculus and is organized as follows: The particular application is rigorously stated and justified, several examples are carefully worked out, and then a number of exercises are given. Roughly 50% of the exercises are accompanied by answers. Most of these applications are discussed in standard text books on complex analysis. There is a chapter on applications to the summation of series—a topic sometimes omitted from standard courses. On the other hand, this book omits entirely the subject of Cauchy principal value.

BURTON RODIN, University of California, San Diego

Number Systems of Analysis. By G. Cuthbert Webber. Addison-Wesley, Reading, Mass., 1966. x+213 pp. $8.50.

The author's purpose in writing this book was to begin with an axiomatic description of the natural numbers, and then to construct successively the integers, rationals, reals, and complex numbers. The book is aimed at students for whom such concise classical treatments as Cohen and Ehrlich's, *The Structure of the Real Number System*, are too sophisticated. The attempt to keep things simple gives rise to the unusual features of this development. Whether they be strengths or weaknesses is largely a matter of taste. Here follow a few of these unusual features.

1. The initial description of the natural numbers is by means of a modified version of the Peano postulates, attributed to Henkin, which describes the natural numbers in terms of a binary operation, $+$, rather than a successor function. This obviates having to prove the difficult existence-uniqueness theorem for $+$ at the very beginning of the course. Later on the classical postulates are stated, the difficult theorem proved, and the modified postulates are proved as theorems. (Incidentally, no general recursion theorem is proved or even stated anywhere in this book.)

2. The construction of successively larger number systems is governed by the author's conviction that it is easier to understand if ". . . each system discussed is an actual subsystem of subsequent systems . . . not just 'up to isomorphism'." This leads to such definitions as the following. The system of integers consists of the set $I=N\cup{}^{-}N\cup\{0\}$ (where N is the set of natural numbers, ${}^{-}N$ is a set in 1-1 correspondence with N such that $N\cap{}^{-}N=\varnothing$, and 0 is any element $\notin N\cup{}^{-}N$) together with the operations $+$ and $\cdot$ defined by $x+y=$(here follow seven alternatives), $x\cdot y=$(here follow five alternatives). And it leads to statements like this one preceding the proof of the distributive law: "There will be thirteen cases. . . . The proof will be given only for two cases."

3. Meticulous "statement-reason" proofs are given for many theorems; weaker students might appreciate this. A possible bad effect is the obscuring in details of the one or two main ideas of the proof. In one instance a sixteen step proof could have been reduced to a line had the theorem been deferred for a few pages.

4. Constructions of the reals from the rationals is by Cauchy sequences.

E. F. KRAUSE, University of Michigan

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)—18 (second graduate year). A boldface star (★) marks a notable book of general interest.

Algebra

The Algebraic Theory of Semigroups. Part 2. By A. H. Clifford and G. B. Preston. Math. Surveys vol. 7. AMS, Providence, R. I., 1967. xv+350 pp. $13.70. Deals with additional branches of the theory rather than deeper development of the topics in volume I. P, L.

Finite Groups. By Daniel Gorenstein (Northeastern Univ.). Harper and Row, New York, 1968. xv+527 pp. $14.95. Presupposing a first year graduate algebra course, this book brings the student toward the bulk of current literature by including the recent work of Feit-Thompson, Suzuki, and the author. T (18), S, P, L.

Noncommutative Rings. By I. N. Herstein (Univ. of Chicago), Carus Math. Monographs, No. 15. Mathematical Association of America. Distributed by Wiley, New York, 1968. xi+199 pp. $6.00. (One copy at $3.00 to members from the MAA.) Not a

treatise but "a certain cross section of ideas techniques and results that will give the reader some inkling of what is going on and what has gone on." Presupposes elementary abstract algebra and linear algebra. A reckless use of letters as proper nouns with no listing in the index or special table will trouble readers. T, S, P, L.

Elementary Linear Algebra. By L. H. Lange (San Jose S.C.). Wiley, New York, 1968. xii+380 pp. $9.50. This is an elementary introduction to vector spaces and matrices, intended for a semester course at the sophomore level and ending with a chapter on linear programming. The author has tried to write so that his book teaches by organizing "opportunities for a student to make his own discoveries." T (13–14).

Matrices and Linear Algebra. By Hans Schneider and George Phillip Barker (Univ. of Wisconsin). Holt, Rinehart and Winston, New York, 1968. ix+385 pp. $7.95. The title correctly suggests a combination of abstract and concrete approaches. The last three chapters are on eigenvalues, inner product spaces and applications to differential equations. Determinants are introduced deftly. T (13–14).

Vorlesungen über Invariantentheorie. By Issai Schur. Edited by Helmut Grunsky. Grund. Math. Wiss., Vol. 143. Springer-Verlag, New York, 1968. viii+134 pp. $8.00. Groups of linear substitutions, projective invariants of binary forms, finiteness question. P.

Modular Lie Algebras. By G. B. Seligman (Yale University). Math. und Grenz. Vol. 40. Springer-Verlag, New York, 1967. ix+165 pp. $9.75. The aim is to summarize the present knowledge of Lie Algebras over fields of prime characteristic. There is a bibliography of 427 titles, intended to be complete. P, L.

Elementary Linear Algebra. By Paul C. Shields (Wayne State Univ.). Worth, New York, 1968. x+349 pp. $8.95. Designed to implement the CUPM recommendation to teach linear algebra as part of the elementary calculus sequence, this book emphasizes interpretation and computation. The last chapter is on linear differential operators. T (13).

Analysis

Vorlesungen über Approximationstheorie. By N. I. Achieser. 2nd. rev. ed. Translated from the Russian. Akademie-Verlag, Berlin, 1967. xii+412 pp. $10.00. P, L.

AMS Translations. Series 2, Vol. 65. Nine papers on Partial Differential Equations and Functional Analysis. AMS, Providence, 1968. iv+296 pp. $15.00. P.

Introduction to Partial Differential Equations and Boundary Value Problems. By Rene Dennemeyer (Calif. S.C. at San Bernardino). McGraw-Hill, New York, 1968. vii+376 pp. $13.75. Besides the usual material on first order, linear second order, elliptic, wave, and heat equations, there are appendices on a special case of the Cauchy-Kowalewski Theorem, Sturm-Liouville problem, and expansions. Presupposes traditional advanced calculus or applied mathematics. T (15–16).

Fourier Series. A Modern Introduction. Vol. II. By R. E. Edwards (Australian National Univ.). Holt, Rinehart and Winston, New York, 1967. ix+318 pp. $9.95. For Vol. I see telegraphic review March 1968. This volume concentrates on more modern aspects and those topics of classical theory that fit most naturally into functional analysis. T (17–18), S, P, L.

Asymptotic Methods in the Theory of Linear Differential Equations. By S. F. Feshchenko, N. I. Shkil' and L. D. Nikolenko (all of Academy of Sciences of the Ukr. SSR.). Translation editor, Herbert Eagle (Univ. of Wisconsin). American Elsevier, New York, 1967. xvi+270 pp. $14.00. Deals with second order linear equations with

slowly varying coefficients, decompositions of a linear system, a case of multiple roots, equations in Banach space and general methods. T (18), P.

Progress in Mathematics, Vol. 2. Mathematical Analysis. Edited by R. V. Gamkrelidze. Translation of Itogi Nauki, Seriya Matematika. Plenum, New York, 1968. viii+161 pp. $15.00. Two articles: Theory and Methods of Investigation of Branch Points of Solutions by M. M. Vainberg and P. G. Aizengendler; Imbedding and Continuation for Classes of Differentiable Functions of Several Variables Defined in the Whole Space by V. I. Burenkov. P.

Analyse mathêmatique. By G. Garsoux. Dunod, Paris, 1968. xviii+582 pp. 68 F. From sets through general topology to elementary functions for students planning to study under Prof. Bourbaki at Paris. P.

Foundations of Potential Theory. O. D. Kellogg (Harvard University). Reprint of the first edition of 1929. Grund. Math. Wiss. Vol. 31. Springer-Verlag, New York, 1967. ix+384 pp. $8.00. P, L.

Kurventheorie. By Karl Menger and Georg Nobeling. Chelsea, Bronx, New York, 1967. iv+374 pp. $12.00. A reprint of the classic, first published in 1932, with some minor revisions and an added diagram of the Universal Curve. P, L.

Uniformisierung. By R. Nevanlinna. 2nd. ed. Grund. Math. Wiss. Vol. 64. Springer-Verlag, New York, 1967. x+391 pp. $12.40. There are only minor corrections to the first edition of 1953, and the author refers those interested in more recent developments to the treatises by Pflüger (1957) and Ahlfors-Sario (1960).

Locally Convex Spaces and Linear Partial Differential Equations. By Francois Treves. Grund. Math. Wiss. Vol. 146. Springer-Verlag, New York, 1967. xii+120 pp. $9.00. Of interest to functional analysts as well as those in the field of partial differential equations. The author is to be commended for his glossary of the main definitions and results. P, L.

Applications

A Mathematical Tool-kit for Engineers. 3rd ed. By H. A. Webb and D. G. Ashwell. Longmans, London, 1967. viii+138 pp. $4.20 (paper). Thin content.

Language and Symbolic Systems. By Yuen Ren Chao (Univ. of Calif. Berkeley). Cambridge, New York, 1968. xv+240 pp. $5.00 (cloth) $1.95 (paper). A nontechnical broad introduction to linguistics, including phonetics, phonemics, semantics, a survey of languages of the world, foreign language study, symbolic systems, communications technology, and application to automatic speech, machine translation and related topics. P, L.

Elementary Classical Hydrodynamics. By B. H. Chirgwin and C. Plumpton (Univ. of London). Pergamon, New York, 1967. viii+224 pp. $5.50 (cloth) $4.00 (paper). An elementary introduction. S, P.

Dynamic Plasticity. By N. Cristescu (Univ. of Bucharest). Vol. 4 of the North-Holland series in Applied Math. and Mechanics. North-Holland, Amsterdam, 1967 and (in the U.S.A. and Canada) Wiley, New York, 1967. xi+614 pp. $25.00. P.

Modern Fluid Dynamics. Vol. 1: Incompressible Flow. By N. Curle and H. J. Davies (both of Univ. of Southampton). Van Nostrand, Princeton, N. J., 1968. xiv+308 pp. $5.95 (paper) $10.50 (cloth). T (16), S, P.

Mathematics for Physicists. By Phillippe Dennery (Univ. of Paris-Orsay) and Andre Krzywicki (École Polytechnique, Paris), Harper and Row, New York, 1967. xiii

+384 pp. \$12.95. Chapters on analytic functions, vector spaces, function spaces and expansions, ordinary and partial differential equations. In a preface beginning with a beautiful quotation from Picasso, the authors state their intention of avoiding both a presentation of pure mathematics and an overemphasis on problem solving. T (16–17).

Theory of Finite Groups. Applications in Physics. Symmetry Groups of Quantum Mechanical Systems. By Laurens Jansen and Michael Boon. North-Holland, Amsterdam, and Wiley, New York, 1967. xi+367 pp. \$19.00. First the theory of groups and representations, then the applications mentioned in the title.

Introduction to Celestial Mechanics. By Jean Kovalevsky. Translated by Express Translation Service. Springer-Verlag, New York, 1967. viii+126 pp. \$6.40. Motivated by and limited to the theory useful in calculating trajectories in space. P, L.

Some Improperly Posed Problems of Mathematical Physics. By M. M. Lavrentiev. Translation editor Robert J. Sacker. Springer Tracts in Natural Philosophy, Vol. 11. Springer-Verlag, New York, 1967. 72 pp. \$5.00. "Properly posed" is related to existence, uniqueness, and continuity of solutions in the sense introduced by Hadamard and since modified in various ways. P.

Quantum Mechanics. By R. A. Newing and J. Cunningham (both of Univ. Coll. of North Wales, Bangor). Oliver and Boyd, London, and Interscience, New York, 1967. ix +225 pp. \$4.50. Intended as an up-to-date mathematical treatment showing the interplay between physics and mathematics. T (16), S, P.

Einführung in die allgemeine Informationstheorie. By J. Peters. Kommunikation und Kybernetik in Einzeldarstellungen. Vol. 6. Springer-Verlag, New York, 1967. xii +266 pp. \$16.00. P, L.

Introduction to Special Relativity. By Herman M. Schwartz (Univ. of Arkansas). McGraw-Hill, New York, 1968. 458 pp. \$14.75. P, L.

Conditional Markov Processes and their Application to the Theory of Optimal Control. By R. L. Stratonovich (Moscow State Univ.). Translated by R. N. and N. B. McDonough. Preface by Richard Bellman. American Elsevier, New York, 1968. xvii+350 pp. \$14.75. The mathematics related to construction of optimal cybernetic systems with emphasis on Bayesian problems, described by the editor as "a major step forward in the current endeavor to create unified mathematical theories with wide-ranging applications . . ." T (17–18), S, P.

★*New Methods of Thought and Procedure.* Edited by F. Zwicky and A. G. Wilson. Contributions to the Symposium on Methodologies sponsored by the Office for Industrial Associates of the California Inst. of Technology and the Society for Morphological Research. Pasadena, California, May 22–24, 1967. Springer-Verlag, New York, 1967. vii+338 pp. \$9.50. Fifteen papers in 6 sections on operations research, systems engineering, dynamic programming, game theory, and morphological research. Whether the work represented is worthy of the tradition of Aristotle, Bacon and Descartes, as suggested by Zwicky in a prologue, and whether methodology is becoming a scientific and technological discipline as suggested by Wilson in his epilogue, must be decided by the reader, but the issues are important. P, L.

General

Fischer Lexikon. Mathematik. Two volumes (29/1, 29/2). Edited by H. Behnke and H. Tietz, *et al.* Fischer Bücherei, Frankfurt, 1964, 1966. Vol. 1. 383 pp. Vol. 2. 397 pp. \$1.60 each (paper). Distributed by Adlers Foreign Books, 162 Fifth Avenue, New

York, N.Y. 10010. The first volume emphasizes elementary mathematics and foundations, the second special topics and applications. The material is grouped in a few fairly long articles such as algebra, algebraic numbers, theory of functions, topology, approximation theory, differential geometry, measure theory, statistics, but very detailed indexes make it possible to use the book for specific reference as well as general orientation. Without a doubt the best bargain on the market in mathematics reference books! It is scandalous that nothing similar exists in English. S, P, L.

Exploring University Mathematics. Edited by N. J. Hardiman. Vol. 2. Pergamon, Toronto, 1968. x+115 pp. $5.50 (cloth), $3.50 (paper). Seven lectures given at the 1966 Easter Conference in Mathematics at Bedford College, London: Fourier Series and the Isoperimetric Problem (J. H. Cohn), The Mathematics of Night Shining Clouds (P. C. Kendall), Numbers made to Measure (B. Fishel), Special Relativity: A Question of Time Reckoning (C. W. Kilmister), Wallpaper Patterns (H. Kestelman), The Mathematics of Gambling (D. M. Burley), Differential Equations (J. S. Griffith). S, P, L.

★*Reprint Series.* Edited by William L. Schaaf. School Mathematics Study Group. Stanford University, 1966–1967. 40¢ per pamphlet. Distributed by Vroman's, 367 So. Pasadena Avenue, Pasadena, Calif., 91105. RS-1: Structure of Algebra. ix+42 pp. RS-2: Prime Numbers and Perfect Numbers. vii+40 pp. RS-3: What is Contemporary Mathematics? vii+36 pp. RS-4: Mascheroni Constructions. vii+31 pp. RS-5: Space, Intuition and Geometry. vii+59 pp. RS-6: Nature and History of Pi. vii+49 pp. RS-7: Computation of Pi. vii+31 pp. RS-8: Mathematics and Music. vii+25 pp. RS-9: The Golden Measure. 46 pp. RS-10: Geometric Constructions. vii+41 pp. Each pamphlet collects historical and expository papers from the *Mathematics Teacher*, *School Science and Mathematics*, the *American Mathematical Monthly*, the *Scientific American* and other journals. Much of this material would be instructive for college students and even for college teachers. More issues are planned. S, P, L.

The Sorting Process. A Study in Mathematical Structure. By T. G. Room and J. M. Mack (Univ. of Sydney). Sydney Univ. Press, and Penn. State Univ. Press, 1966. ix+235 pp. $3.00 (paper). A new pedagogical approach to set theory and many other elementary topics through the notion of sorting, i.e. finding the objects that possess a certain property. TT.

★*Stories about Sets.* By N. Ya. Vilenkin. Academic Press, New York, 1968. xiii+152 pp. $6.50 (cloth), $2.95 (paper). This is a quality popularization addressed to high school and college students but capable of giving pleasure to mature mathematicians also. There are amusing illustrations including one showing two angels perched happily on the head of a pin. S, P, L.

Geometry and Topology

Vector and Tensor Analysis with Applications. By A. I. Borisenko and I. E. Tarapov. Translated and edited by R. A. Silverman. Prentice-Hall, Englewood Cliffs, N.J. 1968. x+257 pp. $10.95. One chapter on vector algebra and four on tensor algebra and analysis with applications. T (15–16).

Introduction to General Topology. By Helen F. Cullen (Univ. of Massachusetts). D. C. Heath, Boston, Mass. 1968. x+427 pp. $12.50. Based on 15 years of teaching topology, this book is directed to the beginning graduate and advanced undergraduate. The last of seven chapters deal with homotopy. T (16–17).

Singularities of Smooth Maps. By James Eells, Jr. Gordon and Breach, New York, 1967. xi+100 pp. $5.50 (cloth) $3.00 (paper). Lecture notes for students already exposed to homotopy and homology and intended "as an introduction to a rapidly developing and far reaching mathematical theory." S, P.

Stereology. Proceedings of the Second International Congress for Stereology, Chicago, 1967. Edited by Hans Elias (Chicago Medical School). Springer-Verlag, New York, 1967. xx+338 pp. $10.00. This is described as "an almost complete synopsis of the entire field" of problems concerning "the recognition of three-dimensional structure, problems which confront those who study materials, rocks, biological systems or heavenly bodies." Since there are no mathematicians among the contributors, this would seem to be a field worth looking at. P.

Studies in Modern Topology. Edited by P. J. Hilton (Cornell Univ.). MAA Studies in Mathematics, vol. 5. Mathematical Association of America. Distributed by Prentice-Hall, 1968. 212 pp. $6.00. (One copy at $3.00 to members from the MAA). Another superb volume in this series of expositions for mathematicians who wish to remain oriented in their rapidly changing world. After a fine introductory survey of modern topology by the editor, there follow: What is a Curve? by G. T. Whyburn; Some Results on Surfaces in 3-manifolds by W. Haken; Semisimplicial Homotopy Theory by V. K. A. M. Gugenheim; Functors of Algebraic Topology by E. Dyer; and the Geometry of Differentiable Manifolds by V. Poénaru. S, P, L.

Topics in Geometry. By Howard Levi (Hunter College). Complementary Series, Vol. 11. Prindle, Weber and Schmidt, Boston, Mass., 1968. vii+104 pp. $2.95 (paper). A collection of essays based on lectures to audiences at various levels. Topics are congruence, line reflections, elliptic geometry, orientability and angle sums, inversions, geometric algebra, hyperbolic geometry, affine geometry, the Euclidean plane, pairs of coordinate systems. S, P, L.

Geometry and Symmetry. By Paul B. Yale (Pomona College). Holden-Day, San Francisco, Calif. 1968. xi+288 pp. $9.75. An introduction to Euclidean, affine, and projective spaces with emphasis on symmetry. Unusual topics include proof of the Polya-Burnside theorem, a chapter on crystallography, and suggestions for term papers. This is a candidate for the needed book to bring undergraduate geometry up to date and prepare for modern courses in differential geometry, algebraic geometry, and topology. T (15–16).

Probability and Statistics

Introduction to Statistics. By J. M. Bevan (Univ. of Oxford). Philosophical Library, New York, 1968. vii+220 pp. $6.00. A high quality elementary exposition covering both descriptive and inferential statistics up through testing hypothesis, estimation, and confidence intervals. S.

A Course in Probability Theory. By Kai Lai Chung (Stanford Univ.). Harcourt, Brace and World, New York, 1968. vii+331 pp. $12.00. Topics include measure theory, law of large numbers, characteristic functions, central limit theorem, random walk, conditioning, Markov property, martingales. There are some good statements in the preface about probability and textbook writing. T (16–17), P, L.

Statistical Dictionary of Terms and Symbols (Facsimile of the 1939 edition). By Albert K. Kurtz, and Harold A. Edgerton. Hafner, New York, 1967. xii+191 pp. $7.50. Obsolete at its first publication nearly 30 years ago, this dictionary is now merely of antiquarian interest as an exhibit of statistics before the development of the subject in the last four decades.

A Selection of Early Statistical Papers of J. Neyman. Univ. of Calif. Press, Berkeley, 1967. viii+429 pp. $14.75. In addition to some 28 papers up to 1946 there is a portrait, a short but nicely written foreword by "Students of J. N. at Berkeley," a one page autobiographical note, and a bibliography of scientific papers by Neyman containing 144 papers and 12 books and monographs from 1916 to 1967. The volume is one of the trilogy consisting of a collection of papers by E. S. Pearson (telegraphic review, July 1967), and a volume of joint papers by Neyman and Pearson (telegraphic review March, 1968). We concur in the hope expressed by the students that there will be a further volume on Neyman's important contributions since the forties. P, L.

Statistics, 3rd ed. By L. H. C. Tippett. Oxford, New York, 1968. 153 pp. $1.20 (paper). A non-mathematical popularization of non-mathematical statistics. Such words as "estimation" and "test" do not appear in the index.

Patterns and Configurations in Finite Spaces. By S. Vajda (Univ. of Birmingham). Hafner, New York, 1967. vii+120 pp. $4.95.

The Mathematics of Experimental Design. Incomplete Block Designs and Latin Squares. By S. Vajda (Univ. of Birmingham). Hafner, New York, 1967. viii+110 pp. $4.75. These two companion volumes are concerned with the mathematics that relates to experimental design rather than with the design itself. The first is more general and covers the abstract algebra, finite planes, finite spaces of higher dimensions, and configurations. The second concentrates on balanced incomplete block designs, latin squares and orthogonal arrays, partially balanced incomplete block designs, and designs with two associate classes. There are bibliographies. T, S, P.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor E. D. Eaves, University of Tennessee, represented the Association at the inauguration of President R. L. Owens III of Knoxville College on March 16, 1968.

Professor D. K. Hughes, Abilene Christian College, represented the Association at the inauguration of President L. D. Vincent of Angelo State College on March 25, 1968.

Professor R. H. Sorgenfrey, UCLA, represented the Association at the inauguration of President R. E. Kennedy of California State Polytechnic College, San Luis Obispo, on April 2 and 3, 1968.

Professor R. J. Troyer, University of North Carolina, Chapel Hill, represented the Association at the inauguration of President N. A. Wiggins of Campbell College on April 6, 1968.

Professor E. F. Wilde, Beloit College, represented the Association at the inauguration of President W. L. Carter of Wisconsin State University, Whitewater, on March 22, 1968.

University of Santa Clara: Assistant Professor G. L. Alexanderson has been promoted to Associate Professor; Mr. L. F. Klosinski has been promoted to Assistant Professor.

Assistant Professor Francine Abeles, Newark State College, has been promoted to Associate Professor.

Dr. H. Margaret Elliott, Washington University, has been appointed Professor at the University of Bridgeport.

Professor Harry M. Gehman, State University of New York at Buffalo, retired on June 30, 1968, with the title of Professor Emeritus.

Dr. Michael Gemignani, SUNY at Buffalo, has been appointed Associate Professor at Smith College.

Mr. D. E. Moxness, General Beadle State College, has been promoted to Assistant Professor.

Associate Professor M. F. Neuts, Purdue University, has been promoted to Professor of Mathematics and Statistics.

Reverend N. E. Nirschl, St. Norbert College, has been promoted from Assistant Professor to Associate Professor.

Assistant Professor Irving Roth, Stonehill College, has been promoted to Associate Professor.

Dr. K. J. C. Smith, University of North Carolina, Chapel Hill, has been appointed Research Assistant Professor.

Professor Emil Amelotti, Villanova University, died on March 3, 1968. He was a member of the Association for nine years.

Professor Bernardo Baidaff, Universidad Nacional Del Sur, Argentina, died on November 8, 1967. He was a member of the Association for forty-seven years.

Capt. R. W. Brower of Stow, Ohio, died on November 9, 1967. He was a member of the Association for six years.

Lt. Col. R. C. Rounding, U.S.A.F. Academy, Colorado, died in April, 1968, while training for Vietnam. He was a member of the Association for nine years.

Professor R. G. Sanger, Kansas State University, died on March 13, 1968. He was a member of the Association for thirty-six years.

Professor Emeritus W. F. Shenton, American University, died on February 26, 1968. He was a Charter Member of the Association.

NSF SEEKS PROPOSALS FOR REGIONAL CONFERENCES IN THE MATHEMATICAL SCIENCES

The National Science Foundation is seeking proposals for five-day regional conferences on subjects of current research interest in the mathematical sciences. The objective of the conferences is to stimulate and broaden mathematical research activity, particularly in regions of the country where such activity needs further development. The organization of the conferences, evaluation of proposals, and arrangements for publication of conference-related expository papers will be carried out by the Conference Board of the Mathematical Sciences (CBMS), Washington, D. C., under contract with the National Science Foundation.

At present ten conferences are projected, each to take place at a host academic institution during a summer week in 1969 or 1970, or possibly within a recess of the intervening academic year. Topics for conferences may be concerned with one or more of the various disciplines of the mathematical sciences, including, in addition to pure mathematics, fields such as applied mathematics, statistics, computer science, operations research, and management science.

Each conference is to have a lecturer (who need not come from the host institution) and about 25 other participants, the latter to be drawn from the broad geographic region around the host institution. It is expected that the lecturer would give two lectures a day during the five days of the conference, with the remainder of the time available for study, informal discussion, and exchange of ideas.

All participants in a conference will receive allowances for travel and subsistence. The lecturer will receive, in addition, a fee for delivering his lectures and for organizing these into a substantial expository paper. The CBMS will subsequently arrange for the editing and publication of these expository papers.

Preliminary inquiries regarding details of these regional conferences may be addressed to the Conference Board of the Mathematical Sciences, 834 Joseph Henry Building, 2100 Pennsylvania Avenue, NW, Washington, D. C. 20037. Proposals by prospective host institutions should be sent to the Mathematical Sciences Section, National Science Foundation, 1800 G Street, NW, Washington, D. C. 20550. Proposals will be evaluated by a panel of the CBMS and awards of conference grants will be made by the National Science Foundation with the advice of the panel.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

The 'HARRY M. GEHMAN INVITED LECTURE' SERIES—UPPER NEW YORK STATE SECTION

In tribute to Professor Harry Gehman as an individual and in appreciation of his many years of service to the Mathematical Association of America, the Upper New York State Section initiated a lecture series in his honor at its May meeting. Entitled the 'Harry M. Gehman Invited Lecture' it will appear annually on the program of the spring meeting of the Section. First speaker in the series was Professor Malcolm F. Smiley of SUNY at Albany lecturing on 'The algebra of rectangular matrices'.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for the three year term July 1, 1968 to June 30, 1971 by a mail vote of the Association in the Sections indicated:

Florida	D. B. Goodner, Florida State University
Illinois	F. E. Hohn, University of Illinois
Iowa	D. E. Sanderson, Iowa State University
Louisiana-Mississippi	Virginia L. Carlton, Centenary College
Maryland-District of Columbia-Virginia	E. E. Floyd, University of Virginia
Michigan	L. M. Kelly, Michigan State University
Minnesota	F. L. Wolf, Carleton College
Philadelphia	B. H. Bissinger, Lebanon Valley College
Southern California	Robert Herrera, University of California, Los Angeles
Texas	D. E. Edmondson, University of Texas, Austin

The highest percentage of voters was 44% in the Iowa Section, followed by the Florida Section with 40%. One candidate was elected by a majority of one vote.

RAOUL HAILPERN, *Associate Secretary*

THE 1968 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-ninth annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 7, 1968. This competition, which is supported by the William Lowell Putnam Intercollegiate Memorial Fund, is under the sponsorship of the Mathematical Association of America. Colleges and universities in the United States and Canada are eligible to register undergraduates in the competition.

Application blanks will be mailed about October 1 to the mathematics department chairmen of the schools on the regular mailing list and also to those who supervised the competition in 1967. If an application blank is not received by October 15, one may be secured by writing the director, James H. McKay, Department of Mathematics, Oakland University, Rochester, Michigan 48063. Your application should be mailed to the director not later than November 1, 1968. Further details are provided in the Announcement Brochure which is mailed with the registration forms.

Reports of the previous competitions, including past examination questions, may be found in the MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August-September 1947; December 1948; August-September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; January 1957; August-September (announcement of winners) and November (questions and solutions) 1957; August-September 1958, 1959; January (questions and solutions for the eighteenth, nineteenth, and twentieth competitions) 1961; August-September 1961; October 1962; August-September 1963; June-July 1964; August-September 1965, 1966, 1967, and in this issue page 732.

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Fall meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at Morgan State College, Baltimore, on December 9, 1967. Professor Avron Douglis, Chairman of the Section, presided over the sixty-three in attendance.

After a short business meeting in which the members agreed to a $1 registration fee at Section meetings, the MAA film, *The Theorem of the Mean*, was shown. This preceded the main lecture, 'The Reverse Cauchy-Schwarz Inequality,' an hour talk by Monroe Martin, Director and Professor, the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

After lunch in the refectory, five members presented papers:

1. *Representations of quadratic forms*, by Sister John Frances Gilman, Saint Joseph College, Emmitsburg, Maryland.

2. *The converse Malfatti problem*, by Michael Goldberg, Washington, D. C.

3. *An experiment in game theory*, by B. L. Schwartz, The Mitre Corporation, Bailey's Crossroads, Virginia.

4. *The energy of a charged capacitor*, by W. J. Young, Emory and Henry College, Emory, Virginia.

5. *Isotonic Boolean functions and finite implication algebras*, by J. C. Abbott, U. S. Naval Academy, Annapolis, Maryland.

S. L. GULICK, *Secretary*

MARCH MEETING OF THE FLORIDA SECTION

The first annual meeting of the newly established Florida Section of the MAA was held on the South Campus of Miami-Dade Junior College, Miami, Florida on March 22–23, 1968. There were one hundred thirty-six persons in attendance, including ninety-nine members of the Association.

Professor Robert Meacham, Chairman of the Section, presided at the business meeting at the luncheon on March 23. The following officers were elected: Chairman, A. R. Bednarek, University of Florida; Vice-Chairman, Roy Mazzagatti, Miami-Dade Junior College, South Campus; Secretary, H. E. Taylor, Florida State University.

An invited address was given on Friday by Professor Laszlo Fuchs, University of Miami. His subject was "On the Factorization of Abelian Groups." On Saturday, Professor E. E. Moise, President of MAA gave an address entitled "How to Tell that a Simple Overhand Knot is Really Knotted."

A session on "The Nature and Teaching of Linear Algebra" was held on Saturday at which Professor J. T. Moore, University of Western Ontario, and Professor T. L. Wade, Florida State University presented 20 minute papers.

A panel composed of Professors Ted Brittan, Chipola Junior College, Eugene Medlin, Stetson University, E. P. Miles, Florida State University, and R. H. Raines, Manatee Junior College, discussed the "Role of Computers in Mathematics Programs in Colleges and Universities."

"The Content of the 4th and 5th Year Mathematics Curriculum in High School" was discussed by a panel consisting of Professors Don Lichtenburg, University of South Florida, Herman Meyer, University of Miami, Louis Nanney, Miami-Dade Junior College, North Campus, and J. E. Snover, Florida State University. The panel was chaired by Mrs. Agnes Y. Rickey, Dade County Public Schools.

The following papers were presented:

1. *On k Spaces*, by Donald Weddington, University of Miami.
2. *Some remarks on the problem of Klee*, by Vadim Komkov, Florida State University.
3. *The torsion of orthotropic beams with various sections*, by C. B. Smith, University of Florida.
4. *Series with sums invariant under rearrangement*, by Charles McArthur, Florida State University.
5. *On a particular functional equation over left groups*, by Y. Matras, University of Florida.
6. *On a generalization of semigroups by Hedrlin*, by H. J. Weinert, University of Florida.
7. *Topological germs in projective planes with weak topology*, by D. P. K. Biallas, University of Florida.
8. *On the basis of prime ideals in polynomial domains*, by J. L. Mott, Florida State University.
9. *A note on semilattices*, by M. M. McWaters, University of South Florida.
10. *Spheres uniformly wedged between balls are tame in E^3*, by H. C. Griffith, Florida State University.
11. *Two operations on families of pseudometrics*, by Ludvik Janos, University of Florida.
12. *On the Laplace transforms of functions of the integral, and of the fractional, parts of x*, by Ira Rosenbaum, University of Miami.
13. *On the general solution of the functional equation $f(x+yf(x))=f(x)f(y)$*, by Petar Javor, University of Florida.
14. *Some results in the theory of algebraic correspondences*, by M. L. Laplaza, University of Puerto Rico.
15. *Topologically equivalent metrics*, by M. Marjanovic, University of Florida.
16. *The nature of the general education mathematics course in junior colleges—the result of a survey*, by E. D. Nichols, Florida State University.
17. *Mathematics for parents of elementary school children*, by R. E. Peinado, University of Puerto Pico.
18. *Some theorems on topological machines*, by E. M. Norris, University of Florida.
19. *The equivalence-problem of quadratic forms over finite fields of characteristic 2*, by Karlhorst Meyer, University of Florida.
20. *The nature of linear algebra*, by J. T. Moore, University of Western Ontario.
21. *On teaching linear and matrix algebra*, by T. L. Wade, Florida State University.

H. E. TAYLOR, *Secretary*

MARCH MEETING OF THE OKLAHOMA-ARKANSAS SECTION

The annual spring meeting of the Oklahoma-Arkansas Section of the MAA was held on March 29–30 at the Federal Aviation Administration Aeronautical Center in Oklahoma City. There were 121 persons registered of whom 97 were members of the Association.

Dr. Herbert Monks of Northeastern State College presided at the business session on Saturday morning. Reports were given on the activities of the Association as follows: Dr. Monks on the meeting of Sectional Officers in August at Toronto; Dr. R. B. Deal on the activities of the Board of Governors; Dr. Lyle Mason on the Annual High School Mathematics Contest sponsored by the MAA, The Society of Actuaries, and Mu Alpha Theta. Dr. Mason reported that 3972 test books were distributed. Drs. Deal and Monks both mentioned the concern of the MAA as related to involvement of teachers at the junior college level.

Officers elected for 1968–69 were: Chairman, John Rieger of Federal Aviation Administration; Vice-Chairman, John Kent of Arkansas State University; Secretary-Treasurer, Harold Huneke of the University of Oklahoma.

On Friday evening a buffet dinner at Oklahoma City University was followed by an invited talk, "Man Power Questions in the Mathematical Sciences," by Dr. John Jewett of Oklahoma State University.

The Saturday session was highlighted by an invited talk by Dr. Richard Anderson of Louisiana State University on the CUPM report, "Qualifications for a College Faculty in Mathematics." A motion was passed asking that CUPM sponsor a conference in this section involving administrators and mathematicians and dealing with the report on Qualifications for a College Faculty in Mathematics.

The following papers were presented:

1. *An infinite field as a reduced product of finite fields*, by Kenneth Loewen, University of Oklahoma, Norman.

2. *Redei's Theory and varieties*, by Robert Schwabauer, University of Oklahoma, Norman.

3. *A proof of a theorem equivalent to the Schroeder-Bernstein's theorem*, by Billy Fullbright, Oklahoma State University, Stillwater.

4. *A characterization of countable compactness*, by J. D. Hansard, University of Arkansas, Fayetteville.

5. *Fatou values of normal subharmonic functions*, by J. L. Meek, University of Arkansas, Fayetteville.

6. *Monics and epics in the category of relations*, by David Simmons, University of Arkansas, Fayetteville.

7. *Strictly isosceles metrics on arcs*, by R. A. Dooley, Oklahoma State University, Stillwater.

8. *Approximation by meromorphic functions*, by Annette Sinclair, University of Oklahoma, Norman.

9. *Tensor product for binary systems*, by Naoki Kimura, University of Arkansas, Fayetteville.

10. *Cotorsion modules*, by G. J. Wimbish, Oklahoma College of Liberal Arts, Chickasha.

11. *On some nonstationary stochastic processes*, by R. B. Deal, University of Oklahoma, Medical Center, Oklahoma City.

12. *The D-classes of the semigroup of stochastic matrices*, by Tetsundo Sekiguchi and Naoki Kimura, University of Arkansas, Fayetteville.

13. *On products in S-admissible classes of topological spaces*, by J. C. Warndof, University of Arkansas, Fayetteville.

14. *Data collection and the data reduction phase of the FAA Flight Inspection Program*, by John Rieger, FAA Center, Oklahoma City.

15. *Orthogonal trees*, by Arthur Bernhart, University of Oklahoma, Norman.

16. *Compounding the multiple runs distribution*, by J. E. Dunn, University of Arkansas, Fayetteville.

H. V. HUNEKE, *Secretary-Treasurer*

MARCH MEETING OF THE SOUTHEASTERN SECTION

East Carolina University, Greenville, North Carolina, was host to the 47th annual meeting of the Southeastern Section of the MAA, March 29–30, 1968. Professor R. E. Wheeler, retiring Chairman of the Section, and Professor Tullio Pignani, Chairman of the Mathematics Department of East Carolina University, presided at the general sessions. Three major addresses were scheduled for the general sessions: "Some Combinatorial Extremum Problems Connected with Convex Polytopes," by V. L. Klee, Jr. (University of Washington), "Varieties of Semigroups—a Topic in Universal Algebra," by Trevor Evans (Emory University), and "CUPM Report: Qualifications for a College Faculty in Mathematics," by Herman Meyer (University of Miami). A report was also presented by Professor C. V. Aucoin (Clemson University) on the CUPM Recommendations on a General Curriculum. The Section continued its policy of encouraging student participation and three special sections were devoted exclusively to presentations by students. With the cooperation of Modern Learning Aids, three films of mathematical interest were shown during the meeting: "Fixed Points" by Solomon Lefschetz, which followed the banquet Friday evening, "Topology" by Raoul Bott and Marston Morse, and "John von Neumann," a documentary on his life and work.

Total registration for this meeting was 313, including 119 nonmembers of the Association. The following officers were elected: Chairman, J. H. Wahab (University of North Carolina at Charlotte); Vice-Chairman, Emilie Haynsworth (Auburn University). The Secretary-Treasurer's first term of office ended with the 1967 meeting. Professor Henry Sharp, Jr., was elected to a second term as Secretary-Treasurer, retroactive to 1967. The invitation from Winthrop College, Rock Hill, South Carolina, to act as host to the 1969 meeting was re-affirmed, and an invitation from Clemson University, Clemson, South Carolina, to act as host to the 1970 meeting was accepted. The following recommendations proposed by the Committee on Special Projects were approved:

1. That in order to encourage participation on the part of our institutions in the Southeastern Section in the William Lowell Putnam contest, the secretary of the Section write a letter to the various department chairmen, at the proper time, to call their attention to this contest.

2. That a committee be appointed to study the matter of the Section offering an award for the best paper published in the MONTHLY, or anywhere else, during a certain period of time.

3. That a committee be appointed to study the matter of the Section electing a second Vice-Chairman to look after the interest of the community and junior colleges in the territory of the Southeastern Section with the aim of encouraging these institutions to participate in the affairs of the Section.

Because of the recent formation of a Florida Section of the Association, the resolution on polling members of the Southeastern Section, introduced in 1965 and amended in 1966, was rescinded. A motion from the floor proposed that the Southeastern Section consider the advisability of endorsing the "Doctor of Arts" degree, as a step toward alleviating the shortage of well-trained faculty in many colleges and junior colleges.

This motion was passed and the question was referred to committee for report at the next meeting.

The following contributed papers were presented:

1. *A note on the weak topology induced by a function*, by D. F. Bailey, East Carolina University.

2. *A lower bound for Ramsey's number* $R(4, 4; 3)$, by D. M. Bardwell, Clemson University.

3. *Maximal matric fields over a finite field*, by J. T. B. Beard, Jr., University of Tennessee and E. D. Dixon, Tennessee Technological University.

4. *The Krull-Schmidt theorem for ternary rings*, by W. E. Bolton, Jr., Western Carolina University.

5. *Equationally complete classes of algebras with two unary operations*, by W. H. Carlisle, Emory University.

6. *Some remarks on Fermat's last theorem*, by R. L. Carroll, Baptist College at Charleston.

7. *Dimension raising maps for which polyhedra are mapped to polyhedra*, by H. J. Charlton, North Carolina State University.

8. *A generalization of the Chinese Remainder Theorem*, by D. M. Clark, Emory University.

9. *Existence of a sequence of plane continua no one of which can be mapped continuously onto another*, by D. L. Cozart, Guilford College.

10. *A generalization of the Dirichlet product*, by K. J. Davis, East Carolina University.

11. *Remote computer usage in the university system*, by L. D. Davis, University of North Carolina at Charlotte.

12. *A characterization of the* T_m *graph*, by T. A. Dowling, University of North Carolina.

13. *Evaluation of a college math instructor*, by J. A. Gore, University of Georgia.

14. *A sorting problem*, by T. A. Gross, East Carolina University.

15. *Undergraduate Research Participation: Report on a Project in* 1967, by W. R. Hare and J. W. Kenelly, Clemson University.

16. *On the construction of pandiagonal magic squares*, by Carolyn B. Hudson, Duke University.

17. *On the greatest length of sequences of consecutive quadratic non-residues*, by R. H. Hudson, Duke University.

18. *An imbedding theorem for continuous affine maps on a compact convex set*, by R. E. Huff, University of North Carolina.

19. *An isomorphism theorem for semigroups*, by R. E. Jamison, Clemson University.

20. *On a compact ring with 1*, by Kwangil Koh, North Carolina State University at Raleigh.

21. *The simulation of lunar trajectories with associated sensitivity and midcourse correction velocity study*, by W. H. Land, Jr., IBM Federal Systems Division, Huntsville.

22. *Eigenvalues of the adjacency matrix of tetrahedral graphs*, by R. C. Bose and Renu Laskar, University of North Carolina at Chapel Hill.

23. *On completing Latin rectangles*, by C. C. Lindner, Emory University.

24. *Successive approximations for a class of functional differential equations*, by G. W. Marrah and T. G. Proctor, Clemson University.

25. *On C^∞ functions analytic on sets of small measure*, by L. E. May, North Carolina State University at Raleigh.

26. *The problem of Milloux*, by J. A. Pfaltzgraff, University of North Carolina at Chapel Hill.

27. *A property equivalent to continuity*, by G. E. Parker, Guilford College.

28. *Unitary convolution subalgebras of incidence algebras*, by L. M. Perry, Jr., North Carolina State University at Raleigh.

29. *On a class of spaces with Lindelöf real compactifications*, by R. T. Ramsay, North Carolina State University at Raleigh.

30. *Homeomorphism groups of spaces*, by H. B. Reiter, Clemson University.

31. *Congruence relations on topological m-groups*, by R. L. Richardson, Appalachian State University.

32. *On generalized hyperbolic sines and cosines*, by C. C. Ross, Jr., Emory University.

33. *A motivated proof of Tychonoff's Theorem*, by Robert Silber, Clemson University.

34. *Imbedding a locally finite partially ordered set in a locally finite distributive lattice*, by D. A. Smith, Duke University.

35. *The rank of the incidence matrix of points and hyperplanes in a finite projective geometry*, by K. J. C. Smith, University of North Carolina at Chapel Hill.

36. *A periodic solution to a periodic Ricatti equation*, by H. H. Suber, Clemson University.

37. *On subsets which determine continuous functions*, by J. P. Thomas, Western Carolina University.

38. *Existence of solutions to certain boundary value problems by a plane separation technique*, by H. K. Wilson, Georgia Institute of Technology.

39. *Computer derivation of part of the twisted tensor product diagonal map*, by J. F. Wirth, East Carolina University.

HENRY SHARP, JR., *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The forty-eighth meeting of the Southern California Section of the MAA was held at Loyola University, Los Angeles, California, on March 9, 1968. The registered attendance was 117, including 101 members of the Association. Professor F. A. Valentine, Chairman of the Section, presided at the morning and afternoon sessions.

At the business meeting Professor D. M. Merriell, Chairman of the Nominating Committee, reported the election of the following officers, who are to serve beginning July 1, 1968: Chairman, Charles Seekins, Occidental College; Vice-Chairman, Edward Posner, JPL/Cal Tech. The Chairman and Vice-Chairman will serve for one year. The following members were elected to the Program Committee for the 1969 meeting of the Section: A. M. Garsia, UC-San Diego; B. R. Gelbaum, UC-Irvine; N. G. Mouck, Jr., Santa Barbara CC; R. T. Sandberg, Cal State-Fullerton; E. M. Scheuer, Rand Corporation.

Professor Charles Halberg reported as Governor for the Section on items and actions of the Board of Governors for the January, 1968 meeting at San Francisco. Dr. J. M Huffman of the Office of Schools for San Diego County reported on the National High School Mathematics Contest. Recognition was given to the two students who received the highest scores in the Section in the 1967 Putnam Contest.

It was announced that the next meeting of the Section is scheduled to be held on March 15, 1969, at California State College, Fullerton.

The following program was presented:

1. *Combinatorial Telemetry*, by E. C. Posner, Jet Propulsion Laboratory.
2. *Effective Decision Processes in Algebra*, by W. M. Lambert, Loyola University.
3. *Ask what Geometry can do for You!*, by Paul Kelly, University of California, Santa Barbara.
4. *Equivalent Forms of the Hahn-Banach Extension Theorem*, by W. A. J. Luxemburg, California Institute of Technology.
5. *Problems of Advanced Placement*, a panel discussion chaired by Courtney Coleman, Harvey Mudd College, together with Gil Peter, Cuesta College, and Mrs. Kathern Layton, Beverly Hills High School.
6. *Remarks on Partitioned Matrices*, by John de Pillis, University of California, Riverside.

D. H. POTTS, *Secretary-Treasurer*

APRIL MEETING OF THE IOWA SECTION

The fifty-fifth regular meeting of the Iowa Section of the MAA was held at Wartburg College, Waverly, Iowa, on April 19, 1968. Chairman C. M. Lindsay presided. Total attendance was 78, including 46 members of the Association. A partial cause of the somewhat below normal attendance was the concurrence of the American Mathematical Society meeting in Chicago.

The following officers were elected: Chairman, Rev. J. L. Friedell, Loras College, Dubuque; Vice-Chairman, Steve Armentrout, State University of Iowa, Iowa City; Secretary-Treasurer, B. E. Gillam, Drake University, Des Moines.

During the morning session, a film, produced by the Calculus Film Project of the MAA, "The Theorem of the Mean," was shown and the following papers were presented:

Scientific and technical communication, by W. J. Jamison, Collins Radio Co., Cedar Rapids (by invitation).

Qualifications for a college faculty in mathematics, by Robert McDowell, Washington University, St. Louis, Missouri.

During the afternoon session, in addition to the business meeting, the following papers were presented:

Magic squares: a note on notation, by E. S. Allen, Wartburg College, Waverly, Iowa.

The mathematics of bowing a violin, and other nonlinear phenomena, by W. S. Loud, University of Minnesota, Minneapolis, Minnesota (by invitation).

The meeting was concluded by the showing of the film, "Infinite Acres," produced by the Calculus Film Project of the MAA.

B. E. GILLAM, *Secretary-Treasurer*

APRIL MEETING OF THE OHIO SECTION

The fifty-second annual meeting of the Ohio Section of the MAA was held at Miami University, Oxford, Ohio, on Friday and Saturday, April 26–27, 1968. Professor Daniel Finkbeiner, Chairman of the Section, presided at the business and dinner meetings, and Professor B. J. Yozwiak, presided at the program sessions. One hundred thirty-four persons registered in attendance including one hundred fourteen members of the Association.

The following officers were elected: Chairman, Professor Arnold Ross, The Ohio State University; Chairman Elect, Professor James Smith, Muskingum College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Program Committee:

Professor R. A. Clark, Case Western Reserve University, Chairman; Professor J. F. Leetch, Bowling Green State University, Professor R. G. Laatsch, Miami University.

Regular reports were given by the Chairman, the Secretary-Treasurer and the Governor for the Ohio Section, Professor Holbrook MacNeille. There were also special reports for the Committee for the MAA High School Test by Professor L. J. Green and for the Committee on Curriculum (CONCUR) by Professor R. A. Roberts.

The following program was presented:

1. *Extremal subadditive functions—the finite case*, by Richard Laatsch, Miami University.

2. *On the rim of an R-group*, by G. J. Sherman, Student, Bowling Green State University.

3. *Non-measurable subsets of a set with positive outer measure*, by Nand Kishore, Toledo University.

4. *Variations on the axioms for a group*, by J. F. Leetch, Bowling Green State University.

5. *Transformation of planes in real Argand six-space*, by Hunter Hardman, Marshall University.

6. *Cut-sets for real functions*, by Foster Brooks, Kent State University.

7. *The history of Ryley's problem: To express a given rational number as a sum of three rational cubes*, by Edward Molnar, Student, Ohio University, introduced by I. A. Barnett, Ohio University.

8. *A note on Ryley's problem*, by I. A. Barnett, Ohio University.

9. *A direct-table test for transitivity of a relation*, by L. D. Rodabaugh, University of Akron.

10. *Non-existence theorems for some configurations related to generalized polygons*, by S. E. Payne, Miami University.

11. *Discussion on calculus*, by Melvin Henriksen, Case Western Reserve University.

12. *The mergences of mathematics*, by D. T. Finkbeiner, Kenyon College (Chairman's address).

13. *Language as a part of mathematics*, by Erwin Engeler, University of Minnesota (invited speaker).

FOSTER BROOKS, *Secretary*

APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the MAA was held on the campus of Texas A & M University, College Station, Texas, on April 19–20, 1968. There were 256 persons registered including 178 members of the Association.

Papers were presented on Friday in three concurrent sessions chaired by Professor R. J. Duffin, Texas A & M University; Professor M. G. Mundt, Texas Lutheran College; Professor J. J. Malone, Texas A & M University; Professor C. M. McLoury, McMurry College; Professor R. M. Whitmore, Southwestern University; Professor R. G. Dean, Stephen F. Austin State College; and Professor G. R. Stewart, Stephen F. Austin State College. Papers were also read in three concurrent sessions on Saturday morning. These sessions were chaired by Professor R. B. Patschke, Tyler Junior College; Professor M. G. Bordelon, College of the Mainland; and Professor Jack Bryant, Texas A & M University.

The welcoming address was given by Vice-President W. C. Hall of Texas A & M University at a dinner meeting April 19, 1968 which was presided over by Professor H. A. Luther, Texas A & M University.

The general session on Saturday, April 20, 1968 was chaired by Professor Dale Maness of Austin College and included reports of all committees, the section governor, and secretary-treasurer. At this session the following officers were elected for the coming year: Chairman, Professor H. A. Luther, Texas A & M University; Vice-Chairman,

Professor Carmon McFerran of Texarkana College; Secretary-Treasurer, Professor B. T. Goldbeck, Jr., Texas Christian University.

The invited speakers were Professor G. S. Young, Tulane University, who spoke on "Topological Aspects of Complex Variables," Dr. R. E. Greenwood, The University of Texas, who spoke on "The Putnam Prize Competition" and Dr. M. L. Curtis, Rice University, who spoke on "What Every Young College Teacher Should Know."

The following papers were presented:

1. *Generalized semirings of quotients*, by J. R. Mosher, Texas Christian University.

2. *Pseudoinverses of positive semidefinite matrices*, by Truman Lewis and Tom Newman, Texas Technological College.

3. *Pseudo-δ-mixing measure-preserving transformations on a probability space*, by C. H. Farmer, TRACOR, Inc.

4. *Invariants associated with plane curves*, by G. L. Shurbet and H. W. Milnes, Texas Technological College.

5. *Matrices in dyadic form*, by Louis Brand, University of Houston.

6. *A class of nonlinear transformations*, by H. L. Gray and T. A. Atchison, Texas Technological College.

7. *Subtended area and determinant integration*, by W. S. McCulley, Texas A and M University.

8. *Concerning weak topologies*, by W. B. Sconyers, Texas Christian University.

9. *Decimal representations of sums and products*, by Margaret R. Wiscamb, University of St. Thomas.

10. *Certain properties of r. k. Hilbert spaces*, by David Drennan, Texas Christian University and Jarvis Christian College.

11. *The endomorphism near-ring on the nonabelian group of order six*, by Carter Lyons, Texas A and M University.

12. *Characterization of Blumberg metric pairs*, by J. C. Bradford, Abilene Christian College.

13. *On duals of spaces of affine continuous functions*, by H. E. Lacey, Nasa-Msc-Houston and University of Texas at Austin.

14. *The Hilbert basis theorem for a class of halfrings*, by H. E. Stone, Texas Christian University.

15. *An example relating to Jacobi's necessary condition in the calculus of variations*, by H. W. Milnes and S. K. Hildebrand, Texas Technological College.

16. *Expansive mappings*, by R. K. Williams, Southern Methodist University.

17. *Cohen's theorem for Noetherian semirings*, by P. J. Allen, University of Alabama.

18. *Polyhedral star-shaped sets*, by F. E. Tidmore, Texas Technological College.

19. *On the universality of diagonal forms over fields*, by Brother Joseph Heisler, St. Edward's University.

20. *A birthday holiday problem*, by Arthur Richert, Jr., The University of Texas at Austin.

21. *A transformation for projecting a functional surface onto a plane*, by M. E. Harris, University of St. Thomas.

22. *Young's inequality for nonunimodular groups*, by Richard O'Neill, The Rice University.

23. *Geometrical characterizations of certain ordered Banach spaces*, by L. F. Guseman, Jr., NASA-Manned Spacecraft Center.

24. *An extension of the simple iteration method for finding roots of an equation*, by Dianne Kleuser, University of St. Thomas.

25. *A technique for obtaining moments for a class of discrete distributions*, by T. L. Boullion, Texas Technological College.

26. *Comparison of two approaches to the teaching of plane and solid geometry*, by Jim Bezdek, North Texas State University.

27. *Near-rings on certain groups*, by H. E. Heatherly, Texas A and M University.

28. *Results of a bilingual program in mathematics in the primary grades*, by Bertha G. Trevino, Laredo Junior College.

29. *Metrization of developable spaces*, by J. R. Boone, Texas Christian University.

30. *Near-ring homomorphisms*, by J. J. Malone, Jr., Texas A and M University.

31. *Borel summability of Fourier series*, by Jack Bryant, Texas A and M University.

32. *Matrix rotation and centro-symmetric matrices*, by Louis Brand and Steve Ligh, University of Houston and Texas A and M University.

33. *Convergence characterizations*, by Norm Howes, Texas Christian University.

34. *Nonnegative matrices with prescribed row and column sum*, by Jean Rogers Edwards, University of Houston.

35. *On product formulas for the gamma function*, by Russell Cowan, Lamar State College of Technology.

36. *A characterization of parabolic Lebesgue spaces*, by C. H. Sampson, Texas A and M University.

37. *Problems involving diagonal sums in nonnegative matrices*, by Mark Hedrick, Houston, Texas

38. *Some characterizations of hemidomains*, by Linda D. Lindsey, Tarleton State College and. Texas Christian University.

39. *Some results in discrete analytic function theory*, by George Berzsenyi, Texas Christian University and Northeast Louisiana State College.

40. *Cardinality of an ultrafilter*, by J. D. Price, North Texas State University.

41. *A reproducing kernel function for discrete analytic functions*, by J. C. Bolen, Texas Christian University.

42. *Integration of the exponential function in n-space: a computational problem*, by R. R. Bunten, USAF Personnel Research Laboratory.

43. *A spectral realization for rectangular matrices*, by Jay Amburgey, Texas Technological College.

44. *Lebesgue classes of functions with parabolic weights*, by Richard Bagby, The Rice University.

B. T. GOLDBECK, JR., *Secretary-Treasurer*

MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held on May 4, 1968 at Ball State University. There were 119 persons in attendance, including 80 members of the Association.

The morning session was held jointly with the Indiana Council of Teachers of Mathematics. The group was welcomed by Dr. J. R. Emens, President of Ball State University, and Professor Donald Foss of the ICTM presided. Professor Franz Hohn of the University of Illinois delivered an address concerning "Problems of the Freshman-Sophomore College Mathematics Program." He was followed by Professor K. B. Henderson of the University of Illinois who spoke on "Implications of Research on the Teaching of Mathematical Concepts."

Professor K. J. Sidebottom, Chairman of the Section, presided at the afternoon session. During the business meeting Professor Rodney Hood reported on the Indiana College Mathematics Competition, and the Secretary-Treasurer reported that Mr. James Mulfur of Notre Dame and Mr. Stephen Spindler of Purdue had each received a one-year membership in the Association in recognition of their achievement in the 28th Putnam Mathematical Competition. Officers for 1968–69 were elected as follows: Chairman, Professor B. E. Rhoades, Indiana University; Vice-Chairman, Professor P. T. Mielke, Wabash College; Secretary-Treasurer, Professor M. J. Mansfield, Purdue University at Fort Wayne.

Following the business meeting, Professor Warren Loud of the University of Minnesota addressed the group on "Nonlinear Phenomena."

M. J. MANSFIELD, *Secretary-Treasurer*

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1968 recipients of these awards, selected by a committee consisting of Ivan Niven, Chairman; C. W. Curtis, and Edwin Hewitt, were announced by President Moise at the Business Meeting of the Association on August 27, 1968, at the University of Wisconsin. The recipients of the Ford Awards for articles published in 1967 were the following:

Frederic Cunningham, Jr., Taking Limits Under the Integral Sign, MATH. MAG., 40 (1967), 179–186.

W. F. Newns, Functional Dependence, MONTHLY, 74 (1967), 911–920.

Daniel Pedoe, On a Theorem in Geometry, MONTHLY, 74 (1967), 627–640.

K. L. Phillips, The Maximal Theorems of Hardy and Littlewood, MONTHLY, 74 (1967), 648–660.

F. V. Waugh and Margaret W. Maxfield, Side-and-diagonal Numbers, MATH. MAG., 40 (1967), 74–83.

H. J. Zassenhaus, On the Fundamental Theorem of Algebra, MONTHLY, 74 (1967), 485–497.

HENRY L. ALDER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fifty-Second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

FLORIDA, Florida Atlantic University, Boca Raton, March 21–22, 1969.

ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.

INDIANA, Butler University, Indianapolis, November 2, 1968.

IOWA, University of Northern Iowa, Cedar Falls, April 18, 1969.

KANSAS

KENTUCKY, Morehead State University, Morehead, Spring 1969.

LOUISIANA-MISSISSIPPI, New Orleans, January 25–27, 1969.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Baltimore, Fall 1968.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, Corcordia College Moorhead, October 26, 1968.

MISSOURI, St. Louis University, St. Louis, April 26, 1969.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 25–26, 1969.

NEW JERSEY, Rutgers—The State University, New Brunswick, November 2, 1968.

NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.

NORTHERN CALIFORNIA, University of Santa Clara, Santa Clara, February 8, 1969.

OHIO

OKLAHOMA-ARKANSAS, Arkansas State University, Jonesboro, March 21–22, 1969.

PACIFIC NORTHWEST, University of Oregon, Eugene, August 1969.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.

ROCKY MOUNTAIN, University of Wyoming, Laramie, May 9–10, 1969.

SOUTHEASTERN, Winthrop College, Rock Hill, South Carolina, March 28–29, 1969.

SOUTHERN CALIFORNIA, California State College at Fullerton, March 15, 1969.

SOUTHWESTERN, Northern Arizona University, Flagstaff, Spring 1969.

TEXAS, Texarkana College, Texarkana, April 18–19, 1969.

UPPER NEW YORK STATE

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, New Orleans, Louisiana, January 23–26, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC, New Orleans, Louisiana, January 22–23, 1969.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, New Orleans, Louisiana, January 25–26, 1969.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, November 6–9, 1968.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Ben Franklin Hotel, Philadelphia, October 21–24, 1968.

MATHEMATICIANS ***Job opportunities in war work are announced in the Notices of the A.M.S., in the Employment Register, and elsewhere. We urge you to regard yourselves as responsible for the uses to which your talents are put. We believe this responsibility forbids putting mathematics in the service of this cruel war.***

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Andrew Adler
Ruth Afflack
Yoshio Akiyama
W. Ambrose
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R. A. ROSENBAUM, *Editor*

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ON THE ZEROS OF THE DERIVATIVE OF AN ENTIRE FUNCTION

MORRIS MARDEN, University of Wisconsin-Milwaukee

1. Introduction. While preparing a course of lectures on entire functions during the past semester, I had the occasion to review what had been done so far in carrying over to entire functions various properties of polynomials. In particular, I was interested in possible extensions of Rolle's Theorem and Lucas' Theorem [1]. By Lucas' Theorem I refer to the one stating that the zeros of the derivative of a polynomial f lie in the convex hull of the zeros of f.

In order to see that extending these theorems to entire functions is not a trivial matter, let us consider the example of the function

$$f(z) = (z+1)e^{z^2/2}. \tag{1.1}$$

Its only zero is the real one $z=-1$, but its derivative

$$f'(z) = (1+z+z^2)e^{z^2/2}$$

has two zeros, the nonreal numbers $e^{2\pi i/3}$ and $e^{4\pi i/3}$. For this example Lucas' Theorem clearly does not hold. The problem is how to modify the theorem so it will become valid for transcendental, as well as polynomial, entire functions.

However, there is a circumstance in which we can extend Lucas' Theorem without any modification. We refer to the case that an entire function f with all its zeros on a closed set T can be approximated uniformly in every bounded domain by a sequence of polynomials $\{f_n\}$ all of whose zeros also lie on T. By Lucas' Theorem the zeros of the derivative f_n' of each f_n would lie in the convex hull $\mathcal{H}(T)$ of T. Since the sequence $\{f_n'\}$ converges uniformly to the derivative f' of f in every bounded domain, the zeros of f' would, by Hurwitz' Theorem, also lie in $\mathcal{H}(T)$. That is, Lucas' Theorem does hold in this case for f and all its derivatives.

Unfortunately, however, not every entire function with all its zeros in an arbitrary set T can be uniformly approximated by sequences $\{f_n\}$ of polynomials all of whose zeros lie in T. For instance, if T is the real axis, any entire function f which can be approximated uniformly by polynomials with real zeros has to have the form

$$f(z) = e^{az^2+bz}\,\Phi(z), \tag{1.2}$$

where Φ is an entire function of genus 0 or 1 and a and b are real constants with $a \leqq 0$. This is according to the work of Laguerre and Pólya [2]. Clearly, function (1.1) is not of the form (1.2), yet has only real zeros. In fact the product of (1.2) by any exponential factor $e^{Q(z)}$, where Q is an entire function, would have all its zeros real, but would not in general have the form (1.2).

There is, of course, a basic reason why Lucas' Theorem does not carry over in general to entire functions. Whereas a polynomial is determined by its zeros uniquely up to a constant multiplier, an entire function is determined by its zeros up to a multiplier which is itself an entire function. In the case of poly-

nomials the constant multiplier does not influence the zeros of the derivative, but in the case of entire functions the multiplier does obviously affect these zeros. In fact according to Weierstrass [3] we may write an entire function with zeros $z=0$ (multiplicity m), $z=a_1, a_2, a_3, \cdots$ in the form

$$f(z) = z^m e^{Q(z)} \prod_{j=1}^{\infty} G(z/a_j, p), \tag{1.3}$$

where p is the smallest nonnegative integer such that

$$\sum_{j=1}^{\infty} |a_j|^{-(p+1)} < \infty, \tag{1.4}$$

where $Q(z)$ is an entire function and where

$$G(u, p) = (1 - u) \exp [u + (u^2/2) + \cdots + (u^p/p)].$$

Furthermore, if f is of finite order ρ, then according to Hadamard [4], Q is a polynomial,

$$Q(z) = c_1 z + c_2 z^2 + \cdots + c_q z^q \tag{1.5}$$

with $p \leqq [\rho]$ and $q \leqq [\rho]$, the symbol $[\rho]$ standing for the largest integer not exceeding ρ. Since

$$G'(u, p)/G(u, p) = (u - 1)^{-1} + 1 + u + u^2 + \cdots + u^{p-1}$$

$$(d/dz)[G(u, p)] = G(u, p)[u^p(u - 1)^{-1}](du/dz),$$

we obtain from (1.3) and (1.5) the expression

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + c_1 + 2c_2 z + \cdots + qc_{q-1} z^{q-1} + \sum_{j=1}^{\infty} \frac{z^p}{a_j^p(z - a_j)} \cdot \tag{1.6}$$

Thus, f' depends, not only upon the given zeros a_j of f, but also upon the parameters $c_1, c_2, \cdots, c_q$ which have no necessary connection with the zeros a_j of f.

The earlier efforts to extend Lucas' Theorem seemed to be confined largely to cases obtained by specializing the order ρ or the parameters $c_1, c_2, \cdots, c_q$. First, Laguerre [5] established its validity for any real entire function f with only real zeros, provided f has an order $\rho = 0$ or 1. Pólya [6] extended the result to entire functions of the form (1.2). Also, Cesàro [7] established its validity for entire functions of the so-called canonical product form corresponding to $c_1 = c_2 = \cdots = c_q = 0$ in (1.5) while Porter [8] extended the result further to the case $c_1 = c_2 = \cdots = c_{q-2} = 0$, $q = p+1$ with $c_q < 0$ if p is odd, but $c_{q-1} > 0$ if p is even. Secondly, as regards Lucas' Theorem for arbitrary entire functions, the only result appears to be that due to Porter [8] establishing its validity for entire functions of order $\rho = 0$.

In 1949, a different representation for the derivative of an entire function

was developed in one of my papers [9]. Like (1.5), this representation does involve some parameters, but, unlike the c_j in (1.5), the parameters have some relevancy to the study of the zeros of f'. This new representation came as an unexpected byproduct of a study of rational functions containing certain arbitrary parameters. It was established after lengthy algebraic computations.

In the present paper we propose to give an improved version of this representation and at the same time a simpler, direct derivation of it. We shall also give some applications which generalize Lucas' Theorem to entire functions of any finite order.

Let us begin by stating the new version and then proceed to its proof.

2. A representation for the derivative of an entire function.

THEOREM (2.1). *Let f be an entire function of finite order ρ, with simple zeros at $a_1, a_2, a_3, \cdots$ and $0<|a_1|\leqq|a_2|\leqq|a_3|\leqq\cdots$, Let f', the derivative of f, have at least $n\geqq[\rho]$ distinct zeros and denote n of these as $\zeta_1, \zeta_2, \cdots, \zeta_n$. Then for any finite $z\neq a_j$, $j=1, 2, 3, \cdots$*

$$f'(z) = f(z)\sum_{j=1}^{\infty}\frac{1}{(z-a_j)}\prod_{k=1}^{n}\frac{(z-\zeta_k)}{(a_j-\zeta_k)}. \tag{2.1}$$

This series is uniformly convergent in every bounded domain.

Proof. If we set

$$\psi(z) = \prod_{k=1}^{n}(z-\zeta_k),$$

so that ψ is a factor of f', we may rewrite (2.1) in the form

$$\frac{f'(z)}{\psi(z)} = f(z)\sum_{j=1}^{\infty}\frac{1}{(z-a_j)\psi(a_j)}. \tag{2.2}$$

With z as an arbitrary fixed point $z\neq a_j$, $j=0, 1, 2, \cdots$, let us choose R so that

$$R \geqq 2\max[|z|, |\zeta_1|, |\zeta_2|, \cdots, |\zeta_n|]. \tag{2.3}$$

Since we wish $f(Re^{i\theta})\neq 0$, let us choose R so that also

$$R \notin A = \{|a_1|, |a_2|, |a_3|, \cdots\}. \tag{2.4}$$

Let us now introduce the circles

$$C\colon |t| = R; \qquad C_j\colon |t-a_j| = k_j, \quad \text{for} \quad |a_j| < R,$$

choosing the k_j so small that the circles C_j lie inside the circle C, no two C_j intersect and each C_j contains only one a_j. Let us denote by γ the boundary of the region

$$\Gamma = \{z\colon |z| < R, |z-a_j| > k_j \quad \text{for all } |a_j| < R\}.$$

Since $f'/(f\psi)$ is analytic in Γ, we have by Cauchy's Integral Formula

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)\psi(z)}\,dz = I_1 - I_2, \tag{2.5}$$

where

$$I_1 = \frac{1}{2\pi i}\int_C \frac{f'(t)dt}{f(t)\psi(t)(t-z)}, \qquad I_2 = \frac{1}{2\pi i}\sum_{|a_j|<R}\int_{C_j}\frac{f'(t)dt}{f(t)\psi(t)(t-z)}. \tag{2.6}$$

Since the residue of the integrand at the simple zero a_j of $f(z)$ is

$$\frac{1}{2\pi i}\int_{C_j}\frac{f'(t)dt}{f(t)\psi(t)(t-z)} = \frac{f'(a_j)/\psi(a_j)}{f'(a_j)(a_j-z)},$$

we may evaluate I_2 at once as

$$I_2 = \sum_{|a_j|<R}\frac{1}{\psi(a_j)(a_j-z)}. \tag{2.7}$$

However, the evaluation of I_1 requires a lengthier analysis. Primarily, we need to study the effect upon I_1 of causing $R\to\infty$ with $R\notin A$. We begin by estimating the order of magnitude of $|f'(Re^{i\theta})/f(Re^{i\theta})|$. From (1.5) with $m=0$,

$$\begin{aligned}\left|\frac{f'(Re^{i\theta})}{f(Re^{i\theta})}\right| \leqq |Q'(Re^{i\theta})| &+ \sum_{0<|a_j|<R}\frac{R^p}{|a_j|^p(R-|a_j|)} \\ &+ \sum_{|a_j|>R}\frac{R^p}{|a_j|^p(|a_j|-R)}.\end{aligned} \tag{2.8}$$

To expedite our proof that $I_1\to 0$ as $R\to\infty$, let us suppose the zeros of f to possess a certain growth property. Specifically, let us assume the existence of positive constants α, δ and $\lambda(>1)$ such that

$$|a_{k+1}| \geqq \lambda[|a_k| + \alpha|a_k|^{1+\delta}], \qquad k = 1, 2, 3, \cdots. \tag{2.9}$$

Choosing $R_N = |a_N| + \alpha|a_N|^{1+\delta}$, we see that the circle $|z| = R_N$ contains only the zeros a_k, $k=1, 2, \cdots, N$ in its interior. Also for $k=1, 2, \cdots, N$,

$$R_N - |a_k| = |a_N| - |a_k| + \alpha|a_N|^{1+\delta} \geqq \alpha|a_N|^\delta|a_k|,$$

whereas for $k=n+1, n+2, \cdots$, $|a_k| \geqq |a_{N+1}| \geqq \lambda R_N$ and so

$$|a_k| - R_N \geqq [1-(1/\lambda)]|a_k|.$$

Consequently, if S_1 and S_2 denote respectively the first and second sums in (2.8), we have for $R=R_N$:

$$S_1 \leqq R^p\sum_{k=1}^N \alpha^{-1}|a_k|^{-p-1}|a_N|^{-\delta}, \qquad S_2 \leqq [\lambda/(\lambda-1)]R^p\sum_{k=N+1}^\infty |a_k|^{-p-1}$$

Since $\sum_{k=1}^{\infty} |a_k|^{-p-1}$ converges, $\sum_{k=1}^{N} |a_k|^{-p-1}$ is bounded whereas $\sum_{k=N+1}^{\infty} |a_k|^{-p-1} < \epsilon$ for any given positive ϵ provided R_N is made sufficiently large. Also from the definition of R_N, we have for large R_N

$$R_N = O(|a_N|^{1+\delta}), \qquad |a_N| = O(R_N^{-1-\delta}).$$

Hence $S_1 \leqq O(R_N^m)$, $S_2 \leqq \epsilon O(R_N^p)$, where $m = p - [\delta/(\delta+1)] < p$. Thus,

$$|f'(R_N e^{i\theta})/f(R_N e^{i\theta})| \leqq O(R^{q-1}) + O(R^m) + O(R^p)\epsilon.$$

Noting further from (2.3) that

$$|R_N e^{i\theta} - z| \geqq (R_N/2, \quad |\psi(R_N e^{i\theta})| = \prod_{k=1}^{n} |R_N e^{i\theta} - \zeta_k| \geqq (R_N/2)^n,$$

we now infer from (2.6) that

$$\begin{aligned} |I_1| &\leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f'(R_N e^{i\theta})| R_N d\theta}{|f(R_N e^{i\theta})\psi(R_N e^{i\theta})(Re^{i\theta} - z)|} \\ &\leqq 2^{n+1} R_N^{-n} [O(R_N^{q-1}) + O(R_N^m) + O(R_N^p)\epsilon]. \end{aligned}$$

Since $n \geqq [\rho]$, $p \leqq [\rho]$, $q \leqq [\rho]$, we can make $|I_1|$ arbitrarily small by taking R sufficiently large with $R \notin A$. Therefore from (2.5) and (2.7) there follows the representation (2.2) or the equivalent (2.1) of Theorem (2.1).

REMARKS. 1. The assumption (2.9) holds only for a restricted class of entire functions. Nevertheless, Theorem (2.1) as stated is valid, as was proved essentially in our previous paper [9] by a method not involving the Residue Theorem. Furthermore, at a later date we expect to publish a generalization of Theorem (2.1) which will be established by a method different from both the present and previous proofs [12].

2. We note that the a_j are removable singularities of the function represented by the right side of (2.1). In fact, from (1.3) we deduce that

$$f_j(z) = f(z)/[1 - (z/a_j)] = \exp[Q(z) + (z/a_j) + \cdots + z^p/pa_j] \prod_{\substack{\nu=1 \\ \nu \neq j}}^{\infty} G(z/a_\nu, p)$$

is an entire function in terms of which we may write (2.1) as

$$f'(z) = -\sum_{j=1}^{\infty} \frac{f_j(z)\psi(z)}{a_j \psi(a_j)}. \tag{2.10}$$

That is, (2.1) or (2.10) represents $f'(z)$ for all finite z.

3. We may regard the right side of (2.1) as a Lagrange interpolation series for $f'(z)/\psi(z)$ since

$$\lim_{z \to a_\nu} f(z) \sum_{j=1}^{\infty} [\psi(a_j)(z - a_j)]^{-1} = f'(a_\nu)/\psi(a_\nu).$$

That is, the right side of (2.1) exhibits $F(z)=f'(z)/\psi(z)$ in terms of the values of $F(z)$ at the specified points a_j, $j=1, 2, 3, \cdots$.

4. We may use the same method of proof as for Theorem (3.1) to obtain a representation for f' when f has multiple zeros including one at $z=0$. But we can accomplish the same also by a limiting process in which we allow m_j zeros of f to coincide at a_j with $0=|a_0|<|a_1|\leqq|a_2|\leqq\cdots$. We thus obtain the formula

$$f'(z) = f(z)\sum_{j=0}^{\infty}\frac{m_j\psi(z)}{(z-a_j)\psi(a_j)}. \tag{2.11}$$

5. If the entire function f has only a finite number $N+1$ of zeros, the corresponding result is clearly

$$f'(z) = f(z)\sum_{j=0}^{N}\frac{m_j\psi(z)}{(z-a_j)\psi(a_j)}. \tag{2.12}$$

We may of course consider (2.12) a limiting case of (2.11) with $m_j=0$ for $j>N$.

3. Location of the zeros of the derivative. We now apply Theorem (2.1) to determining the position of the zeros of the derivative of f in relation to the zeros of f. In this section and the next, we assume that f has at least one zero and that the zeros of f lie on a point set T. We shall use the notation $\mathcal{C}(T)$ for the complement of T in the complex plane and $\mathcal{H}(T)$ for the convex hull of T.

We shall also use the notation $\mathcal{S}(T, \nu)$ for the set of all points from which T subtends an angle of at least ν, where $0\leqq\nu\leqq 2\pi$. The domain $\mathcal{S}(T, \nu)$ is a generalization of the convex hull in that $\mathcal{S}(T, \pi)=\mathcal{H}(T)$ and in that $\mathcal{S}(T, \nu)$ is star-shaped relative to T. By the latter, we mean that each point of $\zeta\in\mathcal{S}(T, \nu)$ can be joined to every point of $a\in T$ by a line segment lying in $\mathcal{S}(T, \nu)$. In fact, the segment from ζ to a lies in the angle subtended at ζ by T and so lies in $\mathcal{S}(T, \nu)$. Finally $\mathcal{S}(T, \nu)$ has the properties

$$\begin{aligned} &T\subset\mathcal{H}(T)\subset\mathcal{S}(T, \nu), &&\text{if}\quad 0<\nu<\pi,\\ &\mathcal{S}(T, \nu_1)\subset\mathcal{S}(T, \nu_2) &&\text{if}\quad \nu_1>\nu_2. \end{aligned}$$

For example, if T is the disk $|z|\leqq a$, $\mathcal{S}(T, \nu)$ is the concentric disc $|z|\leqq a\csc(\nu/2)$. If T is the line segment $-1\leqq x\leqq 1$ of the real axis, $\mathcal{S}(T, \nu)$ is the union of the two domains

$$\begin{aligned} &|z-\cot\nu|\leqq\csc\nu \quad\text{if}\quad \mathcal{I}(z)\geqq 0,\\ &|z+\cot\nu|\leqq\csc\nu \quad\text{if}\quad \mathcal{I}(z)\leqq 0. \end{aligned}$$

We are now ready to state our principal result.

THEOREM (3.1). *Let f be an entire function of finite order ρ. Let all the zeros of f lie on a pointset T with $\mathcal{C}[\mathcal{H}(T)]\neq\varnothing$. Then at most n distinct zeros of f', the derivative of f, lie in the region $K\equiv\mathcal{C}[\mathcal{S}(T, \pi/(n+1))]$ with $n=[\rho]$.*

Proof. Let us suppose on the contrary that f' has more than n distinct zeros in K. Let us choose $\zeta_1, \zeta_2, \cdots, \zeta_{n+1}$ as any $n+1$ of these zeros and let ω be the largest of the angles subtended by T in the points $\zeta_j, j=1, 2, \cdots, n+1$. If we substitute $z=\zeta_{n+1}$ into (2.11), we obtain the equation

$$\sum_{j=0}^{\infty} \frac{m_j}{(\zeta_1 - a_j)(\zeta_2 - a_j) \cdots (\zeta_{n+1} - a_j)} = 0. \tag{3.1}$$

This is an identity that connects any $n+1$ zeros ζ_k of f' with all the zeros a_j of f. Since T subtends an angle less than $\pi/(n+1)$ at each point ζ_k and since the ray from ζ_k to each point of T falls in this angle, we can find a point ξ_k such that, with $t_{jk}=(\zeta_k-\xi_k)/(\zeta_k-a_j)$,

$$0 \leqq \arg t_{jk} \leqq \omega < \pi/(n+1), \qquad j = 1, 2, \cdots.$$

Hence each product $\sigma_j=t_{j1}t_{j2} \cdots t_{jn+1}$ lies in the sector

$$B: \quad 0 \leqq \arg z \leqq (n+1)\omega < \pi. \tag{3.2}$$

Now, the sum of any finite number of nonzero vectors in B is likewise a nonzero vector in B. Hence, if

$$s_h = m_1\sigma_1 + m_2\sigma_2 + \cdots + m_h\sigma_h, \qquad s_N = m_{h+1}\sigma_{h+1} + \cdots + m_N\sigma_N,$$

both s_h, $s_N \in B$ with $s_h s_N \neq 0$. Holding h fixed, we allow $N \to \infty$. Unless $s_\infty=0$, vector $s_\infty \in B$. In any case, if $s=s_h+s_\infty$, we see that $s\in B$, $s\neq 0$.

But the sum s is the left side of (3.1) multiplied by the product $(\zeta_1-\xi_1) \cdot(\zeta_2-\xi_2) \cdots (\zeta_{n+1}-\xi_{n+1})$. The assumption that f' has at least $n+1$ zeros in K has therefore led us to the contradiction $s\neq 0$. Hence, at most n zeros of f' lie in K as asserted in Theorem (3.1).

As an immediate consequence of Theorem (3.1), we now can obtain a generalization of Porter's result by specializing the order ρ to be less than one. In that case $n=[\rho]=0$ so that $\mathcal{S}(T, \pi/(n+1))\equiv \mathcal{H}(T)$. Thus, we find:

COROLLARY (3.1). *If f is an entire function of order ρ with $0\leqq\rho<1$ and if all the zeros of f lie on a set T, then all the zeros of f' lie in $\mathcal{H}(T)$, the convex hull of T.*

For example,

$$f(z) = \prod_{k=0}^{p} \cos(z - ki)^{1/2},$$

an entire function of order 1/2, has its zeros in the semi-infinite strip $\mathcal{R}(z)>0$, $0\leqq\mathcal{I}(z)\leqq p$. By Corollary (3.1) the same is true of the zeros of its derivative.

In effect, Corollary (3.1) says that Lucas' Theorem holds for any entire function of order less than one.

4. Applications to special sets T. First, let us consider entire functions with only a finite number of zeros. One choice of T is a line-segment, which for con-

venience we take as the portion of the real axis $-a \leqq \Re(z) \leqq a$, $a>0$. Another choice of T is a circular disk, which for convenience we take as $|z| \leqq R$. In both cases the corresponding domains $\mathcal{S}(T, \pi/(n+1))$ were determined in Section **3**. We may thus state the following two results.

THEOREM (4.1). *If f is an entire function of the form $f(z)=P(z)e^{Q(z)}$ where P and Q are polynomials and deg $Q=n$, and if all its zeros lie on the line segment $-a \leqq \Re(z) \leqq a$, $(a>0)$ of the real axis, then f' has at most n zeros simultaneously exterior to both of the circles*

$$\left| z \pm ia \cot \left[\pi/(n+1)\right] \right| = a \csc \left[\pi/(n+1)\right].$$

THEOREM (4.2). *If f is chosen as in Theorem* (4.1), *and if all its zeros lie on the disk $|z| \leqq a(a>0)$, then f' has at most n zeros in the domain $|z| > a \csc[\pi/2(n+1)]$.*

Let us note that, if $n=0$, $f(z) \equiv P(z)$ and Theorem (4.2) reduces to Lucas' Theorem.

Secondly, let us consider entire functions with an infinite number of zeros. For T let us make various selections of unbounded domains.

Let us initially take the case that T is a ray emanating from some finite point z_0. For convenience, let us choose $z_0=0$ and T as the positive real axis. Then $\mathcal{S}(T, \pi/(n+1))$ is the sector $|\arg z| \leqq \pi - \pi/(n+1)$. Thus from Theorem (3.1) we deduce

THEOREM (4.3). *If f is an entire function of finite order ρ having only positive real zeros, its derivative has at most $n=[\rho]$ zeros in the sector $|\arg(-z)| < \pi/(n+1)$.*

An example is any function of the form

$$f(z) = e^{Q(z)} z^{-1/2} \sin z^{1/2},$$

where Q is a polynomial of degree $n \geqq 1$.

Theorem (4.3) resembles the following one due to Biernacki [**10**].

Let f be a real entire function of finite genus with only real zeros and let ϕ_0 be an arbitrary number $0<\phi_0<\pi/2$. Then all but a finite number of zeros of f' lie in the double sector

$$|\arg(\pm z)| < \phi_0.$$

However, Theorem (4.3) is somewhat more specific than Biernacki's result and does not require f to be a real function.

Let us next consider the case that T is a convex sector of angular opening not exceeding $\pi/(n+1)$. For convenience, we choose it as $|\arg z| \leqq \alpha < \pi/2(n+1)$, in which case $\mathcal{S}(T, \pi/(n+1))$ is the sector $|\arg z| \leqq \pi - \beta$ where $\alpha+\beta = \pi/(n+1)$. Thus, from (3.1) we deduce

THEOREM (4.4). *If f is an entire function of finite order ρ having all its zeros in the sector $|\arg z| \leqq \alpha < \pi/2(n+1)$ with $n=[\rho]$, then its derivative f' has at most n zeros in the sector $|\arg(-z)| < [\pi/(n+1)] - \alpha$.*

An example is any function of the form

$$f(z) = e^{Q(z)} \prod_{k=-p}^{p} \cos(e^{-k\alpha i/2p} z^{1/2}),$$

where Q is a polynomial of degree $n \geqq 1$.

Let us finally study the case that T is a semi-infinite strip of width $2b$. For convenience let us take the strip to be

$$(4.1) \qquad |\mathcal{I}(z)| \leqq b, \qquad \mathcal{R}(z) \geqq a = b \cot (\pi/(n+1)), \qquad n = [\rho].$$

The strip T subtends an angle of $\pi/(n+1)$ from the points of the circle

$$\Gamma: \quad |z| = (a^2 + b^2)^{1/2} = a \csc \pi/(n+1)$$

exterior to T. The domain $\mathcal{S}(T, \pi/(n+1))$ is the union of T, the closed interior of Γ and the sector

$$|\arg z| \leqq \pi - [\pi/(n+1)].$$

Thus from Theorem (3.1) we conclude the following.

THEOREM (4.5). *If f is an entire function of finite order ρ and if all its zeros lie on the semi-infinite strip T given by* (4.1), *then at most $n = [\rho]$ zeros of f' will lie in the intersection of the sector $|\arg(-z)| < \pi/(n+1)$ with the circular domain $|z| > a \csc \pi/(n+1)$.*

An example is any function of the form

$$f(z) = e^{Q(z)} \prod_{k=-p}^{p} \cos(z - ki - a)^{1/2},$$

where Q is a polynomial of degree $n \geqq 1$ and $b = p$.

5. Application to real entire functions. The following theorem was stated by Laguerre and proved by Borel [11].

Let f be a real entire function of genus p having only a finite number m of nonreal zeros. Then its derivative f' has one zero between each neighboring pair of zeros of f and in addition has at most $p+m$ other zeros, real or nonreal.

Borel's proof is rather lengthy, covering about eight printed pages. We shall give a much simpler proof of a similar theorem in the case $m = 0$, by using Theorem (2.1). Our theorem is the following.

THEOREM (5.1). *Let f be a real entire function of finite order ρ, having only real zeros. Let R be a set formed by choosing exactly one real zero of f' between each pair of neighboring zeros of f. Then at most $n = [\rho]$ real and nonreal zeros of f' do not belong to R.*

To prove Theorem (5.1) we note that the existence of set R is assured by

Rolle's Theorem, but in general there are additional zeros of f' both real and nonreal. Let us assume that it is possible to select from these additional zeros a set $V=\{\zeta_1, \zeta_2, \cdots, \zeta_{n+1}\}$ of $n+1$ distinct zeros of f'. We may separate V into three subsets: V_1 comprised of real ζ_k lying in K, the interval of the real axis containing all the a_j; V_2 comprised of all real $\zeta_k \notin K$, and V_3 comprised of all the nonreal ζ_k. However, any of these subsets might be empty.

Let us consider the factors in the denominators of the terms in (3.1), corresponding to these subsets. By Rolle's Theorem, if $f(z)\neq 0$ for real z, $a_j<z<a_{j+1}$, then an odd number of zeros of f' lie between a_j and a_{j+1}. Since one of these zeros of f' belongs to R, an even number of points $\zeta_k \in V_1$ lie between a_j and a_{j+1}. The corresponding factors in denominators of (3.1) have the form

$$(\zeta_1 - a_\nu)(\zeta_2 - a_\nu) \cdots (\zeta_{2k} - a_\nu) > 0, \qquad \text{for all } \nu.$$

Also any factor $(\zeta_k - a_j)$ with $\zeta_k \in V_2$ is positive for all a_j or negative for all a_j according as $\zeta_k > \max a_j$ or $\zeta_k < \min a_j$ for all j. Finally, since f' is real and thus V_3 consists of conjugate imaginary pairs of ζ_k, the corresponding factors are of the form $(\zeta_k - a_j)(\bar{\zeta}_k - a_j) > 0$. Since therefore all terms in (3.1) have the same sign, the left side of (3.1) cannot vanish. Hence, f' cannot have more than n zeros not in R.

From Theorem (5.1) which we have just proved, we can deduce the following result that does not seem to be derivable from the Laguerre-Borel Theorem.

COROLLARY (5.1). *Let f be a real entire function of order ρ, $0 \leqq \rho < 1$, having only real zeros. Then the derivative f' has one real zero between each pair of neighboring zeros of f and has no other zeros in the complex plane.*

As regards the more general theorem of Laguerre-Borel, we have not yet succeeded in constructing a proof based upon Theorem (2.1), but we expect at a later date to publish a generalization of Theorem (2.1) and upon it to base a simple proof of the general Laguerre-Borel Theorem. [See **12**].

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References

1. M. Marden, Geometry of polynomials, Math. Surveys No. 3, Amer. Math. Soc., 1966.

2. G. Pólya, Über Annäherung durch Polynome mit lauter reellen Wurzeln, Rend. Circ. Mat. Palermo, 36 (1913) 279–295. For a general review of the work in this area, see J. Korevaar, Limits of polynomials whose zeros lie in a given set, Entire Functions and Related Parts of Analysis, Amer. Math. Soc., La Jolla, Cal. 1966.

3. E. Hille, Analytic Function Theory, vol. I, Blaisdell, New York, 1959, pp. 228–229.

4. R. Boas, Entire Functions, Academic Press, 1954, p. 22.

5. See Reference 3, p. 23.

6. See Reference 2.

7. C. Cesàro, Sur les fonctions holomorphes de genre quelconque, C. R. Acad. Sci. Paris, 99 (1884) 26–27.

8. M. B. Porter, On a Theorem of Lucas, Proc. Nat. Acad. Sci. U.S.A., 2 (1916) 247–248, 335–336.

9. M. Marden, On the zeros of rational functions having prescribed poles, etc. Trans. Amer. Math. Soc., 66 (1949) 407–418.

10. M. Biernacki, Sur le déplacement des zéros des fonctions entières par leur dérivation, C. R. Acad. Sci. Paris, 175 (1922) 18–20. Also, M. Alander, Sur les fonctions entières non-réelles, Ark. Mat., Astron. Fysik, 18 (1924) 1–9.

11. E. Borel, Leçons sur les fonctions entières, Gauthier-Villars, Paris, 1900.

12. M. Marden, On the derivative of an entire function, Proc. Amer. Math. Soc., 74 (1968).

ON PAVING THE PLANE

R. B. KERSHNER, Johns Hopkins University

One of the oldest of geometrical problems is the determination of those plane figures which share with the square and regular hexagon the ability to cover the plane, by congruent repetitions, without gaps or overlaps. This problem, called variously the problem of tesselation, plane tiling or plane paving, was brought anew into mathematical prominence by Hilbert [1] in 1900 when he posed it as one of his "Mathematische Probleme."

Particular interest centers on those plane figures which are convex polygons and even for this special case the problem poses ample difficulties. It is readily seen that all triangles and quadrilaterals do pave the plane. It can further be shown, with the help of Euler's equation for the vertices, edges and faces of a polygonal network, that no convex polygon with more than six sides can pave the plane. This has been demonstrated by a number of authors. So the problem reduces to the determination of those convex hexagons and pentagons which can pave the plane.

The author has shown that there are exactly three types of hexagons and eight types of pentagons (of which three are special cases of the three types of hexagons) which can pave the plane.

In order to state the results, let the angles of a hexagon be denoted, consecutively, as A, B, C, D, E, F, and the sides as a, b, c, d, e, f, in such a way that a and b are the sides of A, b and c are the sides of B, etc. Similarly, for a pentagon, let the angles be, consecutively, A, B, C, D, E, and the sides a, b, c, d, e, with a and b the sides of A, etc. Then we can state

THEOREM 1. *A convex hexagon can pave the plane if and only if it is of one of the following three types*:

Hexagon of Type 1: $A+B+C=2\pi$, $a=d$;

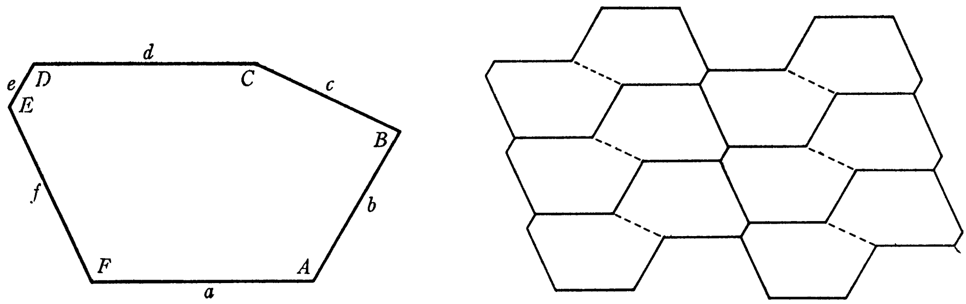
Hexagon of Type 2: $A+B+D=2\pi$, $a=d$, $c=e$;

Hexagon of Type 3: $A=C=E=\frac{2}{3}\pi$, $a=b$, $c=d$, $e=f$.

THEOREM 2. *A convex pentagon can pave the plane if and only if it is one of the following eight types*:

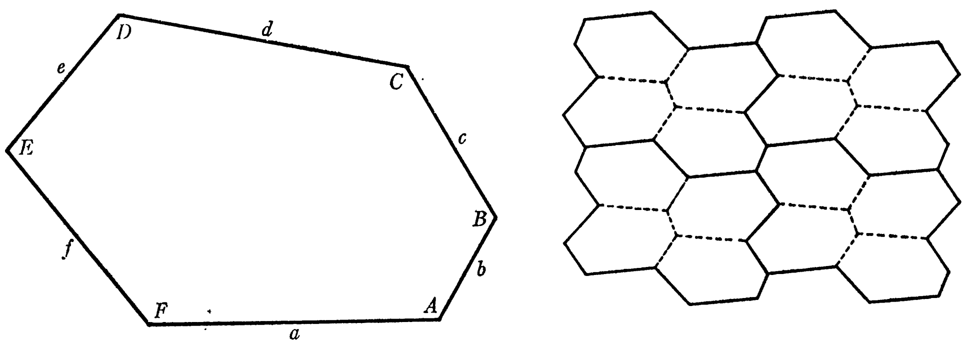
Pentagon of Type 1: $A+B+C=2\pi$;

Pentagon of Type 2: $A+B+D=2\pi$, $a=d$;



(a) Individual hexagon (b) Section of pavement

FIG. 1: Hexagon of Type 1; $A+B+C=2\pi$, $a=d$.



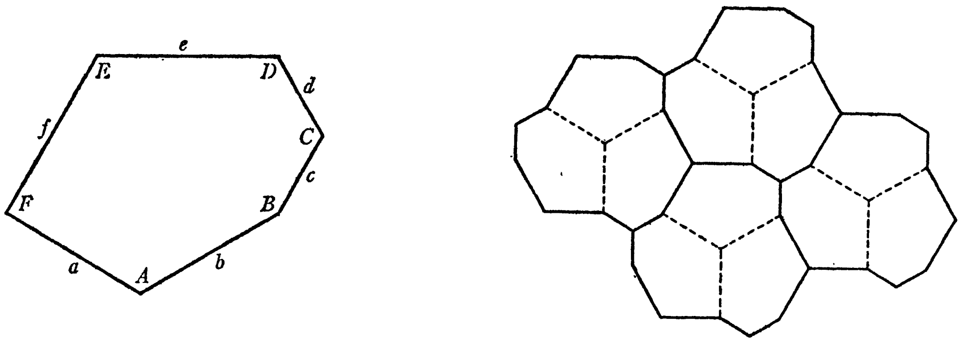
(a) Individual hexagon (b) Section of pavement

FIG. 2: Hexagon of Type 2; $A+B+D=2\pi$, $a=d$, $c=e$.

Pentagon of Type 3: $A=C=D=\frac{2}{3}\pi$, $a=b$, $d=c+e$;
Pentagon of Type 4: $A=C=\frac{1}{2}\pi$, $a=b$, $c=d$;
Pentagon of Type 5: $A=\frac{1}{3}\pi$, $C=\frac{2}{3}\pi$, $a=b$, $c=d$;
Pentagon of Type 6: $A+B+D=2\pi$, $A=2C$, $a=b=e$, $c=d$;
Pentagon of Type 7: $2B+C=2D+A=2\pi$, $a=b=c=d$;
Pentagon of Type 8: $2A+B=2D+C=2\pi$, $a=b=c=d$.

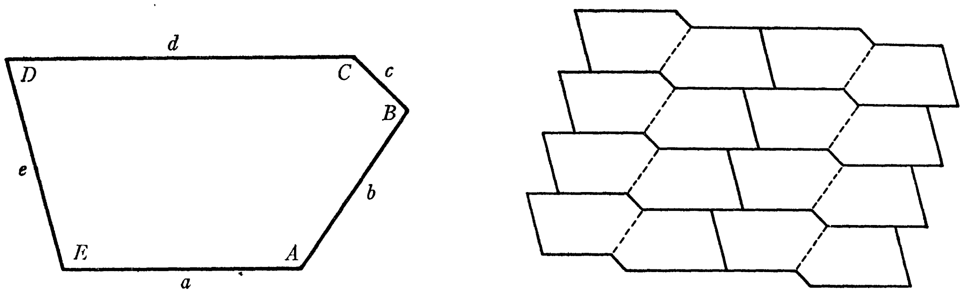
The first three types of pentagons can be considered as special cases of the three types of hexagons and indeed can be converted to hexagons of the desired types by appropriately inserting a vertex along one of the sides. However, the remaining five types of pentagons do not arise as special cases of hexagons which can pave the plane.

The proof that the list in Theorems 1 and 2 is complete is extremely laborious and will be given elsewhere. The fact that these types do pave, however, is quite straightforward and, indeed, is adequately indicated by the accompanying illustrative figures.



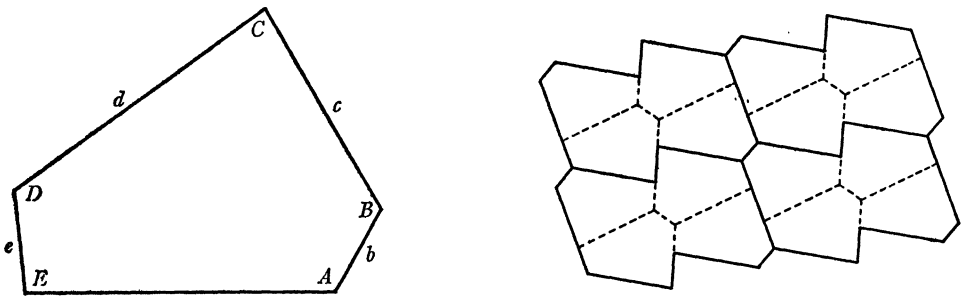
(a) Individual hexagon (b) Section of pavement

FIG. 3: Hexagon of Type 3; $A=C=E=\frac{2}{3}\pi$, $a=b$, $c=d$, $e=f$.



(a) Individual pentagon (b) Section of pavement

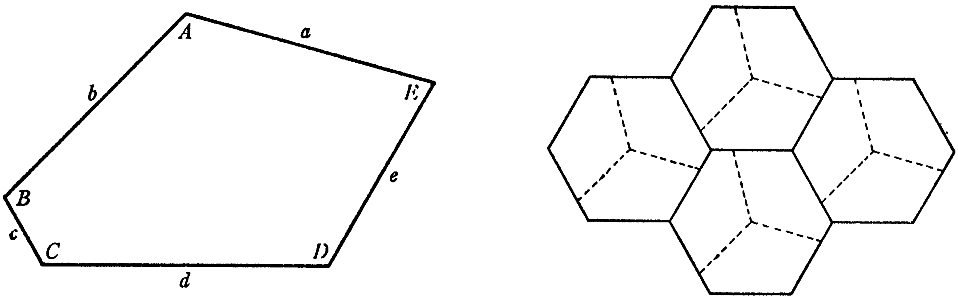
FIG. 4: Pentagon of Type 1; $A+B+C=2\pi$.



(a) Individual pentagon (b) Section of pavement

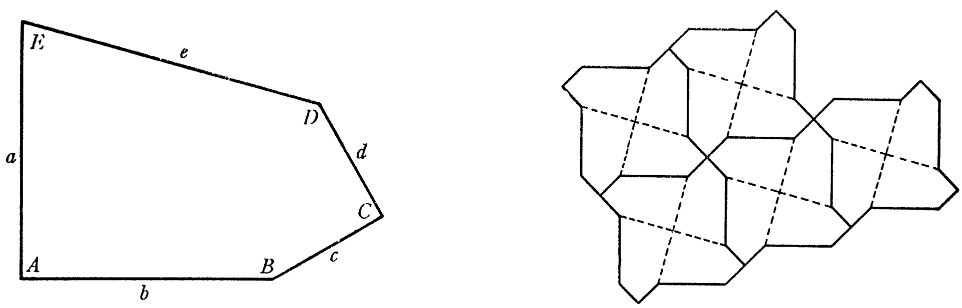
FIG. 5: Pentagon of Type 2; $A+B+D=2\pi$, $a=d$.

It should be noted that the problem of tesselation, not only for convex polygons but for quite general bounded figures, has been treated in detail in a recent book [2]. Indeed, it is stated that the treatment given is complete and that all figures which pave are derived from a general classification scheme developed



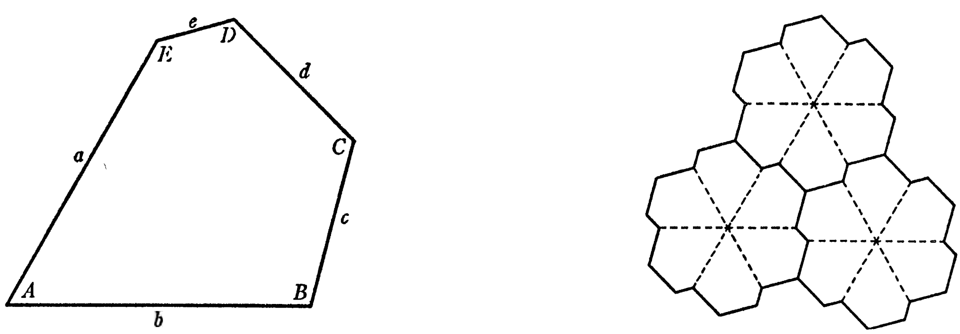
(a) Individual pentagon (b) Section of pavement

FIG. 6: Pentagon of Type 3; $A=C=D=\frac{2}{3}\pi$, $a=b$, $d=c+e$.



(a) Individual pentagon (b) Section of pavement

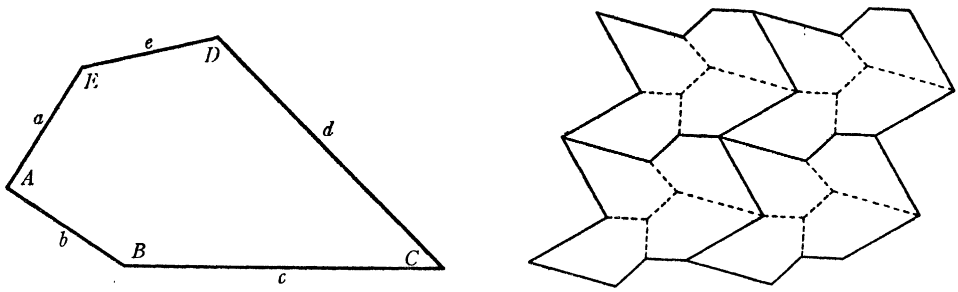
FIG. 7: Pentagon of Type 4; $A=C=\frac{1}{2}\pi$, $a=b$, $c=d$.



(a) Individual pentagon (b) Section of pavement

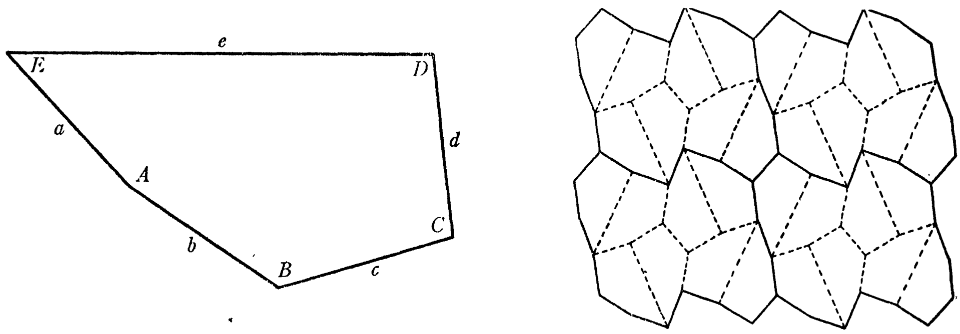
FIG. 8: Pentagon of Type 5; $A=\frac{1}{3}\pi$, $C=\frac{2}{3}\pi$, $a=b$, $c=d$.

by Heinrich Heesch in 1932. Unfortunately this is not the case. When applied to the particular case of convex polygons, the general classification scheme of Heesch yields the three hexagon types of Theorem 1 and the first five pentagon types of Theorem 2 but does not yield the pentagons of Types 6, 7, 8. The three



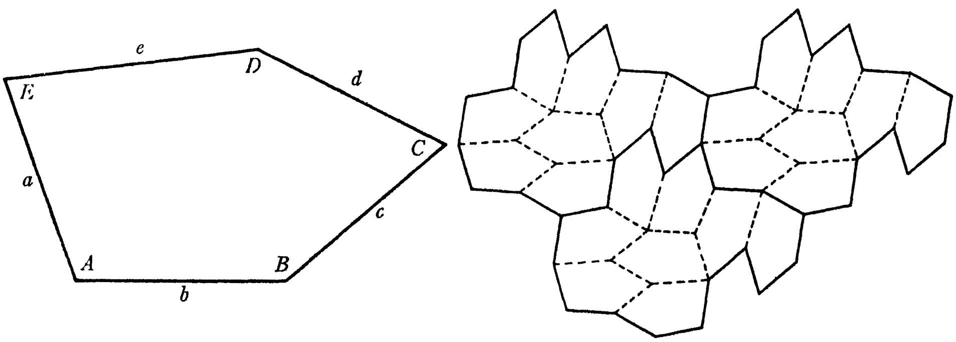
(a) Individual pentagon (b) Section of pavement

FIG. 9: Pentagon of Type 6; $A+B+D=2\pi$, $A=2C$, $a=b=e$, $c=d$.



(a) Individual pentagon (b) Section of pavement

FIG. 10: Pentagon of Type 7; $2B+C=2\pi$, $2D+A=2\pi$, $a=b=c=d$.



(a) Individual pentagon (b) Section of pavement

FIG. 11: Pentagon of Type 8; $2A+B=2\pi$, $2D+C=2\pi$, $a=b=c=d$.

hexagon pavings and the first five pentagon types have also been discovered independently by other authors [**3**, **4**] as well. However, the paving by pentagons of Types 6, 7, 8 appears to be previously unknown.

References

1. D. Hilbert, Pariser Vortrag, "Mathematische Probleme," Problem 18. Nachr. Gesellsch. Wiss. Göttingen, Mathem.-physik. Klasse, 1900.

2. H. Heesch und O. Kienzle, Flächenschluss, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

3. K. Reinhardt, Über die Zerlegung der Ebene in Polygone, Dissertation der Naturwiss. Fakultät, Universität Frankfurt/Main, Borna 1918.

4. B. Bollobas, Filling the plane with congruent convex hexagons without overlapping, Annals Universitatis Scientiarum Budepestinensis, 1963.

5. ———, A sík lefedése egybevágó konvex sokszögekkel, I and II, Különlenyomat Középiskolai Matematikai Lapok, 1961.

SEPARATION OF VARIABLES

OTTO PLAAT, University of San Francisco

1. Introduction. Leibniz's method of separation of variables has had a history of singular neglect. Judging by the literature of the past fifty years, the method has at best been underestimated in respect of its range and power, and at worst has been seriously misunderstood. Considering that tradition has surrounded it with an aura of hesitation and ambiguity, it is not surprising that there is no consensus as to what precisely the method is: to which equations it applies, and what it does. (In lieu of a bibliography, we refer the reader to his three favorite texts.)

Separation of variables involves essentially two indefinite integrations and the inversion of a function. Whether these can be effected in elementary terms—that is, whether the solutions produced by the algorithm are elementary functions—is a question of differential algebra, and concerns us only indirectly. The results in which we are interested refer to the analytic aspect of the algorithm, and particularly to the fact that the method produces solutions in a certain form. The principal purpose of this note is to call attention to Theorem 3, which tells us among other things that whenever the algorithm is effective, i.e., produces elementary functions, the uniqueness or nonuniqueness character of the given differential equation can be deduced from an inspection of the solutions computed. We also append some remarks suggesting uses of the method for obtaining qualitative results in an introductory course on differential equations.

In order to emphasize the elementary and self-contained character of the subject, we shall use no results from the theory of differential equations whatever. It will be necessary to begin by indicating the precise sense in which we employ certain standard notions. The term "S solution" is defined in Section 3.

2. Definitions and Hypotheses. An *interval* will be a connected set in R containing more than one point. We consider the equation

THEOREM 3. (1) *has the uniqueness property if and only if every* S *solution is maximal.*

Proof. Suppose (1) has the uniqueness property. Let u be an S solution. If u is constant, it is obviously maximal. Let u be a nonconstant S solution. We obtain a contradiction by supposing that u has a maximal continuation v. By uniqueness $h(v(t))$ vanishes for no t. It follows from the Lemma that v is itself an S solution, and thus from Theorem 2 that $u=v$. Hence u is maximal.

Conversely, suppose that every S solution is maximal. Let u be a maximal solution. If $h(u(t))$ vanishes for no t, it follows from the Lemma that u is an S solution. We show that if $h(u(t_0))=0$ for some t_0, then u is constant, and therefore also an S solution. If u is not constant, a restriction, v of u, as defined in the Lemma, is an S solution. By hypothesis, v is maximal, so that $u=v$, a contradiction. Hence u is constant. We have shown that every maximal solution is an S solution, and Theorem 2 completes the proof.

4. **Remarks.** The representation $H^{-1}(G(t)+c)$ leads directly to some simple and pedagogically useful results concerning the qualitative behavior of the solutions of (1). The two propositions cited will serve as examples. We suppose that $I_t=I_x=R$.

(A) If $u(t)\equiv x_0$ is an asymptotically stable solution of $dx/dt=h(x)$ and if $G(t)\uparrow\infty$ as $t\rightarrow\infty$, then u is an asymptotically stable solution of (1).

(B) A necessary and sufficient condition that all solutions of (1) are periodic is that G is periodic and every H has range R.

DENSE TOPOLOGIES

NORMAN LEVINE, Ohio State University

1. **Introduction.** We shall call a topology $\mathfrak{I}$ for a set X a *D-topology* (and $(X, \mathfrak{I})$ will be called a *D-space*) whenever every nonempty open set is dense in X.

The main properties of D-spaces and some examples are given in Section 1. In Section 2, D-subspaces of arbitrary topological spaces are studied. Our point of view is shifted in Section 3 where D-subtopologies of arbitrary topologies are probed. We introduce in Section 4 the star subtopology of a given topology and determine its relation to the given topology. Further properties of the star subtopology are developed in Section 5.

Many of the standard spaces in topology are D-spaces. We list here only three.

Example 1. Let $\mathfrak{I}$ be the cofinite topology on an infinite set X.

Example 2. Let x^* be a fixed element in any set X, and let $\mathfrak{I}$ consist of all sets O such that $x^*\in O$ or $O=\emptyset$. Finally

Example 3. Let X be the set of positive integers and let $\mathfrak{I}$ consist of all sets O

such that $O=\emptyset$ or $O=X$ or $O=\{1, \cdots, n\}$ for some n in X.

D-spaces may be characterized as follows:

THEOREM 1. *In a topological space*, $(X, \mathfrak{J})$, *the following are equivalent*: (i) $(X, \mathfrak{J})$ *is a D-space*; (ii) *Every pair of nonempty open sets has a nonempty intersection, and* (iii) *Every open set in X is connected.*

We omit the easy proof. In particular, a D-space is never Hausdorff although it can be a T_1 space.

THEOREM 2. (i) *Every subspace of a D-space is a D-space.* (ii) *The continuous image of a D-space is a D-space.*

Proof of (i): Let $Q=E\cap O$ be a nonempty open set in E; then (using c to denote the closure operator) $c_EQ=E\cap cO=E\cap X=E$, and hence Q is dense in E.

Proof of (ii): If the image is not a D-space, there will exist disjoint nonempty open sets in it by (ii) of Theorem 1. The domain space then will have two nonempty disjoint open subsets and thus not be a D-space.

THEOREM 3. *A product of topological spaces is a D-space iff each of the factor spaces is a D-space.*

Proof. The necessity follows from (ii) of Theorem 2 and the fact that projection maps are continuous and onto. To show the sufficiency, let $(X, \mathfrak{J}) = \times\{(X_\alpha, \mathfrak{J}_\alpha): \alpha\in\Delta\}$ and suppose that O is a nonempty open set in X. Take $x\in O$. There exist then $\alpha_i\in\Delta$ and $U_i\in\mathfrak{J}_{\alpha_i}$ such that $x\in\bigcap\{P_{\alpha_i}^{-1}[U_i]: i=1, \cdots, n\}\subseteq O$. Then

$$\begin{aligned} X &= \bigcap\{P_{\alpha_i}^{-1}[X_{\alpha_i}]: i = 1, \cdots, n\} \\ &= \bigcap\{P_{\alpha_i}^{-1}[c_{\alpha_i}U_i]: i = 1, \cdots, n\} \\ &= c\bigcap\{P_{\alpha_i}^{-1}[U_i]: i = 1, \cdots, n\} \\ &\subseteq cO \subseteq X. \end{aligned}$$

Thus O is dense in X.

There is an invariance principle for certain types of maps into D-spaces. For example

THEOREM 4. *Let* $(X, \mathfrak{J})$ *and* $(Y, \mathfrak{U})$ *be topological spaces and suppose that* $f: X\rightarrow Y$ *(continuity not assumed).*

(i) *If f is one to one and open and if* $(Y, \mathfrak{U})$ *is a D-space, then* $(X, \mathfrak{J})$ *is a D-space.*

(ii) *If f is onto and* $\mathfrak{J}=\{f^{-1}[U]: U\in\mathfrak{U}\}$, *then* $(X, \mathfrak{J})$ *is a D-space iff* $(Y, \mathfrak{U})$ *is a D-space.*

Proof of (i): If $(X, \mathfrak{J})$ is not a D-space, by Theorem 1, there exist nonempty, disjoint, open sets A and B in X. Then $f[A]$ and $f[B]$ are nonempty, disjoint, open sets in Y and $(Y, \mathfrak{U})$ is not a D-space.

Proof of (ii): The necessity follows from (ii) of Theorem 2. To show the sufficiency, suppose $(X, \mathfrak{J})$ is not a D-space. By (ii) of Theorem 1, there exist nonempty, disjoint, open sets A and B in X. Then $\varnothing = A \cap B = f^{-1}[U] \cap f^{-1}[V] = f^{-1}[U \cap V]$ where U and V are open in Y. It follows that U and V are nonempty, disjoint, open sets in Y and thus $(Y, \mathfrak{U})$ is not a D-space.

2. D-subspaces of arbitrary topological spaces.

THEOREM 5. (i) *A space with a dense D-subspace is itself a D-space.* (ii) *If A is a D-subspace of X, then cA is also.*

Proof of (i): Let U and V be nonempty open sets in X and suppose that E is a dense D-subspace of X. Then $E \cap U$ and $E \cap V$ are nonempty open subsets of E. Hence $E \cap U \cap V \neq \varnothing$ by (ii) of Theorem 1 and thus $U \cap V \neq \varnothing$.

Proof of (ii): A is dense in cA.

We now study the D-subspaces of a topological space.

LEMMA 1. *The union of a chain of D-subsets of X is a D-subset of X.*

Proof. Let $\{A_\alpha : \alpha \in \Delta\}$ be a chain of D-subsets of the topological space $(X, \mathfrak{J})$ and suppose $C = \bigcup \{A_\alpha : \alpha \in \Delta\}$. We assert that C is a D-subset of X. If not, there exist sets A and B open in X such that $C \cap A$ and $C \cap B$ are nonempty and disjoint. Take x in $C \cap A$ and y in $C \cap B$. Then $x \in A_\alpha$ and $y \in A_\beta$ for some α, β in Δ. We may assume that $A_\alpha \subseteq A_\beta$. Then $\{x, y\} \subseteq A_\beta$ and it follows that $A_\beta \cap A$ and $A_\beta \cap B$ are nonempty and disjoint. Thus A_β is not a D-subset of X, a contradiction.

THEOREM 6. *Let A be a D-subset of a topological space $(X, \mathfrak{J})$. There exists a maximal D-subset A^* of X which contains A. Furthermore, every maximal D-subset of X is closed.*

Proof: Use Lemma 1 and Zorn's lemma to prove the first assertion. The second assertion follows from (ii) of Theorem 5.

COROLLARY 1. *Every topological space X is the union of its maximal D-subsets.*

Proof: Take x in X. Then $\{x\}$ is a D-subset of X and by Theorem 7, $\{x\}$ is contained in some maximal D-subset of X.

Unlike components in a topological space, maximal D-subsets need not be disjoint. For consider

Example 4. Let $X = \{a, b, c\}$ and $\mathfrak{J} = \{\varnothing, (a), (b), (a, b), X\}$. Then $\{a, c\}$ and $\{b, c\}$ are maximal D-subsets of X as the reader can easily check.

3. D-subtopologies of arbitrary topologies. We now investigate the relationship between a D-subset of a space and certain subtopologies of the given topology. In particular we prove that for each D-subset A of X, there is a maximal D-subtopology of the given topology which induces the given topology on A.

LEMMA 2. *The supremum of a chain of D-topologies for a set X is a D-topology.*

Proof: Let $\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$ be a chain of D-topologies for X and suppose $\mathfrak{J} = \sup\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$. If $\mathfrak{J}$ is not a D-topology, then by (ii) of Theorem 1, there exist disjoint, nonempty, subsets U and V in $\mathfrak{J}$. But $\bigcup\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$ is a base for $\mathfrak{J}$. Thus there exist sets U' and V' in $\bigcup\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$ such that $\emptyset \neq U' \subseteq U$ and $\emptyset \neq V' \subseteq V$. Since $\{\mathfrak{J}_\alpha : \alpha \in \Delta\}$ is a chain of topologies, there exists a β in Δ such that U' and V' are in $\mathfrak{J}_\beta$. Thus U' and V' are nonempty disjoint open sets in $\mathfrak{J}_\beta$ and thus $\mathfrak{J}_\beta$ is not a D-topology.

THEOREM 7. *Let $\mathfrak{U}$ be a D-subtopology of $\mathfrak{J}$. Then $\mathfrak{U}$ is contained in a maximal D-subtopology $\mathfrak{V}$ of $\mathfrak{J}$.*

Proof: Use Zorn's lemma and Lemma 2.

THEOREM 8. *A topology $\mathfrak{J}$ is a D-topology iff $\mathfrak{J}$ contains exactly one maximal D-subtopology.*

Proof: If $\mathfrak{J}$ is a D-topology, then clearly it is the one and only maximal D-subtopology of $\mathfrak{J}$. Conversely, suppose $\mathfrak{J}$ is not a D-topology for the set X. Again using (ii) of Theorem 1, there exist nonempty, disjoint sets O_1 and O_2 in $\mathfrak{J}$. Let $\mathfrak{U}_1 = \{\emptyset, O_1, X\}$ and $\mathfrak{U}_2 = \{\emptyset, O_2, X\}$. Then $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are clearly D-subtopologies of $\mathfrak{J}$ and hence by Theorem 7, there exist $\mathfrak{V}_1$ and $\mathfrak{V}_2$ maximal D-subtopologies of $\mathfrak{J}$ such that $\mathfrak{U}_1 \subseteq \mathfrak{V}_1$ and $\mathfrak{U}_2 \subseteq \mathfrak{V}_2$. Clearly $\mathfrak{V}_1 \neq \mathfrak{V}_2$ since $O_1 \in \mathfrak{V}_1$ but $O_1 \notin \mathfrak{V}_2$.

THEOREM 9. *Let A be a D-set in $(X, \mathfrak{J})$. There exists then a maximal D-subtopology $\mathfrak{V}$ of $\mathfrak{J}$ such that $A \cap \mathfrak{J} = A \cap \mathfrak{V}$.*

Proof: Let $\mathfrak{U} = \{O : O = \emptyset \text{ or } O \in \mathfrak{J} \text{ and } O \cap A \neq \emptyset\}$. Clearly $\mathfrak{U}$ is a D-subtopology of $\mathfrak{J}$. By Theorem 7, there exists a maximal D-subtopology $\mathfrak{V}$ of $\mathfrak{J}$ which contains $\mathfrak{U}$. We assert that $A \cap \mathfrak{J} = A \cap \mathfrak{V}$ or equivalently, that $A \cap \mathfrak{J} \subseteq A \cap \mathfrak{V}$. Let $O \in \mathfrak{J}$. If $A \cap O = \emptyset$, then $A \cap O \in A \cap \mathfrak{V}$. If $A \cap O \neq \emptyset$, then $O \in \mathfrak{U} \subseteq \mathfrak{V}$ and hence $A \cap O \in A \cap \mathfrak{V}$.

4. The star subtopology of a given topology. Each topology $\mathfrak{J}$ on a set X contains a uniquely defined D-subtopology $\mathfrak{J}^*$ (called the star subtopology of $\mathfrak{J}$) where $\mathfrak{J}^* = \{O^* : O^* \in \mathfrak{J} \text{ and } O^* \text{ is dense in } X\}$. We investigate in this section the relation of $\mathfrak{J}^*$ to $\mathfrak{J}$. In general, $\mathfrak{J}^*$ is not a maximal D-subtopology of $\mathfrak{J}$. For consider

Example 5. Let $X = \{a, b\}$ and $\mathfrak{J} = \mathcal{P}(X)$. Then $\mathfrak{J}^* = \{\emptyset, X\}$. It is easy to see that $\mathfrak{J}^*$ is not a maximal D-subtopology of $\mathfrak{J}$.

We will show however that $\mathfrak{J}^*$ is the intersection of all maximal D-subtopologies $\mathfrak{V}$ of $\mathfrak{J}$.

LEMMA 3. *Let $\mathfrak{V}$ be a maximal D-subtopology of $\mathfrak{J}$. Then $\mathfrak{J}^* \subseteq \mathfrak{V}$.*

Proof: Suppose $\mathfrak{J}^* \not\subseteq \mathfrak{V}$; take $O^* \in \mathfrak{J}^* - \mathfrak{V}$. Let $\mathfrak{U} = \{\emptyset, O^*, X\}$ and let $\mathfrak{W} = \sup\{\mathfrak{U}, \mathfrak{V}\}$. It is easy to see that $\mathfrak{W}$ is a D-subtopology of $\mathfrak{J}$ which contains $\mathfrak{V}$ properly, contrary to $\mathfrak{V}$ being a maximal D-subtopology of $\mathfrak{J}$.

THEOREM 10. *The star subtopology $\mathfrak{T}^*$ of $\mathfrak{T}$ is the intersection of all maximal D-subtopologies $\mathfrak{V}$ of $\mathfrak{T}$.*

Proof: By Lemma 3, it suffices to show that if $V \in \mathfrak{V}$ for all maximal D-subtopologies $\mathfrak{V}$ of $\mathfrak{T}$, then $V \in \mathfrak{T}^*$. Suppose that $V \notin \mathfrak{T}^*$. Then there exists a nonempty set U in $\mathfrak{T}$ such that $U \cap V = \emptyset$. Let $\mathfrak{U} = \{\emptyset, U, X\}$. Since $\mathfrak{U}$ is a D-subtopology of $\mathfrak{T}$, by Theorem 7, there exists a maximal D-subtopology $\mathfrak{V}$ of $\mathfrak{T}$ such that $\mathfrak{U} \subseteq \mathfrak{V}$. $V \notin \mathfrak{V}$, a contradiction.

COROLLARY 2. *The star subtopology $\mathfrak{T}^*$ of $\mathfrak{T}$ is a maximal D-subtopology of $\mathfrak{T}$ iff $\mathfrak{T}$ is a D-topology.*

Proof: If $\mathfrak{T}$ is a D-topology, then $\mathfrak{T} = \mathfrak{T}^*$ and hence $\mathfrak{T}^*$ is a maximal D-subtopology of $\mathfrak{T}$. Conversely, if $\mathfrak{T}^*$ is a maximal D-subtopology of $\mathfrak{T}$, then Theorem 10 implies that $\mathfrak{T}^* = \mathfrak{V}$ for all maximal D-subtopologies $\mathfrak{V}$ of $\mathfrak{T}$. By Theorem 8 then, $\mathfrak{T}$ is a D-topology.

5. Further properties of the star subtopology $\mathfrak{T}^*$.

THEOREM 11. *If $\mathfrak{U} \subseteq \mathfrak{T}$ are two topologies for X, then $\mathfrak{T}^* \subseteq \mathfrak{U}^*$ iff $\mathfrak{T}^* \subseteq \mathfrak{U}$.*

Proof: The necessity follows from the fact that $\mathfrak{U}^* \subseteq \mathfrak{U}$. Conversely, let $\mathfrak{T}^* \subseteq \mathfrak{U}$. To show that $\mathfrak{T}^* \subseteq \mathfrak{U}^*$, let O^* be in $\mathfrak{T}^*$ and nonempty. Then $O^* \in \mathfrak{U}$. To show that $O^* \in \mathfrak{U}^*$, let U be in $\mathfrak{U}$ and nonempty. Then $U \in \mathfrak{T}$ and hence $O^* \cap U \neq \emptyset$.

THEOREM 12. *Let $(Y, \mathfrak{U})$ be a subspace of $(X, \mathfrak{T})$. Then* (i) $\mathfrak{U}^* \subseteq Y \cap \mathfrak{T}^*$ *and* (ii) $Y \cap \mathfrak{T}^* \subseteq \mathfrak{U}^*$ *if Y is dense in X.*

Proof of (i): Let $\emptyset \neq U^* \in \mathfrak{U}^*$. Then $U^* \in \mathfrak{U}$ and hence $U^* = Y \cap O$ for some $O \in \mathfrak{T}$. But $U^* = Y \cap (O \cup \mathrm{Int}\ \mathcal{C} Y)$. It suffices to show now that $O \cup \mathrm{Int}\ \mathcal{C} Y \in \mathfrak{T}^*$. Let $\emptyset \neq O' \in \mathfrak{T}$.

Case 1: $O' \cap Y \neq \emptyset$. Then $U^* \cap O' \cap Y \neq \emptyset$ and hence $O \cap O' \neq \emptyset$. Thus $O' \cap (O \cup \mathrm{Int}\ \mathcal{C} Y) \neq \emptyset$.

Case 2: $O' \cap Y = \emptyset$. Then $O' \subseteq \mathrm{Int}\ \mathcal{C} Y$ and hence $O' \cap (O \cup \mathrm{Int}\ \mathcal{C} Y) \neq \emptyset$.

Proof of (ii): Let Y be dense in X. Take $Y \cap O^* \neq \emptyset$ where $O^* \in \mathfrak{T}$. We will show that $Y \cap O^* \in \mathfrak{U}^*$. Let $\emptyset \neq U \in \mathfrak{U}$. Then $U = Y \cap O$ for some $O \in \mathfrak{T}$. But $O^* \cap O \neq \emptyset$ and hence $Y \cap O^* \cap O \neq \emptyset$. Thus $Y \cap O^* \cap U \neq \emptyset$ and $Y \cap O^* \in \mathfrak{U}^*$.

We now lead into the relation between the star subtopology of a product topology and the product of the star subtopologies of the factor topologies.

LEMMA 4. *Let $f\colon X \to Y$ be an open transformation (continuity not assumed), X and Y being topological spaces. If D is dense in Y, then $f^{-1}[D]$ is dense in X.*

Proof: Let O be a nonempty open set in X. It suffices to show that $O \cap f^{-1}[D] \neq \emptyset$. This follows from the fact that $f[O] \cap D \neq \emptyset$.

THEOREM 13. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{U})$ be topological spaces and let $f\colon X \to Y$ be open and continuous. Then f is continuous relative to $\mathfrak{T}^*$ and $\mathfrak{U}^*$.*

Proof: Let U^* be in $\mathfrak{U}^*$ and nonempty. Then U^* is in $\mathfrak{U}$ and is dense in Y relative to $\mathfrak{U}$. It follows then that $f^{-1}[U^*]\in\mathfrak{T}$ since f is continuous and by Lemma 4, $f^{-}[U^*]$ is dense in X since U^* is dense in Y. Thus $f^{-1}[U^*]\in\mathfrak{T}^*$.

THEOREM 14. *Let* $(X, \mathfrak{T}) = \times\{(X_\alpha, \mathfrak{T}_\alpha): \alpha\in\Delta\}$ *where* Δ *is an arbitrary nonempty index set. Let* $(X, \mathfrak{U}) = \times\{(X_\alpha, \mathfrak{T}_\alpha^*): \alpha\in\Delta\}$. *Then* $\mathfrak{U}\subseteq\mathfrak{T}^*$.

Proof: Let $P_\alpha: X\to X_\alpha$ be the projection map for each α in Δ. Now P_α is continuous, open and onto relative to $\mathfrak{T}$ and $\mathfrak{T}_\alpha$. By Theorem 13 then, P_α is continuous relative to $\mathfrak{T}^*$ and $\mathfrak{T}_\alpha^*$ for each α in Δ. But $\mathfrak{U}$ is the smallest topology for X for which P_α is $\mathfrak{T}_\alpha^*$ continuous. Thus $\mathfrak{U}\subseteq\mathfrak{T}^*$.

In general, $\mathfrak{U}\neq\mathfrak{T}^*$ in Theorem 14, for consider

Example 6. Let $X=\{a, b\}$, $Y=\{c, d\}$, $\mathfrak{T}=\mathcal{P}(X)$, $\mathfrak{U}=\{\varnothing, (c), Y\}$. Then $\mathfrak{T}^*=\{\varnothing, X\}$ and $\mathfrak{U}^*=\{\varnothing, (c), Y\}$. Let $(Z, \mathfrak{W})=(X, \mathfrak{T})\times(Y, \mathfrak{U})$ and $(Z, \mathfrak{V}) = (X, \mathfrak{T}^*)\times(Y, \mathfrak{U}^*)$. Then $\{(a, c), (b, c), (a, d)\}\in\mathfrak{W}^*-\mathfrak{V}$.

We will leave the proof of the final theorem to the reader.

THEOREM 15. *Let* $(X, \mathfrak{T})$ *and* $(Y, \mathfrak{U})$ *be topological spaces and suppose that* $f: X\to Y$ *is continuous, open and onto. Then* f *is open relative to* $\mathfrak{T}^*$ *and* $\mathfrak{U}^*$.

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References

1. James Dugundji, Topology, Allyn and Bacon, Boston, Mass., 1966.
2. John L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.

ON CERTAIN DECIDABLE SEMIGROUPS

R. C. BUCK, University of Wisconsin, Madison

A semigroup is a set with an associative multiplication. The standard example is the collection S_X of all mappings of a set X into itself. Every semigroup is a sub-semigroup of S_X for some choice of X. Indeed, given a semigroup S, we can set $X=S\cup\{u\}$ where $u\notin S$, and then associate with $a\in S$ the mapping ϕ_a defined on X by

$$\phi_a(x) = \begin{cases} ax & x \neq u \\ a & x = u. \end{cases}$$

Since $\phi_a\phi_b=\phi_{ab}$, this is a faithful representation of S in S_X.

Clearly, this implies that any finite semigroup can be represented as a semigroup of mappings of a finite set. Suppose that an abstract semigroup S is presented in terms of generators and relations. Can we determine whether or not S is finite? This, of course, is an aspect of the word problem for semigroups, and

the answer depends upon the nature of the imposed relations. The problem is also closely connected with the theory of deductive schemes, and the theory of computers.

The purpose of this note is to examine some elementary cases where it can be shown easily that S is finite and where in fact we can determine the exact order. The interest lies not so much in these particular semigroups, as in their use in testing the power of certain computer algorithms, or in exploring conjectures about the classification of relations. The central result is the following:

THEOREM. *Let r be an integer $\geqq 1$, and let S be the abstract semigroup generated by a and b and the pair of relations*

$$aba = b \tag{1}$$

$$bab = aa \cdots a = a^r. \tag{2}$$

Then, S is finite and has order exactly $5r+3$.

We prove this by describing an algorithm for reducing any word in a and b to one of a list of $5r+3$ words, using (1) and (2) and relations deduced from them. We then show that no further reductions are possible by proving that there is a representation of S in which the list gives rise to $5r+3$ distinct mappings. Except in the special case $r=1$ (when S coincides with the quaternion group) there is no unit. (A generalization of this class is studied in [2].)

Our first step is to derive a number of further relations from the basic relations (1) and (2):

$$b^2 = a^{r+1}. \tag{3}$$

For $b^2 = b(aba) = (bab)a = a^r a = a^{r+1}$. We note that in consequence, b^2 and a^{r+1} commute with any word in S.

$$a^n b a^n = b \qquad \text{for } n = 1, 2, 3, \cdots. \tag{4}$$

From (1) we have $a^2ba^2 = a(aba)b = aba = b$ and induction gives the general case.

$$a^{3r+2} = a^r. \tag{5}$$

From (2), (4) and (3), $a^r = bab = ba(a^rba^r) = ba^{r+1}ba^r = a^{r+1}b^2a^r = a^{r+1}a^{r+1}a^r = a^{3r+2}$.

$$b = ba^{2r+2} = a^{2r+2}b. \tag{6}$$

From (4) and (3), $b = a^{r+1}ba^{r+1} = ba^{r+1}a^{r+1} = ba^{2r+2}$.

$$ba^s = a^tb \qquad \text{for } 0 \leqq s, \quad 0 \leqq t, \quad s + t = 2r + 2. \tag{7}$$

If $s \leqq r+1$, then by (4) and (3),

$$ba^s = abaa^s = \cdots = a^{r+1-s}ba^{r+1-s}a^s = a^{r+1-s}ba^{r+1} = a^{r+1-s}a^{r+1}b = a^tb.$$

A similar argument works if $s \geqq r+1$.

Any word composed of a and b has the form of a sequence of blocks of a and blocks of b. We describe a sequence of steps which make use of the reduction formulae (1)–(7) to replace any word w by an equivalent word w' in a finite list $\mathfrak{L}$.

STEP 1. *Every block of b's can be either eliminated or changed to a single occurrence of b, by introducing additional a's.*

For, by (3) and (4)

$$\begin{aligned} b^{2m} &= (a^{r+1})^m \\ b^{4m+1} &= b^{2m}bb^{2m} = (a^{r+1})^m b (a^{r+1})^m = b \\ b^{4m+3} &= b^2 b^{4m+1} = a^{r+1}b. \end{aligned}$$

At this stage, the general word is replaced by one in which no b block is longer than 1.

STEP 2. *Blocks of a separated by an occurrence of b can be merged by introducing more a's.*

For, again by (4) and (3), any segment of the form $a^n b a^m$ is equivalent either to $a^k b$, or ba^k or b, and b^2 is replaced by a^{r+1}. As an illustration,

$$\begin{aligned} \cdots ba^7ba^{12} \cdots &= \cdots ba^7ba^7a^5 \cdots \\ &= \cdots bba^5 \cdots \\ &= \cdots a^{r+6} \cdots . \end{aligned}$$

At the end of a sequence of Step 2 reductions, the general word has been reduced to one of the following five forms

$$b,\ a^m,\ ba^m,\ a^m b,\ ba^m b. \tag{8}$$

STEP 3. *Using* (5), *we can reduce the exponents m until $m \leqq 3r+1$.*

We note that at this stage, we have shown S to be finite, with order $(S) \leqq 1 + 4(3r+1) = 12r+5$.

STEP 4: *Using* (6), *any exponent m in words of the form ba^m, $a^m b$ or $ba^m b$ can be reduced until $m \leqq 2r+2$.*

STEP 5. *Using* (7) *and* (3), *any remaining word of the form $ba^m b$ can be replaced by a word of the form a^k.*

For, $ba^m b = a^{2r+2-m}bb = a^{3r+3-m}$.

STEP 6. *Using* (7) *any remaining word of the form ba^m or $a^m b$ can be replaced by a word of one of these forms, with $m \leqq r+1$.*

Finally, observing that $ba^{r+1} = a^{r+1}b$, we have shown that any word in a and b may be replaced by one of the following words:

$$\begin{aligned} &b,\ a,\ a^2,\ a^3,\ \cdots,\ a^{3r+1} \\ &ba,\ ba^2,\ \cdots,\ ba^{r+1} \\ &ab,\ a^2b,\ \cdots,\ a^rb. \end{aligned} \tag{9}$$

Thus, the order of S is at most $1+(3r+1)+(r+1)+r = 5r+3$.

To complete the proof, we must show that this list (9) is irreducible, that no use of (1) and (2), or relations derived from them, can show equality among these $5r+3$ words.

The case $r=1$ is special. Here, (9) becomes a list of 8 words. If the multiplication table is constructed, it is easily seen to be that of a group, with a^4 as the unit. The group is readily identified as the so-called quaternion group (see [1], p. 138).

When $r \geqq 2$, S is not a group. To show that the list $\mathcal{L}$ in (9) is irreducible, we create a representation of S as a semigroup of mappings, and then verify that all the maps corresponding to words in the list $\mathcal{L}$ are distinct.

Let $X=\{0, 1, 2, \cdots, 5r+3\}$, and set up the following correspondence between the words in $\mathcal{L}$ and $X-\{0\}$.

$$\begin{array}{lll}
 & b \leftrightarrow 1 & \\
 & a^k \leftrightarrow k+1 & \text{for } k = 1, 2, \cdots, 3r+1 \\
 & ba \leftrightarrow 3r+3 & \\
(10) & ba^k \leftrightarrow 3r+k+2 & \text{for } k = 1, 2, \cdots, r+1 \\
 & ab \leftrightarrow 4r+4 & \\
 & a^kb \leftrightarrow 4r+k+3 & \text{for } k = 1, 2, \cdots, r.
\end{array}$$

If we choose any word w in the list $\mathcal{L}$, then we could define a mapping ϕ_w of X into itself by:

$$\begin{aligned}
\phi_w(0) &= \text{the index number of } w, \\
\phi_w(x) &= \text{the index number of } wA, \quad \text{where } A \text{ is the word indexed by } x.
\end{aligned}$$

If we do this for $w=a$, the resulting map $\phi_w=f$ acts on X as shown below:

$$\begin{tikzcd}
 & & & & r+2 \arrow[r] & r+3 \arrow[r] & \cdots \arrow[r] & 2r \arrow[d] \\
0 \arrow[r] & 2 \arrow[r] & 3 \cdots \arrow[r] & r \arrow[r] & r+1 \arrow[u] & & & 2r+1 \arrow[d] \\
 & & & & 3r+2 \arrow[u] & 3r+1 \arrow[l] & \cdots \arrow[l] & 2r+2 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & 4r+4 \arrow[r] & 4r+5 \arrow[r] & \cdots \arrow[r] & 5r+3 \arrow[d] \\
3r+3 \arrow[u] & 3r+4 \arrow[l] & \cdots \arrow[l] & 4r+2 \arrow[l] & 4r+3. \arrow[l]
\end{tikzcd}$$

If we choose $w=b$, then the map $\phi_w=g$ acts on X as shown below:

$$\begin{tikzcd}
r+1 \arrow[r] & 4r+2 \arrow[d] \\
4r+4 \arrow[u] & 2r+2 \arrow[l]
\end{tikzcd}$$

$$\begin{array}{ccccc}
0 \to & 1 & \to & r+2 & \\
 & \uparrow & & \downarrow & \\
 & 2r+3 & \leftarrow & 4r+3 & \\
k \to & 3r+k+1 & \to & r+k+1 & \\
 & \uparrow & & \downarrow & \quad \text{for } k = 2, 3, \cdots, r \\
 & 2r+k+2 & \leftarrow & 5r+5-k. &
\end{array}$$

Direct examination shows that $fgf=g$ and that $gfg=f^r$, corresponding to (1) and (2). Thus the sub-semigroup of S_X that is generated by f and g is a homomorph of the original semigroup S. To show that this is an isomorphism (and therefore that the order of S is exactly $5r+3$) we must show that each word w in the list $\mathfrak{L}$ yields a distinct map of X. This is done by directly verifying that if w is a word with index number k, and F is the mapping obtained from w by replacing each occurrence of a by f, and b by g, then $F(0)=k$. Since different words have different index numbers, the corresponding mappings F are distinct.

References

1. I. Grossman and W. Magnus, Groups and their graphs, Singer, Syracuse, N. Y., 1964.
2. R. C. Buck, Decidable semigroups, to appear in Bull. Amer. Math. Soc.

THE SPECTRAL THEOREM FOR A NORMAL OPERATOR

ROBERT WHITLEY, University of Maryland and University of California at Irvine

We present a simple proof of the Gelfand-Naimark form of the spectral theorem for a normal operator on a Hilbert space which uses only the Stone-Weierstrass theorem for continuous functions defined on a compact set in the plane and elementary Hilbert space results. In this way this basic result can be presented, towards the end of a one-semester course in Hilbert space, to graduate students whose background in analysis consists only of advanced calculus. This goal was suggested by Berberian's paper [1]. The proof of the spectral theorem for a normal operator is harder than the proof for a self-adjoint, i.e., Hermitian, operator, in fact comments to this effect in [1], [3] and [5] motivated this work. Below we point out the well-known source of this difficulty and have tried to make it clear how one is led to the solution.

An operator will always be a bounded linear operator mapping a complex Hilbert space H into itself and the spectrum of an operator T will be denoted by $\sigma(T)$.

The Gelfand-Naimark version of the spectral theorem for a self-adjoint operator is discussed in [1]. We will sketch the results here to remind the reader of the basic idea and to remind him of those details of the theorem which we will need.

Let A be a self-adjoint. Then its spectrum $\sigma(A)$ is a nonvoid compact set of real numbers. For any real polynomial p, i.e. one with real coefficients, $p(A)$ is a self-adjoint operator with $\|p(A)\| = \sup\{|p(\lambda)| : \lambda \text{ in } \sigma(A)\}$ [6, Theorem 3, page 55]. Then the map ϕ which takes the operator $p(A)$ into the function p defined on $\sigma(A)$ is a norm-preserving map into the space $C(\sigma(A), R)$ of real-valued continuous functions f defined on $\sigma(A)$ with the sup norm $\|f\| = \sup\{|f(\lambda)| : \lambda \text{ in } \sigma(A)\}$. The procedure for extending ϕ to the norm closure $\mathfrak{A}$ of the set of real polynomials in A is standard [1]. Given a sequence $p_n(A)$ of real polynomials in A converging to B in $\mathfrak{A}$, we define $\phi(B) = \lim \phi(p_n(A))$. The existence of the limit follows from the completeness of the normed linear space $C(\sigma(A), R)$ and it is easy to show that it is independent of the choice of polynomials in A which converge to B. This extended mapping ϕ, which we call the Gelfand map, has several obvious properties when restricted to real polynomials in A. For example,

$$\phi(p(A) + q(A)) = \phi(p(A)) + \phi(q(A)) \quad \text{and} \quad \phi(p(A)q(A)) = \phi(p(A))\phi(q(A)),$$

for real polynomials p and q. Taking limits establishes these properties for $\mathfrak{A}$, noting in the process that for B and C in $\mathfrak{A}$, $B+C$ and BC are also in $\mathfrak{A}$. Passing to the limit also yields the formula $\|B\| = \|\phi(B)\|$ for B in $\mathfrak{A}$ and in the same way we see that if S is any operator which commutes with A then S commutes with each member of $\mathfrak{A}$.

Since $\mathfrak{A}$ is a closed real subspace of the complete space of all operators on H [2, Theorem 2, page 101] and ϕ preserves norm, the image $\phi(\mathfrak{A})$ must be closed in $C(\sigma(A), R)$. However, $\phi(\mathfrak{A})$ contains all the polynomials in $C(\sigma(A), R)$ and so must be all of $C(\sigma(A), R)$ by the Stone-Weierstrass theorem [8, Theorem A, page 160]. Thus the map ϕ is onto and for any real-valued continuous function f we define $f(A)$ to be the operator B in $\mathfrak{A}$ with $\phi(B) = f$.

The facts above constitute the Gelfand-Naimark form of the spectral theorem for a self-adjoint operator A. The real closed subalgebra $\mathfrak{A}$ generated by A and I can be mapped onto the space $C(\sigma(A), R)$ in a way which preserves addition, multiplication and norm.

What is the difficulty with a normal operator? To find out we attempt to apply the same procedure, as given above for a self-adjoint operator, to a normal operator T. The first thing we notice is that the spectrum of T, a compact set in the plane, may not be real and so the polynomial $p(z) = z$, which will correspond to T under the Gelfand map, will be complex-valued. Consequently we must take as range space the space $C(\sigma(T), C)$ of complex-valued continuous functions on $\sigma(T)$ with the sup norm. We now need the relation

$$\|T\| = \sup\{|\lambda| : \lambda \text{ in } \sigma(T)\} \tag{1}$$

for T normal, which, incidentally, shows that $\sigma(T)$ is not void. Bernau and Smithies have recently given an amazingly simple proof of (1) and for this we refer the reader to their paper [4] and to Theorem 1 of [3]. Continuing to imitate the method which worked for a self-adjoint operator and using (1) we obtain

a norm-preserving linear correspondence between the closure of the set of complex polynomials in T and a closed subalgebra $\mathcal{B}$ of $C(\sigma(T), C)$ which contains the polynomials. All the subtlety in the distinction between the result for self-adjoint and the result for normal operators occurs now. $\mathcal{B}$ is the closure of the polynomials but from the Stone-Weierstrass theorem for complex-valued functions [8, Theorem B, page 161] we see that before we can conclude that $\mathcal{B}$ is all of $C(\sigma(T), C)$ we need to know that for f in $\mathcal{B}$ the conjugate function $\bar{f}$ is also in $\mathcal{B}$. This may fail even for the simple function $p(z) = z$ in $\mathcal{B}$ [8, page 160]. All is not lost, however; we have seen that what we must do is arrange to have the range of the Gelfand map contain polynomials in λ and $\bar{\lambda}$. We now return to the beginning and try to find such a map.

For a one-dimensional Hilbert space any operator T is multiplication by a scalar α and the adjoint T^* of this operator is multiplication by $\bar{\alpha}$. It is natural to guess that the Gelfand map in this special case will send the operator to its associated scalar; thus T to α and T^* to $\bar{\alpha}$. For a larger Hilbert space we make an analogous guess: For a polynomial $p(x, y)$ in two variables with complex coefficients we send the normal operator $p(T, T^*)$ to the function whose value at λ in $\sigma(T)$ is $p(\lambda, \bar{\lambda})$. In order that the correspondence so defined be norm-preserving, we need to know that $\|p(T, T^*)\| = \sup\{|p(\lambda, \bar{\lambda})| : \lambda \text{ in } \sigma(T)\}$. Using relation (1) it will certainly suffice to establish

$$\sigma(p(T, T^*)) = \{p(\lambda, \bar{\lambda}) : \lambda \text{ in } \sigma(T)\}. \tag{2}$$

There is an elementary proof of (2) in [3], but the proof is complicated and computational. The purpose of our Lemma 1 and Lemma 2 is to present a simpler conceptual proof of (2). The alert reader will notice that the source of the main idea in the proof of Lemma 2 is a remark on page 484 of [3].

LEMMA 1. *Let T be a normal operator whose spectrum contains zero. Given $\varepsilon > 0$ there is a closed nonzero subspace M with the property that any operator which commutes with T^*T is reduced by M and that the operator T has restriction to M, $T|_M$, with $\|T|_M\| \leqq \varepsilon$.*

Proof. Let $A = T^*T$. Since 0 is in $\sigma(T)$ there are vectors x_n of norm one with $Tx_n \to 0$ [6, Theorem 2, page 51]. Thus $Ax_n \to 0$ and the self-adjoint operator A has 0 in its spectrum.

Given $\varepsilon > 0$ we consider the continuous function f, a truncated pyramid, defined for all real t by

$$f(t) = \begin{cases} 1 & \text{for } |t| \leqq \varepsilon/2 \\ 2(1 - |t/\varepsilon|) & \text{for } \varepsilon/2 \leqq |t| \leqq \varepsilon \\ 0 & \text{for } |t| \geqq \varepsilon. \end{cases}$$

We use the Gelfand-Naimark theorem for a self-adjoint operator to define the self-adjoint operator $f(A)$.

Let $M = \{x : f(A)x = x\}$, a closed subspace. As we have noted, if B is an

operator which commutes with A then B commutes with $f(A)$. Consequently, for any x in M, $Bx=Bf(A)x=f(A)Bx$, which shows that M is invariant under B. Since B^* also commutes with A, M is invariant under B^* and so reduces B.

Using the norm-preserving property of the Gelfand map, we see that for any x of norm one in M, $\|Ax\|=\|Af(A)x\|\leqq\|Af(A)\|=\sup\{|\lambda f(\lambda)|:\lambda \text{ in } \sigma(A)\}\leqq\mathcal{E}$. Thus $\|Tx\|^2=(Ax|x)\leqq\mathcal{E}$ and so $\|T|_M\|\leqq(\mathcal{E})^{1/2}$.

It remains to show that M is not (0). We compute

$$\|(I-f(A))f(2A)\| = \sup\{\,|\,(1-f(\lambda))(f(2\lambda))\,|\,:\lambda \text{ in } \sigma(A)\} = 0,$$

since $f(\lambda)$ is one whenever $f(2\lambda)$ is not zero. Hence every element in the range of the operator $f(2A)$ lies in M and this range is not zero because $\|f(2A)\|=\sup\{|f(2\lambda)|:\lambda \text{ in } \sigma(A)\}\geqq|f(0)|=1$.

LEMMA 2. *Let $p(x, y)$ be a polynomial in two variables and T a normal operator. Then $\sigma(p(T, T^*)=\{p(\lambda,\bar\lambda):\lambda \text{ in } \sigma(T)\}$.*

Proof. We write $p(x, y)=\sum a_{nm}x^ny^m$. Let λ be in $\sigma(T)$. Then there are vectors x_j of norm one with $(\lambda-T)x_j\to 0$. By the normality of T, $(\bar\lambda-T^*)x_j\to 0$ [6, Theorem 1, page 42]. Then

$$\begin{aligned}(p(T,T^*)-p(\lambda,\bar\lambda))x_j &= \sum_{n,m} a_{nm}(T^nT^{*m}-\lambda^n\bar\lambda^m)x_j\\ &= \sum_{n,m} a_{nm}(T^n(T^{*m}-\bar\lambda^m)x_j+\bar\lambda^m(T^n-\lambda^n)x_j)\\ &= \sum_{n,m} a_{nm}(T^n(T^{*m-1}+\cdots+\bar\lambda^{m-1})(T^*-\bar\lambda)x_j\\ &\quad+\bar\lambda^m(T^{n-1}+\cdots+\lambda^{n-1})(T-\lambda)x_j)\end{aligned}$$

which converges to zero as j tends to infinity. We conclude that $p(\lambda,\bar\lambda)$ is in $\sigma(p(T, T^*))$.

Now choose μ in $\sigma(p(T, T^*))$. The operator $B=p(T, T^*)-\mu I$ is normal and has 0 in its spectrum. By Lemma 1, for each n there is a closed nonzero subspace M_n which reduces B with $\|B|_{M_n}\|\leqq 1/n$, and, since T commutes with $B*B$, M_n reduces T. Thus $T|_{M_n}$ is normal [2, Corollary 1, page 159]. From relation (1) we know that $\sigma(T|_{M_n})$ is nonvoid and so we choose λ_n in $\sigma(T|_{M_n})$. For this λ_n there is a vector y_n of norm one in M_n with $\|(\lambda_n-T)y_n\|\leqq 1/n$. The sequence $\{\lambda_n\}$ is bounded by $\|T\|$ and so contains a subsequence converging to λ; by re-indexing we may suppose that the sequence $\{\lambda_n\}$ converges to λ.

The point λ is in $\sigma(T)$, since $\|(\lambda-T)y_n\|\leqq|\lambda_n-\lambda|+\|(\lambda_n-T)y_n\|$ and so $(\lambda-T)y_n\to 0$. As we saw in the beginning of this proof, $(\lambda-T)y_n\to 0$ implies that $\{(p(T, T^*)-p(\lambda,\bar\lambda))y_n\}$ must also converge to zero. The vector y_n is in the subspace M_n and so $\{(p(T, T^*)-\mu)y_n\}$ also converges to 0. It follows that $\mu=p(\lambda,\bar\lambda)$ which completes the proof.

We have now laid a firm foundation for the spectral theorem. The Gelfand map is properly defined on polynomials in T and T^*, the extension of the map

to the closure of these polynomials presents no difficulties and the rest follows as in the case of a self-adjoint operator. We omit the details and state the result.

Gelfand-Naimark Theorem for a Normal Operator. Let T be a normal operator on H. Let $\mathcal{A}$ be the closure in norm of polynomials $p(T, T^*)$ in T and T^*. Then $\mathcal{A}$ is a closed subspace of the space of all operators on H which consists of normal operators and contains the sum, product and adjoint of operators in $\mathcal{A}$. ($\mathcal{A}$ is sometimes called the closed *-subalgebra generated by T, T^* and I.) The Gelfand map ϕ maps $\mathcal{A}$ onto the space $C(\sigma(T), C)$ of continuous complex-valued functions on $\sigma(T)$ with the sup norm and has the properties:

(a) The map ϕ is linear and $\phi(AB)=\phi(A)\phi(B)$ for A and B in $\mathcal{A}$,

(b) $\|\phi(A)\|=\|A\|$, and

(c) $\phi(A^*)=\overline{\phi(A)}$ for A in $\mathcal{A}$.

(d) For a polynomial $p(x, y)$ in two variables,

$$\phi(p(T, T^*))(\lambda) = p(\lambda, \bar{\lambda}) \qquad \text{for } \lambda \text{ in } \sigma(T).$$

(e) If the operator S commutes with T and T^*, then S commutes with each operator in $\mathcal{A}$.

The last statement (e) can be improved by using the Fuglede commutativity theorem: *If S commutes with a normal operator T, then S commutes with T^*.* An interesting discussion of this result and of a particularly elegant version of the spectral theorem is found in [5]. The reader should also look at the beautiful proof of Fuglede's theorem given in [7].

We close with an extension of Lemma 2.

Spectral Mapping Theorem for a Continuous Function of a Normal Operator:

Let T be a normal operator and f a complex-valued function which is continuous on $\sigma(T)$. Then $\sigma(f(T))=f(\sigma(T))$.

Proof. The operator $f(T)$ is, of course, the operator in $\mathcal{A}$ which the Gelfand map takes into f.

Suppose that μ is in $\sigma(T)$ with $f(\mu)=\lambda$. Choose polynomials $p_n(T, T^*)$ converging to $f(T)$. Then $p_n(\mu, \bar{\mu})I-p_n(T, T^*)$ converges to $f(\lambda)I-f(T)$. By Lemma 2, $p_n(\mu, \bar{\mu})I-p_n(T, T^*)$ is not invertible and so [6, Theorem 1, page 52] $f(\lambda)I-f(T)$ is not invertible. We have shown that $f(\sigma(T))\subseteq\sigma(f(T))$.

Take λ not in $f(\sigma(T))$. Then $\lambda-f$ is never 0 on $\sigma(T)$ and so the function $g=1/(\lambda-f)$ is continuous there. Using the properties of the Gelfand map, $\phi(g(T)(\lambda I-f(T)))=\phi((\lambda I-f(T))g(T))=1$. Since ϕ also takes the identity I into 1 we must have $g(T)(\lambda I-f(T))=(\lambda I-f(T))g(T)=I$. Hence $\lambda I-f(T)$ is invertible and λ cannot be in $\sigma(f(T))$.

References

1. S. Berberian, The spectral mapping theorem for a Hermitian operator, this MONTHLY, 70 (1963) 1049–1051.

2. ———, Introduction to Hilbert Space, Oxford University Press, New York, 1961

3. S. Bernau, The spectral theorem for normal operators, J. London Math. Soc., 40 (1965) 478–486.

4. Bernau and F. Smithies, A note on normal operators, Proc. Camb. Phil. Soc., 59 (1963) 727–729.
5. P. Halmos, What does the spectral theorem say?, this MONTHLY, 70 (1963) 241–247.
6. ———, Introduction to Hilbert Space and the Theory of Spectral Multiplicity, Chelsea, New York, 1951.
7. M. Rosenblum, On a theorem of Fuglede and Putnam, J. London Math. Soc., 33 (1958) 376–377.
8. G. Simmons, Introduction to Topology and Modern Analysis, McGraw-Hill, New York, 1963.

MATHEMATICAL NOTES

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

DOMINATING SET AND CONVERSE DOMINATING SET OF A DIRECTED GRAPH

YUMIN FU, Institute of Technology, Southern Methodist University

Let G be a directed graph with the vertex set $V(G)$ and the edge set $E(G)$. We say that G is finite if both $V(G)$ and $E(G)$ are finite. The notation $e(a, b)$ will be used to represent an edge directed from vertex a to vertex b and we say that a dominates b and b is dominated by a. Also, we call a the initial vertex of edge e and b the end vertex of e. A subset D of V is a dominating set for G when every vertex not in D is the end vertex of some edge from a vertex in D. Similarly, a subset D' of G is a converse dominating set for G when every vertex not in D' is the initial vertex of some edge toward a vertex in D'. A minimal dominating set is a dominating set such that no subset has this property. Problems related to dominating sets of nondirected graphs appear in various puzzle questions [1], [2]. O. Ore [3] has given an intensive study of dominating sets in his book. One of his important theorems is introduced here for reference:

THEOREM 1. *When G is locally finite, i.e. when each vertex of G is associated with a finite number of edges, any dominating set contains a minimal one.*

He also proved several theorems about dominating set in undirected graphs. These, however, cannot be applied or even extended to directed graphs. He proposed the study of dominating sets in directed graph as a research problem.

The following theorem gives a sufficient condition to find a minimal dominating set:

THEOREM 2. *A dominating set D_0 is a minimal dominating set if for each $d_0 \in D_0$ there is no edge $e(d_1, d_0)$ such that $d_1 \in D_0$.*

Proof. Since D_0 is a dominating set, every vertex in its complement $\overline{D}_0$ is an end vertex of some edge which has an initial vertex in D_0. By assumption that

in the complement of D_0-d_0, there is a vertex d_0 which is not an end vertex of any edge initiated from a vertex in $D-d_0$, hence D_0-d_0 is not a dominating set. Therefore, D_0 must be a minimal dominating set.

This condition is certainly not necessary. It can be seen from the following examples.

Example 1: Let G be a directed path $P(a, b)$ containing vertices in the following order: $a=v_0, v_1, \cdots, v_n=b$. When n is odd, the minimal dominating set is $D=\{v_0, v_2, \cdots, v_{n-1}\}$. When n is even, there are $(1+\frac{1}{2}n)$ minimal dominating sets. Each of them contains $1+\frac{1}{2}n$ vertices and they are:

$$\{v_0, v_2, \cdots, v_{n-4}, v_{n-2}, v_n\},$$
$$\{v_0, v_2, \cdots, v_{n-4}, v_{n-2}, v_{n-1}\}, \cdots,$$
$$\{v_0, v_1, \cdots, v_{n-5}, v_{n-3}, v_{n-1}\}.$$

In the first case, the complement $\overline{D}$ is a converse dominating set; while in the second case, no complement of the $1+\frac{1}{2}n$ minimal dominating sets is a converse dominating set.

Example 2: Suppose G is a directed circuit with vertices $v_1, \cdots, v_n$ in order. When n is even, both the set $\{v_1, v_3, \cdots, v_{n-1}\}$ and its complement, $\{v_2, v_4, \cdots, v_n\}$ are dominating sets and also converse dominating sets. When n is odd, the minimal dominating set is not unique. They always contain $(n+1)/2$ vertices and their complements are neither dominating sets nor converse dominating sets.

We next consider the relationship between a dominating set and its complement. We shall be interested in the case when its complement is also a dominating set. The next theorems follow directly from definitions.

THEOREM 3. *In order that a directed graph G have a dominating set D such that its complement $\overline{D}$ is also a dominating set, it is necessary and sufficient that each vertex in D is dominated by a vertex in $\overline{D}$ and each vertex in $\overline{D}$ is dominated by a vertex in D.*

THEOREM 4. *In order that a directed graph G have a dominating set D such that its complement $\overline{D}$ is a converse dominating set, it is necessary and sufficient that each vertex in D dominates at least one vertex in $\overline{D}$.*

Now we turn to cyclic graphs. A directed graph G is cyclic if every two vertices in G are contained in a directed circuit. Subsequently, G is acyclic if no two vertices in G are contained in a directed circuit. Some necessary and sufficient conditions for the existence of dominating sets of cyclic graphs have been found.

THEOREM 5. *In order that a finite cyclic graph G have a dominating set D such that its complement $\overline{D}$ is also a dominating set, it is necessary and sufficient that G contains a directed circuit of even length, i.e., a directed circuit containing even number of edges.*

Proof: For the necessity part, we assume that G has a dominating set D and its complement $\overline{D}$ is also a dominating set. Without loss of generality, we start to investigate G from an arbitrary vertex d_0 in D_0. By the assumption that d_0 is dominated by vertices in $\overline{D}$, there is at least an edge $e(\bar{d}_0, d_0)$ from a vertex $\bar{d}_0$ in $\overline{D}$ to d_0. Mark this edge and vertices d_0, $\bar{d}_0$ with a color pencil. Similarly, $\bar{d}_0$ must be dominated by some vertices, say d_1, in D and there is an edge $e(d_1, \bar{d}_0)$ from the vertex d_1 to $\bar{d}_0$. Again mark this edge and vertices with a color pencil. Continue this process, we shall obtain a growing colored directed path. Since the graph is finite, the growing path must return to some vertex which we have colored. Stop at this vertex, we then have a directed circuit of colored edges and vertices. With successive vertices alternately belonging to D and $\overline{D}$, this circuit is therefore of even length.

To prove the sufficiency, we need only to show that there is a way to assign all the vertices of G to either D or $\overline{D}$ provided that G contains a directed circuit of even length and that both D and $\overline{D}$ are dominating sets. Let us start with a directed circuit C_0 of even length and assign its vertices alternately to D and $\overline{D}$. Thus each vertex in this circuit is in a dominating set and is also dominated by some vertex in the complement of this dominating set. If C_0 contains all the vertices of G, then the theorem is proved. Otherwise, there must be some unassigned vertices. Since the graph is cyclic, there is at least one vertex b dominated by some vertex a of C_0. Starting from the edge $e(a, b)$, we may find a directed path ending at some vertex which has already been assigned to either D or $\overline{D}$. Assign the vertices of this path alternately to D and $\overline{D}$ by starting from b with a and b belonging to different sets. This path may end up with two consecutive vertices being assigned to the same set, but each vertex, thus far assigned, is always in a dominating set and is also dominated by at least one vertex in the set which is the complement of this dominating set. The rest of the unsigned vertices will be signed by the same way, and the theorem is thus proved.

The following two corollaries will be stated without proof since they clearly follow proofs similar to that of the theorem.

COROLLARY 1. *In order that a cyclic graph G have a dominating set D such that its complement $\overline{D}$ is also a dominating set, furthermore that both D and $\overline{D}$ are converse dominating set, it is necessary and sufficient that each vertex in G is in some circuit of even length.*

COROLLARY 2. *In order that a cyclic graph G have a dominating set D such that its complement is a converse dominating set, it is sufficient that G contains a circuit of even length.*

References

1. F. Scheid, Some packing problems, this MONTHLY, 67 (1960) 231–235.
2. W. Rouse-Ball, Mathematical Recreations and Problems, London 1892, Macmillan, London, 1944.
3. O. Ore, Theory of Graphs, Amer. Math. Soc. Coll. Publ., 38 (1962).

A NOTE ON QUADRATIC EUCLIDEAN DOMAINS

P. J. Arpaia, C. W. Post College

To begin with, by an integral domain we shall always mean a commutative ring *with identity* such that $ab=0$ implies $a=0$ or $b=0$. By a subdomain of an integral domain we will mean a subset of an integral domain that forms an integral domain under the given operations—hence *must possess an identity* element. By a Euclidean domain we mean an integral domain that is Euclidean. We now pose our question. Do there exist Euclidean domains having subdomains that are not Euclidean? We propose to answer this question in the affirmative.

It is a well-known fact that the integers under addition and multiplication form a Euclidean domain. Furthermore, there are many subrings of the integers that are *not* Euclidean. For example $2Z=\{n\in Z\colon n=2k,\ k\in Z\}$. We have $36=2\cdot 18=6\cdot 6$. Since factorization into irreducible elements in a Euclidean ring must be unique, $2Z$ is not Euclidean. However, $2Z$ does not contain an identity element hence is not an integral domain. Furthermore, since Z is a cyclic group under addition and since 1 generates Z, Z can possess no *subdomain* that is not Euclidean. Hence we must look elsewhere for Euclidean domains possessing subdomains that are not Euclidean.

In [1], LeVeque states that $R[\sqrt{d}]$ is a quadratic Euclidean domain if and only if d has one of the 21 values:

$$-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57 \quad \text{or} \quad 73.$$

We will now show that each of these domains possesses a subdomain that is not Euclidean.

Before we proceed recall that

$$N(a+b\sqrt{d}) = a^2 - db^2 \quad \text{and} \quad N((a+b\sqrt{d})(e+f\sqrt{d})) = N(a+b\sqrt{d})N(e+f\sqrt{d}).$$

We will need the following lemmas:

Lemma 1. *Let $R[\sqrt{d}]$ be any quadratic domain and define D by $D=\{a+b\sqrt{d}\in R[\sqrt{d}]\colon a, b\in Z, b\equiv 0 \pmod{p}, p$ prime$\}$. Then D is a subdomain of $R[\sqrt{d}]$.*

Proof. For $a+b\sqrt{d}$, $e+f\sqrt{d}\in D$ we have $(a+b\sqrt{d})+(e+f\sqrt{d})\in D$. Now

$$(a+b\sqrt{d})(e+f\sqrt{d}) = (ae+dbf)+(af+be)\sqrt{d}.$$

Hence $(a+b\sqrt{d})(e+f\sqrt{d})\in D$. Therefore D is a subdomain of $R[\sqrt{d}]$. Further, $1=1+0\sqrt{d}\in D$.

Lemma 2. *Let D be defined as in Lemma 1. If $a+b\sqrt{d}\in R[\sqrt{d}]$ and $N(a+b\sqrt{d})=p$, p a prime; then $a+b\sqrt{d}\notin D$.*

Proof. Suppose $a+b\sqrt{d}\in D$. Now $a^2-db^2=p$. Since $b\equiv 0 \pmod{p}$, $b^2=p^2n^2$. Hence $a^2=p(1+dpn^2)$. But then $a^2\equiv 0 \pmod{p}$. Hence $a\equiv 0 \pmod{p}$. Therefore

$a^2=p^2m^2$. But then $p^2(m^2-dn^2)=p$. This implies that p divides 1. Hence $a+b\sqrt{d}\notin D$.

LEMMA 3. *If $p=a^2-db^2$, then p^3 has the two distinct factorizations ppp and $(pa+pb\sqrt{d})(pa-pb\sqrt{d})$ in D.*

Proof. Since $N(p)=p^2$, p is a prime in D by Lemma 2. Furthermore, since $N(pa+pb\sqrt{d})=N(pa-pb\sqrt{d})=p^3$, each of $pa+pb\sqrt{d}$ and $pa-pb\sqrt{d}$ is prime in D by Lemma 2. Finally, since $pa\pm pb\sqrt{d}=p(a\pm b\sqrt{d})$; p is not an associate of $pa+pb\sqrt{d}$ or of $pa-pb\sqrt{d}$.

THEOREM. *Each of the Quadratic Euclidean domains $R[\sqrt{d}]$ possesses a subdomain that is not Euclidean.*

Proof. In view of Lemma 3, all we need show is that for each of the 21 values of d stated in the first paragraph, there exists a solution to $a^2-db^2=p$, p a prime. The following table suffices:

d	$p=a^2-db^2$	d	$p=a^2-db^2$
-11	$47=36-(-11)(1)$	13	$3=16-(13)(1)$
-7	$11=4-(-7)(1)$	17	$-13=4-(17)(1)$
-3	$7=4-(-3)(1)$	19	$-3=16-(19)(1)$
-2	$3=1-(-2)(1)$	21	$-17=4-(21)(1)$
-1	$2=1-(-1)(1)$	29	$-13=16-(29)(1)$
2	$-7=1-(2)(4)$	33	$-29=4-(33)(1)$
3	$-2=1-(3)(1)$	37	$107=144-(37)(1)$
5	$11=16-(5)(1)$	41	$-37=4-(41)(1)$
6	$-5=1-(6)(1)$	57	$-53=4-(57)(1)$
7	$-3=4-(7)(1)$	73	$-37=36-(73)(1)$
11	$-7=4-(11)(1)$		

Reference

1. W. J. LeVeque, Topics in Number Theory, Vol. 2, Addison-Wesley, Reading, Mass., 1956.

A NOTE ON ORIENTATION OF GRAPHS

A. LEMPEL, Technion—Israel Institute of Technology

The graphs considered in this note are finite, connected, with no self-loops and with at most one arc between any pair of vertices. It is shown that every planar graph with the above properties, may be oriented (or reoriented) in such a way that the number of arcs directed away from any vertex of the graph is not greater than 3.

A graph is denoted by G or the pair (X, A), where X and A denote the vertex and arc sets of G, respectively.

Let $v=|X|$, $\ell=|A|$ and let $X=\{x_1, x_2, \cdots, x_v\}$. For a directed graph G, the *outward degree* $d_0(x_i)$ of $x_i\in X$ is the number of arcs directed away from x_i. The *mean* outward degree $\bar{d}_0$ of the graph is defined as

$$\bar{d}_0 = \frac{1}{v}\sum_{i=1}^{v} d_0(x_i). \tag{1}$$

Lemma. *Let $G=(X, A)$ be a directed planar graph. If there is a vertex $x_i \in X$ with $d_0(x_i) \geqq 3$, then there exists in G a directed path from x_i to a vertex $x_j \in X$ with $d_0(x_j) \leqq 2$.*

Proof. It is well known that for a planar graph (see [1], p. 78)

$$\ell \leqq 3(v-2). \tag{2}$$

Since

$$\ell = \sum_{i=1}^{v} d_0(x_i) = \bar{d}_0 v \tag{3}$$

we have $\bar{d}_0 \leqq 3-(6/v)$ or

$$\bar{d}_0 < 3. \tag{4}$$

Therefore, if there is a vertex $x_i \in X$ with $d_0(x_i) \geqq 3$, then there must be at least one vertex $x_j \in X$ with $d_0(x_j) \leqq 2$. Suppose there is a vertex x_i with $d_0(x_i) \geqq 3$. Let $Y_i(\subset X)$ be the set of all vertices to which there exists a directed path from x_i. If $Y_i = X$, we are through. Suppose Y_i is a proper subset of X. Then, every arc connecting a vertex $x_k \in Y_i$ with a vertex $x_m \in (X-Y_i)$ is directed toward x_k. The set of all such arcs forms a directed cut-set $A_i \subset A$ separating the subgraph G_i corresponding to Y_i from the vertices belonging to $X-Y_i$. Delete A_i and consider the subgraph G_i. Since G_i is planar and the outward degree of every vertex in G_i is the same as in G, Y_i must contain a vertex x_j with $d_0(x_j) \leqq 2$.

Theorem 1. *Every planar graph $G=(X, A)$ may be oriented (or reoriented) in such a way that for each $x_i \in X$ there is $d_0(x_i) \leqq 3$.*

Proof. Assign an arbitrary orientation to each arc of G. If there is any vertex $x_i \in X$ with $d_0(x_i) > 3$, there exists, by the lemma, a directed path from x_i to a vertex $x_j \in X$ with $d_0(x_j) \leqq 2$.

Reverse the orientation of every arc on this path. The only vertices whose outward degree is changed thereby, are x_i and x_j. $d_0(x_i)$ is decreased by 1 and $d_0(x_j)$ is still not greater than 3.

Repeated application of the above procedure will result in an orientation of G with $d_0(x_k) \leqq 3$ for each $x_k \in X$.

This theorem may be generalized as follows:

Let $\mathcal{G} = \{G_k = (X_k, A_k)\}$ be the set of all subgraphs of a given graph $G=(X, A)$. Let $|X_k| = v_k$ and let $d_k(x_i)$ denote the number of arcs incident at x_i in G_k. The mean degree δ_k of a subgraph $G_k \in \mathcal{G}$ is defined as

$$\delta_k = (1/v_k) \sum_{x_i \in X_k} d_k(x_i).$$

Theorem 1*. *Every graph $G=(X, A)$ may be oriented (or reoriented) in such a way that for each vertex $x_i \in X$ there is $d_0(x_i) \leqq [(\delta_{\max}+1)/2]$, where $\delta_{\max} = \max_k \{\delta_k\}$.*

The proof of this theorem proceeds along the same lines as that of Theorem 1, with the argument of planarity replaced by the definition of $\delta_{\max}$.

It is also to be noted that the least upper bound on the possible outward degree, as given by Theorem 1*, cannot be improved (consider the complete 4-gon).

Reference

1. R. G. Busacker and T. L. Saaty, Finite Graphs and Networks: An Introduction with Applications, McGraw-Hill, New York, 1965.

A NOTE ON SECOND ORDER BOUNDARY VALUE PROBLEMS

K. W. SCHRADER, University of Missouri, Columbia

1. Introduction. Let $P(y)=\sum_{i=0}^{n} a_i y^i$, $n\geqq 0$, $a_n\neq 0$, each a_i a real constant. We shall be concerned with the boundary value problem

$$(1)\qquad \begin{aligned} y'' &= P(y) \\ y(a) &= \alpha,\ y(b) = \beta, \end{aligned}$$

where $a<b$.

THEOREM. *Boundary value problem* (1) *is solvable for all choices of* a, b, α *and* β *if and only if one of the following conditions holds*:

(i) $n=0$,

(ii) n *is odd and* a_n *is positive*,

(iii) n *is odd*, $n>1$ *and* a_n *is negative*.

Notice that this theorem yields an interesting corollary.

COROLLARY. *If* n *is odd and* a_n *is negative then boundary value problem* (1) *is solvable for all choices of* a, b, α *and* β *if and only if equation* (1) *is not linear*.

The author wishes to point out that no attempt has been made to attain complete generality and that existence proofs for wider classes of boundary value problems with $P(y)$ replaced by a continuous function $f(x, y, y')$ satisfying certain other conditions are given in [1] and [2]. It is difficult, however, to find results in the literature of a negative nature for nonlinear equations and the discussion in case (c) indicates how one can establish the nonsolvability of certain boundary value problems by using the general existence theorems in [2] and some known facts about simple differential equations.

2. Proof. We need only to consider the five cases:

(a) $n=0$ or n is odd and $a_n>0$,

(b) $n=1$ and $a_1<0$,

(c) n is even and $n\geqq 2$ with $a_n>0$,

(d) n is even and $n\geqq 2$ with $a_n<0$,

(e) n is odd and $n\geqq 3$ with $a_n<0$.

(a) In this case it follows from Corollary 3.2 of [2] that boundary value problem (1) is always solvable.

(b) It is well known that boundary value problem (1) is solvable for all values of α and β if and only if $\sqrt{-a_1}(b-a)\neq k\pi$ where k is an integer. Moreover, if $\sqrt{-a_1}(b-a)=k\pi$ where k is an integer then a necessary and sufficient condition that (1) be solvable is that $\alpha=\beta$ if k is even and that $\alpha+\beta=-(2a_0)/a_1$ if k is odd.

(c) We will show that for every interval $[a, b]$, boundary value problem (1) is not solvable for some values of α and β.

Let K be the minimum value of $P(y)$. For λ chosen so that $\lambda=8\pi/(b-a)$ we may choose $N_0>0$ large enough so that $|N_1|>N_0$ implies $P(N_1)>\lambda^2|N_1|$. Denote by $y(x)$ the solution of the boundary value problem

$$y'' = K - 1, \qquad y(a) = y(b) = M \tag{2}$$

and note that

$$y(x) = \frac{(K-1)}{2}\left(x - \frac{a+b}{2}\right)^2 + M - \frac{(K-1)}{8}(b-a)^2.$$

Choose M such that $y(x)\leqq -N_0$ for $x\in[a, b]$. Now assume that there exists a solution $y_0(x)$ to boundary value problem (1) with $\alpha=\beta=M$. The function $y_0(x)$ satisfies $y_0''(x)=P(y_0(x))\geqq K$ and hence $y_0(x)\leqq y(x)$ for $x\in[a, b]$, for otherwise the function $y_0(x)-y(x)$ would have an interior maximum on $[a, b]$ at which its second derivative was positive. From this we see that

$$|y_0(x)| \geqq |y(x)| \geqq N_0$$

and hence that

$$P(y_0(x)) > \lambda^2 |y_0(x)| \geqq -\lambda^2 y_0(x).$$

It follows that $\phi(x)\equiv y_0(x)$ and $\psi(x)\equiv 0$ satisfy

$$\phi''(x) = P(\phi(x)) \geqq -\lambda^2\phi(x) \quad \text{and} \quad \psi''(x) = 0 \leqq -\lambda^2\psi(x).$$

Consider the boundary value problem

$$\begin{aligned} y'' &= -\lambda^2 y \\ y(a) &= \phi(a),\ y(b) = \phi(b) \end{aligned} \tag{3}$$

and note that this has a solution $z(x)$ with $\phi(x)\leqq z(x)\leqq\psi(x)$ by Corollary 3.1 of [2]. This would imply the equation

$$y'' = -\lambda^2 y \tag{4}$$

has a solution that is not identically zero and is never positive on $[a, b]$. This cannot be since all solutions are of the form

$$A \sin\left(\frac{8\pi}{b-a}x + \delta\right)$$

and have period $(b-a)/4$. We conclude that the solution $y_0(x)$ cannot exist.

(d) Exactly the same as (c)—replace y by $-y$.

(e) It follows as a special case of *Existenzsatz* 1 [1, p. 173] that boundary value problem (1) is always solvable and in fact always has infinitely many solutions.

This proves the theorem.

References

1. H. Ehrmann, Über die Existenz der Lösungen von Randwertaufgaben bei gewöhnlichen nichtlinearen Differentialgleichungen zweiter Ordnung, Math. Ann., 134 (1957) 167–194.

2. K. Schrader, Boundary Value Problems for Second-Order Ordinary Differential Equations, J. Diff. Equations, 3 (1967) 403–413.

ON CONTINUOUS IMAGES OF CANTOR SPACES

W. B. Johnson, Iowa State University

By noting that a Hausdorff space can be represented as a space of dyadic functions, we offer a new, simple proof of a theorem in [1, p. 166]. The following result of Taimanov [2] is used:

Theorem. *Let X be dense in Z. Then a necessary and sufficient condition that a continuous function, f, from X into a compact T_2 space Y have a continuous extension, $\bar{f}: Z \to Y$, is that for each two disjoint closed sets K and L in Y, $\mathrm{Cl}_Z f^{-1}(K)$ and $\mathrm{Cl}_Z f^{-1}(L)$ be disjoint.*

Proof: Necessity is obvious.

Sufficiency: For each $z \in Z$, let A_z be the open neighborhood system of z. Let $K_z = \{\mathrm{Cl}_Y f(0): 0 \in A_z\}$. Since A_z is closed under finite intersections and since X is dense in Z, K_z has the finite intersection property. Since Y is compact, $\cap K_z \neq \emptyset$. We show that $\cap K_z$ is singleton.

Suppose $t \in \cap K_z$ and U is a (not necessarily open) neighborhood of t. Then for each $0 \in A_z$, $f(0) \cap U \neq \emptyset$, and hence $0 \cap f^{-1}(U) \neq \emptyset$. Thus for any neighborhood U of t,

$$z \in \mathrm{Cl}_Z f^{-1}(U). \tag{1}$$

Now suppose t and s are distinct points of $\cap K_z$. By using the T_3 property of the compact T_2 space Y, we can find disjoint closed neighborhoods C and D of t and s, respectively. By (1), $z \in \mathrm{Cl}_Z f^{-1}(C) \cap \mathrm{Cl}_Z f^{-1}(D)$, which contradicts the hypothesis. Thus $\cap K_z$ is singleton, for each $z \in Z$.

Let $\bar{f}: Z \to Y$ be defined by $\bar{f}(z)$ is the unique element of K_z. Then $\bar{f}$ is an extension of f to all of Z.

We show that $\bar{f}$ is continuous. Suppose $\{S_\gamma\}$ is a net in Z which converges

to $z\in Z$. If $\{\bar{f}(S_\gamma)\}$ does not converge to $\bar{f}(z)$, we can find an open neighborhood U of $\bar{f}(z)$ and a subnet $\{S_{k(\gamma)}\}$ of $\{S_\gamma\}$ such that $\{S_{k(\gamma)}\}$ is in $Y-U$. Let $\bar{f}(z)\in C \subset V$, where C is a closed neighborhood of $\bar{f}(z)$ and V is an open neighborhood of $\bar{f}(z)$ such that $\mathrm{Cl}_Y V\subset U$. Then C and $Y-V$ are disjoint closed sets. Further,

$$\{\bar{f}(S_{k(\gamma)})\} \tag{2}$$

is in the interior of $Y-V$.

By (1), $z\in\mathrm{Cl}_Z f^{-1}(C)$. By (1) and (2), for each $k(\gamma)$, $S_{k(\gamma)}$ is in $\mathrm{Cl}_Z f^{-1}(Y-V)$. Since $\{S_{k(\gamma)}\}$ converges to z, $z\in\mathrm{Cl}_Z f^{-1}(Y-V)$. That is, $z\in\mathrm{Cl}_Z\, f^{-1}(C)\cap \mathrm{Cl}_Z f^{-1}(Y-V)$, which contradicts the hypothesis. Hence, f is continuous and the theorem is established.

REMARK: Taimanov actually assumed that Z is a T_1 space. Our proof shows that this restriction was not necessary.

DEFINITION. Suppose (X, T) is a topological space each of whose elements is a function with domain D, $S=\{S_\gamma: \gamma\in Q, >\}$ is a net in X, and $d\in D$. The statement that "S eventually has property P in the dth coordinate" means that "there is $q\in Q$ such that if $\gamma>q$, then $S_\gamma(d)$ has property P."

THEOREM. *Every compact T_2 space (X, T) is the continuous image of a closed subset of a Cantor space.*

Proof: Let $\{0_d: d\in D\}$ be an open base for (X, T). For each $x\in X$, define $f_x: D\to\{0, 1\}$ by.

$$f_x(d) = \begin{cases} 1, & \text{if } x\in 0_d \\ 0, & \text{if } x\notin 0_d. \end{cases}$$

Since X is T_2, distinct points in X correspond to distinct dyadic functions with domain D, and X can be naturally identified with a subset X' of $\{0, 1\}^D$. Let T' be the topology on X' induced by T under the natural correspondence $x\to f_x$. Then (X, T) and (X', T') are homeomorphic. Let (C, β) be the Cantor space $\{0, 1\}^D$ with the product topology. A net S in X' T'-converges to $f_x\in X'$ if and only if $f_x(d)=1$ implies S is eventually 1 in the dth coordinate. A net S in C β-converges to $f\in C$ if and only if for each $d\in D$, S is eventually $f(d)$ in the dth coordinate. Consequently, $\beta|X'$ convergence is stronger than T' convergence, so the identity function $I: (X', \beta|X')\to(X', T')$ is continuous. Let K and L be disjoint closed sets in (X', T'). We must show that $\mathrm{Cl}_{(C,\beta)}K\cap\mathrm{Cl}_{(C,\beta)}L=\emptyset$. Suppose S and U are nets in K and L, respectively, which β-converge to $f\in C$. Since (X', T') is compact, S and U have T'-convergent subnets. For each $d\in D$, S and U are eventually equal to $f(d)$ in the dth coordinate, so S and U are themselves T' convergent and T'-converge to the same point, f_x, of X'. K and L are closed in (X', T'), so $f_x\in K\cap L$, and a contradiction is reached. Hence $\mathrm{Cl}_{(C,\beta)}K \cap\mathrm{Cl}_{(C,\beta)}L=\emptyset$, and by Taimanov's Theorem, I has a continuous extension to $\mathrm{Cl}_{(C,\beta)}X'$.

In case (X, T) is a metric space, a countable base $\{0_i: i=1, 2, \cdots\}$ may be chosen. Since every closed subset of the Cantor set $\{0, 1\}^{\aleph_0}$ is a retract of the space, we have as a special case a theorem of P. Alexandroff and Urysohn: "Every compact metric space is the continuous image of the Cantor set."

The author wishes to thank James Cornette and the referee, who made useful suggestions concerning this paper.

References

1. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1964.
2. A. D. Taimanov, On extensions of continuous mappings of topological spaces, (Russian) Mat. Sb., 31 (1952) 459–463.

A PROBLEM IN MATHEMATICS AND MUSIC

JOHN ROGERS, University of New Hampshire, AND BARRY MITCHELL, Bowdoin College

Consider a finite sequence $a_1, a_2, \cdots, a_m$ in an abelian group A. For each integer k satisfying $1 \leqq k \leqq m-1$, we shall call the m expressions $a_{i+k}-a_i$ (subscripts mod m) the kth *order differences* of the given sequence. Thus the kth order differences are the negatives of the $m-k$th order differences. Notice that for each k, the kth order differences must sum to 0. We shall call the sequence a *perfect m-sequence* in A if for each k, the kth order differences are distinct and nonzero. Their being nonzero is equivalent to the terms of the original sequence being distinct. We shall say that the sequence is *near perfect* if its terms are distinct, and if the first order differences are distinct. Given an abelian group A, one can ask for which m there exist perfect (or near perfect) m-sequences. Of particular interest is the case where $A=Z_n$ (integers mod n), and in fact this question was suggested by some pieces of Stravinsky which are based on certain 6-sequences in Z_{12} (see [1], [2], [6], and [7]). There are compositions by John Rogers based on perfect 5 and 6-sequences entitled "Rotational Arrays for Woodwind Quintet" and "Symmetries for Instrumental Ensemble" respectively.

PROPOSITION 1. *Let A be Z_n. If n is even and $\geqq 4$, then there are no near perfect $n-1$ sequences. If n is odd and $\geqq 5$, then there are no near perfect $n-2$ sequences. If n is even and $\geqq 6$, then there are no perfect $n-2$ sequences.*

Proof: In a near perfect $n-1$ sequence in Z_n, the first order differences are precisely the numbers $1, 2, \cdots, n-1$. The sum is $(n-1)n/2$ which is congruent to $n/2$ mod n if n is even. Hence, if n is even, there are no near perfect $n-1$ sequences.

If n is odd, then $(n-1)n/2$ is 0 mod n. Hence the sum of all but one of the numbers $1, 2, \cdots, n-1$ is not 0 mod n. Consequently there can be no near perfect $n-2$ sequences if n is odd.

Now suppose that n is even and $\geqq 6$, and that $a_1, a_2, \cdots, a_{n-2}$ is perfect. Since each set of differences must consist of $n-2$ distinct nonzero members of Z_n, and since they must all sum to 0, they must all omit $n/2$. But then we see

that the numbers $a_1, a_2, \cdots, a_{n-2}$ are distinct mod $n/2$, which is impossible since $n-2>n/2$ if $n\geqq 6$.

PROPOSITION 2. *Let R be a ring with identity, and let d be a generator for a multiplicative cyclic group of order m in R. Denoting $d^m=e$ (which need not be the identity of the ring), suppose that $e-d$ has a left inverse in R. Then considering R as an additive group, there is a near perfect m-sequence in R whose k-th order differences are distinct for each k such that $e-d^k$ has a left inverse in R.*

Proof: We have $(e-d)\ (e+d+d^2+\cdots+d^{m-1})=0$. Since $e-d$ has a left inverse, this implies that the sum is 0. Therefore we can find a sequence whose first order differences are $e, d, d^2, \cdots, d^{m-1}$ in that order. The k-th order differences are given by

$$d^i + d^{i+1} + \cdots + d^{i+k-1}, \qquad 0 \leqq i \leqq m-1.$$

Multiplying these differences on the left by $e-d$, we obtain

$$(e - d^k)d^i. \tag{1}$$

Multiplying (1) by d^{m-i}, we obtain $e-d^k$. Consequently no kth order difference is 0 for $1\leqq k\leqq m-1$, and therefore we have constructed a near perfect m-sequence. Furthermore it follows from (1) that the kth order differences are distinct for each k such that $e-d^k$ has a left inverse in R.

COROLLARY 1. *For each odd prime p, there are perfect $p-1$ sequences in Z_p. Furthermore for $t\geqq 1$, there are near perfect $p^{t-1}(p-1)$ sequences in Z_n, where $n=p^t$.*

Proof: It is well known that for odd primes p and $t\geqq 1$, the group of units in Z_n is cyclic. (See [4, page 53].) Let d be a generator. Under the natural map $Z_n\to Z_p$, the units are mapped onto the units, and it follows that the image of d is a generator for the group of units in Z_p. From this we can conclude that $1-d$ is invertible in Z_n, and consequently our results follow from proposition 2.

COROLLARY 2. *If F is a finite field with n elements, then there are perfect $n-1$ sequences in F.*

Proof: The nonzero elements of a finite field constitute a cyclic group under multiplication.

Question: Are the only perfect $n-1$ sequences in F of the type found above?

Proposition 1 shows that in Z_{12} there are no perfect 11 or 10-sequences. The IBM 1620 found no perfect 9 or 8 sequences. The first perfect 7-sequence (of which there are plenty) is 0, 1, 3, 7, 2, 5, 4.

The situation regarding $n-1$ sequences in Z_n when n is odd and composite is so far a mystery. The first such n is 9, and the computer found no perfect 8-sequences. However, there are lots of near perfect 8-sequences, for example 0, 1, 3, 2, 7, 4, 8, 6. The next such n is 15, which was too large for the computer to test for perfect 14-sequences. However, it was very quick in finding a near perfect 14-sequence, namely 0, 1, 3, 2, 5, 9, 4, 10, 7, 14, 12, 8, 13, 6.

As for perfect $n-3$ sequences in Z_n, we can only say that there are such for $n=9$, but not $n=10$, 11, 12. Someone with a larger computer may wish to check this for $n=13$, 14, and possibly 15, but we conjecture that for $n>9$, there are no perfect $n-3$ sequences in Z_n.

References

1. M. Babbitt, The use of computers in musicological research, Perspectives of New Music, Spring-Summer (1965), 77–78.

2. ———, Remarks on Recent Stravinsky, Perspectives of New Music, Spring-Summer (1964), 53–54.

3. E. Krenek, New developments of the 12-tone technique, Music Review, May, 1943.

4. W. J. LeVeque, Topics in Number Theory, Vol. I, Addison Wesley, Reading, Mass., 1956.

5. J. Rogers, Some properties of rotational arrays, To appear.

6. C. Spies, Notes on Stravinsky's "Abraham and Isaac." Perspectives of New Music, Spring-Summer (1965), 116–120.

7. ———, Stravinsky's "Variations." Perspectives of New Music, Fall–Winter (1965), 62–74.

FURTHER UNIQUENESS THEOREMS FOR n-TH ORDER DIFFERENTIAL EQUATIONS

J. B. GARNER, Louisiana Polytechnic Institute

Let $S=[a, b]\times R^n$ and $(\alpha_1, \alpha_2, \beta_1, \cdots, \beta_n)\in[a, b]\times[a, b]\times R^n$, where R is the set of real numbers and $\alpha_1\leqq\alpha_2$. Let $Y=(y, y^{(1)}, \cdots, y^{(n-1)})$ where $y^{(i)}=d^iy/dx^i$. The purpose of this note is to obtain sufficient conditions for $f(x, Y)$ in S for the uniqueness of solutions of

$$y^{(n)} = f(x, Y) \tag{1}$$

satisfying either

$$y^{(k)}(\alpha_p) = \beta_{k+1}, y^{(n-1)}(\alpha_q) = \beta_n \tag{2}$$

or

$$y^{(k)}(\alpha_p) = \beta_{k+1}, y^{(n-2)}(\alpha_q) = \beta_n, \alpha_1 \neq \alpha_2, \tag{3}$$

where $k=0, 1, \cdots, n-2$, where p and q are fixed integers, $1\leqq p, q\leqq 2$, and where $\alpha_1=\alpha_2$ if $n=1$. Dual statements are made throughout the paper by the conventional use of parentheses.

The basic uniqueness theorems for the initial value problem (1), (2), where $\alpha_1=\alpha_2$ require a Lipschitz condition on $f(x, Y)$. This condition is replaced here by monotoneity conditions. Often it suffices to have uniqueness of the initial value problem on the right or uniqueness on the left (e.g., see [2]). In this note a condition is imposed on $f(x, Y)$ for $x\in[a, \alpha_1]$ $(x\in[\alpha_2, b])$ which assures uniqueness to the left (right). The results of this note generalize the corresponding results of [1]. Also, by letting $n=1$, the Peano Uniqueness Theorem is obtained.

THEOREM 1. *Let $f(x, Y)$ be continuous in S. If n is even (odd) let $f(x, Y)$ be nonincreasing (nondecreasing) in $y^{(i)}$, i even, and nondecreasing (nonincreasing) in $y^{(i)}$, i odd, for each fixed $x\in[a, \alpha_1]$ and for $i\in I=\{0, 1, \cdots, n-1\}$. Let $f(x, Y)$ be nondecreasing in each $y^{(i)}$ for each fixed $x\in[\alpha_1, \alpha_2]$ and nonincreasing in each $y^{(i)}$ for each fixed $x\in[\alpha_2, b]$, $i\in I$. Then there is at most one solution of* (1), (2) *or* (1), (3) *when $p=1$ and $q=2$.*

Proof. Let $p=1$, $q=2$, and assume $y_1(x)$ and $y_2(x)$ are two solutions of (1), (2) ((1), (3)) such that $y_2(c)>y_1(c)$ for some $c\in[a, b]$. Assume $\alpha_1\neq\alpha_2$ and $c\in[\alpha_1, \alpha_2]$. From (2) ((3)) and, in particular, from the fact that $(y_2-y_1)^{(n-2)}$ vanishes at $x=\alpha_1$ and its derivative vanishes at $x=\alpha_2((y_2-y_1)^{(n-2)}$ vanishes at $x=\alpha_1$ and $x=\alpha_2)$ and from the existence of $(y_2-y_1)^{(n-1)}$, it follows that there exist t_1, t_2 of $[\alpha_1, \alpha_2]$ such that

$$(4)\qquad (y_2(t_2)-y_1(t_2))^{(n-1)}=0,\ (y_2(x)-y_1(x))^{(i)}>0$$

for $x\in(t_1, t_2)$ and $i\in I$. From (1)

$$y^{(n-1)}(x)=y^{(n-1)}(d)+\int_d^x f(u, Y(u))du$$

and thus

$$(5)\quad (y_2(x)-y_1(x))^{(n-1)}=(y_2(d)-y_1(d))^{(n-1)}+\int_d^x[f(u, Y_2)-f(u, Y_1)]du.$$

As a consequence of (4) and the hypotheses, the left member of (5) is positive and the right member is nonpositive for $d=t_2$ and $x\in(t_1, t_2)$. Hence $c\notin[\alpha_1, \alpha_2]$.

Assume $a\neq\alpha_1$ and $c\in[a, \alpha_1]$. From (2) if $\alpha_1=\alpha_2$ and from the preceding argument if $\alpha_1\neq\alpha_2$, $(y_2(\alpha_1)-y_1(\alpha_1))^{(i)}=0$ for $i\in I$. Now there exists $r_2\leqq\alpha_1$ such that $(y_2-y_1)^{(i)}=0$, $x\in[r_2, \alpha_1]$ and $y_2-y_1>0$ on $(r_2-\delta, r_2)$ for some $\delta>0$. Thus $(y_2-y_1)^{(1)}<0$ to the immediate left of $x=r_2$. This, in turn, implies that $(y_2-y_1)^{(2)}>0$ on $(r_2-\delta_2, r_2)$ for some $\delta_2>0$. Continuing in this manner and making use of the continuity of $(y_2-y_1)^{(i)}$, we have for some r_1, r_2 of $[a, \alpha_1]$

$$(y_2(r_2)-y_1(r_2))^{(i)}=0,\qquad (-1)^i(y_2(x)-y_1(x))^{(i)}>0$$

for $x\in(r_1, r_2)$ and $i\in I$. This and the hypotheses imply that the left member of (5) is negative (positive) while the right member is nonnegative (nonpositive) for n even (odd) when $d=r_2$ and $x\in(r_1, r_2)$. Hence $c\notin[a, \alpha_1]$.

Now $(y_2(\alpha_2)-y_1(\alpha_2))^{(i)}=0$ for $i\in I$. If $\alpha_2\neq b$ and $c\in[\alpha_2, b]$ then there exist s_1, s_2 such that

$$(y_2(s_1)-y_1(s_1))^{(i)}=0,\qquad (y_2(x)-y_1(x))^{(i)}>0$$

for $x\in(s_1, s_2)$ and $i\in I$. However, this and the hypotheses imply that the left member of (5) is positive and the right member is nonpositive for $d=s_1$ and $x\in(s_1, s_2)$. Thus $c\notin[\alpha_2, b]$ and the theorem follows.

THEOREM 2. *Let* $f(x, y, y^{(1)})$ *satisfy the conditions of Theorem* 1 *when* $n=2$ *and* $\alpha_1=\alpha_2$, *and when S is replaced by*

$$D = \{(x, y, y^{(1)}) \mid x \in [a, b], y \leqq \beta_1 + \beta_2(x - \alpha_1), \mid y^{(1)} \mid < \infty\},$$

and let $f(x, y, y^{(1)}) \leqq 0$ *in D. Then there exists at most one solution of* (1), (2) *where* $\alpha_1=\alpha_2$.

Proof. If $f(x, y, y^{(1)}) \leqq 0$ in D then $y^{(2)} \leqq 0$ there. Thus any solution $y(x)$ of (1), (2), with $\alpha_1=\alpha_2$, has the property

$$y(x) \leqq \beta_1 + \beta_2(x - \alpha_1), \qquad x \in [a, b].$$

The proof now follows by Theorem 1.

THEOREM 3. *Suppose the conditions of Theorem* 1 *hold except that for each fixed* $x \in [\alpha_1, \alpha_2]$ *and for* $i \in I$, $f(x, Y)$ *is nondecreasing* (*nonincreasing*) *in* $y^{(i)}$, *i even, and nonincreasing* (*nondecreasing*) *in* $y^{(i)}$, *i odd, for n even* (*odd*). *Then there is at most one solution of* (1), (2) *or* (1), (3) *when* $p=2$ *and* $q=1$.

Proof. Let $p=2$, $q=1$, and assume two solutions $y_1(x)$ and $y_2(x)$ as in Theorem 1. Assume $c \in [\alpha_1, \alpha_2]$. From (2) ((3)) and, in particular, from the fact that $(y_2-y_1)^{(n-2)}$ vanishes at $x=\alpha_2$ and its derivative vanishes at $x=\alpha_1$ ($(y_2-y_1)^{(n-2)}$ vanishes at $x=\alpha_2$ and $x=\alpha_1$), it follows that there exist t_1, t_2 of $[\alpha_1, \alpha_2]$ such that

$$(y_2(t_1) - y_1(t_1))^{(n-1)} = 0, \; (-1)^i(y_2(x) - y_1(x))^{(i)} > 0$$

for $x \in (t_1, t_2)$ and $i \in I$. This and the hypotheses imply that the left member of (5) for the $y_1(x)$ and $y_2(x)$ here is negative (positive) while the right member is nonnegative (nonpositive) for n even (odd) when $d=t_1$ and $x \in (t_1, t_2)$. Thus $c \notin [\alpha_1, \alpha_2]$. By Theorem 1, $c \notin [a, \alpha_1]$ and $c \notin [\alpha_2, b]$; hence the theorem.

References

1. J. B. Garner, Uniqueness theorems for a differential equation, this MONTHLY, 73 (1966) 752–754.

2. Leonard Jackson and Keith Schrader, On second order differential inequalities, Proc. Amer. Math. Soc., 17 (1966) 1023–1027.

AN EXTENSION OF BERNOULLI'S INEQUALITY

LEON GERBER, St. John's University

The following inequality is quite useful and does not seem to have appeared in print. Let

$$F(k, a, x) = 1 + ax + C(a, 2)x^2 + \cdots + C(a, k)x^k$$

be the kth partial sum of the binomial series for $(1+x)^a$ where $x > -1$. Note that the series need not be convergent. Then

(a) if the first term omitted is positive, then $(1+x)^a > F(k, a, x)$,

(b) if the first term omitted is zero, then $(1+x)^a = F(k, a, x)$,

(c) if the first term omitted is negative, then $(1+x)^a < F(k, a, x)$.

The case of equality is easily verified: if $C(a, k+1)x^{k+1} = 0$, then $x = 0$ or a is a nonnegative integer $\leqq k$. The inequality is obvious for $k = 0$. Suppose the inequality true for $k-1$, all (real) a, and all $x > -1$. Let

$$G(k, a, x) = (1 + x)^a - F(k, a, x);$$

then

$$G(k, a, x) = a \int_0^x G(k - 1, a - 1, t)dt = aI.$$

Case I: The first omitted term is positive, i.e. $C(a, k+1)x^{k+1} = a(a-1) \cdots (a-k)x^{k+1}/(k+1)! > 0$.

A. $x > 0$. If $a > 0$, then the first omitted term of $F(k-1, a-1, x)$ is

$$C(a - 1, k)x^k = (a - 1) \cdots (a - k)x^k/k! = C(a, k + 1)x^{k+1} \cdot (k + 1)/ax$$

which is also positive. By induction hypothesis, $G(k-1, a-1, t) > 0$, $I > 0$, and $aI > 0$.

If $a < 0$, then the first omitted term of $F(k-1, a-1, x)$ is negative,

$$G(k - 1, a - 1, t) < 0, I < 0, \quad \text{and} \quad aI > 0.$$

B. $x < 0$. If $a > 0$, the first omitted term of $F(k-1, a-1, x)$ is negative, $G(k-1, a-1, t) < 0$, $I > 0$ (since the lower limit exceeds the upper) and $aI > 0$.

If $a < 0$, the first omitted term of $F(k-1, a-1, x)$ is positive, $G(k-1, a-1, t) > 0$, $I < 0$, and $aI > 0$.

Case II: The first omitted term of $F(k, a, x)$ is negative. The subcases and arguments are similar. Bernoulli's inequality is the case $k = 1$.

These inequalities can be used to generate inequalities for functions which are integrals of binomials such as the logarithm, the inverse trigonometric functions and elliptic integrals.

INTERSECTING CURVES ON A SURFACE

M. S. Klamkin, Ford Motor Company, Dearborn, Michigan

It is well known that two great circles of the same sphere always intersect in two points which are antipodal. The usual proof follows by noting that the line of intersection of the two planes determined by two great circles contains a diameter. Another proof, which suggests a generalization of this result, is that since each circle divides the surface into two congruent regions, the circles must intersect. The generalization whose proof is identical except for the replacing of "circles" by "curves" is

Theorem. *If two simple closed centrosymmetric curves lie on a simple closed centrosymmetric surface, all having the same center, then the two curves intersect.*

Also, by centrosymmetry, the points of intersection will occur in pairs of antipodal points.

It is to be noted that the latter result is not valid for curves on a multiconnected surface, e.g., a torus.

On looking back at the previous proof it will be seen that the condition of centrosymmetry is not necessary. It was more essential that the surface and the two curves each be mapped into themselves by a reflection through the common center. We now formalize these ideas.

Let O denote an interior point to a convex three-dimensional region with boundary S and consider a mapping f of S into itself such that each point of S goes into its antipodal point with respect to O. The previous theorem can now be generalized to

THEOREM. *If C_1 and C_2 are two simple closed curves on S such that C_1 and C_2 each map into themselves under the mapping f, then C_1 and C_2 intersect (in pairs of antipodal points with respect to O).*

Proof: The curve C_1 divides the surface S into two disjoint open regions A and B, each of which maps into the other under f. For a justification of this intuitively obvious fact see the following theorem. If C_2 did not intersect C_1, it would lie wholly in either A or B. But this would lead to a contradiction, since then C_2 would not map into itself under f.

On looking back again, it will be seen that the last theorem can be extended. It isn't necessary that S be the boundary of a convex region, it suffices if S is starlike with respect to O. A still wider generalization is given by

THEOREM.† *If C_1 and C_2 are two simple closed curves on a simple closed surface S and f is a continuous mapping, without fixed points, of S into itself such that $C_1=f(C_1)$ and $C_2=f(C_2)$ then $C_1\cap C_2\neq\varnothing$.*

Proof: $S-C_1$ consists of two components A and B each homeomorphic to open disks while $A\cup C_1$ and $B\cup C_1$ are homeomorphic to closed disks.

Some point of A must map under f into a point of B. Otherwise, $f(A\cup C_1)\subset A\cup C_1$ and since f has no fixed points this contradicts the Brouwer* fixed point theorem. Thus some point of A maps into a point of B and since $f(A)$ is connected,

$$f(A)\subset B\cup C_1;\qquad \text{similarly,}\quad f(B)\subset A\cup C_1. \tag{1}$$

Now suppose C_1 and C_2 do not intersect. Then C_2 lies wholly in either A or B. Since this contradicts (1), C_1 and C_2 intersect.

The previous results are now immediate corollaries of the latter theorem.

It would be of interest if further generalizations can be given.

† The author is grateful to the referee for suggesting this generalization.

* R. H. Bing notes in his paper "The Elusive Fixed Point Property" (unpublished as yet) that the work of Brouwer was preceded by that of P. Bohl, "Über die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage," J. Reine Angew. Math., 127 (1904) 179–276.

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the Monthly. Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.

ON A CLASS OF DEFINITE INTEGRALS

David Zeitlin, Minneapolis, Minnesota

J. Chaudhuri [1] gave recursions, with $p>0$ and $q>0$ as rational numbers, for $n!(-1)^n I(n, p, q) = \int_0^{\pi/2} (\log \sin^p\theta \cos^q\theta)^n d\theta$, $n=0, 1, \cdots$. An explicit representation for $I(n, p, q)$ as a series, with p and q as arbitrary parameters, is now given by

$$(1)\quad I(n,p,q) = \sum_{m=0}^{n} (-1)^m p^{n-m} q^m \sum_{k=m}^{\infty} \left((-1)^k \sum_{i=m}^{k} \binom{i}{m} S_{k+1}^{i+1} 2^{-i}\right) \Big/ (k!(2k+1)^{n+1-m}),$$

with S_{k+1}^{i+1}, first kind Stirling numbers, defined by $b(b-1)(b-2)\cdots(b-k) = \sum_{i=0}^{k} S_{k+1}^{i+1} b^{i+1}$.

Proof. The generating function for $I(n, p, q)$ (see [1, p. 546, (4)]) is

$$(2)\quad \sum_{n=0}^{\infty} (-1)^n I(n, p, q) x^n = \Gamma((px+1)/2)\Gamma((qx+1)/2)/(2\Gamma((px+qx+2)/2)).$$

Since $\Gamma(b+1) = b\Gamma(b)$, the identity $\Gamma(a)\Gamma(b+1)/\Gamma(a+b) = \sum_{k=0}^{\infty} (-1)^k b(b-1) \cdots (b-k)/(k!(a+k))$, with $a=(px+1)/2$ and $b=(qx+1)/2$, gives a power series in x for the right-hand side of (2).

For $n, m=0, 1, \cdots$, we have the additional result

$$(3)\quad \int_0^{\pi/2} (\log \sin\theta)^n (\log\cos\theta)^m d\theta = n!m!(-1)^n \sum_{k=m}^{\infty} \left((-1)^k \sum_{i=m}^{k} \binom{i}{m} S_{k+1}^{i+1} 2^{-i}\right) \Big/ (k!(2k+1)^{n+1}).$$

Since $\int_0^{\pi/2} \log^{2n+1}\tan\theta d\theta = 0$, $n=0, 1, \cdots$, (1), with $q=-p$, $p>0$, gives the class of identities

$$(4)\quad \sum_{m=0}^{2n+1} \sum_{k=m}^{\infty} \left((-1)^k \sum_{i=m}^{k} \binom{i}{m} S_{k+1}^{i+1} 2^{-i}\right) \Big/ (k!(2k+1)^{2n+2-m}) = 0 \qquad (n=0, 1, \cdots).$$

Since $I(n, 1, 0) = I(n, 0, 1)$, we obtain from (1) the class of identities

$$\sum_{k=0}^{\infty}\binom{2k}{k}\Big/\left(2^{2k}(2k+1)^{n+1}\right) \tag{5}$$
$$= (-1)^n\sum_{k=n}^{\infty}\left((-1)^k\sum_{i=n}^{k}\binom{i}{n}S_{k+1}^{i+1}2^{-i}\right)\Big/\left(k!(2k+1)\right) \qquad (n=0,1,\cdots).$$

For $n=0$ and $n=1$, the common value in (5) is $\pi/2$ and $(\pi/2)\log 2$, respectively. Since

$$\int_0^{\pi/2}\log^{2n}\tan\theta d\theta = (\pi/2)^{2n+1}\cdot|E_{2n}| \qquad (n=0,1,\cdots),$$

where E_n are Euler numbers, (1), with $q=-p$, $p>0$, gives the class of identities

$$\sum_{m=0}^{2n}\sum_{k=m}^{\infty}\left((-1)^k\sum_{i=m}^{k}\binom{i}{m}S_{k+1}^{i+1}2^{-i}\right)\Big/\left(k!(2k+1)^{2n+1-m}\right) \tag{6}$$
$$= (\pi/2)^{2n+1}\cdot|E_{2n}|/(2n)! \qquad (n=0,1,\cdots).$$

Mail address: 1650 Vincent Avenue North, Minneapolis, Minnesota 55411.

Reference

1. J. Chaudhuri, Some special integrals, this MONTHLY, 74 (1967) 545–548.

A GENERALIZED NOTION OF CONVEXITY IN A FIELD

M. C. GEMIGNANI, Smith College

The purpose of this paper is to present an algebraic approach to generalizing the notion of convexity. Our method is to generalize the standard unit interval.

DEFINITION 1: Let F be any field. A subset I of F is said to be a *unit interval* for F if I satisfies the following properties:

(1) $1\in I$.

(2) If $t\in T$, then $1-t\in I$.

(3) If s and t are in I, then $st\in I$.

(4) If s and t are in I, then either $s^{-1}t$ or $(1-s)^{-1}(1-t)$ is in I.

(5) If $s\in I$, $s\neq 0$, then $-s\notin I$.

The standard line $|01|$ of F is defined by $|01| = \{w\in F|\, w, w^{-1}, \text{or } (1-w)^{-1}\in I\}$.

(6) The nonzero elements of $|01|$ form a multiplicative group.

DEFINITION 2: If V is a vector space over a field F with unit interval I and a and b are distinct elements of V, we set $\overline{ab}=\{ta+(1-t)b|\ t\in I\}$ and $|ab| = \{wa+(1-w)b|\ w\in|01|\}$. We call $\overline{ab}$ the *segment* joining a and b and $|ab|$ the *line* determined by a and b.

It is shown that segments and lines have the order properties one would expect them to have.

A unit interval enables us to define a "natural" geometry and topology on

any vector space over the field containing the unit interval. The topology in the case of R^m and $[0, 1] \subset R$ is not the usual topology for R^m, $m \geqq 2$. The topology is the finest topology which gives lines the order subspace topology.

Mail Address: M. C. Gemignani, 23 Arlington St., Northampton, MA 01060.

A NOTE ON THE INTEGRAL FUNCTIONS OF TWO COMPLEX VARIABLES

ARUN KUMAR AGARWAL, Lucknow University, India

1. Let

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2} \tag{1.1}$$

be an integral function of the two complex variables z_1 and z_2 holomorphic for $|z_t| \leqq r_t$, $t=1, 2$. The maximum modulus of $f(z_1, z_2)$ is denoted as

$$M(r_1, r_2) = \max_{|z_t| \leqq r_t} |f(z_1, z_2)| \qquad t = 1, 2. \tag{1.2}$$

Following Bose and Sharma ([1], pp. 213–215), for every value of r_1, keeping m_2 and r_2 fixed, i.e. the m_2th column in the expansion of the moduli of the terms of the double series (1.1) of $f(z_1, z_2)$, there is, one term which is greater than or equal to all the rest and is denoted by $\mu(m_2; r_1, r_2)$, the maximum term in that column. If there are more than one such term, then the term of the highest rank will be regarded as the maximum term of this column and the rank will be denoted by $\nu_1(m_2; r_1)$. Give different values to m_2 and suppose the greatest term occurs in the ν_2th column, then the maximum term with respect to columns is denoted by $\mu(\nu_2; r_1, r_2)$ and the rank by $\nu_1(\nu_2; r_1)$. Similarly, considering the rows instead of columns of (1.1), the terms $\mu(\nu_1; r_1, r_2)$ and $\nu_2(\nu_1; r_2)$ can be defined. The maximum of $\mu(\nu_1; r_1, r_2)$ and $\mu(\nu_2; r_1, r_2)$ is denoted by $\mu(r_1, r_2)$, the maximum term and the rank by $\nu(r_1, r_2)$.

The finite order ρ of an integral function $f(z_1, z_2)$ is defined as ([1], p. 219)

$$\overline{\lim_{r_1, r_2 \to \infty}} \left\{ \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} \right\} = \rho. \tag{1.3}$$

In this note we have deduced some inequalities connecting the maximum term $\mu(r_1, r_2)$ and the ranks $\nu_1(\nu_2; r_1)$ and $\nu_2(\nu_1; r_2)$.

2. Let

$$\overline{\lim_{r_2 \to \infty}}\ \overline{\lim_{r_1 \to \infty}} \left\{ \frac{\nu_1(\nu_2; r_1)}{(r_1 r_2)^{\rho} \phi(r_1, r_2)} \right\} = \frac{c_1}{d_1}, \qquad \overline{\lim_{r_1 \to \infty}}\ \overline{\lim_{r_2 \to \infty}} \left\{ \frac{\nu_1(\nu_2; r_1)}{(r_1 r_2)^{\rho} \phi(r_1, r_2)} \right\} = \frac{c_2}{d_2}, \tag{2.1}$$

$$\overline{\lim_{r_2 \to \infty}}\ \overline{\lim_{r_1 \to \infty}} \left\{ \frac{\nu_2(\nu_1; r_2)}{(r_1 r_2)^{\rho} \phi(r_1, r_2)} \right\} = \frac{a_1}{b_1}, \qquad \overline{\lim_{r_1 \to \infty}}\ \overline{\lim_{r_2 \to \infty}} \left\{ \frac{\nu_2(\nu_1; r_2)}{(r_1 r_2)^{\rho} \phi(r_1, r_2)} \right\} = \frac{a_2}{b_2}, \tag{2.2}$$

$$\overline{\lim_{r_2 \to \infty}}\ \overline{\lim_{r_1 \to \infty}} \left\{ \frac{\log \mu(r_1, r_2)}{(r_1 r_2)^{\rho} \phi(r_1, r_2)} \right\} = \frac{p_1}{q_1}, \qquad \overline{\lim_{r_1 \to \infty}}\ \overline{\lim_{r_2 \to \infty}} \left\{ \frac{\log \mu(r_1, r_2)}{(r_1 r_2)^{\rho} \phi(r_1, r_2)} \right\} = \frac{p_2}{q_2}, \tag{2.3}$$

where $\phi(r_1, r_2)$ is a "slowly changing function," i.e. $\phi(r_1, r_2)>0$ and is a continuous function of both the variables for $r_1>r_1^0$, $r_2>r_2^0$ and $\phi(lr_1, mr_2)\sim\phi(r_1, r_2)$ for large r_1 and r_2, where l and m are any constants.

THEOREM 1. *Let $f(z_1, z_2)$ be an integral function of order $\rho(0<\rho<\infty)$, then*

$$(b_1 + d_1) \leqq \rho q_1 \leqq \rho p_1 \leqq (a_1 + c_1), \tag{2.4}$$

$$(b_1 + d_1) \leqq \rho q_2 \leqq \rho p_2 \leqq (a_1 + c_1), \tag{2.5}$$

$$d_1 \leqq 2e\rho q_1 - b_1, \qquad c_1 \leqq 2e\rho p_1 - b_1, \quad \textit{and} \tag{2.6}$$

$$d_2 \leqq 2e\rho q_2 - b_2, \qquad c_2 \leqq 2e\rho p_2 - b_2. \tag{2.7}$$

THEOREM 2. *Let $f(z_1, z_2)$ be an integral function of order $\rho(0<\rho<\infty)$, then*

$$e^2\rho p_1 \leqq 2\{\rho q_1 + (a_1 + c_1)e^2\}, \qquad e^2\rho p_2 \leqq 2\{\rho q_2 + (a_2 + c_2)e^2\}; \tag{2.8}$$

$$2e^2\rho q_1 \geqq \{\rho p_1 + 2(b_1 + d_1)e^2\}, \qquad 2e^2\rho q_2 \geqq \{\rho p_2 + 2(b_2 + d_2)e^2\}. \tag{2.9}$$

The proof of the above two theorems can be deduced by using Lemma 5 [2] in (4.2) and (4.3) of [1].

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References

1. S. K. Bose and Devendra Sharma, Integral functions of two complex variables, Compositio Math., Fasc. 2, 15 (1963) 210–226.

2. G. H. Hardy and W. W. Rogosinski, Notes on Fourier Series III, Quart. J. Math. Oxford Ser., 16 (1945) 49–58.

CAUCHY-RIEMANN RELATIONS IN n DIMENSIONS

T. E. PHIPPS, JR., NOL, White Oak

Generalized Cauchy-Riemann (GCR) equations appropriate to 3-dimensional function theory were obtained by P. W. Ketchum [Trans. Amer. Math. Soc., 30 (1928), pp. 641–667, Am. J. Math., 51 (1929), pp. 179–188] through use of an infinite-dimensional hypervariable. The corresponding n-dimensional GCR equations,

$$\frac{\partial w_0}{\partial z_0} = \frac{1}{(n-1)} \sum_{m=1}^{n-1} \frac{\partial w_m}{\partial z_m}, \qquad n > 1,$$

$$\frac{\partial w_0}{\partial z_j} = -\frac{1}{(n-1)} \frac{\partial w_j}{\partial z_0}, \qquad j = 1, 2, \cdots, (n-1),$$

$$\frac{\partial w_j}{\partial z_k} = \frac{\partial w_k}{\partial z_j} \qquad \text{for all } k > j > 0,$$

can be deduced by elementary finite-dimensional methods. Here the components w_i, $i=0, 1, \cdots, (n-1)$, of the ordered n-tuplet of reals, $w=(w_0, w_1, \cdots, w_{n-1})$, depend functionally on similar z-components. Harmonicity, $\sum_{j=0}^{n-1} \partial^2 w_i/\partial z_j^2=0$, $i=0, 1, \cdots, (n-1)$, follows. In three dimensions polynomials homogeneous of degree k, obeying the GCR equations, such as

$$w_0^{(k)} = \sum_{q=0}^{[k/2]} \sum_{p=0}^{[(k-2q)/2]} \frac{(-1)^{p+q} k! z_0^{k-2p-2q} z_1^{2p} z_2^{2q}}{2^{2(p+q)} p! q! (p+q)! (k-2p-2q)!},$$

$$w_1^{(k)} = \sum_{q=0}^{[k/2]} \sum_{p=1}^{[(k-2q+1)/2]} \frac{(-1)^{p+q-1} k! z_0^{k-2p-2q+1} z_1^{2p-1} z_2^{2q}}{2^{2(p+q-1)} (p-1)! q! (p+q)! (k-2p-2q+1)!},$$

$$w_2^{(k)} = \sum_{q=1}^{[(k+1)/2]} \sum_{p=0}^{[(k-2q+1)/2]} \frac{(-1)^{p+q-1} k! z_0^{k-2p-2q+1} z_1^{2p} z_2^{2q-1}}{2^{2(p+q-1)} p! (q-1)! (p+q)! (k-2p-2q+1)!},$$

may be introduced. These permit definitions of "power of z," e.g., $w=z^k \equiv (w_0^{(k)}, w_1^{(k)}, w_2^{(k)})$, and of "derivative," $dw/dz \equiv (\partial w_0/\partial z_0, \partial w_1/\partial z_0, \partial w_2/\partial z_0)$, by means of which power series can be formed and formally differentiated term-by-term, $dw/dz=d(z^k)/dz=kz^{k-1}$, as well as line-integrated with familiar results such as a Cauchy theorem. As in two dimensions, analyticity may be defined in terms of either power series or first-order partial differential relationships; and orthogonality relationships hold among the w-components.

Mail Address: Naval Ordnance Laboratory, White Oak, Maryland 20910.

A PROBABILITY FUNCTION FOR PARTITIONS

LEON BERNSTEIN, Syracuse University

In this paper a probability problem of applicational significance is treated by numbertheoretic methods. Consider a fragile stick of n units length (n a natural number) on which the integral units are marked by deep cuts. The stick, when thrown against solid ground, will break along the cuts into x_1 pieces of one unit length, into x_2 pieces of two units length, $\cdots$, into x_n pieces of n units length, so that

$$(1)\quad x_1 + 2x_2 + \cdots + nx_n = n \qquad (x_i \text{ nonnegative integers}; \quad i = 1, \cdots, n).$$

The question in how many ways the stick may break into pieces, thus reduces to the usual problem of partitioning a natural number n into nonnegative summands without restrictions, and the answer to the question is $p(n)$. There could be restrictions on the fragility of the stick caused by its material nature, to the extent that the maximal length of a piece that is not breakable equals m, $m<n$. The number of possible partitions in this case is $p_m(n)$, and instead of (1) the equation holds

$$(2)\quad x_1 + 2x_2 + \cdots + mx_m = n \qquad (x_i \text{ nonnegative integers}; \quad i = 1, \cdots, n).$$

A specific solution of (2), viz. $x_i=c_i$ $(i=1, \cdots, m)$ is the partition vector

$$C = (c_1, c_2, \cdots, c_m; n). \tag{3}$$

The number of ways of breaking such a fragile stick of n units length into c_1 pieces of length one, c_2 pieces of length two, $\cdots$, c_m pieces of length m, where $c_1+2c_2+\cdots+mc_m=n$, equals (by a well-known counting process)

$$(c_1 + c_2 + \cdots + c_m)!/(c_1!c_2!\cdots c_m!). \tag{4}$$

Thus, to every discrete partition vector (3) a real number (4) is assigned, and a discrete vector function is generated; for the mapping of (3) onto (4) we shall use the notation

$$\begin{aligned} g(X_i) &= (x_{1,i} + x_{2,i} + \cdots + x_{m,i})!/(x_{1,i}!\,x_{2,i}!\cdots x_{m,i}!), \\ X_i &= (x_{1,i}, x_{2,i}, \cdots, x_{m,i}; n); \ \sum_{j=1}^{m} jx_{j,i} = n; \quad i = 1, \cdots, p_m(n). \end{aligned} \tag{5}$$

We shall call $g(X_i)$ the probability weight of the partition vector X_i; then the ratio

$$f(X_i) = g(X_i) \Big/ \sum_{i=1}^{p_m(n)} g(X_i) \tag{6}$$

is a probability function for the fragility of a material stick of n units length, breakable into parts each consisting of an integral number of units not exceeding m.

As is known, there exists no explicit formula for the sum of the probability weights in (6). This sum will be calculated here by means of the Jacobi-Perron Algorithm which was modified by the author in [1] in the following way: Let V_{n-1} $(n\geqq 2)$ be a Euclidean vector space of real numbers of dimension $n-1$, and let

$$a^{(v)} = (a_1^{(v)}, \cdots, a_{n-1}^{(v)}), \quad b^{(v)} = (b_1^{(v)}, \cdots, b_{n-1}^{(v)}) \qquad (v = 0, 1, \cdots) \tag{7}$$

be two sequences of vectors in V_{n-1}, with $a^{(0)}$ a fixed, given vector, while the vectors $a^{(v+1)}$, $b^{(v)}$ $(v=0, 1, \cdots)$ are derived from $a^{(0)}$ in the following way: Let T be a transformation of V_{n-1} into itself defined by

$$\begin{aligned} Ta^{(v)} &= (a_1^{(v)} - b_1^{(v)})^{-1}(a_2^{(v)} - b_2^{(v)}, \cdots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1) = a^{(v+1)}, \\ &\qquad (a_1^{(v)} \neq b_1^{(v)}, \quad v = 0, 1, \cdots). \end{aligned} \tag{8}$$

We call $T^k a^{(0)} = T(T^{k-1}a^{(0)})$ $(k=2, 3, \cdots)$ an algorithm of the vector $a^{(0)}$ and say the algorithm is periodic, if there exist integers q, m

$$q \geqq 0, m \geqq 1 \quad \text{with} \quad \min q = Q, \quad \min m = M \quad \text{such that} \tag{9}$$

$$T^{v+M} = T^v \qquad (v = Q, Q + 1, \cdots). \tag{10}$$

The vectors $a^{(v)}$ $(v=0, 1, \cdots, Q-1)$ are called the primitive pre-period of the algorithm, and the vectors $a^{(v)}$ $(v=Q, Q+1, \cdots, Q+M-1)$ its primitive period; if $Q=0$, the algorithm is called purely periodic. The transformation matrix of T^k is generated by the numbers $A_i^{(v)}$ from the recursion formulas

$$\text{(11)} \quad \begin{aligned} &A_i^{(i)} = 1;\ A_i^{(v)} = 0; && (i \neq v;\ i, v = 0, \cdots, n-1) \\ &A_i^{(v+n)} = \sum_{j=0}^{n} b_j^{(v)} A_i^{(v+j)}; && (b_0^{(v)} = 1;\ i = 0, \cdots, n-1;\ v = 0, 1, \cdots). \end{aligned}$$

In [1] the author has invented the following method of deriving the vectors $b^{(v)}$ from the vectors $a^{(v)}$: Let $K(w)$ be an algebraic number field generated by the irrational w; let the $n-1$ components of $a^{(0)}$ be (algebraic) irrationals in $K(w)$; if the $b_i^{(v)}$ are rationals, the components of the $a^{(v)}$ are irrationals too, viz. $a_i^{(v)}=a_i^{(v)}(w)$; $(i=1, \cdots, n-1;\ v=0, 1, \cdots)$ then the $b_i^{(v)}$ are obtained by

$$\text{(12)} \qquad b_i^{(v)} = a_i^{(v)}([w]), \qquad (i = 1, \cdots, n-1;\ v = 0, 1, \cdots),$$

where $[x]$ denotes the greatest integer function. Jacobi and Perron used the formula $b_i^{(v)} = [a_i^{(v)}]$ to derive the $b^{(v)}$ from the $a^{(v)}$. In [1] the author proved the following

THEOREM. *Let w $(0<|w|<1)$ be a real root of an irreducible monic polynomial with integral coefficients*

$$\text{(13)} \qquad w^n + \sum_{i=1}^{n-1} k_{n-i} w^{n-i} - d = 0, \qquad (d,\ k_1 \neq 0;\ n \geqq 2).$$

The algorithm $T^k a^{(0)}$ where $a^{(0)}$ has the components

$$\text{(14)} \qquad a_s^{(0)} = w^s + \sum_{i=1}^{s} k_{n-i} w^{s-i} \qquad (s = 1, \cdots, n-1)$$

is purely periodic, and its length $M=n$ for $d \neq 1$, $=1$ for $d=1$; if $k_{n-i} \geqq 0$, $d>0$, $k_1>0$, then the algorithm is convergent in the sense

$$\text{(15)} \qquad w = \lim_{s\to\infty} \left(A_0^{(sn)} / A_0^{(sn+1)}\right),$$

where the $A_0^{(v)}$ are calculated from the $b^{(v)}$ of the period, having the form

$$\text{(16)} \quad \begin{gathered} b^{(0)} = (k_{n-1}, \cdots, k_1); \qquad b^{(v)} = (k_{n-1}, \cdots, k_{v+1}, k_v', \cdots, k_1'); \\ (k_v' = k_v/d;\ v = 1, \cdots, n-1). \end{gathered}$$

If we define the partition polynomials F by

$$\text{(17)} \qquad F(sn+r) = \sum \left(\prod_{j=1}^{n} k_j^{x_{j,i}} \right) g(X_i),$$

$\sum$ extended over all nonnegative $x_{j,i}$ of $\sum_{j=1}^{n} j x_{j,i} = sn+r$,

$$(s = 0, 1, \cdots; r = 1, \cdots, n-1; i = 1, \cdots, p_n(sn+r)),$$

then the basic formula is proved

$$A_0^{((s+1)n+r)} = k_n F(sn+r), \quad (k_n = b_0^{(v)}; s = 0, 1, \cdots; r = 1, \cdots, n-1), \tag{18}$$

where the $A_0^{((s+1)n+r)}$ are calculated from (11) by means of the k_i from (16). If we substitute in (17) $k_j = 1$ $(j = 1, \cdots, n)$, we obtain, in virtue of (18),

$$F(sn+r) = \sum g(X_i) = A_0^{((s+1)n+r)}, \tag{19}$$

so that, in virtue of (19), formula (6) takes the form

$$f(X_i) = g(X_i)/A_0^{((s+1)n+r)}, \qquad (i = 1, \cdots, p_n(sn+r)). \tag{20}$$

From (11), (16) we calculate easily, for $k_i = 1$ $(i = 1, \cdots, n-1)$

$$\begin{aligned} &A_0^{(n+k)} = 2^{k-1}; \; A_0^{(2n+k)} = 2^{n+k-1} - (k+1)2^{k-2}; \\ &A_0^{(3n+k)} = 2^{2n+k-1} - (n+k+1)2^{n+k-2} + (k-1)(k+2)2^{k-4}, \\ &\qquad\qquad (\text{in all cases } k = 1, \cdots, n). \end{aligned} \tag{21}$$

If we substitute in (13) $k_i = 1$, $(i = 1, \cdots, n-1)$, then, as was proved by the author in [2], the formula for the root w holds

$$2w = \lim_{t\to\infty} \frac{\sum_{i=0}^{t} (-1)^i \binom{(n+1)i + t - i}{(n+1)i} 2^{(n+1)i}}{\sum_{i=0}^{t} (-1)^i \binom{(n+1)i + t - i + 1}{(n+1)i + 1} 2^{(n+1)i}}. \tag{22}$$

It was further shown in [2] that the ratio of the two series in (22) converges comparatively very quickly. In [1] it was proved that, for $k_i = 1$, $(i = 1, \cdots, n-1)$, and for s sufficiently large, the approximation formula holds

$$w = A_0^{((s+1)n+r)}/A_0^{((s+1)n+r+1)} \qquad (r = 1, \cdots, n-1; s \geqq s_0), \tag{23}$$

so that, from (23), the formula can be derived

$$A_0^{((s+1)n+k)} = A_0^{((t+1)n+k)} w^{(t-s)n}, \qquad t < s. \tag{24}$$

Taking for $A_0^{((t+1)n+k)} = E$ an initial value from (21), and w from (22), we can state, on basis of (6) and (19), the main result of this paper in

THEOREM. *The probability that a fragile stick of $sn+k$ units length, breakable into pieces, each consisting of an integral number of units not exceeding n, would break into x_1 pieces of one unit length, into x_2 pieces of two units length, $\cdots$, into x_n pieces of n unit length, equals*

$$f(X_i) = g(X_i)E^{-1}w^{(s-t)n}. \tag{25}$$

Mail Address: Dept. of Math., Illinois Institute of Technology, Chicago, Ill. 60616.

References

1. Leon Bernstein, The modified algorithm of Jacobi-Perron, Memoirs of the American Math. Soc., 67 (1966) 1–44.
2. ———, The linear Diophantine equation in n variables and its application to generalized Fibonacci Numbers, The Fibonacci Quarterly, Special Issue, (June 1968) 1–52.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

THE SLIDING INTERVAL TECHNIQUE

R. J. EASTON, Indiana State University AND S. G. WAYMENT, Utah State University

1. Introduction. We present here a rather novel proof of the ordinary mean value theorem for the derivative of a function of one variable. The proof depends on the nested interval theorem rather than on Rolle's Theorem. The technique is also used to establish an interesting result concerning the derived function.

2. The Mean Value Theorem. We proceed in two steps.

THEOREM 1. *If $f'(x)$ exists at each point of the open set G and $[a, b]=I$ is a closed interval contained in G, then there exists a point $\xi \in I$ such that $f'(\xi) = [f(b)-f(a)]/[b-a]$.*

Proof: We establish the following notation: if $[a, b]=I$ is a closed interval, then let $f(b)-f(a)=F(I)$ and $b-a=|I|$. We observe that if $a \leqq x \leqq b$ and $I' = [a, x]$, $I''=[x, b]$, then $F(I')+F(I'')=F(I)$ and $|I'|+|I''|=|I|$. Let $h=|I|$. We now proceed with the proof.

Divide the interval $I=I_1$ into two equal intervals $I_{11}=[a, a+h/2]$ and $I_{12}=[b-h/2, b]$. Without loss of generality we may suppose $F(I_{11}) \leqq (1/2)F(I_1)$ and $F(I_{12}) \geqq (1/2)F(I_1)$. Define the auxiliary function

$$g(t) = f\left(a + \frac{h}{2} + t\right) - f(a + t).$$

Then g is a continuous function of t for $0 \leqq t \leqq h/2$ and $g(0)=F(I_{11})$ and $g(h/2) = F(I_{12})$. Hence, from the intermediate value theorem there is a t_0 between 0 and $h/2$ such that $g(t_0)=(1/2)F(I_1)$. This value t_0 determines an interval $I_2 \subset I_1$ such that $|I_2|=(1/2)|I_1|$ and $F(I_2)=(1/2)F(I_1)$. This gives

$$[f(b) - f(a)]/(b - a) = F(I_1)/|I_1| = F(I_2)/|I_2|.$$

We proceed inductively to obtain a nested sequence of closed intervals $\{I_j\}$ such that

(i) $|I_{j+1}| = (1/2)|I_j|$ and (ii) $F(I_{j+1}) = (1/2)F(I_j)$. If we let $\xi = \bigcap_{j=1}^{\infty} I_j$, then

$$f'(\xi) = \lim_{j\to\infty} [F(I_j)/|I_j|] = F(I_1)/|I_1|.$$

We note that we have not yet proved the ordinary mean value theorem, since we have required $f'(x)$ to exist at $x=a$ and $x=b$, i.e., we have not prevented the possibility $\xi=a$ or $\xi=b$. We now remove this restriction. We shall use the same notation as in Theorem 1.

THEOREM 2. *If $f'(x)$ exists for each $x\in(a, b)$ and f is continuous from the right at $x=a$ and continuous from the left at $x=b$, then there is a point $\xi\in(a, b)$ such that $f'(\xi)=F(I)/|I|$.*

Proof: In view of Theorem 1 it suffices to show there exists a closed interval $I_0\subset(a, b) = \operatorname{Int} I$ such that $F(I_0)/|I_0| = F(I)/|I|$.

Divide $I=I_1$ into I_{11} and I_{12} as in Theorem 1. If $F(I_{11}) < (1/2)F(I_1)$ then $F(I_{12}) > (1/2)F(I_1)$ and the procedure described in the proof of Theorem 1 gives an $I_2 = I_0 \subset \operatorname{Int} I_1$ with the desired property. If $F(I_{11}) > (1/2)F(I_1)$, then the procedure of Theorem 1 again obtains the desired result. If $F(I_{11}) = (1/2)F(I_1)$, then let $I_{11} = I_2$ and divide I_2 into two equal intervals I_{21} and I_{22}. If $F(I_{21}) < (1/2)F(I_2)$ or if $F(I_{21}) > (1/2)F(I_2)$, then again the procedure of Theorem 1 gives $I_3 = I_0 \subset \operatorname{Int} I_1$ with the desired property. If $F(I_{21}) = (1/2)F(I_2)$, then $F(I_{22}) = (1/2)F(I_2)$ and we choose $I_{22} = I_0 \subset \operatorname{Int} I_1$.

It is well known that the image of a connected set by a continuous function is a connected set. We now establish a similar result for $f'(x)$. Note that the only connected subsets of the line are open, half open, and closed intervals. We allow the usual usage of $-\infty$ and $+\infty$ for endpoints of intervals and consider a point as a degenerate closed interval. We only require one-sided derivatives at end points.

THEOREM 3 (Darboux's Theorem). *If $f'(x)$ exists for $a\leqq x\leqq b$ and if $f'(a) < f'(b)$, then $[f'(a), f'(b)]\subseteq f'([a, b])$.*

Proof: For $0<\epsilon<(f'(b)-f'(a))/2$ there exists an h such that $0<h<b-a$ and

$$|[f(a+h)-f(a)]/h - f'(a)| < \epsilon \quad\text{and}\quad |[f(b)-f(b-h)]/h - f'(b)| < \epsilon.$$

Let $F(t) = [f(a+h+t) - f(a+t)]/h$. Then $F(t)$ is continuous on $W = \{t \mid 0\leqq t\leqq b-a-h\}$ and since $F(0) < f'(a)+\epsilon$ and $F(b-a-h) > f'(b)-\epsilon$ we conclude that $F(t)$ attains every value in the closed interval $[f'(a)+\epsilon, f'(b)-\epsilon]$. By the mean value theorem, for each $t\in W$ there is a $t'\in(a+t, a+h+t)$ such that $f'(t') = F(t)$. Thus $[f'(a)+\epsilon, f'(b)-\epsilon]\subseteq f'([a, b])$ and since ϵ is arbitrary we have $(f'(a), f'(b))\subseteq f'([a, b])$ and hence $[f'(a), f'(b)]\subseteq f'([a, b])$.

THEOREM 4. *If $f'(x)$ is defined at each point of a connected set H, then $f'(H)=K$ is a connected set.*

Proof: Let $A=\text{g.l.b.}\ \{f'(x): x\in H\}$ and $B=\text{l.u.b.}\ \{f'(x): x\in H\}$. If $A=B$ this theorem is trivial, hence suppose that $A<B$. For $0<\epsilon<(B-A)/2$ there is an $x_1\in H$ such that $f'(x_1)<A+\epsilon$ and an $x_2\in H$ such that $f'(x_2)>B-\epsilon$. We may assume without loss of generality that $x_1<x_2$. Then we have from Theorem 3

$$[f'(x_1), f'(x_2)]\subseteq f'([x_1, x_2])\subseteq f'(H)\subseteq [A, B].$$

Since ϵ was arbitrary we conclude $(A, B)\subseteq f'(H)\subseteq [A, B]$.

Note: All four cases ($f(H)=(A, B)$, $(A, B]$, $[A, B)$, or $[A, B]$) are possible.

REMARK: The techniques may be used to prove similar results for a generalization of the notion of a derivative to functions of n-variables.

SOME APPLICATIONS OF THE METHOD OF DETACHED COEFFICIENTS

M. L. CHU, New Asia College, Hong Kong

The method of detached coefficients has found a wide range of applications in algebra and calculus, synthetic division being one of the elementary yet important instances. This note presents a few applications of this method in integral calculus and differential equations.

1. Integration of $I=\int P(x)\, e^{hx}dx$.

$$\text{(a)} \qquad P(x) = \sum_{k=0}^{n} a_k x^{n-k} \qquad (a_0 \neq 0).$$

Using the method of symbolic operators, we have

$$I = D^{-1}[e^{hx}P(x)] = (e^{hx}/h)[1 - D/h + D^2/h^2 - D^3/h^3 + \cdots]P(x) + C$$

$$= (e^{hx}/h) \sum_{k=0}^{n} (-1)^k (P^{(k)}(x)/h^k) + C \quad (\text{with } P^{(0)}(x) = P(x)).$$

Letting

$$I = (e^{hx}/h)Q(x) + C = (e^{hx}/h) \sum_{r=0}^{n} b_r x^{n-r} + C,$$

we find the relationship between the a's and the b's after differentiation and elimination of the common factor e^{hx}:

$$b_0 = a_0; \quad b_k = a_k - \frac{n-k+1}{h} b_{k-1}, \qquad (k = 1, 2, \cdots, n).$$

A tabulated procedure for finding the b's, analogous to synthetic division, can be constructed as follows:

power of x	n	$n-1$	$n-2\ \cdots$	$n-k$	$\cdots$	1	0
coeff. of $P(x)$	a_0	a_1	$a_2\ \cdots$	a_k	$\cdots$	a_{n-1}	a_n
$-\dfrac{n-k+1}{h}b_{k-1}$		$-\dfrac{n}{h}b_0$	$-\dfrac{n-1}{h}b_1\cdots$	$-\dfrac{n-k+1}{h}b_{k-1}$	$\cdots$	$-\dfrac{2}{h}b_{n-2}$	$-\dfrac{1}{h}b_{n-1}(+$
coeff. of $Q(x)$	b_0	b_1	$b_2\ \cdots$	b_k	$\cdots$	b_{n-1}	b_n

As an illustrative example let us consider the integral $\int x^n e^{hx}dx$, $(n\geqq 0)$. With proper identification of the a's the integral is readily found by the above scheme to be (cf. [1]):

$$e^{hx}\sum_{k=0}^{n}(-1)^k\frac{(n;\,-1;\,k)}{h^{k+1}}x^{n-k}+C,$$

where $(n;\ -1;\ k)=n(n-1)(n-2)\cdots(n-\overline{k-1})$.

(b) $$P(x)=\sum_{k=1}^{n}\frac{a_{n-k+1}}{x^{n-k+1}}\qquad(a_n\neq 0,\quad n\geqq 1).$$

Application of integration by parts to the integral I would result in a double sum and is therefore comparatively involved. The method of detached coefficients can again be advantageously used here. Letting

$$I=e^{hx}R(x)+b_0\int\frac{e^{hx}}{x}dx=-e^{hx}\sum_{k=2}^{n}\frac{b_{n-k+1}}{x^{n-k+1}}+b_0\int\frac{e^{hx}}{x}dx$$

and comparing the a's and b's after differentiation, we obtain the relationship between them as:

$$\begin{aligned}(n-1)b_{n-1}&=a_n,\\ (n-k)b_{n-k}&=a_{n-k+1}+hb_{n-k+1},\qquad(k=2,3,\cdots,n-1)\\ b_0&=a_1+hb_1.\end{aligned}$$

The procedure for finding the b's is tabulated as before:

power of $1/x$	n	$n-1$	$\cdots$	r	$\cdots$	2	1
coeff. of $P(x)$	a_n	a_{n-1}	$\cdots$	a_r	$\cdots$	a_2	a_1
hb_r		hb_{n-1}	$\cdots$	hb_r	$\cdots$	hb_2	hb_1
coeff. of $-R(x)$	$b_{n-1}=\dfrac{a_n}{n-1}$,	$b_{n-2}=\dfrac{a_{n-1}+hb_{n-1}}{n-2}$,	$\cdots$,	$b_{r-1}=\dfrac{a_r+hb_r}{r-1}$,	$\cdots$,	$b_1=a_2+hb_2$,	$b_0=a_1+hb_1$.

As an illustrative example, the integral $\int(e^{hx}/x^n)dx$ is readily found by the above procedure to be (cf. [1]):

$$-e^{hx}\sum_{k=1}^{n-1}\frac{h^{k-1}}{(n-1;-1;k)x^{n-k}}+\frac{h^{n-1}}{(n-1)!}\int\frac{e^{hx}}{x}dx.$$

Integrals of the forms $\int P(x)\cos(qx)dx$, $\int P(x)\sin(qx)dx$, $\int P(x)e^{px}\cos(qx)dx$ and $\int P(x)e^{px}\sin(qx)dx$ can be handled in the same way, by considering h as a complex number $p+iq$; the manipulations, however, become somewhat involved.

2. Solution of exact differential equations. For a linear differential equation of order n

$$L_n y = \sum_{k=0}^{n} P_k(x)D^{n-k}y = \chi(x), \tag{1}$$

the necessary and sufficient condition for exactness is (cf. [2], for instance)

$$Q_n(x) = \sum_{k=0}^{n}(-1)^k P_{n-k}^{(k)}(x) \equiv 0. \tag{2}$$

The equation (1) has the solution

$$\frac{dF_{n-1}y}{dx} = L_n y = \chi(x), \tag{3}$$

where

$$F_{n-1}y = \sum_{k=1}^{n} Q_{k-1}(x)D^{n-k}y, \tag{4}$$

when condition (2) is satisfied. Substitution of (4) into (3) will give the recurrence relations:

$$Q_0(x) = P_0(x), \qquad Q_k(x) = P_k(x) - Q_{k-1}'(x), \quad k = 1, 2, 3, \cdots, n. \tag{5}$$

The procedure is similar to that of the previous section:

order of derivative	n	$n-1$	$n-2\cdots$	$n-k\cdots$	1	0
coeff. of $L_n y$	$P_0(x)$	$P_1(x)$	$P_2(x)\cdots$	$P_k(x)\cdots$	$P_{n-1}(x)$	$P_n(x)$
$-Q_{k-1}'(x)$		$-Q_0'(x)$	$-Q_1'(x)\cdots$	$-Q_{k-1}(x)\cdots$	$-Q_{n-2}'(x)$	$-Q_{n-1}'(x)$ (+
$Q_k(x)$	$Q_0(x)$	$Q_1(x)$	$Q_2(x)\cdots$	$Q_k(x)\cdots$	$Q_{n-1}(x)$	$Q_n(x)$

When the last remainder $Q_n(x)$ is identically 0, equation (1) is exact. Integrating, we obtain $F_{n-1}y=\sum_{k=1}^{n} Q_{k-1}(x)D^{n-k}y=\int\chi(x)dx+C$. This process can be continued, as the following example illustrates:

$$L_4 y = (e^x + 2x)y'''' + 4(e^x+2)y''' + 6e^x y'' + 4e^x y' + e^x y = Ae^{ax}\cos(bx).$$

order	4	3	2	1	0	
coeff. of L_4y	e^x+2x	$4(e^x+2)$	$6e^x$	$4e^x$	e^x	
$-Q_{k-1}'(x)$		$-(e^x+2)$	$-3e^x$	$-3e^x$	$-e^x$	(+
1st integral	e^x+2x	$3(e^x+2)$	$3e^x$	e^x	0	
		$-(e^x+2)$	$-2e^x$	$-e^x$	(+	
2nd integral	e^x+2x	$2(e^x+2)$	e^x	0		
		$-(e^x+2)$	$-e^x$	(+		
3rd integral	e^x+2x	e^x+2	0			
		$-(e^x+2)$	(+			
4th integral	e^x+2x	0				

The general solution is therefore

$$(e^x+2x)y = C_0 + C_1x + C_2x^2 + C_3x^3 + A\int\int\int\int e^{ax}\cos(bx)dx^4$$

$$= C_0 + C_1x + C_2x^2 + C_3x^3 + [Ae^{ax}/(a^2+b^2)^2]\cos(bx - 4\tan^{-1} b/a).$$

The method can also be used in seeking an integrating factor, as the next example shows:

$$L_3y = x^3y''' + 5x^2y'' + (2x - x^3)y' - (2+x^2)y = e^x(2/x - 2 + x).$$

order	3	2	1	0	
coeff. of L_3y	x^3	$5x^2$	$2x-x^3$	$-(2+x^2)$	
$-Q_{k-1}'(x)$		$-3x^2$	$-4x$	$2+3x^2$	(+
$Q_k(x)$	x^3	$2x^2$	$-2x-x^3$	$2x^2 \not\equiv 0$	

The fact that the coefficients of L_3y are polynomials in x suggests that there may be an integrating factor of type x^m. The number m can be determined by this method:

order	3	2	1	0	
coeff. of x^mL_3y	x^{m+3}	$5x^{m+2}$	$2x^{m+1}-x^{m+3}$	$-(2x^m+x^{m+2})$	
$-Q_{k-1}'(x)$		$-(m+3)x^{m+2}$	$(m^2-4)x^{m+1}$	$-(m+1)(m^2-2)x^m+(m+3)x^{m+2}$	(+
$Q_k(x)$	x^{m+3}	$-(m-2)x^{m+2}$	$(m^2-2)x^{m+1}-x^{m+3}$	$-m(m-1)(m+2)x^m+(m+2)x^{m+2}$	

Equating the remainder to zero yields $m = -2$ and the first integral is

$$F_2 y = xy'' + 4y' + (2/x - x)y = C + \int e^x(2/x^3 - 2/x^2 + 1/x)dx$$
$$= C + e^x(1/x - 1/x^2).$$

References

1. W. Gröbner and N. Hofreiter, Integraltafel, Erster Teil: Unbestimmte Integrale, 3rd ed., Springer-Verlag, Vienna, 1961.

2. A. R. Forsyth, A Treatise on Differential Equations, 6th ed., St. Martin's Press, New York, 1961.

3. R. A. Frazer, W. J. Duncan, and A. R. Collar, Elementary Matrices, Cambridge University Press, Cambridge, 1955, pp. 103–104.

Editor's Note. Referring to the article by W. Phillips, *On the definition of even and odd permutations*, this MONTHLY, 74 (1967) 1249–1251, and to the article by E. L. Spitznagel, Jr. entitled *Note on the alternating group*, this MONTHLY, 75 (1968) 68–69, Dr. J. L. Brenner has pointed out that similar ideas were discussed in his paper *A new proof that no permutation is both even and odd*, this MONTHLY, 64 (1957) 499–500, and Professor J. M. Thomas has indicated that these considerations were presented some years ago in his book, *Theory of Equations*, McGraw-Hill, New York, 1938, Chap. 2.

Professor S. A. Naimpally has informed us that the construction exhibited in O. Hájek's *Metric completion simplified*, this MONTHLY, 75 (1968) 62–63, was given by K. Kuratowski in *Quelques problèmes concernant les espaces métriques non-séparables*, Fund. Math., 25 (1935) 534–545, and has been described more recently by W. J. Pervin in *Foundations of General Topology*, Academic Press, New York, 1964, p. 122, Theorem 7.2.1. The extension of this construction to uniform spaces has been discussed by R. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math., 6 (1956) 397–403, and by S. A. Naimpally, *A generalization of the Kuratowski embedding theorem*, Nieuw Arch. Wisk. (3), 13 (1965) 113–115.

Commenting on two articles by I. I. Kolodner, *A simple proof of the Schröder-Bernstein theorem*, this MONTHLY, 74 (1967) 995–996, and *The compact graph theorem*, this MONTHLY, 75 (1968) 167, Professor R. C. Buck states that similar arguments appear in his *Advanced Calculus*, 2nd ed., McGraw-Hill, New York, 1965, pp. 479–480, p. 75, Theorem 12, and p. 277, Theorem 22.

Concerning the note by V. W. Bryant, *A remark on a fixed-point theorem for iterated mappings*, this MONTHLY, 75 (1968) 399–400, Professor K. L. Singh has remarked that Bryant's theorem appears in Gustave Choquet's *Topology*, Academic Press, New York, 1966, p. 88, Remark 2, and Professor J. B. Diaz has informed us that the same result appears as Theorem 3 of the article by S. C. Chu and J. B. Diaz, *Remarks on a generalization of Banach's principle of contraction mappings*, Jour. Math. Analysis and Applications, 11 (1965) 440–446.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John G. Harvey, Dept. of Math., University of Wisconsin, Madison, WI 53706.

FIRST WESTERN NEW YORK REGIONAL MEETING OF UNDERGRADUATE MATHEMATICS CLUBS

D. S. MARTIN, State University of New York at Buffalo

The First Western New York Regional Meeting of Undergraduate Mathematics Clubs was held at the State University of New York at Buffalo on March 23, 1968, John Kohl, President of the host club, presiding. Forty-five students from eight schools were in attendance. Three students, Sam Bisignano (Senior, State University College at Buffalo), Steven Gagola (Junior, State University of New York at Buffalo), and Paul Miller (Senior, State University College at Fredonia) spoke on their mathematical research. In addition, Robert Daigler (Senior, Assumption College) spoke on Mathematics Clubs. Special guest lecturer was Professor A. E. Danese (State University College at Fredonia). A panel on the 'Problems facing Mathematics Clubs and possible solutions' underscored the need for better faculty cooperation to combat the problem, faced by all the clubs attending, of poor attendance. It was decided to hold a second Regional meeting in November, 1968, at Niagara University, Niagara University, New York, 14109. The meeting ended with a tour of the State University of New York at Buffalo Computing Center.

REFLECTIONS OF A TEACHER

I. A. BARNETT, Ohio University, Athens, and University of Cincinnati

In looking back over the past fifty years as a teacher, I realize that I could have profited by some guidance at the beginning of my career. I hope that some of my remarks may be of benefit to those just entering the teaching profession.

The most fortunate teachers are those with the kind of personality to which students react favorably. However, even the fortunate ones must not be lulled into a false sense of security, since there remain many hurdles to overcome. Teaching is one of the most complex of human endeavors. From the very first moment the teacher must present material to the class to arouse an interest and curiosity in the subject.

Mathematicians have neglected the "descriptive" mathematics because of the feeling that every statement must be proved immediately. In Analytic Geometry, for example, the properties of such curves as the cycloid and the catenary may be introduced, even though it may be necessary to leave some of the proofs for a later time. In Number Theory, the instructor may discuss

the "famous problems of antiquity," which the Greeks attempted to solve—the Greeks knew nothing about the relation between the constructions of the regular polygons and Fermat numbers. They would have been amazed to learn that a 9-sided regular polygon is not constructible, while one of 17 sides is.

To gain the interest of the student is a necessary first step. Along with that the teacher must utilize the maximum capability and potential of the students, making each student aware of the powers he possesses. The confidence the student builds within himself is cumulative—the more he achieves the more faith he acquires for further achievement. I have in mind a student who started out at the bottom of the class. Before the semester ended, she had, by sheer will and encouragement, struggled through enough of the subject matter to feel confident of her power to achieve and she became one of the top students in the class. She could scarcely believe that this could have happened to her.

The devices by means of which this confidence can be developed in a student are many and varied. It is not enough to be content with seeing that a student does his daily assignments. Teaching must go beyond that. The student must be led to think further about the subject matter. He should obtain an insight into the implications and consequences of the problems. Where possible, problems should be put into the framework of an abstraction, of which the problems under consideration are merely special instances.

The teacher need not present a finished performance in the classroom. He must encourage and stimulate the class in the formulation of ideas. The emphasis must be on that alone. One of the great weaknesses in teaching is the natural desire of the teacher to present his subject with elegance. But this very elegance may obscure the thinking processes. The effect on the students is often inversely proportional to the smoothness of the presentation. The smooth lecture which hangs together beautifully is not necessarily the most effective. At one time or another, every teacher has experienced a sense of frustration from a well prepared lecture which failed to register with his students. On the other hand, the most satisfying class sessions usually come as a result of questions and ideas originating with the students themselves. Perhaps you too know an outstanding teacher whose lectures are so polished and clear that he attracts many students. They come to hear his lectures and enjoy them. But how many students who have listened to these lectures have been stimulated to further study? Perhaps the students are discouraged from attempting such perfection.

The teacher must be aware of the recent developments in his field and he must be able to transmit the spirit of such progress. Only by extending his own knowledge can the teacher hope to instill enthusiasm for the subject. It is this aspect that distinguishes the good teacher from the average graduate assistant who may have a limited knowledge of the broader field from which to draw examples.

From what I have said, it is clear that I am in favor of having the most experienced and scholarly teachers, for the freshmen and sophomores. Only in this way can a student be led to discover his abilities at an early date and then

be encouraged to pursue his chosen field. This is an ideal situation which the universities must try to achieve.

I come now to the subject of examinations. What is it that the teacher wishes to learn from an examination? Does he want the student to repeat what he has heard in the classroom or read in a particular book? Is the student expected to come to the examination room with his head crammed with formulas and data which he will probably forget as soon as the examination is over?

The examination is intended to ferret out knowledge which is fundamental. Every course, properly taught, can be developed on the basis of a small number of well defined principles. This is true also in history, literature, economics and science. One such principle is that of generalization which permeates all fields of knowledge. It enables the student to unify many special results and methods and then to put his conclusions into a concise and abstract form.

How can an examination which tests the student on this type of knowledge be developed? What is wrong with allowing the student to bring into the examination room any notes or books he would wish to consult during the examination? After all, these will always be available to him whenever he needs them. To be sure, this will require a new approach and new techniques in daily instruction. The teacher will have to find ways and means of determining if the students understand the application of general principles to new and old situations. This type of examination will then be written in a more relaxed atmosphere. At the same time it will give the student an opportunity to convey his own thoughts without depending on his memory or trying to get help from fellow students. Until this reform in teaching is accomplished, we will fail in all other objectives of education.

USE OF SCHOOL MATHEMATICS STUDY GROUP MATERIALS IN CHILE

C. H. Clemens, Universidad Tecnica del Estado, Santiago, Chile

During the school year (March through December) of 1967, a small experiment was conducted in the teaching of mathematics at the seventh and eighth grade level in four public schools in Santiago, Chile. Goals of the experiment were to introduce into eight classrooms:

(1) Systematic use by students of modern textbooks.

(2) Occasional novel classroom experiences.

The project had as its director one of the leading mathematics educators in Chile, Profesora Amalia Villarroel of the Chilean Ministry of Education. Numbered among its collaborators were Professor Edward Begle of Stanford University, who donated the four hundred copies of the Spanish translation of the SMSG series *Mathematics for Junior High School* used in the experiment, and Professor Bruce Vogeli of Columbia University, who guided the initial planning.

The following are some of the conclusions indicated by the ten-month experience:

1. The SMSG Spanish language texts are an excellent guide to secondary school teachers. They are conducive to an ordered, unified presentation of class material—thus remedying a serious defect of present practices.

2. The average Chilean seventh and eighth grade student is totally unfamiliar with the use of textbooks in mathematics education. The introduction of individual copies of a comprehensive text did not alter this situation. Perhaps what is needed is a three- or four-year graduated program to introduce the use of textbooks to students. The first step would be the occasional use of short pamphlets to simplify one or two elusive concepts.

3. Two types of "occasional novel classroom experiences" were tried. These were talks by working mathematicians, the success of which depended mostly on the mathematician's ability to relate to thirteen-year-olds, and math laboratories, which were used for the first time in Chile and met with enthusiastic student response.

4. The time is still not ripe in Chile for new materials development in mathematics on a national scale. Small materials preparation projects should be encouraged, then ruthlessly evaluated by controlled classroom use.

5. It is as easy to mechanize and ritualize new teaching methods as old ones, and even the well-intentioned often fall prey to this deadening tendency.

Anyone interested in more detail about the experiment may write Profesora Villarroel in care of the Dirección de Enseñanza Secundaria, Ministerio de Educación, Santiago de Chile.

FURTHER TECHNIQUES IN THE THEORY OF BIG GAME HUNTING

PATRICIA L. DUDLEY, G. T. EVANS, K. D. HANSEN and I. D. RICHARDSON,
Carleton University, Ottawa

Interest in the problem of big game hunting has recently been reawakened by Morphy's paper in this MONTHLY, Feb. 1968, p. 185. We outline below several new techniques, including one from the humanities. We are also in possession of a solution by means of Bachmann geometry which we shall be glad to communicate to anyone who is interested.

1. (Moore-Smith method) Letting A = Sahara Desert, one can construct a net in A converging to any point in $\overline{A}$. Now lions are unable to resist tuna fish, on account of the charged atoms found therein (see Galileo Galilei, *Dialogues Concerning Tuna's Ionses*). Place a tuna fish in a tavern, thus attracting a lion. As noted above, one can construct a net converging to any point in a bar; in this net enmesh the lion.

2. (Method of analytical mechanics) Since the lion has nonzero mass it has moments of inertia. Grab it during one of them.

3. (Mittag-Leffler method) The number of lions in the Sahara Desert is finite, so the collection of such lions has no cluster point. Use Mittag-Leffler's theorem to construct a meromorphic function with a pole at each lion. Being a

tropical animal a lion will freeze if placed at a pole, and may then be easily taken.

4. (Method of natural functions) The lion, having spent his life under the Sahara sun, will surely have a tan. Induce him to lie on his back; he can then, by virtue of his reciprocal tan, be cot.

5. (Boundary value method) As Dr. Morphy has pointed out, Brouwer's theorem on the invariance of domain makes the location of the hunt irrelevant. The present method is designed for use in North America. Assemble the requisite equipment in Kentucky, and await inclement weather. Catching the lion then readily becomes a Storm-Louisville problem.

6. (Method of moral philosophy) Construct a corral in the Sahara and wait until autumn. At that time the corral will contain a large number of lions, for it is well known that a pride cometh before the fall.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, Maine 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before February 28, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2115. *Proposed by K. M. Brown, Cornell University*

Let a sequence $\{S_n\}$ be defined by

$$S_n = \frac{n+1}{2^{n+1}} \sum_{i=1}^{n} 2^i/i, \qquad n = 1, 2, \cdots.$$

Show that $\lim_{n\to\infty} S_n$ exists and find the value of this limit.

E 2116. *Proposed by Erwin Just, Bronx Community College, New York*

Prove that every integer, modulo a prime, may be expressed either as a sum of two cubes, as a sum of two fourth powers, or as a sum of a cube and a fourth power.

E 2117. *Proposed by Michael Schulz, Bell Telephone Laboratories, Murray Hill, N. J.*

Show that the following relation is valid for all real $n \geqq 1$:

$$\int_0^n \frac{n^{[x]}}{[x]!}\, dx \geqq e^{n-1}.$$

Compare E 1583 [1964, 208].

E 2118. *Proposed by S. W. Golomb, University of Southern California*

For $N \geqq 1$, define

$$f(N) = \max A_1^{A_2^{\cdot^{\cdot^{\cdot^{A_k}}}}}$$

where the maximum is extended over all partitions of N into positive integers, $N = A_1 + A_2 + \cdots + A_k$. Thus $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$, $f(5) = 9$, $f(6) = 27$, $f(7) = 512$, etc. Determine $f(N)$ in general.

E 2119. *Proposed by J. Garfunkel, Forest Hills High School, N. Y.*

If A, B, C are the angles of an acute triangle ABC, prove that

$$\frac{\sqrt{1+8\cos^2 B}}{\sin A} + \frac{\sqrt{1+8\cos^2 C}}{\sin B} + \frac{\sqrt{1+8\cos^2 A}}{\sin C} \geqq 6.$$

E 2120. *Proposed by Edward B. Wright, Western Washington State College*

Find the greatest common divisor of the set $\{k^n - k: k = 2, 3, \cdots\}$ where n is a given, fixed positive integer.

E 2121. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Consider rectangles inscribed in a given rectangle R (i.e., having a vertex on each side of R) and suppose two such inscribed rectangles R_1 and R_2 have a common vertex on one side of R. Show that the sum of the areas of R_1 and R_2 equals the area of R.

E 2122. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let D, E, and F be points in the plane of a nonequilateral triangle ABC so

that triangles BDC, CEA, and AFB are directly similar. Prove that triangle DEF is equilateral if and only if the three triangles are isosceles (with a side of triangle ABC as base) with base angles of 30°. (The "if" part, Napoleon's theorem, is known. See the *Mathematics Magazine*, 1966, p. 166.)

E 2123. *Proposed by A. C. Williams, Mobil Research, Princeton, N. J.*

Define the generalized nth derivative as in problem E 1992, [1968, 900] i.e., $f^{(n)}(x) = \lim_{h\to 0} \Delta^n f/h^n$, where

$$\Delta^n f(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{n-j} f(x+jh).$$

Show that the existence of the nth generalized derivative does not imply the existence of lower order derivatives. Show, in fact, that for each subset S of the positive integers, there exists a function such that $f^{(n)}(0)$ exists for $n \in S$ and fails to exist for $n \notin S$.

E 2124. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Construct on the sides BC, CA, AB of a triangle ABC, exteriorly, the squares $BCDE$, $ACFG$, $BAHK$ and build the parallelograms $FCDQ$, $EBKP$. Show that APQ is an isosceles right triangle.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Diophantine Equation

E 1991 [1967, 590]. *Proposed by J. M. Khatri, Baroda, India*

Find integral solutions of

$$(1) \qquad (a^2 - x)(b^2 - x) = c^2 - x.$$

I. *Solution by C. C. Oursler, Southern Illinois University.* We have given

$$(2') \qquad x^2 + (1 - a^2 - b^2)x + a^2b^2 = c^2.$$

The solutions are immediate in case $1 - a^2 - b^2 = \pm 2ab$, or in case $c^2 = a^2b^2$, or in case $x = a^2$ (or b^2) or in case $x = a^2 + 1$. These give

(i) For all a, x, $b^2 = (a+1)^2$, $c^2 = (a^2 + a - x)^2$;
(ii) For all a, x, $b^2 = (a-1)^2$, $c^2 = (a^2 - a - x)^2$;
(iii) For all a, b, $c^2 = a^2b^2$, $x = 0$;
(iv) For all a, b, $c^2 = a^2b^2$, $x = a^2 + b^2 - 1$;
(v) For all a, b, $x = a^2 = c^2$;
(vi) For all a, b, $c^2 = b^2$, $x = a^2 - 1$.

We may put (1) in the form

$$(3') \qquad c^2 - (a^2 - x)b^2 = x(1 + x - a^2).$$

Let a, x be chosen arbitrarily. If $a^2 - x$ is positive and not a square, then (3′) is

a Pell equation in b and c, admitting solutions given by (i) and (ii). By the well-known theory of the Pell equation, there are infinitely many other pairs b, c which also satisfy (3′) and these may be obtained by standard procedure.

In case a^2-x is a square, or $a^2-x\leqq 0$, the number of additional solutions, if any, is finite.

II. *Solution by D. Ž. Djoković, University of Waterloo, Canada.* From $(ab+c)(ab-c)=x(a^2+b^2-x-1)$, which implies $ab+c=pq$, $ab-c=rs$, $x=pr$, $a^2+b^2-x-1=qs$, we infer

$$x = pr, \qquad c = \tfrac{1}{2}(pq - rs), \tag{2}$$

and $(a+b+1)(a+b-1)=(p+s)(r+q)$, $(a-b+1)(a-b-1)=(p-s)(r-q)$. Therefore

$$\begin{aligned} &a + b + 1 = \alpha\beta, \quad &a + b - 1 = \gamma\delta, \quad &p + s = \alpha\gamma, \quad &r + q = \beta\delta,\\ &a - b + 1 = \alpha'\beta', \quad &a - b - 1 = \gamma'\delta', \quad &p - s = \alpha'\gamma', \quad &r - q = \beta'\delta' \end{aligned} \tag{3}$$

There follow at once

$$a = \tfrac{1}{2}(\alpha\beta + \alpha'\beta') - 1, \qquad b = \tfrac{1}{2}(\alpha\beta - \alpha'\beta'), \tag{4}$$

$$\alpha\beta - \gamma\delta = \alpha'\beta' - \gamma'\delta' = 2. \tag{5}$$

Putting the values of p, q, r, s found from (3) into (2) we get also

$$c = \tfrac{1}{4}(\alpha'\gamma'\beta\delta - \alpha\gamma\beta'\delta'), \qquad x = \tfrac{1}{4}(\alpha\gamma + \alpha'\gamma')(\beta\delta + \beta'\delta'). \tag{6}$$

Since the values are to be integral, we have also

$$\text{(7)}\ \alpha\beta \equiv \alpha'\beta', \qquad \alpha\gamma \equiv \alpha'\gamma', \qquad \beta\delta \equiv \beta'\delta' \pmod 2, \qquad \alpha'\gamma'\beta\delta \equiv \alpha\gamma\beta'\delta' \pmod 4.$$

Hence, the general solution in integers is given by (4) and (6), where α, $\cdots$, δ' must satisfy (5) and (7).

Also solved by Joseph Arkin, Leon Bankoff, R. L. Browning, Lindley Burton, Michael Goldberg, A. E. A. Hunt, Donald Jeffords, Lew Kowarski, D. C. B. Marsh, Charles McCracken, Erich Michalup (Venezuela), Norman Miller, R. A. Moore, Steven Russ, D. R. Stark, Julius Vogel, Charles Wexler, and the proposer.

Editorial Comment. This problem, in a form suitable for high school students, was proposed by Mannis Charosh and appears as problem 265 in the January 1967, *Mathematics Student Journal.* In the published solutions, a and b are consecutive integers.

If the right member of (1) is squared, we have problem 5020 [1963, 574], still not completely solved.

Generalized Derivative

E 1992 [1967, 590]. *Proposed by Eric Mendelsohn, University of Manitoba*

If we define a generalized nth derivative as $f^{(n)}(x)=\lim_{n\to 0}\Delta^n f/h^n$, where $\Delta f=f(x+h)-f(x)$, we have

$$f''(x) = \lim\,[f(x + 2h) - 2f(x + h) + f(x)]/h^2,$$

and so on. Is there a function which is infinitely differentiable at a limit point of its discontinuities?

Solution by Sidney Spital, California State College at Hayward. Yes. Consider the function defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x = 1/n \ (n = 1, 2, \cdots) \\ 0 & \text{otherwise,} \end{cases}$$

at $x=0$. From the binomial-like expansion of the nth difference,

$$\left| \Delta^n f(0) \right| < 2^n B_n e^{-1/nh},$$

where $\binom{n}{k} \leqq B_n$ for $k=1, 2, \cdots, n$. Since $\lim_{h\to 0}(e^{-1/nh})/h^n = 0$ for any n, all $f^{(n)}(0)=0$.

Also solved by Dan Marcus, Jürg Rätz (Switzerland), J. J. Swetik, A. C. Williams, and the proposer.

Vanishing Determinant

E 1993 [1967, 590]. *Proposed by R. O. Davies and A. J. Wiseman, the University, Leicester, England*

Prove that for $n=1, 2, \cdots$, the following $(n+1)\times(n+1)$ determinant vanishes:

$$\begin{vmatrix} 1 & 2\binom{2n-1}{1} & 2^2\binom{2n-1}{2} & \cdots & 2^n\binom{2n-1}{n} \\ 1 & \binom{n+1}{1} & \binom{n+1}{2} & \cdots & \binom{n+1}{n} \\ 0 & 1 & \binom{n+1}{1} & \cdots & \binom{n+1}{n-1} \\ 0 & 0 & 1 & \cdots & \binom{n+1}{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \binom{n+1}{1} \end{vmatrix}.$$

I. *Solution by Rampurkar Manohar, University of Saskatchewan.* Multiply the $(k+1)$-th column by $(-1)^k\binom{2n-k}{n-k}$ for $k=0, 1, \cdots, n$ and add to the last column. The desired result follows immediately due to the following two known identities:

$$\sum_{k=0}^{r} (-1)^k \binom{n+1}{k}\binom{n+r-k}{r-k} = 0, \quad \sum_{k=0}^{n} (-1)^k 2^k \binom{2n-1}{k}\binom{2n-k}{n-k} = 0.$$

II. *Solution by Leonard Carlitz, Duke University.* Consider the more general determinant $\Delta_n = \Delta_n(a_0, a_1, \cdots, a_n)$ which results when the first row is replaced by $a_0, a_1, a_2, \cdots, a_n$, where the a_j are arbitrary functions of n. Let C_j denote the jth column of Δ_n, $0 \leqq j \leqq n$. We perform the following operations. Replace C_n by

$$C_n - C_{n-1} + \cdots + (-1)^n C_0,$$

then replace C_{n-1} by $C_{n-1} - C_{n-2} + \cdots + (-1)^{n-1} C_0$, and so on. Call the new determinant Δ_n' and let C_j' denote the jth column of Δ_n'. Now repeat the above operations to get Δ_n''. We continue this until we get $\Delta_n^{(n+1)}$. It is easily verified that $\Delta_n^{(n+1)}$ is of the following form:

$$\Delta_n^{(n+1)} = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} & b_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix} = (-1)^n b_n.$$

Denote the elements of the first row of Δ_n' by a_j' those of the first row of Δ_n'' by a_j'', and so on. Then we have

$$\begin{aligned} a_j' &= \sum_{i=0}^{j} (-1)^{j-i} a_i, \\ a_j'' &= \sum_{k=0}^{j} (-1)^{j-k} a_k' = \sum_{k=0}^{j} (-1)^{j-k} \sum_{i=0}^{k} (-1)^{k-i} a_i \\ &= \sum_{i=0}^{j} (-1)^{j-i} a_i \sum_{k=i}^{j} 1 = \sum_{i=0}^{j} (-1)^{j-i} (j - i + 1) a_i, \\ a_j''' &= \sum_{k=0}^{j} (-1)^{j-k} a_k'' = \sum_{k=0}^{j} (-1)^{j-k} \sum_{i=0}^{k} (-1)^{k-i} (k - i + 1) a_i \\ &= \sum_{i=0}^{j} (-1)^{j-i} \binom{j - i + 2}{2} a_i, \end{aligned}$$

and generally

$$a_j^{(k+1)} = \sum_{i=0}^{j} (-1)^{j-i} \binom{j - i + k}{k} a_i = \sum_{i=0} (-1)^i \binom{i + k}{k} a_{j-i}.$$

In particular then

$$b_n = a_n^{(n+1)} = \sum_{i=0}^{n} (-1)^i \binom{n + i}{n} a_{n-i},$$

so that

$$\Delta_n = \Delta_n^{(n+1)} = (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n+i}{n} a_{n-i}.$$

If now we take $a_j = \binom{2n-1}{j} a^j$, we get

$$\begin{aligned}
\Delta_n &= (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n+j}{j} \binom{2n-1}{n-j} a^{n-j} \\
&= (-1)^n \binom{2n-1}{n} \sum_{j=0}^{n} (-1)^j (n+j) \binom{n}{j} a^{n-j} \\
&= (-1)^n \binom{2n-1}{n} \{ n(a-1)^n - n(a-1)^{n-1} \} \\
&= (-1)^n n \binom{2n-1}{n} (a-1)^{n-1}(a-2).
\end{aligned}$$

In particular Δ_n vanishes when $a=2$ and when $a=1$ and $n>1$.

Also solved by Peter Ash & J. Marshall Ash, D. Ž. Djoković, M. G. Greening (Australia), Eldon Hansen, D. C. B. Marsh, C. B. A. Peck, and the proposers.

Limit of a Recursive Sequence

E 1994 [1967, 591]. *Proposed by T. M. K. Davison, University of Waterloo, Ontario*

Let $\{\alpha_n\}$ be a sequence of positive numbers. Put $a_1 = 1$, $a_{n+1} = a_n + \alpha_n a_n^{-1}$ for $n \geqq 1$. Prove that as $n \to \infty$, lim a_n exists if and only if $\sum_{n=1}^{\infty} \alpha_n$ is finite.

Solution by Beatriz Margolis, Fundación Bariloche, Buenos Aires, Argentina. We observe first of all: $a_{n+1} > a_n \geqq 1$. Assume that $\lim_{n\to\infty} a_n = a$. Hence, for all n, $a_n \leqq a$. Therefore

$$\sum_{n=1}^{N} (a_{n+1} - a_n) = a_{N+1} - a_1 = \sum_{n=1}^{N} \frac{\alpha_n}{a_n} \geqq \frac{1}{a} \sum_{n=1}^{N} \alpha_n.$$

Hence $\sum_{n=1}^{\infty} \alpha_n \leqq a(a-1) < \infty$.

Conversely, assume that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then

$$\sum_{n=1}^{N} (a_{n+1} - a_n) = a_{N+1} - a_1 = \sum_{n=1}^{N} \frac{\alpha_n}{a_n} \leqq \sum_{n=1}^{N} \alpha_n.$$

Therefore $\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} \alpha_n$.

Also solved by P. R. Atwood, H. T. Banks, Stephen Berman, P. M. Brady, Jr., Peter Bundschuh (Germany), T. J. Burke, Lindley Burton, Roxanne M. Byrne, Neil Cameron (New Zealand), Red Cougar, T. J. Cullen, D. Ž. Djoković, R. J. Driscoll, M. A. Ettrick, J. A. Ewell, Neal Felsinger, Bengt Fornberg (Sweden), Toyomasa Fujinawa (Japan), H. A. Greenbaum, M. G. Greening (Australia), Marvin Gruber, C. P. Gupta, Donald Jeffords, S. S. Kapur & D. R. Chand, B. G. Klein, Kenneth Kramer, J. F. Leetch, Douglas Lind, Dan Marcus & Jerry Fischer, Norman Miller, Steven Minsker, P. L. Montgomery, Barbara W. Nason, P. G. Pantelidakis, C. B. A.

Peck, Dale Peterson, Jernej Polajnar (Yugoslavia), Stanley Rabinowitz, Simeon Reich (Israel), Henry Ricardo, Steven Russ, Perry Scheinok, Rajinder Singh, Al Somayajulu, Sidney Spital, Hugo Sun, J. J. Swetik, Stephen Tanny, Brian Warrack & David Leeming, J. E. Wilkins, Jr., Jang Mei Wu (Japan), and the proposer.

Inequalities in m-space

E 1995 [1967, 718]. *Proposed by L. J. Mordell, St. John's College, Cambridge, England*

$ABCD$ is a tetrahedron T, and P is any point of T. The circumsphere of T has center O and radius R, and a concentric sphere of radius $R_1 \geqq \frac{1}{2}R$ cuts OA in A_1, etc. Let $|P|$ be the least of the distances PA_1, PB_1, PC_1, PD_1. Prove that $|P| \leqq R_1$, with equality only if $R_1 = \frac{1}{2}R$ and P is distant $\frac{1}{2}R$ from A_1, B_1, C_1 or D_1.

A similar result holds for a simplex and then, when $R_1 = R$, it is due to M. J. Godwin, J. London Math. Soc., 40 (1965) 699.

Solution by E. C. Milner, University of Calgary, Alberta, Canada. Reference to the circumcenter seems to be irrelevant. It is just as easy to prove the following more general statement: *Let $A_1, \cdots, A_n$ be n points in m-space with convex hull T. Let O be any point of T and let T' be the homothetic image of T in which O is the center of similitude and Q' corresponds to Q if $OQ' = 2OQ$. If P is any point of T', then $PA_i \leqq OA_i$ for some i $(1 \leqq i \leqq n)$. There is equality if and only if P is the projection from O onto one of the associated subsimplexes of T'.* (If T' is an r-simplex, there are $2^{r+1}-1$ of these associated subsimplexes with dimensions O through r.)

Proof: If P coincides with O the result is trivial (and in this case P is the projection from O onto T'). We now assume O and P are distinct points. The orthogonal projection of T' onto OP is a segment containing O and P. Hence, if α_1, α_1', etc., denote the projections of A_1, A_1', etc., there is some $i(1 \leqq i \leqq n)$ so that P is a point of the segment $O\alpha_i'$. Since $O\alpha_i = \frac{1}{2}O\alpha_i'$, it follows that $P\alpha_i \leqq O\alpha_i$ and hence $PA_i \leqq OA_i$. Since $P \neq O$, there is equality only if $P = \alpha_i'$. In this case the set $J = \{j: \alpha_j' = P\}$ is nonempty and P is the projection from O onto the subsimplex of T' formed by the convex hull of the A_j' $(j \in J)$.

Also solved by L. J. Burton, Charles Chouteau, Michael Goldberg, Bohuslav Míšek (Czechoslovakia), and the proposer.

An Abelian Group

E 1996 [1967, 719]. *Proposed by Erwin Just, Bronx Community College*

Let G be an infinite group in which no element other than the identity has finite order. If $(xy)^2 = (yx)^2$ for all $x, y \in G$, must G be abelian?

Solution by Mary R. Embry, University of North Carolina at Charlotte. We answer in the affirmative and strengthen the proposition as follows: *Let G be a group in which no element has order 2. If $(xy)^2 = (yx)^2$ for all $x, y \in G$, then G is abelian.* Note first that for all $x, y \in G$ we have

$$x^2 = ((xy^{-1})y)^2 = (y(xy^{-1}))^2 = yx^2y^{-1},$$

so that $x^2y = xy^2$.

Further we have $x^{-1}y^{-1}x = x(x^{-1})^2y^{-1}x = xy^{-1}(x^{-1})^2x = xy^{-1}x^{-1}$. Similarly $y^{-1}x^{-1}y = yx^{-1}y^{-1}$. Setting $z = xyx^{-1}y^{-1}$, we have

$$\begin{aligned} z^2 &= xy(x^{-1}y^{-1}x)yx^{-1}y^{-1} = xy(xy^{-1}x^{-1})yx^{-1}y^{-1} = xyx(y^{-1}x^{-1}y)x^{-1}y^{-1} \\ &= xyx(yx^{-1}y^{-1})x^{-1}y^{-1} = (xy)^2(yx)^{-2} = (yx)^2(yx)^{-2} = e. \end{aligned}$$

Therefore $z = e$ and $yx = xy$.

Also solved by I. K. Abruob, W. O. Alltop, J. C. Barron, F. B. Cannonito, David M. Cohen, D. Ž. Djoković, Thomas Elsner, N. J. Fine, W. F. Fox, E. R. Gentile (Argentina), M. G. Greening (Australia), K. D. Joshi, John Kieffer, D. S. Lawrence, C. C. Lindner, Zachary Martin, W. G. McArthur, J. J. McClor, Peter Miletta & Richard Gisselquist, P. L. Montgomery, Wanda J. Mourant, T. N. Murphy, W. H. Patterson, Jr., D. E. Penney, Peter Perkins, Stanley Poreda, S. N. Rao, Frederick Sipinen, Erick Smith, R. C. Steinlage, Hugo Sun, Z. Z. Uoiea, and the proposer.

Concerning $\prod p_i - 1$

E 1997 [1967, 719]. *Proposed by D. Rameshwar Rao, Osmania University, India*

If P is a product of n distinct primes, then there exist distinct positive integers $a_i (i = 1, 2, \cdots, n)$ such that

$$P - 1 = \sum a_1 + \sum a_1a_2 + \cdots + \sum a_1a_2 \cdots a_{n-1} + a_1a_2 \cdots a_n.$$

The set of n a's is unique.

Solution by Frank R. Olson, State College at Fredonia, N. Y. We write

$$\begin{aligned} P = \prod_{i=1}^{n} p_i &= \prod_{i=1}^{n} ((p_i - 1) + 1) = \prod_{i=1}^{n} (a_i + 1) \\ &= 1 + \sum a_1 + \sum a_1a_2 + \cdots + a_1a_2 \cdots a_n, \end{aligned}$$

where the p_i are the prime factors of P and $a_i = p_i - 1$. Since P is the product of n distinct primes, P can be written as the product of n factors, each > 1, in only one way and consequently the a_i and distinct and unique.

Also solved by I. K. Abruob, M. G. Beumer (Netherlands), Lindley Burton, R. M. Caron, Mannis Charosh, Jerome Cherniack, Charles Chouteau, David M. Cohen, D. Ž. Djoković, Ragnar Dybvik (Norway), Thomas Elsner, J. A. Ewell, Neal Felsinger, H. M. Gehman, Thomas Givnish, Ray Glenn, Michael Goldberg, M. G. Greening, Clair Haberman, J. E. Homer, Jr., Aughtum Howard, Donald Jeffords, John Kieffer, Kenneth Kramer, Douglas Lind, R. Manohar & B. S. Lalli, D. C. B. Marsh, Helen M. Marston, R. M. Meredith, Michael Merscher, B. Míšek (Czechoslovakia), Barbara W. Nason, C. B. A. Peck, Dale Peterson, Stanley Rabinowitz, D. P. Roselle, Ira Rosenholtz, Peter Salamon, Perry Scheinok, A. J. Sommese, H. E. Thomas, Jr., A. M. Vaidya (India), C. S. Venkataraman (India), Julius Vogel, Jang Mei Wu (Japan), Gregory Wulczyn, Kenneth Young, David Zeitlin, and the proposer.

Editorial Comment. Beumer raises the problem of finding the number of distinct factorizations (aside from order) of $N=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$. He notes that no simple solution can be given, but that the similar but easier problem of finding all possible factorizations (taking order into account) of N into a product of k factors is solved in his paper, *The Arithmetical Function* $T_k(N)$, this MONTHLY, 69 (1962), 777–781.

Primes Having Given Properties

E 1998 [1967, 719]. *Proposed by C. F. Moppert, Melbourne University, Australia*

Characterize the rational primes p for which

$$(p+1)\phi(p-1)=\phi(p^2-1)\qquad \text{if } p\equiv -1 \pmod 4,$$
$$(p-1)\phi(p-1)=\phi(p^2-1)\qquad \text{if } p\equiv 1 \pmod 4,$$

where ϕ is Euler's totient function.

Solution by Wayne Sherrer, Student, Grade 10, *Phillipsburg* (*N. J*). *High School, and NSF summer program at Lehigh University.* The first formula holds if and only if p is a prime of the form 2^b-1, a Mersenne prime. The second formula holds if and only if p is a prime of the form $2q-1$, where q is also prime.

Substitute $p=4k+3$ into the first given formula. Standard manipulations, taking into account the fact that $8(k+1)$ and $2k+1$ must be relatively prime, yield $\phi[8(k+1)]=4(k+1)$. If k is even, this yields $\phi(k+1)=k+1$, hence $k=0$, so $p=3$. If k is odd, say, $k=2^at-1$, with t odd, this yields $\phi(t)=t$ and so $t=1$. Thus $p=2^b-1$, $b\geqq 2$.

Let $p=4k+1$ and substitute into the second given formula. This leads to $\phi[8k(2k+1)]=4k\phi(4k)$. If k is odd, this gives $\phi(2k+1)=2k$, hence $2k+1$ must be prime. If k is even, say $k=2^at$, with t odd, an argument similar to the one in the first part above leads to $\phi(2k+1)=2k$ and so $2k+1$ is prime.

These arguments show that the given forms of p are sufficient as well as necessary.

Also solved by I. K. Abruob, Merrill Barnebey, Joe Bechely, Lindley Burton, Mannis Charosh, D. Ž. Djoković, H. M. Edgar, J. A. Ewell, N. J. Fine, R. E. Giudici, M. G. Greening (Australia), Donald Jeffords, Lew Kowarski, D. C. B. Marsh, M. J. Merscher, S. S. Muchnick, Dale Peterson, D. P. Roselle, Ira Rosenholtz, A. J. Sommese, A. M. Vaidya (India), and the proposer.

Summations of Products of Binomial Coefficients

E 1999 [1967, 719]. *Proposed by L. Carlitz, Duke University*

$$\text{I. Put } S_n=\sum_{i+j+k=n}\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k}$$

where the summation is over all nonnegative i, j, k such that $i+j+k=n$. Show that $S_n-S_{n-1}=\binom{2n}{n}$. A combinatorial proof would be preferred.

$$\text{II. Put } R_{m,n}=\sum_{i+j+k\leqq\min(m,n)}\binom{i+j}{i}\binom{j+k}{k}\binom{m-i-j}{k}\binom{n-j-k}{i}.$$

Show that $R_{m,n}-2\,R_{m-1,n-1}=\binom{m+n}{m}$.

I. *Solution by M. G. Greening, The University of New South Wales, Kensington, Australia.* Consider the $\binom{2n}{n}$ ordered arrangements $R_t(t=1,2,\cdots,\binom{2n}{n})$ of n indistinguishable objects "a" and n indistinguishable objects "b". If among the first m elements of R_t there are r elements a, set $\alpha(m)=r$, $\beta(m)=m-r$. If it is possible to find nonnegative integers i, j, k $(i+j+k=n)$ such that $\alpha(i+j)=i$, $\alpha(n+j)=i+j$ for a particular R_t, say that this R_t *belongs* to the partition (i, j).

LEMMA A. *Each R_t belongs to some (i, j).*

Proof. Given a particular R_t note that $m_1>m_2$ implies $\alpha(m_1)\geqq\alpha(m_2)$, $\beta(m_1)\geqq\beta(m_2)$ and $0\leqq\alpha(m)$, $\beta(m)\leqq m$. For $0\leqq m\leqq n$, set $f(m)=m-\alpha(n+m-\alpha(m))=m-\alpha(n+\beta(m))$. $f(0)=-\alpha(n)$; $f(n)\geqq 0$ as $n-\alpha(d)\geqq 0$ for all d. But $f(m+1)-f(m)=1$ or 0, so $f(m')=0$ for some m'. Set $\alpha(m')=i$, $\beta(m')=j$, so that $\alpha(n+j)=m'=i+j$.

LEMMA B. *If R_t belongs to both (i, j) and (i', j'), then $i=i'$.*

Proof. Take $i'+j'>i+j$. Then (1) $i'=\alpha(i'+j')\geqq\alpha(i+j)=i$. (2) $j'=\beta(i'+j')\geqq\beta(i+j)=j$. Then $\beta(n+j')\geqq\beta(n+j)$ by (2), i.e., $n-i'=j'+k'=\beta(n+j')\geqq\beta(n+j)=j+k=n-i$; so $i\geqq i'$. This, with (1), gives $i=i'$.

Each R_t belongs to $\gamma(t)$ different partitions, $(i(t), j(t))$, $(i(t), j(t)+1)$, $\cdots$, $(i(t), j(t)+\gamma(t)-1)$, $1\leqq\gamma(t)\leqq k_t+1$. Say R_t belongs *properly* to the partition $(i(t), j(t))$. We have now

$$\sum_t\gamma(t)=\sum_{i+j+k=n}\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k}=S_n.$$

The right hand side counts the number of R_t belonging to each (i, j) with repetitions; the left hand side counts the number of repetitions for each t. Then $S_n-\binom{2n}{n}=\sum_t\{\gamma(t)-1\}$. The nonzero components of the right hand sum arise from those R_t belonging at least to both $(i(t), j(t))$ and $(i(t), j(t)+1)$. When summed over those R_t belonging to both (i_1, j_1) and (i_1, j_1+1) it is seen that $\sum\{\gamma(t)-1\}$ is equal to $\sum\gamma(t')$ summed over those ordered arrangements $R_{t'}$ of $(n-1)$ a's and $(n-1)$ b's belonging properly to (i_1, j_1). Noting that $i_1+j_1\leqq n-1$, we obtain $S_n-\binom{2n}{n}=\sum_{t'}\gamma(t')$ and this equals S_{n-1} as every $R_{t'}$ $(t'=1, 2,\cdots,\binom{2n-2}{n-1})$ belongs properly to some (i, j), $i+j\leqq n-1$. The removal of the (j_1+1)-th b and the (i_1+j_1+1)-th a yields an $R_{t'}$ with $\alpha(i_1+j_1)=i_1$, $\alpha(n-1+j_1)=i_1+j_1$, $\alpha(2n-2)=n-1$ with $i_1+j_1+k_1=n-1$. Thus $S_n-\binom{2n}{n}=S_{n-1}$.

II. *Solution by the proposer.* We have

$$\begin{aligned}
&\sum_{m,n=0}^{\infty}R_{m,n}x^my^n\\
&=\sum_{i,j,k=0}^{\infty}\binom{i+j}{i}\binom{j+k}{k}(xy)^{i+j+k}\cdot\sum_{m=0}^{\infty}\binom{m+k}{k}x^m\sum_{n=0}^{\infty}\binom{n+i}{i}y^i\\
&=\sum_{i,j,k=0}^{\infty}\binom{i+j}{i}\binom{j+k}{k}(xy)^{i+j+k}(1-x)^{-k-1}(1-y)^{-i-1}
\end{aligned}$$

$$= (1-x)^{-1}(1-y)^{-1}\sum_{j=0}^{\infty}(xy)^j\sum_{i=0}^{\infty}\binom{i+j}{i}\left(\frac{xy}{1-y}\right)^i\sum_{k=0}^{\infty}\binom{j+k}{k}\left(\frac{xy}{1-x}\right)^k$$

$$= (1-x)^{-1}(1-y)^{-1}\sum_{j=0}^{\infty}(xy)^j\left(1-\frac{xy}{1-y}\right)^{-j-1}\left(1-\frac{xy}{1-x}\right)^{-j-1}$$

$$= (1-x-xy)^{-1}(1-y-xy)^{-1}\left\{1-\frac{xy}{[1-xy/(1-x)][1-xy/(1-y)]}\right\}^{-1}$$

$$= \left\{(1-x-xy)(1-y-xy)-xy(1-x)(1-y)\right\}^{-1}$$

$$= \left\{(1-xy)^2-(1-xy)(x+y)+xy(x+y)-x^2y^2\right\}^{-1}$$

$$= (1-2xy)^{-1}(1-x-y)^{-1},$$

so that $(1-2xy)\sum_{m,n=0}^{\infty}R_{m,n}x^my^n=(1-x-y)^{-1}$. Therefore $R_{m,n}-2R_{m-1,n-1}=\binom{m+n}{m}$.

Editorial Note. Greening gave a lengthy solution of part II by extending the combinatorial argument given in I above. The proposer gave an algebraic solution of I.—M. Nassif (University of Khartoum, Sudan) also solved I.

The result shows that the summation in I can be put in the simple form $S_n=\sum_{i=0}^{n}\binom{2i}{i}$.

Polynomial Divisor

E2000 [1967, 719, 1132]. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let q be any prime number >2, and let m be any positive integer. Then show that

$$\sum_{n=1}^{q}x^{q-n}\,\Big|\,\left\{\sum_{m,n=1}^{q-1}x^{mn}+q-1\right\}.$$

I. *Solution by H. S. Hahn, West Georgia College.* Since the roots of the divisor are $r_k=\exp(2k\pi i/q)$ $(k=1,2,\cdots,q-1)$, it is enough to show that each satisfies the dividend. And it does, for

$$\sum_{m,n=1}^{q-1}x^{mn}=\sum_{n=1}^{q-1}\sum_{m=1}^{q-1}(x^n)^m=\sum_{n=1}^{q-1}\left(\frac{1-x^{nq}}{1-x}-1\right)$$

$$=\sum_{n=1}^{q-1}(-1)\qquad(\text{for } x=r_k)$$

$$=-q+1.$$

II. *Remark by L. Carlitz, Duke University.* The result is true for arbitrary q if we replace the divisor by the cyclotomic polynomial

$$F_q(x)=\sum_{rs=q}(x^r-1)^{\mu(s)}.$$

The proof is exactly the same as in the special case.

Another result of a similar kind is

$$(*)\qquad F_q(x)\Bigg|\left\{\sum_{m,n=1}^{q-1}{}' x^{mn}+\phi(q)\right\},$$

where $\phi(q)$ is the Euler ϕ-function and the summation is restricted to m, n prime to q. To prove (*) let ζ be a primitive qth root of unity and note that

$$\sum_{m,n=1}^{q-1}{}' \zeta^{mn}+\phi(q)=\phi(q)\sum_{k=1}^{q-1}\zeta^k+\phi(q)=\phi(q)\sum_{k=0}^{q-1}\zeta^k=\phi(q)\frac{1-\zeta^q}{1-\zeta}=0.$$

Also solved by Arnold Adelberg, Anders Bager (Denmark), Peter Bundschuh (Germany), D. Ž. Djoković, M. G. Greening (Australia), Emil Grosswald, Donald Jeffords, John Kieffer, Peter Kornya, E. S. Langford, D. C. B. Marsh, H. F. Mattson, Jr., Stanley Rabinowitz, Simeon Reich (Israel), D. P. Roselle, Al Somayajulu, Stephen Spindler, E. W. Trost (Switzerland), Gregory Wulczyn, K. L. Yocom, and the proposer.

Integration by Probability Considerations

E 2001 [1967, 719]. *Proposed by Harry Lass, California Institute of Technology*

Show from elementary probability considerations that

$$\int_0^1 x^k(1-x)^l dx=\frac{k!l!}{(k+l+1)!}\cdot$$

Solution by Jorge Dou, Barcelona, Spain. Let $k+l+1$ points be distributed independently and uniformly along the segment $[0, 1]$. The probability that the $(k+1)$-st point lies in $[x, x+dx]$ is

$$f(x)dx=(k+l+1)\cdot dx\cdot\binom{k+l}{k}\cdot x^k\cdot(1-x)^l.$$

Now the relation $\int_0^1 f(x)dx=1$ implies the desired result.

Also solved by J. L. Allen, R. A. Bell, L. J. Burton, E. M. Clark, N. L. Crawford, D. Ž. Djoković, E. J. Dudewicz, N. J. Fine, Michael Goldberg, Rudolf Gorenflo (Germany), M. G. Greening (Australia), J. E. Hafstrom, J. E. Homer, Jr., A. E. A. Hunt, W. D. Markel, G. B. Miller, Steven Minsker, Dale Peterson, F. J. Samaniego, Perry Scheinok, H. D. Shane, Eugene Sloane, W. B. Smith, Franz Streit, Julius Vogel, H. L. Walton, and the proposer.

A Triangle Construction

E 2002 [1967, 720, 860]. *Proposed by F. Luenberger, Feldmeilen, Switzerland*

Construct a triangle ABC, having given the two sides AB and AC, and knowing that the line joining the circumcenter and the incenter is perpendicular to the interior angle bisector of angle A.

Solution by J. Steinig, Zurich, Switzerland. It is known that the locus of all points in a triangle whose distances from the three sides have the same sum is a segment perpendicular to the line joining the triangle's incenter I to its circum-

center O. [E. J. F. Primrose, Math. Gaz., 45 (1961) 231–232.] Therefore, if IO is perpendicular to the interior angle bisector through A, we must have $h_a = 3r$, where r is the inradius of ABC, and h_a is the altitude through vertex A. Then, since $ah_a = r(a+b+c)$, we have $a = \frac{1}{2}(b+c)$, and the construction of ABC follows immediately.

Also solved by Leon Bankoff, H. Demir (Turkey), Michael Goldberg, M. G. Greening (Australia), C. V. Subbarama Iyer (India), Henrik Meyer (Denmark), Bohuslav Míšek (Czechoslovakia), J. R. Purdy, V. V. Rao (India), V. R. S. Raghavan (India), Dimitrios Vathis (Greece), C. S. Venkataraman (India), Oswald Wyler, and the proposer.

Venkataraman notes that, in the triangle as described, the line joining the centroid and the incenter is parallel to side BC.

Several solvers missed the corrected statement of the problem and submitted correct discussions of the original misprinted (trivial) version. Slobodan Ćuk (Yugoslavia) and Simeon Reich (Israel) solved the problem assuming that IO is parallel to the external bisector of angle A. Of course, this condition is equivalent to the correct version above.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate signed sheets and should be mailed before April 30, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5604 [1968, 686], Correction. *Proposed by R. A. Struble, North Carolina State University at Raleigh*

The second integrand in the formula should read $e^{a(t-s)}f(s)ds$ instead of $e^{-a(t-s)}f(s)ds$.

5620. *Proposed by Simeon Reich, The Technion, Haifa, Israel*

What is the nature of those natural numbers N with the following property: if $f(x)$ is any polynomial with integer coefficients such that $f(m) \equiv 0 \pmod{N}$ for all integers m, then N divides each coefficient of $f(x)$?

5621. *Proposed by J. P. Williams, Indiana University*

Let T be a bounded linear operator on a complex Hilbert space whose numerical radius

$$|W(T)| = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$$

does not exceed 1. Show that each eigenspace corresponding to an eigenvalue of modulus 1 decomposes T.

5622. *Proposed by D. S. Lawrence, Brooklyn Polytechnic Institute, New York*

Let X be a collection of sets of real numbers, each of which is well-ordered

by magnitude. Suppose also that X is simply ordered by inclusion. Show that X is denumerable.

5623. *Proposed by J. M. S. Simões Pereira, Gulbenkian Scientific Computing Center, Lisbon, Portugal*

Let $\{f_n\}$ be a sequence of continuous (real) functions such that $\sum_{n=0}^{\infty} f_n(x)$ is conditionally convergent in a neighborhood of a number c but the sum is not continuous at c. Is it possible that some rearrangement of the series will have a sum which is continuous at c?

5624. *Proposed by A. A. Mullin, Taegu, Korea*

Let $\boldsymbol{A}$ be the set of all nonconstant analytic functions, let C be the set of all complex numbers, and let N be the set of all nonnegative integers. Put F^2 for $F(F(\cdot))$ and let F^n be defined recursively. Let $F^{(n)}$ be the nth derivative of F. Does there exist a function $F\in\boldsymbol{A}$ such that $F^2=F^{(1)}$? Indeed, does there exist $F\in\boldsymbol{A}$ such that for each $n\in N$, $F^{n+1}(z)=F^{(n)}(z)$ for every $z\in C$?

5625. *Proposed by R. C. Freiwald, University of Rochester*

Let Y be an arbitrary topological space. We say Y has the G_σ property if and only if for every topological space X and every function $f: X\rightarrow Y$, the points of continuity of f are a G_σ set in X. (a) Prove that if Y is metric, then Y has the G_σ property. (b) Show that a compact T_2 space may fail to have the G_σ property. (c) Suppose Y is a normal T_1 space having the G_σ property, is Y necessarily metrisable? (d) Characterize those spaces Y having the G_σ property.

5626. *Proposed by D. S. Mitronović, University of Belgrade, Yugoslavia*

1. Determine those algebraic functions $A_k(x)$, $k=2, 3, \cdots$, which enjoy the following properties:

$$
\begin{aligned}
&\log x/(x-1) \leqq A_k(x) && (x>0),\\
&A_k(x) \sim x^{-1/k} && (x\rightarrow 0^+),\\
&xA_k(x) \sim x^{1/k} && (x\rightarrow +\infty),\\
&A_k(x)-\log x/(x-1) \sim a_k(x-1)^{2k-2} && (x\rightarrow 1),
\end{aligned}
$$

where a_k is independent of x.

J. Karamata (Bull. Soc. Math. Phys. Serbie, *I*(1949), p. 77–78) and D. Blanusa (*ibid.*, I(1949), p. 156–157) have given, without proof, the forms of $A_2(x)$, $A_3(x)$, and $A_4(x)$. For example,

$$A_3(x) = (1+\sqrt[3]{x})/(x+\sqrt[3]{x}).$$

2. Find also algebraic functions $A_k(x)$ $(k=2, 3, \cdots)$ having the following properties:

$$\log x/(x-1) \leqq A_k(x) \quad (x>0), \qquad A_m(x) \geqq A_n(x) \quad (x>0),$$

with $2 \leqq m < n$.

5627. *Proposed by Hansjoachim Groh, University of Florida*

Let K be a commutative topological field. Let S be the set of squares in K, and let A be the equivalence relation on $K \times K$ defined by $(k_1, k_2)A(k_1' k_2')$ if and only if $\{k_1, k_2\} = \{k_1', k_2'\}$. Is the square root function $f\colon S \to K \times K/A$, defined by $f(k^2) = \{k, -k\}$ always continuous?

5628. *Proposed by Raymond Redheffer, University of California, Los Angeles*

If z is complex and p is a nonnegative integer, prove that

$$\log \left| (1-z)\exp(z + \tfrac{1}{2}z^2 + \cdots + (1/p)z^p) \right| \leqq c_p \min(|z|^p, |z|^{p+1}),$$

where $c_p = 1 + 2\log(1+p)$. Can the optimum c_p be determined?

SOLUTIONS OF ADVANCED PROBLEMS

The Automorphism Group of D_n

5521 [1967, 1014]. *Proposed by I. K. Abruob.*

Prove that the automorphism group of D_n (the dihedral group of order $2n$) is isomorphic to the holomorph of Z_n (the integers, mod n).

Solution by Kenneth Yanosko, Ohio State University. Let $D_n = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b\rangle$ and let $\alpha \in \mathrm{Aut}\, D_n$. Then $a^\alpha = a^s b^t$ and $b^\alpha = a^u b^v$ where the orders of $a^s b^t$ and $a^u b^v$ are n and 2 respectively. Hence if $n > 2$ we must have $(s, n) = 1$, $t = 0$, $v = 1$. If we define automorphisms σ_i and τ^j with $0 < i < n$, $(i, n) = 1$, and $0 \leqq j < n$ such that

$$a^{\sigma_i} = a^i, \quad a^{\tau^j} = a, \quad b^{\sigma_i} = b, \quad b^{\tau^j} = a^j b,$$

we have $\alpha = \sigma_s \tau^u$ so that $\mathrm{Aut}\, D_n = \{\sigma_i \tau^j\}$.

Now $\{\tau^j\}$ is a cyclic group of order n and we may identify $\{\sigma_i\}$ with $\mathrm{Aut}\{\tau^j\}$ by defining $(\tau^1)^{\sigma_i} = \tau^i$. It is easy to check that

$$(\sigma_i \tau^j)(\sigma_k \tau^m) = \sigma_i \sigma_k (\tau^j)^{\sigma_k} \tau^m,$$

so that $\mathrm{Aut}\, D_n$ is the semidirect product of $\{\tau^j\}$ and $\{\sigma_i\}$, which is isomorphic to the semidirect product of Z_n and $\mathrm{Aut}\, Z_n$, which is the holomorph of Z_n.

Also solved by D. S. Brouder, M. G. Greening (Australia), Rudolf Kochendörffer (Germany), Harsh Pittie, Hugo Sun, George Whitson, and Oswald Wyler.

Sun comments that the present problem is a special case of a theorem on p. 169 of Miller, Blichfeldt and Dickson, *Theory and Application of Finite Groups*, Dover, 1961.

Note. When this problem appeared it was erroneously attributed to the communicant. We have designated the correct proposer's name above.

Cyclotomic Polynomials of Degree $2p$

5525 [1967, 1014]. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let p be a prime such that $2p+1$ is not a prime. Show that there is no cyclotomic polynomial of degree $2p$.

Solution by Stanley F. Robinson, Eastern Washington State College. Since the degree of the nth cyclotomic polynomial is $\phi(n)$, the question is then one concerning the solution of $\phi(x)=2p$. C. L. Klee, in a note *On the equation* $\phi(x)=2m$ [this MONTHLY, 53 (1946), 327] proved that $\phi(x)=2m$ has no solution if m has no divisor $d>1$ for which $2d+1$ is prime. This suffices to establish the present result.

Also solved by Anders Bager (Denmark), M. G. Beumer (Netherlands), Benedict Carlat, H. M. Edgar, Michael Goldberg, M. G. Greening (Australia), Erwin Just, M. S. Klamkin, R. B. Lakein, Eric Langford, Douglas Lind, Roger Lyndon, W. G. McArthur, Wanda J. Mourant, Simeon Reich (Israel), Stephen Tice, A. M. Vaidya (India), W. C. Waterhouse, Steven Weintraub, Oswald Wyler, and the proposer.

Beumer and Vaidya prove that if $2p+1$ is not a prime then $\phi(x)=2p^s$ has no solution. These are included among results given by Alois Pilcher for which Langford has provided the reference, Dickson, *History of the Theory of Numbers*, Vol. I, p. 135. Lind refers to problem 601 in the Mathematics Magazine, 39 (1966), p. 190, which includes a solution of the problem above. Beumer provides references which contain recent results on the equation $\phi(x)=a$: problem 4995 [1963, 101]; A. Schenzel, *Sur l'équation* $\phi(x)=m$, Elemente der Mathematik, XI (1956), 75, 78.

Beumer also asks (i) for which numbers b are there no solutions for $\phi(x)=2b$? (ii) are there infinitely many primes q for which $\phi(x)=2q$ has a solution?

Packing Circles in a Unit Disk

5528 [1967, 1015]. *Proposed by L. A. Steen, Saint Olaf College, Northfield, Minn.*

Let $\{D_i\}_1^\infty$ be a sequence of pairwise disjoint open disks contained inside the unit disk; let r_i denote the radius of D_i. Prove that

$$\sum_{i=1}^{\infty} r_i^2 < 1 \quad \text{whenever} \quad \sum_{i=1}^{\infty} r_i < \infty.$$

(In other words: Whenever you remove from the interior of the closed unit disk a sequence of pairwise disjoint open disks whose boundaries are of finite total length, the remaining set has positive Lebesgue measure.)

Solution by Roger A. Horn, University of Santa Clara, California. The analog of this problem in n dimensions is solved by Oscar Wesler in *An infinite packing theorem for spheres*, Proc. Amer. Math. Soc., 11 (1960), 324–326. The theorem of this problem was first given by S. N. Mergelyan, and relevant references may be found in Wesler's paper.

Also solved by R. B. Crittenden, D. A. Hejhal, H. E. Reinhardt, J. B. Wilker, Lawrence Zalcman, and the proposer.

For a more complete reference list on this problem and its function-theoretic implications see L. Zalcman, *Analytic Capacity and Rational Approximation* (Springer, Lecture Notes in Mathematics).

$$\int \prod_j \frac{\sin k_j(x-a_j)}{x-a_j}\,dx$$

5529 [1967, 1015]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Evaluate

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{\sin k_j(x-a_j)}{x-a_j}\,dx,$$

with k_j, a_j, $j=1, 2, \cdots, n$ real numbers.

Solution by M. L. Glasser and R. P. Kenan, Battelle Memorial Institute. Let I represent the given integral and assume, without changing the problem, that $k_1 \geqq k_2 \geqq \cdots \geqq k_n \geqq 0$. Note that

$$(1) \qquad z^{-1}\sin kz = \frac{1}{2}\int_{-k}^{k} \exp(izy)dy, \qquad (2) \qquad \int_{-\infty}^{\infty} \exp(izx)dx = 2\pi\delta(z).$$

(For the methods of operation with the Dirac δ-function see, e.g., M. J. Lightfoot, *Fourier Analysis and Generalized Functions*, Cambridge, 1958.) Then by (1) and (2),

$$\begin{aligned} I &= \frac{1}{2^n}\int_{-\infty}^{\infty} dx \int_{-k_1}^{k_1} dy_1 \cdots \int_{-k_n}^{k_n} dy_n \exp\left(i\sum_{j=1}^{n}(x-a_j)y_j\right) \\ &= \frac{\pi}{2^{n-1}}\int_{-k_1}^{k_1} dy_1 \cdots \int_{-k_n}^{k_n} dy_n \delta(y_1+\cdots+y_n)\exp\left(-i\sum_{j=1}^{n} a_j y_j\right). \end{aligned}$$

By integration over y_1,

$$I = \frac{\pi}{2^{n-1}}\int_{-k_n}^{k_n} dy_n \cdots \int_{-k_2}^{k_2} dy_2 \exp\left(i\sum_{j=2}^{n}(a_1-a_j)y_j\right),$$

where $|y_2+\cdots+y_n|$ is to be $\leqq k_1$ in the integration since $k_2 \leqq k_1$. Then

$$I = \frac{\pi}{2^{n-1}}\int_{-k_n}^{k_n} dy_n \cdots \int_{-k_3}^{k_3} dy_3 \int_{-l}^{l} dy_2 \exp\left(i\sum_{j=2}^{n}(a_1-a_j)y_j\right),$$

where $l = k_2 - \sum_{j=3}^{n} y_j$, and again $|y_3+\cdots+y_n|$ is to be $\leqq k_2$ in the integration. The integration over y_2 can now be done and we find

$$I = \pi\,\frac{\sin k_2(a_1-a_2)}{a_1-a_2}\,\frac{1}{2^{n-2}}\int_{-k_n}^{k_n} dy_n \cdots \int_{-k_3}^{k_3} dy_3 \exp\left(i\sum_{j=3}^{n}(a_2-a_j)y_j\right),$$

with $|y_3+\cdots+y_n|\leqq k_2$. Proceeding in this way, we find

$$I=\pi\frac{\sin k_2(a_1-a_2)}{(a_1-a_2)}\frac{\sin k_3(a_2-a_3)}{(a_2-a_3)}\cdots\frac{\sin k_n(a_{n-1}-a_n)}{(a_{n-1}-a_n)}$$
$$\pi\prod_{j=2}^{n}\frac{\sin k_j(a_{j-1}-a_j)}{a_{j-1}-a_j}.$$

Also solved by M. A. Ettrick, and by Chang Sung-sheng (Taiwan).

Monotone Decreasing Functions

5530 [1967, 1143]. *Proposed by S. S. Mitra and J. R. Porter, University of Oklahoma*

Let I be the unit interval, and let $f: I\to I$ be a continuous function such that $f(f(x))\equiv x$. Prove or disprove that if f is monotone decreasing, then $f(x)\equiv 1-x$.

Solution by Wanda J. Mourant, Denison University, Ohio. If a is an interior point of I and if g is any continuous 1-1 monotone decreasing function which maps $[0, a]$ onto $[a, 1]$ then g has an inverse, g^{-1}, which is continuous, 1-1 and monotone decreasing and maps $[a, 1]$ onto $[0, a]$. If $f(x)=g(x)$ for $x\in[0, a]$, and $f(x)=g^{-1}(x)$ for $x\in(a, 1]$, then $f: I\to I$ is continuous, monotone decreasing and $f(f(x))\equiv x$.

Also solved by D. T. Adams, B. C. Anderson, D. R. Anderson, K. W. Anderson & D. W. Hall, Anders Bager (Denmark), I. N. Baker (England), Dean Bandes, C. L. Bandy, S. L. Bloom, W. D. Bouwsma, G. R. Cash & R. H. Raines, R. E. Chandler, V. R. Chandran (India), P. R. Chernoff & W. C. Waterhouse, Hu Chi-ping (Taiwan), Charles Chouteau, R. A. Christiansen, John Cobb, Alexander Cramer, Peter Csontos & Peter Renz & Bert Schreiber, Ted Cullen, R. J. Cormier, W. A. Darbro, L. W. Deaton, P. De Munter (Jordan), E. J. Denes, D. Ž. Djoković, W. G. Dotson, Jr., R. C. Entringer, N. J. Fine, W. F. Fox, Charles Hanna, D. A. Hejhal, Robert Heller, Dennis Henkel, Ellen S. Hertz, G. A. Heuer, John Kelly, Michael Keyton, C. J. Knight (England), G. A. Kraus, E. S. Langford, Mrs. R. S. Lee, Andrzej Makowski (Poland), Dan Marcus, Osvaldo Marrero, W. D. Maurer, M. D. Mavinkurve (India), R. F. McDermot, Renate McLaughlin, A. Meir, Ka Menehune, B. W. Miller, Bohuslav Míšek (Czechoslovakia), George Mitchell, T. A. Mossman, M. E. Muldoon, P. J. Murray, Hugh Noland, G. B. Parrish & A. S. Galbraith, C. B. A. Peck, George Piranian, S. J. Poreda, Bryan Powers, C. M. Price, E. J. F. Primrose & A. Weinmann (England), Henry Ricardo, Ira Rosenholtz, F. G. Schmitt, Jr., Bl. Sendov (Bulgaria), Michael Skalsky, J. F. Smith, William Smythe, Hugh Sun, Alberto Torchinsky, Zalman Usiskin, R. W. Wagner, K. E. Whipple, W. T. Whitley, A. C. Williams, J. C. Williams, S. V. Witt, R. S. C. Wong, Oswald Wyler, and the proposers.

For a general discussion of the problem reference is made to Anderson and Hall, *Sets, Sequences and Mappings*, pp. 136–141. See also Fine and Schweigert, Annals of Math., 62 (1955), pp. 237 ff. Sendov, referring to Charles Babbage as the "first computer man" provides the reference to Babbage's *Examples of the Solutions of Functional Equations*, London 1820.

Terms in a Convergent Series

5531 [1967, 1143]. *Proposed by R. O. Davies, The University, Leicester, England*

Prove that if $\sum u_m$ is a convergent series of nonzero terms and u_m-u_{m+1} is decreasing, then $u_{m+1}^{-1}-u_m^{-1}\to\infty$ as $m\to\infty$.

Solution by A. Meir, University of Alberta, Edmonton. Since $\sum(u_m - u_{m+1})$ is convergent and $u_m - u_{m+1}$ is decreasing, it follows that $u_m > u_{m+1} > 0$. Then

$$0 < \frac{u_m u_{m+1}}{u_m - u_{m+1}} \leq \frac{u_m^2}{u_m - u_{m+1}} = \frac{1}{u_m - u_{m+1}} \cdot \sum_{k=m}^{\infty} (u_k^2 - u_{k+1}^2)$$

$$\leq \sum_{k=m}^{\infty} (u_k + u_{k+1}) \to 0$$

as $m \to \infty$. Therefore $u_{m+1}^{-1} - u_m^{-1} \to \infty$.

Also solved by D. Borwein, Peter Bundschuh (Germany), R. A. Christiansen, D. F. Dawson, D. Ž. Djoković, Bengt Fornberg (Sweden), D. A. Hejhal, O. P. Lossers (Netherlands), G. V. McWilliams, R. A. Moore, Hugh Noland, Jonathan Ryshpan, Allen Saleski, J. P. Schroeter, Michael Skalsky, S. B. Weinberg, Chi Song Wong, P. H. Young, and the proposer.

Endomorphism of an Abelian p-group

5532 [1967, 1143]. *Proposed by Ka Menehune, University of Hawaii*

Let G be an abelian p-group, and f an endomorphism of G. If $f/G[p] = 1$, where $G[p] = \{x : x \in G, px = 0\}$, then f is an automorphism of G. Is f necessarily the identity of G?

Solution by Jonathan Ryshpan, University of Wisconsin. A. Suppose $x \neq 0$, but $f(x) = 0$. Let k be the least integer for which $p^{k+1}x = 0$. Then $p^k x \neq 0$ and $p \cdot (p^k x) = 0$. We have $f(p^k x) = p^k x \neq 0$, whereas $f(p^k x) = p^k f(x) = p^k \cdot 0 = 0$. So $\ker f = 0$ and f is an automorphism.

B. f need not be the identity, for the following reason. Let G be any abelian p-group in which there is an x for which $px \neq 0$, (e.g. $G = Z_{p^2}$). Put $f(x) = (p+1)x$. Then $f(x) = x \Leftrightarrow (p+1)x = x \Leftrightarrow px = 0$.

Also solved by Frank Castagna & Ray Mines, C. R. Chandran (India), D. Ž. Djoković, M. G. Greening (Australia), R. B. Hardin, Jr., E. C. Hook, Rudolf Kochendörffer (Germany), Dan Marcus, K. M. Rangaswamy, D. P. Sumner, Hugo Sun, Earl Taft, Antonio Villanueva, W. C. Waterhouse, Chi Song Wong, Oswald Wyler, Kenneth Yanosko, and the proposer.

The Solution of a Matrix Equation with Doubly-Stochastic Coefficients

5534 [1967, 1144]. *Proposed by Richard Sinkhorn, University of Houston*

Let A and B be $n \times n$ irreducible doubly stochastic matrices. Show that any matrix X satisfying $AX = XB$ has constant row and column sums.

Solution by T. L. Markham, University of North Carolina at Charlotte. Assume $X \neq 0$, and $AX = XB$. Let e denote the $n \times 1$ vector with each coordinate equal to one. Then $Ae = e$ and $Be = e$. Moreover, we have $A(Xe) = (AX)e = (XB)e = X(Be) = Xe$, and Xe is a characteristic vector of A corresponding to the maximal simple root, 1. Since A is irreducible, $\{Xe, e\}$ is a linearly dependent set, $Xe = \alpha e$. Hence the row sums of X are constant. Similarly the column sums of X are constant, and summing all the elements of X we find these constants to be equal.

Also solved by A. J. Bosch (Netherlands), A. S. Householder, M. S. Lynn, Henryk Minc, M. F. Neuts, P. J. Nikolai, Stephen Pierce, Dan Richman, R. C. Thompson, and the proposer.

The result of the problem appears in M. Marcus, H. Minc and B. Moyls, *Some Results on Non-Negative Matrices*, Journal of Research of the National Bureau of Standards, Vol. 65B, No. 3, p. 207.

Probability on the Size of a Population

5536 [1967, 1144]. *Proposed by Herbert Robbins, Purdue University*

We confront an urn containing an unknown number N of similar balls. We are allowed to draw as many balls as we like, one at a time, marking each ball drawn (so we will recognize it if we ever draw it again) and replacing it before the next ball is drawn. We must stop eventually and guess the value of N. Is there a procedure for doing this with the property that the probability that our guess will be exactly correct is $\geqq .999$ uniformly for all $N \geqq 1$?

Solution by F. G. Schmitt, Jr., Berkeley, California. The following procedure will suffice. Continue sampling from the urn only as long as there are at least k distinct balls in the first $5k^2$ samples, $k=1, 2, 3, \cdots$. When the sampling stops, there are then, let us say, exactly X distinct balls in the first $5(X+1)^2$ samples. The random variable X takes values between 1 and N. For n, $1 \leqq n \leqq N-1$, we have

$$\begin{aligned} P(X = n) &\leqq \binom{N}{n}\left(\frac{n}{N}\right)^{5(n+1)^2} \leqq \left(\frac{n^n}{n!}\right)\left(\frac{n}{N}\right)^{5(n+1)^2-n} \\ &< e^n\left(\frac{n}{n+1}\right)^{5(n+1)^2-n} = e^n\left(1-\frac{1}{n+1}\right)^{5(n+1)^2}\left(1+\frac{1}{n}\right)^n \\ &< e^n e^{-5(n+1)} e = e^{-4(n+1)}. \end{aligned}$$

Therefore

$$P(X < N) = \sum_{n=1}^{N-1} P(X = n) < \sum_{n=1}^{\infty} e^{-4(n+1)} = \frac{e^{-8}}{1-e^{-4}} < .001,$$

so $P(x=N) > .999$, uniformly for all $N \geqq 1$.

Also solved by W. O. Alltop, D. R. Barr, E. J. Dudewicz, Michael Goodman, Dennis Henkel, Ellen S. Hertz, Hugh Noland, Bart Park, and the proposer.

Dudewicz informs us that a solution may be found in Darling and Robbins, *Finding the size of a finite population*, Annals of Mathematical Statistics, vol. 38 (1967).

Euler's Constant

5537 [1967, 1144]. *Proposed by Sidney Heller, Brookhaven National Laboratory*

Show that

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{A_n}{n} = C, \quad \text{Euler's constant,}$$

where A_n is the determinant $|a_{ij}|$ such that

$$a_{ij} = \begin{cases} (i-j+2)^{-1} & \text{for } i-j+2>0 \\ 0 & \text{for } i-j+2 \leqq 0. \end{cases}$$

Solution by R. E. Shafer, University of California at Livermore. Expanding the determinants, we obtain $A_1 = \frac{1}{2}$,

$$A_n = \frac{1}{2} A_{n-1} - \frac{1}{3} A_{n-2} + \frac{1}{4} A_{n-3} + \cdots + \frac{(-1)^n}{n} A_1 + \frac{(-1)^{n+1}}{n+1},$$

$n=2, 3, 4, \cdots$. If we formally multiply the two functions $\log(1-Z)$ and $F(Z) = (1 - A_1 Z + A_2 Z^2 - A_3 Z^3 + \cdots)$ and apply the recurrence relation, then $F(Z)\log(1-Z) = -Z$. The desired sum is

$$\int_0^1 \frac{1-F(Z)}{Z}\, dZ = \int_0^1 \left\{ \frac{1}{Z} + \frac{1}{\log(1-Z)} \right\} dZ = C.$$

The convergence of the integral presents no problems.

Also solved by M. G. Beumer (Netherlands), L. Carlitz, E. S. Langford, J. H. van Lint (Netherlands), and the proposer.

Beumer goes further to exhibit the expression

$$. C = -\lim_{h\to\infty} \begin{vmatrix} 0 & 1 & \frac{1}{2} & \cdots 1/h \\ \frac{1}{2} & 1 & 0 & \cdots 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1/(h+1) & 1/h & 1/(h-1) & \cdots 1 \end{vmatrix}.$$

He also observes that the series which results from the problem,

$$C = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \cdots$$

is more rapidly convergent than Vacca's series for C: $\sum_{h=2}^{\infty} (h^2 \log h)^{-1}$.

Products of Elements in a Hamel Basis

5538 [1967, 1144]. *Proposed by L. F. Meyers, Ohio State University*

Is it possible to find a Hamel basis of the real numbers over the rational numbers which is closed under ordinary multiplication of real numbers?

I. *Solution by R. D. Berlin, General Electric Company, Syracuse, N. Y.* No, for suppose there did exist such a basis, $\{h_i\}$. An algebraic number different from 1 could not belong since closure would require that each of its distinct powers belong also; whereas a sufficient number of successive powers of an algebraic number must (by definition) be linearly dependent. 1 must belong, for otherwise in

$$(1) \qquad 1 = \sum_i \alpha_{k_i} h_{k_i} = \left(\sum \alpha_{k_i} h_{k_i} \right)^2$$

at least the squares of the smallest and largest of the h_i would appear once as a basis element in the last sum but not at all in the first.

Let $1=h_0$ and let Z be an irrational number with a rational square. Since a rational multiple of Z cannot belong to the basis we have

$$Z^2 = Z^2\cdot h_0 = \left(\sum_i \beta_{k_i} h_{k_i}\right)^2, \tag{2}$$

where the sum is the representation of Z in the basis. The remark following (1) applied now to (2) leads to the required contradiction.

II. *Solution by L. J. Lardy, Syracuse University.* It is impossible. If $[b_i: i\in I]$ were such a Hamel basis, then the map ϕ defined for each real number $x=\sum r_i b_i$, r_i rational, by $\phi(\sum r_i b_i)=\sum r_i$ would be a nontrivial ring homomorphism from the reals onto the rationals. Such a homomorphism does not exist.

Also solved by B. C. Anderson, P. R. Chernoff, N. J. Fine, D. A. Hejhal, Dennis Henkel, Erwin Just, R. B. Killgrove & E. J. Eckert, Dan Markus, Ka Menehune, Sidney Penner, Donald Quiring, Charles Riley, D. P. Sumner, Oswald Wyler, and the proposer.

A. Abian and the proposer state and prove the following generalization: *Let A be a simple associative algebra over a nontrivial ring R with unit element e. Suppose that A has some nonempty basis over R which is closed under multiplication. Then A is one-dimensional over R.*

The Equation $f(f(x))=k^2x$

5539 [1967, 1144]. *Proposed by Robert Breusch, Amherst College*

Find all the continuous, nonnegative valued, strictly increasing functions f, defined for $x\geqq 0$, such that for all $x\geqq 0$,

$$f(f(x)) = k^2x, \quad \text{with } k>1.$$

Solution by M. E. Muldoon, York University, Toronto. It is necessary and sufficient that:

(a) $f(0)=0$;

(b) for some $\alpha>0$, $f(\alpha)=k\alpha$ and $f(k\alpha)=k^2\alpha$;

(c) f is continuous and strictly increasing on the interval $\alpha\leqq x\leqq k\alpha$;

(d) for each interval of the sequence $k^n\alpha\leqq x\leqq k^{n+1}\alpha$, $n=1, 2, \cdots$, f is defined in terms of its values in the preceding interval of the sequence by

$$f(x) = k^2f^{-1}(x);$$

(e) for each interval of the sequence $k^{-n-1}\alpha\leqq x\leqq k^{-n}\alpha$, $n=0, 1, 2, \cdots$, f is defined in terms of its values in the preceding interval of the sequence by

$$f(x) = f^{-1}(k^2x).$$

The proof of the sufficiency of these conditions is direct. The condition (c) is clearly necessary. We note that if $f(x)\not\equiv kx$, then for some $\beta>0$, $f(\beta)<k\beta$ or $f(\beta)>k\beta$. In the former case we have $f(k\beta)>f(f(\beta))=k(k\beta)$ so that there exists an α, $\beta<\alpha<k\beta$, for which $f(\alpha)=k\alpha$. Similarly such an α exists if $f(\beta)>k\beta$. Now

the condition $f(f(x))=k^2x$ shows that $f(k\alpha)=k^2\alpha$. Thus the condition (b) is necessary. The necessity of (d) and (e) follows from the condition $f(f(x))=k^2x$, together with (b) and (c). Finally, (a) follows from (e) and the continuity of f.

Also solved by A. S. Adikesavan (India), W. D. Bouwsma, D. K. Cohoon & G. A. Kraus, Ted Cullen, D. Ž. Djoković, N. J. Fine, R. C. Entringer, D. A. Hejhal, W. D. Maurer, M. D. Mavinkurve (India), A. Meir, Ka Menehune, Charles Riley, R. W. Wagner, K. E. Whipple, A. C. Williams, Oswald Wyler, and the proposer.

Editorial Notes. Mavinkurve offers the example $f(x)=2x[1+\frac{1}{4}(x-1)(2-x)]$ for $1\leqq x\leqq 2$ and extended over the real line by the construction given above to exhibit a solution in the case $k=2$. Menehune observes that if $k=1$, then the only solution is $f(x)\equiv x$; but a construction is possible when $k^2<1$. Cohoon and Kraus and Maurer observe that the problem may be generalized with similar results by replacing k^2x with $h(x)$, a strictly increasing function $>x$. Adikesavan also generalizes by offering the equation $f(f(f\cdots(x)))=k^nx$, using n iterates of f. For additional results in the direction of the problem, Fine refers us to the paper, Fine and Schweigert, *On the group of homomorphisms of an arc*, Annals of Math., 62 (1955) 237, ff.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. All unsigned material is by the editor. Correspondence about Reviews will be welcome.

Beginning with the January 1969 issue, film reviews will be edited by Seymour Schuster, Carleton College, Northfield, Minnesota 55057. All correspondence concerning films should be sent to him.

Foundations of Constructive Analysis. By Errett Bishop (Univ. of California, San Diego). McGraw-Hill, New York, 1967. xiii+370 pp. \$12.00. (Telegraphic Review, May 1968.)

In the second half of the nineteenth century, Leopold Kronecker made a determined attempt to turn mathematics away from its trend of ever increasing abstraction. His approach was based on the principle that in order to be meaningful, an existential assertion has to be buttressed by the actual construction of the object in question. Thus, a procedure which leads us to infer the existence of a mathematical object from purely formal-deductive considerations, e.g. by the use of the principle of the excluded middle, is regarded as inadequate or even misleading. Kronecker lent substance to his point of view by actually realizing the constructive approach in his lectures. Later, Brouwer, whose approach was based on the same attitude, went beyond Kronecker by developing a theory of the continuum which may be called conditionally constructive since it accepts the idea of a sequence of free choices as the basis of the theory of real numbers (just as even the most restrictive point of view accepts the unlimited counting process as the basis of arithmetic). Brouwer's school of thought—intuitionism

—has remained the most vigorous of the several constructivist trends that have developed since Kronecker. Among those who have shown a consistently constructivist attitude we may mention Heyting and, more recently, Lorenzen and Fitch. Others, such as Kreisel and Kleene have carried out searching investigations of intuitionism without committing themselves to its philosophy, while still other mathematicians have come out in favor of a constructivist philosophy without living up to their principles in their mathematics.

The present author's point of view is essentially Kronecker's. He rejects the formalized versions of intuitionism produced by Heyting and others, mentioned above, as well as Brouwer's theory of the continuum. On this basis, he provides a constructive development of some of the most important areas of classical and modern analysis and uses his great knowledge and power as an analyst to cope constructively with topics as advanced as the duality theory of locally compact abelian groups and the theory of Banach algebras.

The sections of the book that attempt to describe the philosophical and historical background of this remarkable endeavor are more vigorous than accurate and tend to belittle or ignore the efforts of others who have worked in the same general direction.

ABRAHAM ROBINSON, Yale University

Studies in Global Geometry and Analysis. Edited by S. S. Chern. Vol. 4, Studies in Mathematics. Mathematical Association of America, distributed by Prentice-Hall, Englewood Cliffs, N. J., 1967. 197 pp. $6.00. ($3.00 for a single copy to members of the MAA) (Telegraphic Review, January 1968.)

The book, like the others in the same series, is intended as supplementary reading and introduction to a rapidly expanding field of research. The exposition is throughout of high quality. For texts intended to stimulate the independent study of the field, the bibliographies of the first two papers are inadequate. Some editorial coordination of the articles also would have helped. These criticisms notwithstanding, the book is highly recommended for every mathematician. In particular, the paper by Flanders is a model for the combination of elegance and explicitness that should be ideal in mathematical exposition.

H. W. GUGGENHEIMER, Polytechnic Institute of Brooklyn

Lecture Notes on Elementary Topology and Geometry. By I. M. Singer (M.I.T.) and John A. Thorpe (Haverford College). Scott, Foresman, Glenview, Ill., 1967. 214 pp., $6.25 (paper). (Telegraphic Review, November 1967.)

This well organized and stimulating book is a set of notes designed for a one year course with prerequisites of at least one semester of modern algebra and one semester of advanced calculus "done right." In my opinion, such a course would be an excellent introduction to modern differential geometry and topology; it could well replace the standard courses in differential geometry and general topology. It would certainly fulfill the authors' objectives of revealing to the undergraduate student some of the unity and excitement of mathematics.

In order to use this book for the course the authors describe, both students and instructor should be well prepared. Spivak's book, *Calculus on Manifolds*, (W. A. Benjamin, Inc., New York), would be an excellent background. The instructor should be ready to supply motivating examples and applications. For instance, the book stops short of completely classifying covering spaces and computing the automorphism group of an arbitrary covering. These would make good exercises, but someone must suggest them. (There are frequent stimulating remarks, but no formal exercises.) After Chapter 7, very little more work is required to prove, for example, that every compact surface of constant curvature is covered by one of the standard models. This would tie in nicely with the chapter on covering spaces. Applications of de Rham's theorem should be given.

The authors cannot be faulted for these omissions, since they modestly call their book "notes." Nevertheless, I would encourage them to expand their excellent Notes into what would be a most useful book.

M. W. HIRSCH, University of California, Berkeley

Stories About Sets. By N. Ya. Vilenkin. Academic Press, New York, 1968. xiii +152 pp. $6.50 (cloth), $2.95 (paper).

Professor Vilenkin has written a delightful book which can be enjoyed by college freshmen as well as by mature mathematicians. Beginning with the development of the notion of cardinality and employing an erudite interstellar traveller, Ion the Quiet, the author explores the properties of infinite sets and the existence and arithmetic of transfinite numbers. Using these concepts as a basis, he develops the notions of function and curve by "a stroll through a mathematical art museum" ending with P. S. Urysohn's inductive definition of dimension.

In presenting advanced mathematics from an elementary point of view, with the treatment of topics at a high level of integrity, this volume fulfills a need in the literature. Even mathematicians not usually attracted by popularizations will delight in its elegant style and wealth of examples, and many will find themselves adopting Ion's viewpoint and enlarging upon his imaginary experience. Though there may be no royal road to the transfinite, Professor Vilenkin's small book provides a princely path to those regions. It is recommended for all libraries and as exciting reading for high school and college students of mathematics.

H. S. TROPP, Humboldt State College

Introduction to Real Analysis. By Casper Goffman (Purdue University). Harper and Row, New York, 1966. xi+160 pp. $7.50.

This well-written textbook covers the theory of functions on the real line through the Riemann-Stieltjes integral and Fourier series. It appears suitable for undergraduate students who have had two or three semesters of college level elementary calculus. A course based on this text would give such students a solid understanding of the theory of the pre-Lebesgue calculus of one variable

and might well replace one semester of a traditional junior year advanced calculus course. The chapter titles are: 1. The Real Numbers; 2. Topology of the Reals; 3. Infinite Series; 4. Continuous Functions; 5 Special Functions (linear, exponential, logarithmic, and trigonometric); 6. Sequences and Series of Functions (including S. Bernstein's proof of the Weierstrass Polynomial Approximation Theorem); 7. Differentiation; 8. Integration; 9. Power Series; 10. Fourier Series.

Many of the theorems of chapters 2 and 4 can be extended to functions of several variables. Professor Goffman does not discuss these extensions in this book, but he develops the topology of the reals from nested sequences of decreasing intervals in a way which enables him to give proofs which can be easily extended to higher dimensions. For example, instead of proving that the reals are uncountable in the usual way, using decimal representation, he starts with an arbitrary sequence $a_1, a_2, a_3, \cdots$, of real numbers and then constructs a decreasing, nested sequence $I_1, I_2, \cdots$ of closed, bounded intervals such that a_n is not in I_n for all n. In chapter 9 he gives a complete treatment of the Riemann and Riemann-Stieltjes integrals including a proof that a bounded function on a closed interval is Riemann integrable if and only if its set of points of discontinuity is the union of a countable number of sets each of content zero.

The problems, which are well selected including both straightforward and difficult ones, are collected at the end of each chapter, but are numbered in a way which indicates to which section they pertain. The typography is very clear.

Note: This book is quite different in content from a book of similar title by M. E. Munroe which was recently reviewed in this Monthly, 73 (1966), 682–683.

L. E. PURSELL, University of Missouri at Rolla

Introduction to Measure and Probability. By J. F. C. Kingman and S. J. Taylor. Cambridge Univ. Press, 1966. x+401 pp. $12.50. (Teleg. Rev., March 1967.)

Two thirds of this book is devoted to measure theory and one-third to probability theory. The first part covers theory of sets, point set topology, set functions, construction and properties of measures, definitions and properties of the integral, related spaces and measures, the space of measurable functions, linear functionals and structure of measures in special spaces. The approach throughout is a classical one by now. The proofs are given in detail. This part corresponds to a serious first graduate course of one semester in measure theory.

The probability theory part deals with random variables, characteristic functions, independence, finite collections of random variables and stochastic processes (18 pages). The reviewer is somewhat unhappy about this part of the book: there is enough measure theory in the first part to go much more deeply into the basic concepts and fundamental results than is done. In fact, he feels that it is more of a collection of introductions to some of the simpler parts of probability theory than an integrated and systematic introduction to mathematical probability theory to be based on the measure-theoretic part of the book. In particular, conditioning is not analyzed. Markov property appears

only in an exercise, and martingales seem to be missing altogether. However, this book would be an excellent text for a course in measure theory with a *choice* of applications to probability theory.

M. LOÈVE, University of California, Berkeley

The Linear Hypothesis: A General Theory. By G. A. F. Seber. Edited by M. G. Kendall. Hafner, New York, 1966. viii+115 pp. $4.15.

This little book is number 19 of Griffin's Statistical Monographs and Courses. It maintains the generally high level already established in this series.

The topics covered include estimation and tests of linear hypotheses, power and robustness of the F test, orthogonality with reference to the classical experimental designs, analysis of covariance, "missing" observations, multivariate hypotheses, linear and non-linear regression, and large sample theory for non-linear hypotheses. Necessarily some of the topics are treated only briefly but the author provides over 100 references to published articles over half of which are dated 1960 or later. The author has used a vector space approach throughout which allows for considerable geometrical interpretation. This approach also simplifies many of the proofs.

A minimum background for reading this book would be a good first course in mathematical statistics plus a basic knowledge of matrix theory and vector spaces. The condensed style, inadequate exercises (only 13), and complete lack of numerical examples, makes the book unsuitable as a text. However, it provides an excellent up-to-date summary of results on the theory of linear statistical models. As stated in the preface, the book is intended for ". . . the mathematically minded reader who wishes to understand some of the basic ideas behind the subject with a minimum attention to detail. . . ."

H. J. ARNOLD, Oakland University

Complex Analysis: An introduction to the theory of analytic functions of one complex variable, 2nd ed. By Lars V. Ahlfors. McGraw-Hill, New York, 1966, xiii+317 pp. $8.95.

The first edition of this book appeared in 1953. The main changes in the second edition are the addition of a section on conformal mapping of polygons, a chapter on elliptic functions and a section on Picard's theorem on entire functions. Some further changes and additions are as follows: Chapter 2 now contains a section on elementary theory of power series and a section on the exponential and trigonometric functions which are defined by means of power series. The introductory material on point set topology has been rewritten; it now includes the fundamental properties of metric spaces and a detailed discussion of compactness. The section on normal families has been revised and expanded, and Chapter 6 includes a new section giving the Schwarz-Christoffel formula and the triangle functions of Schwarz. Chapter 7 is new and contains sections on simply periodic and doubly periodic functions. The properties of Weierstrass $\wp$-function, the associated functions $\zeta(z)$ and $\sigma(z)$, and the modular function $\lambda(\tau)$ are

given. The monodromy theorem, which was proved earlier, and the modular function enable the author to give a proof of Picard's theorem.

One misprint on page 206, formula 45, (where $\bar{a}_i$ in $(z-\bar{a}_i)$ should be replaced by a_i) was noticed. The reviewer considers this book as one of the best on the subject. It is rigorous, readable and has a number of challenging exercises, some with hints. It is a suitable textbook for a two semester course on complex analysis for first-year graduate students.

S. M. SHAH, University of Kentucky

Mathematics and Statistics for Chemists. By C. J. Brookes, I. G. Betteley, and S. M. Loxston. Wiley, London and New York, 1966. vii+418 pp. $10.00.

In its first eight chapters, this text "covers" most of the standard topics of calculus. It then presents a chapter on Fourier series and follows this with seven chapters on probability and statistics. There are a few exercises at the end of each chapter.

The preface to the book implies that only trigonometry and algebra are prerequisites for reading the book. However, on the first page in Chapter 1, the exponential function is defined by means of an "absolutely convergent" infinite series. The presentation, especially in the first portion of the book, is very terse, although a good number of examples are worked out in detail and many of these illustrate applications in chemistry or chemical engineering.

In many sections of the text there are items that a mathematician (and, probably, a bright student) would find objectionable. Some of the proofs given are simply incorrect—an instance is that given for the fundamental theorem of calculus. The flavour of the lack of mathematical care is perhaps indicated by the following quote from the discussion of the problem of testing whether or not a coin is true: "With no bias, after an infinite number of trials, we would expect an equal number of heads and tails, i.e., $H_0 \equiv \text{heads} = \text{tails}$."

The book is a good source of examples of mathematical applications in chemistry and might be helpful to chemists as a kind of cookbook. But it has severe shortcomings as a text to be used in a course.

F. L. WOLF, Carleton College

Lectures on Real and Complex Vector Spaces. By Frank S. Cater. Saunders, Philadelphia, 1966. x+167 pp. $5.00.

The title justifies the author's terseness and suggests why the book is not "inundated by trivial arguments." With carefully selected exercises, the first three of the five parts cover in a nontrivial manner the fundamentals of finite-dimensional vector spaces over the complex numbers. Direct sums and quotient spaces are introduced early followed by a chapter devoted to the fundamental theorems on projections and culminating with the Jordan canonical form. The last two parts, infinite-dimensional vector spaces and finite-dimensional unitary spaces, merit attention in that the author has selected with care topics normally associated with the theory of modules, rings, and finite-dimensional Hilbert

spaces. As a result of making the underlying theme of the book consistent, an emphasis is made on structural concepts within the complex vector space free of more sophisticated surroundings. The end result is a clear concise timely nugget for the maturation of first year graduate students. For a teacher who wants a reference on which to base a course, it should not be overlooked as a text, even for advanced undergraduates. Examples, motivation, and the numerical type exercise must be supplied and it is not meant for the light-hearted. However this alone makes the book unique in a day when pedagogic approaches and uniqueness of notation, terminology, and devices often destroy a book as an effective tool for an individualized instructor.

HOMER BECHTELL, University of New Hampshire

Fundamentals of Digital Machine Computing. By Guenther Hintze. Springer, New York, 1966. ix+225 pp. $6.40.

The book provides a first class practical introduction to the organization and programming of a conventional binary digital computer for numerical applications. It is precisely and clearly written and has well-designed problems to assist students in understanding the material.

Although in the preface the author indicates his appreciation for the broad applicability of computers and their extensive social implications, the student is led by the text to think of computing as assembly language programming for the solution of numerical problems. No mention is made of simulation or of string or list processing. Little insight is provided into possible variations of computer design. The final chapter on automata programming is an exceptionally clear introduction to the structure of programming languages and compilers using ALGOL as an example, but the significance of such languages is not evident from the text.

D. C. EVANS, University of Utah

Homology Theory: A first course in algebraic topology. By S.-T. Hu (University of California at Los Angeles) Holden-Day, San Francisco, 1966. xii+247 pp. $11.00.

This is a unique and highly useful text. In Chapter I the Eilenberg-Steenrod axioms are presented for homology and cohomology theories. Immediately, enough consequences are derived from the axioms to give a new short proof (based on Puppe's work) of the uniqueness of homology theories on the category of all finite CW-complexes. Chapter II develops from the axioms more machinery; this is used in Chapter III to compute the homology groups of graphs and surfaces, lens spaces, etc. Chapter IV offers applications: invariance of dimension, fixed point theorems, vector fields on spheres, the fundamental theorem of algebra, etc. The techniques of the uniqueness proof are used in Chapter V to construct the cellular homology groups of finite CW-complexes, under the continuing assumption that homology theories exist. (Cellular theories, first studied by G. W. Whitehead, are simpler than the classical simplicial theories since

orientations of cells are built in by means of the attaching maps. On the other hand, they are more useful than simplicial theories since cellular decompositions of polyhedra may have fewer cells; hence it may be easier to compute the ranks and invariant factors of incidence matrices.) Only finally, in Chapter VI, is a homology theory realized: singular theories are constructed there and the axioms, except for excision and homotopy, verified.

A courageous instructor could certainly lead a senior or first-year graduate class straight through this book, justifying its faith during the final month by producing one of the objects he has been talking about for the year. However, the author has made his exposition of singular theories depend only upon Chapter I, and many instructors will choose to follow I with VI. In fact, there is much additional flexibility in this text: each of Chapters III, IV and V depends only on II; hence a short course might contain only I, VI, II, and IV, for instance.

The exposition is good, with arithmetically lucid and thorough (if occasionally pedestrian) proofs throughout. The instructor is often left to provide geometric insight on the one hand, or a more algebraic overview on the other; although the choice of topics seems in general very good, in one instance I was surprised to be unable to find the "five" lemma anywhere (special cases of it appear as problems). The level of this exposition is compellingly more digestible to its intended student than is that of the classic, *Foundations of Algebraic Topology* by Eilenberg and Steenrod. Hence we now have a good introductory text for homology theory which totally avoids the cumbersome arithmetic and tedious detail of simplicial theory. This axiomatic approach, plus singular theory, provide much of the homological information needed for advanced courses in algebraic topology and do so in the appropriate abstract, categorical spirit. After statement of the axioms, cohomology theory is relegated to the problems (singular cohomology is made explicit in the last chapter), with no mention of its cup product.

The prerequisites for this course are, nominally, general topology and modern algebra. Actually, the student who has not used Hu's texts for these two courses will need to be told about finite CW-complexes and some elementary categorical algebra before his education in homology can begin. And, as he proceeds in this text, he probably will need to glean a few new facts about abelian groups. The author has provided ample reference citations for these prerequisites; in fact some of these refer to a line or so of proof. However, this reviewer strongly recommends to the author and the publisher that another edition of this text be made available, one which contains a chapter introducing finite CW-complexes and a chapter teaching the prerequisite categorical algebra.

GEORGE MCCARTY, University of California at Irvine

A First Course in Stochastic Processes. By Samuel Karlin. Academic Press. New York, 1966. xi+502 pp. $11.75.

This well-written work gives a vivid and rather systematic account of the theory of stochastic processes at an intermediate level. The prerequisites for

most of the book are a first course in probability and familiarity with the elements of matrix theory. In several places a knowledge of advanced calculus would be helpful. Owing to its expository clarity, abundance of examples, and large variety of problems of varying degrees of difficulty, the book could easily be used for self study. As a text it could be used in several ways. In its entirety it would serve as a stimulating book for a year course at the senior level which could be addressed to engineering and science majors as well as math majors. In part, the more elementary portions could be used during the last part of a basic undergraduate probability course, while the more advanced portions could be used to supplement a graduate probability course.

The processes treated in the book are mostly Markov processes. There is no discussion of stationary processes, Gaussian processes, or martingales. Chapter 1 contains the fundamental definitions and classification of stochastic processes. A rather complete treatment of Markov chains is developed in the next three chapters. Here such topics as the renewal theorem and the basic limit theorem, various ratio limit theorems, and algebraic methods are considered. There is no treatment of the potential or boundary theory for Markov chains. Chapter 6, which is somewhat specialized in nature, treats some of the aspects of random walks from the point of view of Markov chains. The next two chapters are devoted to continuous time Markov chains in which the Poisson process, birth and death processes, and finite state space processes are discussed as well as some of the more advanced aspects of these processes. Chapter 9 is again of a special nature. In this chapter the relationship between Poisson processes and order statistics is developed and used to give results on the ballot problem and the empirical distribution function. The highlights of one-dimensional Brownian motion are sketched in the brief chapter 10. The basic properties of branching processes are developed in chapter 11. Chapter 12 discusses the multi-dimensional Poisson process and various population growth models. A nice introduction to genetic and ecological models forms the contents of chapter 13. The final chapter 14 gives an account of the theory of queues.

SIDNEY PORT, University of California, Los Angeles

Logic and Algorithms, With Applications to the Computer and Information Sciences. By Robert R. Korfhage (Purdue University). Wiley, New York, 1966. xii+194 pp. $7.95.

The professed aim of the author is to present logic from the standpoints of philosophy, mathematics, and engineering and to apply it to algorithm theory. Such an ambitious undertaking, with adequate motivation for students, would require a somewhat longer presentation than this, which consists of about 150 pages, including many tables and figures. The book is carefully written; a list of references is found at the end of each chapter; and there are many illustrative examples and extensive sets of exercises with answers. The historical references are adequate, and there is a historical summary at the end. However, some of

the definitions are long and complex, and it is often tedious, or even difficult, to proceed with the exercises. Although very little mathematical background is required, it appears that the book's usefulness would be restricted to a reasonably sophisticated class of computer science students. Without a well-qualified instructor, the class may fail to discover the significance of the subject matter. This book should be a valuable addition to every computer science library.

C. H. CUNKLE, Slippery Rock State University

FILMS

Measures and Set Theory. A film by Stanislaw Ulam of the University of Colorado, 47 minutes running time, black and white, 16 mm. Individual lectures project of the Committee on Educational Media of the Mathematical Association of America. Produced and directed by A. N. Feldzamen, University of Wisconsin. Distributed by Modern Learning Aids.

This fine film shows Professor Ulam in his office at the Los Alamos Laboratory, talking informally and engagingly about measure theory. Prerequisites for understanding it are a knowledge of Boolean algebra, a little about ordinal and cardinal numbers, and an abstract point of view. Well-informed seniors majoring in mathematics at U. S. universities should be able to follow most of it. Graduate students in mathematics should find it easy to follow. It is interesting for any mathematician to hear and see Professor Ulam discoursing on one of his favorite topics.

The theme of the film is the construction (when possible) of set functions m on all or some of the subsets of a given set E such that:

(1) $m(A) \geqq 0$ for all A for which m is defined, $m(E)=1$, and $m(\{p\})=0$ for every point $p \in A$;

(2) $m(A \cup B)=m(A)+m(B)$ if A and B are disjoint. Sometimes (2) is replaced by

(2*) $m(\bigcup_{n=1}^{\infty} A_n)=m(A_1)+m(A_2)+\cdots+m(A_n)+\cdots$ if the A_n's are mutually disjoint.

Sometimes a notion of congruence is postulated for subsets of E, and then one can ask for

(3) $m(A)=m(B)$ if A and B are congruent.

Professor Ulam refers to Lebesgue measure without defining it, and sketches Vitali's proof that there is no measure on all subsets of $[0, 1]$ satisfying (1), (2*), and (3) for congruence under translations. He points out that (1) and (2) can be satisfied for all subsets of any infinite set, with a measure assuming only the values 0 and 1. He sketches his own proof (Fund. Math. Vol. 16) that there is no measure on all subsets of a set of cardinal number $\aleph_1$ satisfying (1) and (2*). He emphasizes over and over the importance of Paul Cohen's proof that the continuum hypothesis is independent of the standard axioms of set theory, and the new interest that Cohen's theorem lends to abstract measure theory. It was not clear to the reviewer how this new interest arises.

Turning to property (3), Ulam points out that (1), (2), and (3) can be satisfied for all subsets of the line or the plane (congruence being Euclidean motion in both cases), as Banach proved. He states the Banach-Tarski paradox, with nine subsets, and shows that this proves the impossibility of constructing a measure with (1), (2), and (3) on all subsets of 3-dimensional Euclidean space. He hints at some ideas for constructing measures on subfamilies of a set of cardinal number $\aleph_1$; these were far from clear to the reviewer.

He closes his lecture by stating an unsolved problem. Given a compact metric space X (he evidently means an uncountable metric space), does there exist a measure on all Borel subsets of X satisfying (1) and (2*) and (3) for Borel subsets A and B for which there is an isometry of A carrying A onto B? He says, "I could not solve this and perhaps now it is worthy of some thought."

The film is warmly recommended for mature audiences.

EDWIN HEWITT, University of Washington

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)—18 (second graduate year). A boldface star (★) marks a notable book that might be overlooked.

Analysis

Nine Papers on Functional Analysis and Partial Differential Equations. A.M.S. Translations, Ser. 2, Vol. 67. AMS., Providence, R.I., 1968. iv+288 pp. $14.60. P.

Singular Integrals. Edited by Alberto P. Calderon. Proceedings of Symposia in Pure Mathematics. Vol. X. AMS, Providence, 1967. vi+375 pp. $11.40. The Symposium was held at the University of Chicago, April 20–22, 1966. The volume is dedicated to Antoni Sygmund whose portrait is included. There are twenty papers relating to singular integrals, differential equations, and operators. P.

Nonlinear Two Point Boundary Value Problems. By Paul B. Bailey, Lawrence F. Shampine (both of Sandia Corp., New Mexico), and Paul E. Waltman (Univ. of Iowa). Academic Press, New York, 1968. xiii+171 pp. $9.50. Though much of the material has not previously appeared in textbooks, the level is about that of "introductory courses in differential equations which treat initial value problems." There are historical remarks and annotated bibliographies at the ends of the chapters. T, S, P, L.

Expansions in Eigenfunctions of Selfadjoint Operators. By Ju. M. Berezanskii. Translation Math. Mon. No. 17, AMS, Providence, R. I., 1968. x+810 pp. $31.70. A very substantial and detailed survey of "the application of the general theory of selfadjoint operators to spectral problems for differential and difference equations." Notes on the literature and thirty page bibliography. P, L.

Vorlesungen über reelle Funktionen. By Constantin Caratheodory. Chelsea, New York, 1968. x+718 pp. $12.00. This is described as the "third (corrected) edition" because the publishers "have gone to considerable lengths to search out all the errors" that appeared in the famous second edition of 1927 and were only partially corrected in the reprint of 1948, so that this great classic "is now available substantially free of error." (According to a letter to the editor from Mr. A. Galuten, President of Chelsea). P, L.

Elementary Differential Equations. By Donald L. Kreider (Dartmouth College), Robert G. Kuller (Wayne State Univ.), and Donald R. Ostberg (Northern Illinois Univ.). Addison-Wesley, Reading, Mass., 1968. xiv+492 pp. $10.95. In order to rescue the differential equations course "from the wasteland of unrelated techniques and dreary formalism in which it has all too long been lost," the authors have chosen to take linear algebra as a starting point and linear differential equations as a major theme. The necessary linear algebra is covered. Topics include the Laplace transform, series solution, existence, uniqueness, and stability. Applications are not neglected. T (14–15).

Basic Real and Abstract Analysis. By John F. Randolph (Univ. of Rochester). Academic Press, New York, 1968. ix+515 pp. $14.00. After preliminaries on real numbers, sets, spaces, sequences and series, major topics are Lebesgue integration ("time is overdue for Lebesgue theory to come into its own at an early stage without an initial high abstraction"), measure theory, continuity, derivatives (beginning with Dini derivatives), and Stieltjes integrals (ending with Lebesgue decomposition and Radon-Nikodym theorems). T (16–17), P.

Differential Equations with Applications. By Paul D. Ritger and Nicholas J. Rose (both of Stevens Inst. of Tech.). McGraw-Hill, New York, 1968. xiv+545 pp. $9.50. An outgrowth of courses given by the authors during the last decade, this book is addressed primarily to engineers and others interested in applications, but the authors believe that "a knowledge and appreciation of the basic theory of differential equations are important for the scientist and engineer as well as the mathematician." Topics include systems of linear differential equations, non-linear differential equations, linear difference equations, numerical methods, boundary value problems, and a short introduction to partial differential equations. May be adapted to a one or two semester course. T (14–15).

Real Analysis. 2nd. ed. By H. L. Royden (Stanford Univ.). Macmillan, New York, 1968. xii+349 pp. $11.95. The first edition of 1963 was favorably reviewed by Truman Botts in this Monthly, November 1964. It has been here corrected, extensively revised and substantially augmented. The treatment of Lebesgue measure and integration before the more general theory has been maintained. T (16–17).

An Introduction to Partial Differential Equations for Science Students. By G. Stephenson (Imperial College, London). Longmans, Don Mills, Ontario, 1968. vii+142 pp. $4.80 (paper). More elementary than most textbooks, this concentrates on second order equations that arise in the simpler problems of science and engineering. T (15–16), S.

Modern Analysis: An Introduction. By A. J. White (Univ. of Aderbeen, Scotland). Addison-Wesley, Reading, Mass., 1968. vii+244 pp. $8.75. Designed for a second course (or a rigorous first course) in calculus of functions of a single variable, this book's novelty is the early introduction and use of the notion of a metric space. T (14–15).

Applications

Leçons sur la Théorie des Groupes et les Symétries des Particules Élémentaires. By H. Bacry (Univ. of Marseille). Dunod, Paris, and Gordon and Breach, New York, 1967. xvi+449 pp. $22.00. Group theory with applications to physics. P.

The Economics of Uncertainty. By Karl Henrik Borch (Norwegian School of Economics and Business Administration, Bergen, Norway). Princeton Univ. Press, N. J., 1968. vii+227 pp. $8.50. Number two of the Princeton Studies in Mathematical Economics, edited by O. Morgenstern, H. W. Kuhn, and D. Gale. S, P.

Mathematical Physics. By Eugene Butkov (St. John's Univ. New York). Addison-Wesley, Reading, Mass., 1968. xi+735 pp. $17.50. May be useful to teachers of analysis, as a supplement, or as a textbook in courses in applied mathematics. In spite of the title, the chapter headings suggest a book in advanced calculus for applied mathematicians. T (16), S, P.

Foundations of Mathematics with Applications to the Social and Management Sciences. By Grace A. Bush and John E. Young (both of Kent State Univ.). McGraw-Hill, New York, 1968. 466 pp. $8.95. A freshman level survey of elementary mathematics through a little calculus. An obvious novelty is to use a pale green ink for various purposes including facsimile reproduction of the author's handwritten (and sometimes illegible) notes in the margin. The use of "foundations" to describe elementary introductions is unfortunate since it conflicts with a quite different and universal usage by mathematicians. P.

Mathematical Model Building in Economics and Industry. Being the collected papers of a conference organized by CEIR Ltd., held in London on 4th to 6th July 1967. Hafner, New York, 1968. vii+165 pp. $6.30. Included are papers by M. G. Kendall and H. O. Wold. P.

Statistical Selection of Business Strategies. By John Forester (California S. C., Long Beach). Irwin, Homewood, Illinois, 1968. viii+220 pp. $7.00. The Bayesian approach from a computational and practical point of view. T, P.

Potential Theory and its Applications to Basic Problems of Mathematical Physics. By N. M. Gunter. Translated by J. R. Schulenberger. Ungar, New York, 1967. xi+338 pp. $12.50. Included are changes and supplements to the original published in Paris in 1934, an historical foreword by V. I. Smirnov, and a ten page biography of Gunter by Smirnov and S. L. Sobolev. P, L.

Vector Geometry and Linear Algebra for Engineers and Scientists. By M. Jeger and B. Eckmann (both of Swiss Federal Inst. of Tech. Zurich). Translated from the German by Scripta Technica of London. Wiley, New York, 1967. 259 pp. $9.50. "This book is based on lectures given to students of engineering with a view to introducing them to vector methods and their geometrical applications, and to the simplest concepts of linear algebra." T (13–14).

Mathematical Methods. Vol. 1: Linear algebra, Normed spaces, Distributions, Integration. By Jacob Korevaar. (Univ. of Calif. San Diego). Academic Press, New York, 1968. x+505 pp. $14.00. Designed for students in physical sciences and engineering and for mathematics majors with interest in applications, this book presupposes advanced calculus. The second volume is to cover orthogonal series, linear operators in Hilbert space, integral equations, and Sturm-Liouville problems. T (15–16).

Latent Structure Analysis. By Paul F. Lazarsfeld (Columbia Univ.) and Neil W. Henry (Cornell Univ.). Houghton-Mifflin, Boston, Mass., 1968. ix+294 pp. $9.50. A survey of this field which, like factor analysis, is concerned with models of probabilistic relations between observable and assumed unobservable "latent structures." Here the latent structure is usually simply a division of a set into two parts. The book begins with a nice quotation from Robert Frost: "We dance round in a ring and suppose, But the Secret sits in the middle and knows." P, L.

Looking at History Through Mathematics. By N. Rashevsky. M.I.T. Press, Cambridge, Mass., 1968. xvi+199 pp. $10.50. The title is misleading, since in the preface the author describes his book as illustrating "in a number of different, sometimes almost

disconnected, examples how mathematical reasoning could in principle be used in attempted explanations of some historical phenomena." Like the author's work on mathematical biophysics, this appears to be rather far removed from the field of applications. P.

Aspects of Planometrics. By Alfred Zauberman. Yale Univ. Press, New Haven, 1967. xiii+318 pp. $9.50. This is about planning not planes. It covers mathematical economics in the Soviet Union since the respectability of the mathematical approach was admitted a few years ago. P, L.

Computers etc.

Mathematical Aspects of Computer Science. Proceedings of Symposia in Applied Math. Vol. 19. AMS, Providence, R.I., 1967. v+224 pp. $6.80. Eleven papers including J. A. Robinson on automatic theorem-proving, M. O. Rabin on mathematical theory of automata, and M. Minsky and S. Papert on linearly unrecognizable patterns. Author and subject indexes. P, L.

Information Processing Journal. A reference journal devoted to the theory, fabrication, and application of computers. Cambridge Communications Corporation, 1612 K St. N.W., Washington, D. C. 20006. Quarterly. Annual subscription, U.S.A. and Canada, $90.00. This is an abstracting journal. Abstracts are arranged under nearly 300 general topics. They are also indexed by author, by source, by subject (multiply for each subject indicated in the abstract) and by acronym. Each issue contains addresses of sources. Over 6000 abstracts per year. Volume 5 is dated 1967. P, L.

Fundamental Algorithms. Volume I. of the Art of Computer Programming. By Donald E. Knuth (Calif. Inst. of Tech.). Addison-Wesley, Reading, Mass., 1968. xxi+634 pp. $19.50. Since this is merely the first of seven volumes it is not surprising that the preface devotes 12 pages to describing what is to come and include a tongue in cheek flow chart for reading the twelve chapters to be contained in the entire work. The two chapters in this volume cover mathematical preliminaries, a description of MIX, fundamental programming technique, information structures, some history and bibliographies. There is a combined index and glossary. S, P, L.

Recognizing Patterns. Studies in Living and Automatic Systems. Edited by Paul A. Kolers and Murray Eden. M.I.T. Press, Cambridge, Mass., 1968. xi+237 pp. $11.00. Eight papers, including J. Weizenbaum on contextual understanding by computers. P.

Computers, System Science, and Evolving Society. The Challenge of Man-Machine Digital Systems. By Harold Sackman (System Development Corp.) Wiley, New York, 1967. xviii+638 pp. $14.50. Discusses such interesting topics as man-machine digital systems, computer-serviced society, division of labor between men and the computer, simulation and training and man-computer dialogue. Glossary. S, P.

A Handbook of Numerical Matrix Inversion and Solution of Linear Equations. By Joan R. Westlake (I.B.M.) Wiley, New York, 1968. vii+171 pp. $10.95. Designed as a comprehensive single reference for computer programmers with background equivalent to a mathematics major. S, P, L.

Education

Contemporary Arithmetic. By Thomas C. Crooks and Harry L. Hancock (both of Contra Costa College). Macmillan, New York, 1968. ix+336 pp. $5.95. Though intended for

college students and adults, this book is at a lower level conceptually than many elementary school textbooks and is contemporary with the nineteenth century in its terminology and approach.

Delta-Epsilon. A journal of undergraduate mathematics edited by Roger B. Kirchner. Carleton College, Northfield, Minnesota 55057. Volume 8 is being published during 1967–1968. Backfiles available. Undergraduate contributions are welcome and must involve original work by the authors but not necessarily previously unknown results.

Mathematical Education in the Americas II. A report of the Second Inter-American Conference on Mathematical Education, Lima, Peru, December 4–12, 1966. Edited by Howard F. Fehr (Columbia Univ.). Teachers College Press, New York, 1967. v+482 pp. $3.25 (paper). The first Inter-American Conference was held in 1961 to bring to the attention of all participants the need for curricular reform. This second conference addressed itself to a review of current problems of Latin America, examination of curricula for secondary and undergraduate university study, and the problem of training teachers for school and university. This volume contains conference information, addresses, summaries of mathematical development in different countries 1961–1966, and conclusions of the conference. TT, P, L.

Journal of Recreational Mathematics. Edited by Joseph S. Madachy (Kettering, Ohio). Editorial Board: J. A. H. Hunter, Howard C. Saar, Leo Moser. Quarterly. Greenwood Periodicals, Inc., 211 East 43rd Street, New York, N. Y. 10017. Annual Subscription; $9.00 plus $1.00 for foreign subscription. Students at all levels will find interesting material here. L.

Elements of Mathematics. By Bruce E. Meserve (Univ. of Vermont) and Max A. Sobel (Montclair State College). Prentice-Hall, Engelwood Cliffs, N. J., 1968. xi+303 pp. $7.95. Numbers, sentences in one and two variables, elementary and analytic geometry, structures, probability and statistics, functions and relations. T (one semester for non-science students). TT.

Ideas in Mathematics. By M. Evans Munroe (Univ. of New Hampshire). Addison-Wesley, Reading, Mass., 1968. vii+264 pp. $8.95. A sampling for the non-mathematician including sets, logic, calculus, probability, linear algebra, programming, abstract algebra, and computers. T.

Proceedings of National Conference on Needed Research in Mathematics Education. Journal of Research and Development in Education. Volume 1, No. 1, Fall, 1967. Published by the College of Education, University of Georgia. Thirteen papers (including one by Suppes on models for mathematical learning) and several summaries of sections and of the entire conference. Primarily addressed to problems at the secondary level, but many of the ideas will still be of interest to college teachers. $2.00 per copy. Available through the National Council of Teachers of Mathematics, 1201 16th St. N.W., Washington, D. C. 20036.

Modern Mathematics for Elementary Teachers, 2nd ed. By Jay D. Weaver and Charles T. Wolf (both of Millersville State College). International Textbook, Scranton, Pa., 1968. xvii+274 pp. $7.50. TT.

Logic and Foundations

Truth Functions and the Problem of their Realization by Two-Terminal Graphs. By A. Adam. Akademiai Kiado, Budapest, 1968. 206 pp. $7.80. A comprehensive survey of research on Boolean function, ending with a brief appendix on unsolved problems and a short bibliography. P.

Problems in the Philosophy of Science. Proceedings of the International Colloquium in the Philosophy of Science, London, 1965, vol. 3. Edited by Imre Lakatos and Alan Musgrave. North-Holland, Amsterdam, 1968. ix+448 pp. \$15.40. Of interest to mathematicians in this volume are the papers and discussions on formal logic and the development of knowledge (Suszko), relativity (Bergmann and Juhos), information and choice (Suppes). P, L.

Logique mathématique. By Daniel Ponasse (Univ. of Lyon). O.C.D.L., Paris, 1967. 164 pp. 18 F. The author deals with the propositional calculus and with the predicate calculus of the first order from syntactic, semantic, set theoretic, algebraic, and topological points of view. S, P.

Theory of Recursive Functions and Effective Computability. By Hartley Rogers, Jr. (M.I.T.). McGraw-Hill, New York, 1967. xiv+482 pp. \$14.75. Although the author does not claim to have written either a comprehensive or definitive treatise, he gives very substantial coverage to this important and growing field. He surveys past work, describes current areas of interest, and includes new results, exercises and a bibliography. The style is informal by current standards. T (16–17), S, P, L.

★*The Number System.* By H. A. Thurston (Univ. of British Columbia). Dover, New York, 1967. vii+134 pp. \$1.50. When this book was first published in 1956, R. L. Goodstein wrote ". . . Professor Broadbent said that Landau had done the job of setting up the number system once and for all; Dr. Thurston has shown that the job was worth doing again, and very well it has been done." However, it was not much noticed on this side of the Atlantic. Now available in this very modestly priced reprint it is a strong candidate for courses relating to the structure of the number system and for supplementary reading by undergraduates. T, TT, S.

Probability and Statistics

Selected Translations in Mathematical Statistics and Probability. Vol. 7. IMS and AMS, Providence, R. I.,1968. v+308 pp. \$15.40. Twenty-four papers ending with A. N. Kolmogorov on the definition of "amount of information." P.

Probability. By Leo Breiman. Addison-Wesley, Reading, Mass., 1968. ix+421 pp. \$13.50. Presupposing elementary real variable and measure theory but not a previous course in probability, the book includes such topics as martingales, the ergodic theorem, Markov chains, Brownian motion, and diffusions. T (17).

Computational Handbook of Statistics. By James L. Bruning (Ohio Univ.), and B. L. Kintz (Western Washington S.C.). Scott, Foresman, Glenview, Illinois, 1968. 269 pp. \$4.95 (cloth), \$3.25 (paper). Methods covered include the standard tests, analysis of variance, correlation, nonparametric tests, and indices of relationship. There are twelve numerical tables and two sample computer programs for analysis of variance. P, L.

Introduction to Business Statistics. By G. Hadley (Univ. of Hawaii). Holden-Day, San Francisco, 1968. x+463 pp. \$9.75. Both the classical and Bayesian approaches are covered so that the course may emphasize one or both. Prerequisites are minimal. Coverage includes basic probability and random variables, estimation, hypothesis testing, decision theory, quality control, regression, time series and forecasting. T.

Statistical Theory, 2nd ed. By B. W. Lindgren (Univ. of Minnesota). Macmillan, New York, 1968. xi+521 pp. \$9.95. Like the first edition, this presupposes calculus and has

a broad coverage. The revisions consist of expansions, clarifications, minor rearrangements, and corrections. T (15–16).

Statistical Problems with Nuisance Parameters. By Yu. V. Linnik. Translations of Math. Monographs. Vol. 20. AMS, Providence, R. I., 1968. x+262 pp. $12.00. There is a thirty page supplement on recent results and a seven page bibliography. S, P.

Topics in the Theory of Random Noise. By R. L. Stratonovich (Moscow State Univ.). Revised English Edition. Translated by Richard A. Silverman. Vol. I. General Theory of Random Processes. Nonlinear Transformations of Signals and Noise. Vol. II. Peaks of Random Functions and the Effect of Noise on Relays. Nonlinear Self-Excited Oscillations in the Presence of Noise. Gordon and Breach, New York, 1963, 1967. Vol. I. xi+292 pp. $14.50. Vol II. xiv+329 pp. $17.50. P.

Problems in Probability Theory, Mathematical Statistics and the Theory of Random Functions. Edited by A. A. Sveshnikov. Translated by Scripta Technica. Edited by Bernard R. Gelbaum. Saunders, Philadelphia, 1968. ix+481 pp. $14.50. The Russians excel at producing substantial problem collections that can be used as supplementary material (or even alone with adequate lectures). Each section gives basic information and sample solutions followed by problems whose answers are given at the back of the book. This volume should be very useful to both professors and students. T, S, P.

Introduction to Statistics. By Ronald E. Walpole (Roanoke College). Macmillan, New York, 1968. xii+365 pp. $7.95. At the pre-calculus level for which this book is designed, there are already too many books, including a very small number that are mathematically satisfactory.

NOTABLE ARTICLES

Logische Probleme im Mathematikunterricht I, II, III. Edited by Hans-Georg Steiner. In *Der Mathematikunterricht* (Ernst Klett Verlag, Stuttgart.), Vol. 5 (1959) Heft 4; Vol. 7 (1961) Heft 1; Vol. 13 (1967) Heft 5. This series of interesting articles is relevant to current educational problems in mathematics. The last collection includes an article by Hans Freudenthal on logic as subject and method.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dr. A. A. Albert, Dean of the Division of the Physical Sciences and Distinguished Service Professor of Mathematics at the University of Chicago, received an honorary Doctor of Science degree at Yeshiva University's 37th annual commencement on June 13, 1968.

Professor Dean C. Benson, South Dakota School of Mines and Technology, represented the Association at the inauguration of President M. N. Freeman of Black Hills State College on April 27, 1968.

Professor Dorothy L. Bernstein, Goucher College, represented the Association at the inauguration of President M. B. Perry, Jr. of Goucher College on May 3, 1968.

Professor C. M. Braden, Macalester College, represented the Association at the inauguration of President M. C. Moos of the University of Minnesota on May 8 and 9, 1968.

Professor R. B. Crouch, Drexel Institute of Technology, represented the Association at the inauguration of President P. R. Anderson of Temple University on May 1, 1968.

Professor Gene Levy, University of Oklahoma, represented the Association at the inauguration of President R. L. Martin of Oklahoma College of Liberal Arts on April 20, 1968.

Professor C. M. Lindsay, Coe College, represented the Association at the inauguration of President S. E. Stumpf of Cornell College on May 4, 1968.

Professor Harriet F. Montague, State University of New York at Buffalo, represented the Association at the inauguration of President E. K. Fretwell, Jr. of the State University of New York College at Buffalo on May 10, 1968.

Professor Everett Pitcher, Lehigh University, was the recipient of the 1968 R. R. and E. C. Hillman Award for outstanding teaching and service.

Professor W. A. Raab, University of South Dakota, represented the Association at the inauguration of President H. P. Bowes of General Beadle State College on May 17, 1968.

Professor T. D. Reynolds, Duke University, represented the Association at the inauguration of President J. R. Scales of Wake Forest University on April 11, 1968.

Professor H. C. Saar, Bloomington High School, represented the Association at the inauguration of President S. E. Braden of Illinois State University on May 11, 1968.

Professor Henry Sharp, Jr., Emory University, represented the Association at the inauguration of President F. C. Davison of the University of Georgia on May 11, 1968.

Professor C. J. Vanderlin, University of Wisconsin, represented the Association at the inauguration of President J. T. Middaugh of Carroll College on April 27, 1968.

Professor F. A. Varrichio, Saint Peter's College, represented the Association at the inauguration of President J. O. Fuller of Fairleigh Dickinson University on May 9, 1968.

University of Georgia: Dr. J. W. Wilson, Project Coordinator, SMSG Research and Analysis Section, Stanford University, has been appointed Assistant Professor; Dr. W. D. McKillip has been promoted to Associate Professor.

Wesleyan University: Drs. W. L. Reddy, State University of New York at Albany, and C. M. Stanton, Stanford University, have been appointed Assistant Professors.

Dr. Walter Feibes, State University of New York at Buffalo, has been appointed Associate Professor at Western Kentucky University.

Dr. Newman Fisher, San Francisco State College, has been named Chairman of the Mathematics Department.

Dr. J. L. Gross, Dartmouth College, has been appointed to an Instructorship at Princeton University.

Mr. R. A. Herrmann, American University, has been appointed Assistant Professor at the United States Naval Academy.

Dr. H. N. Laden, Director of Research Services in the Planning Department C&O-B&O, has been appointed Assistant Vice President-Research of the Chesapeake & Ohio-Baltimore & Ohio railroads.

Dr. Fazlollah Reza has been appointed Chancellor of the Aria-Mehr University of Technology at Tehran, Iran.

Dr. George Van Zwalenberg, Fresno State College and NSF Science Faculty Fellow at the University of California at Berkeley, has been appointed Professor at Calvin College.

Associate Professor B. A. Amira, Hebrew University, Jerusalem, Israel, died on January 20, 1968. He was a member of the Association for eight years.

Dr. Angeline J. Brandt, Wheaton College, died on March 25, 1968. She was a member of the Association for seventeen years.

Associate Professor Lindley J. Burton, Lake Forest College, Visiting Lecturer at the University of Melbourne, died on May 21, 1968. He was a member of the Association for nineteen years.

Assistant Professor Louise F. Hanson, Olivet College, died on March 20, 1968. She was a member of the Association for sixteen years.

Associate Professor Emeritus L. A. Hopkins, University of Michigan, died on April 28, 1968. He was a member of the Association for forty-eight years.

Associate Professor Emeritus G. H. Hunt, University of California at Los Angeles, died in February, 1968. He was a member of the Association for forty-five years.

Mr. L. M. Klauber, San Diego Gas and Electric Company, died on May 8, 1968. He was a member of the Association for forty-eight years.

Associate Professor Leon LeBlanc, University of Montreal, died on April 18, 1968. He was a member of the Association for nine years.

Associate Professor W. K. Morrill, Johns Hopkins University, died on April 11, 1968. He was a member of the Association for forty-one years.

Associate Professor Emeritus E. E. Nash, Rensselaer Polytechnic Institute, died on April 8, 1968. He was a member of the Association for thirty-two years.

Assistant Professor G. K. Overholtzer, Los Angeles State College, died on April 12, 1968. He was a member of the Association for fourteen years.

Professor Emeritus Meyer Salkover, University of Cincinnati, died on March 15, 1968. He was a member of the Association for sixteen years.

ABSTRACT OF ANNUAL REPORT FOR 1967-1968 DIVISION OF MATHEMATICAL SCIENCES—NATIONAL RESEARCH COUNCIL

The report of the Committee on Support of Research in the Mathematical Sciences has been completed and received the approval of the Academy's Committee on Science and Public Policy. The manuscript is being prepared for publication in three volumes: Volume I contains the COSRIMS Report and the rationale for the Committee's conclusions. Volume II is a supplement setting out the extensive study made by the Panel on Undergraduate Education and contains certain additional recommendations on this aspect of the mathematical sciences. Volume III consists of a collection of 22 essays by specialists on mathematics and its applications. Each essay deals with a typical aspect of the mathematical sciences in a manner designed to provide non-mathematicians with some insight into the relevant ideas and mathematical methods. Publication is expected by September.

The first of the three volume report of the Mathematics Survey, (also called the Young Survey; Vol. I, 164 pp., $1.75, available from the CBMS, Joseph Henry Bldg., 2100 Pennsylvania Ave. N.W., Suite 834, Washington, D. C. 20037), carried out under the Conference Board of the Mathematical Sciences (CBMS) with support from the Ford Foundation, appeared in January, 1968. This volume, *Aspects of Undergraduate Training in the Mathematical Sciences*, sets out data relevant to the COSRIMS supplement on undergraduate education. The complete report is expected by December, 1968.

The proceedings of the Second Inter-American Conference on Mathematical Education, Lima, 1966, have appeared in the English version. The Portuguese and Spanish versions are to appear in the summer of 1968. A permanent self-sustaining office for the Inter-American Committee for Mathematical Education is being set up. The Division is participating in initiating the office by allocating a modest sum from the Ford Foundation grant supporting the Lima Conference.

The Committee on Travel Grants cooperated with the International Federation for Information Processing (IFIP) through its U. S. member, the American Federation of Information Processing Societies (AFIPS) in support of the third triennial congress in Edinburgh, August 5–10, 1968. The Committee set up the Subcommittee on Travel Grants for Computer Science which reviewed 281 applications. With funds from the National Science Foundation, the Air Force Office of Scientific Research and private sources, 66 grants of $300 each have been made for travel to IFIP Congress 68.

The Executive Committee has undertaken to report to the 1969 Annual Meeting of the Division on a representation structure reflecting contemporary relationships among the mathematical sciences. This report will take into account the 1953 recommendations of the Committee on Membership and modifications indicated by subsequent developments. As an interim measure the 1968 Annual Meeting voted that one additional representative in the Division be invited from the Association for Computing Machinery and the Society for Industrial and Applied Mathematics.

During the year, 47 mathematicians represented the Division on panels and boards to provide advice on the selection of candidates for fellowships in mathematics. They evaluated 1748 candidates for fellowships and other awards in seven programs sponsored by various agencies. The total number of awards was 471 in the mathematical sciences. The Division distributed in September its annual brochure on "Fellowship and Research Opportunities in the Mathematical Sciences" to some 900 Chairmen of Departments of Mathematics. About 400 copies were mailed during the year in response to individual requests.

Copies of the complete Annual Report of the Division of Mathematical Sciences may be obtained by writing to: Division of Mathematical Sciences, National Research Council, 2101 Constitution Avenue N.W., Washington, D. C. 20418.

INTERNATIONAL PRIZE IN MEMORY OF RAPHAEL SALEM

An International Prize in memory of Raphael Salem and his mathematical work has been established. This prize of 5,000 French francs will be awarded yearly to a young mathematician who is judged to have contributed the best work on Fourier Series and related topics.

The first recipient of the prize is Dr. Nicholas Th. Varopoulos of the Faculté des Sciences, and Trinity College, Cambridge. The award was made in June 1968, and the jury consisted of Professor J.-P. Kahane, Professor C. Pisot, and Professor A. Zygmund.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE METROPOLITAN NEW YORK SECTION

The twenty-seventh annual meeting of the Metropolitan New York Section of the MAA was held on March 16, 1968, at Staten Island Community College, Staten Island, New York. There were 102 persons in attendance, including 82 members of the Association. Dr. Arthur Kaufman, Assistant to the President of Staten Island Community College, welcomed the group.

Professor Meyer Jordan of Brooklyn College, Chairman of the Section, presided at the morning session. The following talks were given:

1. *Computer films as a new educational tool*, by E. E. Zajac, Bell Telephone Laboratories.

2. *Ergodic theory as applied to number theory*, by Louis Auslander, The City University of New York.

Professor Joseph Houle of Pace College, Vice-Chairman for Senior Colleges, presided at the afternoon session. At the business meeting, Professor Abraham Schwartz of the City College of New York reported on some of the highlights of the meeting of the Board of Governors at San Francisco in January, 1968. Professor James Eastham of Queensborough Community College reported on the activities of the Speakers' Bureau, while Mr. Aaron Shapiro of Midwood High School gave the Treasurer's report. Professor Charles Salkind of Polytechnic Institute of Brooklyn gave the report of the Chairman of the Committee on Contests and Awards. The following are the highlights of his report:

1. The MAA Committee on High School Contests, at its August 1967 meeting, voted to modify the annual high school mathematics examination, from a three-part structure with 40 problems to a four-part examination with 35 problems. The highest possible score remains at 150.

2. As of 1968 the National Council of Teachers of Mathematics joins the MAA, the Society of Actuaries, and Mu Alpha Theta in sponsoring the activity.

3. Participation for 1968 is at an all-time high: approximately 7200 schools with over 300,000 contestants.

4. A miniature 'English Olympiad' will be held in May between a small group of participants from upper New York State and a competing group from England, both groups being selected on the basis of performance in the 1968 examination.

Beginning this year, the Metropolitan New York Section is inaugurating an award program for local Putnam Competition participants falling in the rank of Honorable Mention. Two awards are being made this year, consisting of a specially-designed Certificate of Merit and a year's subscription to the MONTHLY.

At the conclusion of the Business meeting, there was a panel discussion of 'The First Two Years of Collegiate Mathematics.' Professor M. L. Richter of Staten Island Community College served as chairman of the panel which consisted of Professor James Eastham, Queensborough Community College; Professor Allan Gewirtz, Pace College; Professor C. C. Robusto, St. John's University.

THERESA J. BARZ, *Secretary*

APRIL MEETING OF THE KENTUCKY SECTION

The fifty-first annual meeting of the Kentucky Section was held at the University of Kentucky, Lexington, on April 27, 1968, Chairman W. C. Royster presiding. A total of 102 persons registered at the meeting, including 52 members of the Association.

At the business meeting these officers were elected: Chairman: B. R. Nail, Morehead State University; Secretary-Treasurer: W. H. Spragens, University of Louisville; Chairman for High School Mathematics Contest: Stephen Puckette, University of Kentucky.

Professor P. T. Bateman of the University of Illinois gave the invited address on *Some Recent Advances in Number Theory.*

The morning session was devoted to two parallel programs. One of these was a panel on the senior high school mathematics program, moderated by Martin Brown, Northern Community College, Covington, with the following participants: Mrs. Forest Mercer, Henry Clay High School, Lexington; Denton Cormany, Woodford County High School, Versailles; Robert Lamkin, Pleasure Ridge Park High School, Pleasure Ridge Park; Rev. Charles Rusterholtz, Bellarmine College; Stephen Puckette, University of Kentucky; Walter Gerlach, Southeast Community College, Cumberland.

At the other session the following papers were presented:

1. *Approximation in the L^1 metric*, by C. N. Kellog, University of Kentucky.

2. *Integration on certain spaces of continuous functions*, by Jean T. Sells, University of Louisville.

3. *Derivatives, indefinite integrals, and the fundamental homomorphism theorem*, by M. C. Rayburn, University of Kentucky.

4. *Polygonomials*, by A. S. Howard, Eastern Kentucky University.

5. *Necessary and sufficient conditions for the central simplicity of the derivation algebra of a finite dimensional inseparable extension field*, by B. R. Nail, Morehead State University.

6. *Entire functions of bounded index and Bessel functions*, by B. S. Lee and S. M. Shah, University of Kentucky.

7. *On closest unitary matrices*, by R. L. Causey, University of Louisville.

W. H. SPRAGENS, *Secretary-Treasurer*

APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the MAA was held at Lindenwood College on April 27, 1968. Professor R. W. Murdock presided at the morning session and Professor Guido Weiss, Vice-Chairman of the Section, presided at the afternoon session and business meeting. One hundred and one persons attended the meeting.

At the business meeting the following officers were elected: Chairman: Professor Waldo Vezeau, St. Louis University; Vice Chairman: Professor Troy Hicks, University of Missouri-Rolla; Secretary-Treasurer: Mrs. Virginia Kern, St. Louis University.

The following papers were presented at the morning session:

1. *What is a truth table?*, by F. B. Wright, Tulane University (invited address).

2. *Cross-ratio in geometry*, by C. E. Kelley, Central Missouri State College.

3. *Generalizations of Krull domains*, by Elbert Pirtle, University of Missouri, Kansas City.

4. *Elementary linkage analysis of research competencies in the sciences*, by Ron Moss, Northwest Missouri State College.

The afternoon session consisted of a discussion of the CUPM report, *Qualifications for a college faculty in mathematics*, led by R. H. McDowell of Washington University.

R. J. MIHALEK, *Secretary-Treasurer*

APRIL MEETING OF THE NEBRASKA SECTION

The forty-fourth annual meeting of the Nebraska Section of the MAA was held on Saturday, April 27, 1968, at the Nebraska Center for Continuing Education, Lincoln, Nebraska, in conjunction with the seventy-eighth annual meeting of the Nebraska Academy of Sciences. Professor A. W. Zechmann, Chairman of the Section, presided. There were some seventy-five persons present at both the morning and afternoon sessions of whom fifty were members of the Association. Mathematical films were shown at both sessions.

The following officers were elected for 1968–1969: Chairman, Professor D. M. Mesner, University of Nebraska; Vice-Chairman, Professor A. W. Zechmann, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska. Professor J. M. Earl was continued as Chairman of the Nebraska-South Dakota High School Mathematics Contest Committee.

Invited lecturers were: Professor D. H. Lehmer, University of California, and Professor Victor Klee, University of Washington.

The following papers were presented:

1. *The Nebraska-South Dakota Mathematics Contest*, by J. M. Earl, University of Omaha, and H. M. Cox, University of Nebraska.

2. *A commutativity theorem*, by R. L. Tangeman, University of Nebraska.

3. *Factorizable semigroups*, by K. W. Tolo, University of Nebraska.

4. *The off-line computer concept*, by D. H. Lehmer (invited lecture).

5. *Nonlinear boundary value problems and the equation of first variation*, by Lynn Erbe, University of Nebraska.
6. *Some aspects of the theory of extensors*, by P. S. Morey, Jr., University of Omaha.
7. *The dependence of solutions of second order differential equations on their boundary conditions*, by G. A. Klassen, University of Nebraska.
8. *Unsolved problems in intuitive geometry*, by Victor Klee (invited lecture).

HENRY M. COX, *Secretary*

MAY MEETING OF THE ILLINOIS SECTION

The forty-seventh annual meeting of the Illinois Section of the MAA was held on the Edwardsville campus of Southern Illinois University on May 10–11, 1968. Dr. R. D. Boswell, Section Chairman, presided. There were eighty-five persons in attendance.

Professor Daniel Zelinsky of Northwestern University spoke on the topic, "Algebra is the study of functions too" and this address was followed by the presentation of five papers:

1. *Homomorphism topologies on abelian groups*, by B. F. Hobbs, Olivet Nazarene College.
2. *Abelian surfaces*, by Nancy Fincke, Western Illinois University.
3. *Approximation of Banach-valued multidimensional complex functions*, by W. J. Neath, Northern Illinois University.
4. *On a special linear transformation*, by Carl Townsend, Southern Illinois University (Carbondale).
5. *Why not divide by zero?* by William Bennewitz, Southern Illinois University (Edwardsville).

The dinner address was presented by Dr. H. K. Farahat, Senior Lecturer, University of Sheffield, England, and currently Visiting Associate Professor of Mathematics at the University of Illinois. His topic was "Mathematical education in England." Professor A. A. Albert, Dean of the Division of Physical Sciences, University of Chicago, spoke on Saturday morning on "Finite Projective Planes." The meeting concluded with a panel on "Computers in the Undergraduate Curriculum" in which the participants were Professors Alphonso DiPietro, Eastern Illinois University, Donald Herrick, Northern Illinois University, Jurg Nievergelt, University of Illinois, and William Rippenberger, Knox College.

At the annual business meeting following sessions on Friday afternoon the following officers of the Section were elected for 1968–1969: Chairman, Professor Arnold Wendt, Western Illinois University; Vice-Chairman, Professor Hiram Paley, University of Illinois; and Secretary-Treasurer, Professor Howard Saar, Shawnee Community College.

HOWARD SAAR, *Secretary-Treasurer*

MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The fifty-first annual meeting of the Rocky Mountain Section of the MAA was held at the University of Denver, Colorado, on May 10 and 11, 1968. There were 110 registrants, including Professor F. M. Stein of Colorado State University, the Sectional Governor, and Professor Kenneth Noble of the University of Denver, the Section Chairman. The invited address was delivered by Dr. W. S. Dorn, Watson Research Center, IBM Corporation, who spoke on 'Computer Extended Instruction'. Chancellor M. B. Mitchell of the University of Denver welcomed the Section at the banquet on Friday evening.

At the business meeting, the nominating committee was instructed to consider the advisability of amending the By-Laws of the Section to provide for the election of a second vice chairman to look after the interests of the junior colleges. The committee is to make its recommendation at next year's meeting of the Section. The Section also voted to

sponsor a program of high school lecturers, and a committee consisting of Professor Robert McKelvey, University of Colorado, Chairman; Professor William Scott, University of Utah and Professor Verne Varineau, University of Wyoming, was set up to implement the proposal.

The following officers were elected: Chairman, Jerrold Bebernes, University of Colorado, Boulder, Colorado; Vice Chairman, Ray Hanna, University of Wyoming, Laramie, Wyoming; Secretary-Treasurer, C. R. Wylie, Jr., University of Utah, Salt Lake City, Utah.

The following papers were read at the meeting:

1. *Numerical invariants in noncommutative orders*, by D. W. Ballew, South Dakota School of Mines and Technology.

2. *The Duplication of the sphere*, by Robert Bitts, Arapahoe Junior College.

3. *Edge diffraction for parabolic differential equations*, by Jack Cohen and David Hector, University of Denver.

4. *Student grades as a multiple Markov chain*, by Major R. L. Eisenman, USAF Academy.

5. *Continuous dependence for two-point boundary-value problems*, by R. E. Gaines, Colorado State University.

6. *Eigenvalue studies for second order differential equations using invariant bedding*, by Frank Hagin, University of Denver.

7. *Closed factors of Chebyshev polynomials $S_n(x)$*, by C. A. Halijak, University of Denver.

8. *An L_q approximate solution of the Ricatti equation*, by M. S. Henry and F. M. Stein, Colorado State University.

9. *A maximal ideal radical class*, by T. L. Jenkins, University of Wyoming.

10. *When is a curve a curve?*, by A. J. Kempner, University of Colorado.

11. *The spin—an algebraic and probabilistic toy*, by Jean-Paul Marchand, University of Geneva, Visiting Professor in Mathematics and Physics, University of Denver.

12. *A senior seminar topic*, by D. C. B. Marsh, Colorado School of Mines.

13. *Generalized quadratic forms in a finite field*, by A. D. Porter, University of Wyoming.

14. *Nonlinear difference methods for ordinary differential equations*, by D. P. Squier, Colorado State University.

15. *Some random hydrodynamics*, by J. W. Thomas, University of Wyoming.

16. *A class of positive definite functions on noncommutative groups*, by R. C. Weger, South Dakota School of Mines and Technology.

17. *A model for projective spaces with three points on every line*, by A. Zirakzadeh, University of Colorado.

In addition to these papers, the program included a panel discussion on 'Computer Extended Mathematics' in which the participants were W. S. Dorn, Watson Research Center, IBM Corporation, E. R. Kreuger, University of Colorado, Ruth Hoffman, University of Denver, John Skelton, University of Denver, Larry Blevins, Northeastern Junior College.

C. R. WYLIE, JR., *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-Second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

FLORIDA, Florida Atlantic University, Boca Raton, March 21–22, 1969.

ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.

INDIANA, Butler University, Indianapolis, November 2, 1968.

IOWA, University of Northern Iowa, Cedar Falls, April 18, 1969.

KANSAS, Wichita State University, March 1969

KENTUCKY, Morehead State University, Morehead, Spring 1969.

LOUISIANA-MISSISSIPPI, New Orleans, January 25–27, 1969.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Baltimore, November 23, 1968.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1969.

MINNESOTA

MISSOURI, St. Louis University, St. Louis, April 26, 1969.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 25–26, 1969.

NEW JERSEY, Rutgers—The State University' New Brunswick, November 2, 1968.

NORTHEASTERN, University of Bridgeport, Connecticut, November 30, 1968.

NORTHERN CALIFORNIA, University of Santa Clara, Santa Clara, February 8, 1969.

OHIO

OKLAHOMA-ARKANSAS, Arkansas State University, Jonesboro, March 21–22, 1969.

PACIFIC NORTHWEST, University of Oregon, Eugene, August 1969.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 23, 1968.

ROCKY MOUNTAIN, University of Colorado. Boulder, Colorado, May 9–10, 1969.

SOUTHEASTERN, Winthrop College, Rock Hill, South Carolina, March 28–29, 1969.

SOUTHERN CALIFORNIA, California State College at Fullerton, March 15, 1969.

SOUTHWESTERN, Northern Arizona University, Flagstaff, Spring 1969.

TEXAS, Texarkana College, Texarkana, April 18–19, 1969.

UPPER NEW YORK STATE, Rensselaer Polytechnic Institute, Troy, November 9, 1968.

WISCONSIN, Oshkosh, May 2–3, 1969.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968

AMERICAN MATHEMATICAL SOCIETY, New Orleans, Louisiana, January 23–26, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC, New Orleans, Louisiana, January 22–23, 1969.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, New Orleans, Louisiana, January 25–26, 1969.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, November 6–9, 1968.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

MATHEMATICAL ASSOCIATION OF AMERICA

New Address for Central Office

A. B. WILLCOX has succeeded H. M. GEHMAN as Executive Director of MAA. The central office of the Association has been moved from Buffalo to Washington.

All correspondence for the central office should be sent hereafter to:

Mathematical Association of America
1225 Connecticut Ave. N.W.
Washington, D.C. 20036

In particular send to the central office payments for dues and subscriptions, orders for publications, and changes of address.

Manuscripts and other editorial correspondence for the AMERICAN MATHEMATICAL MONTHLY and for MATHEMATICS MAGAZINE should be sent to their respective editors-elect:

For the MONTHLY: HARLEY FLANDERS
Purdue University
Lafayette, IN-47907

For MATHEMATICS MAGAZINE: S. A. JENNINGS
University of Victoria
Victoria, BC, CANADA

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R. A. ROSENBAUM, *Editor*

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, A. B. WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D. C. 20036.

NOTICE OF CHANGE OF ADDRESS

Effective November 1, 1968, the address of the business office of the Association is:

Mathematical Association of America
1225 Connecticut Avenue, N. W.
Washington, D. C. 20036.

Payments for dues and subscriptions and notices of changes of address should be sent to the above address.

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UNIQUE FACTORIZATION

PIERRE SAMUEL, Institut Henri Poincaré, Paris

1. Introduction. It is well known that every ordinary integer is, in a unique way, a product of prime numbers. With an eye to generalizations it is better to state this unique factorization property in the *ring* Z of rational (i.e., >0 or <0) integers. Thus, if we denote by P the set of all prime numbers, every nonzero element x of Z may be written, in a unique way, as

$$x = \pm 1 \prod_{p \in P} p^{v_p(x)}, \tag{1}$$

where the exponents $v_p(x)$ are positive integers, almost all 0 (i.e., equal to 0 except for a finite number of them) in order that formula (1) makes sense. The somewhat abstract formulation given by (1), with its seemingly infinite product, has the great advantage of indicating how the exponents $v_p(x)$ depend on x. If we allow negative exponents, we see that (1) holds also for all *nonzero* rational numbers x. Furthermore, for any pair x,y of nonzero rational numbers, we see that we have

$$v_p(xy) = v_p(x) + v_p(y), \qquad v_p(x + y) \geqq \inf(v_p(x), v_p(y)). \tag{2}$$

Algebraists express formulae (2) by saying that the mapping $v_p\colon Q^* \to Z$ is a *discrete valuation* of the field of rational numbers.

More generally, we define a *factorial ring* (or a "unique factorization domain," U.F.D.) to be an integral domain A for which there exists a subset P of A such that every nonzero element x of A may be written, in a *unique* way, as

$$x = u \prod_{p \in P} p^{v_p(x)} \tag{1'}$$

where u is a *unit* (i.e., an invertible element) in A, and where the exponents $v_p(x)$ are positive integers, almost all 0. It can easily be proved that the subset P is uniquely determined up to units; more precisely the set $(Ap)_{p \in P}$ of principal ideals is uniquely determined, and coincides with the set of all maximal principal ideals distinct from A. Let us notice that a principal ideal Ab of a domain A is maximal (among principal ideals distinct from A) iff every divisor d of b is either a unit or is such that db^{-1} is a unit; such an element b is called an *irreducible* element of A.

For a ring A, factoriality is a very useful property. At least for multiplicative questions, the *arithmetic* in a factorial ring A is as nice as in the ring Z of ordinary integers. It may be recalled that, in the 19th century, arithmeticians like Kummer and Dedekind noticed that some rings of algebraic integers failed to be factorial; e.g., the formulae

$$2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}), \qquad 3 \cdot 3 = (\sqrt{10} + 1)(\sqrt{10} - 1)$$

show that the rings $Z[\sqrt{-5}]$, $Z[\sqrt{10}]$ are not factorial; this led Kummer and

Dedekind to introduce the important notion of an *ideal*, and to replace the unique factorization of elements by the unique factorization of ideals, thus inaugurating the theory of rings which we now call "Dedekind rings." Lack of time prevents me from talking more about this important and beautiful theory.

The interest of factorial rings does not come only from arithmetical reasons. Factoriality has also a very simple *geometric* interpretation. In geometry, more precisely in the study of algebraic, analytic or formal varieties, a ring A occurs as a ring of functions (algebraic or analytic, as the case may be) on some variety V, or in the neighborhood of some point of V. To say that A is a domain means that V is irreducible. Denoting by n the dimension of V, the factoriality of A then means, roughly speaking, that every subvariety of W of dimension $n-1$ of V may be defined *by a single equation*; more precisely the functions $f \in A$ which vanish on W form an ideal $p(W)$ in A, and factoriality means that these ideals $p(W)$ (for dim $W=n-1$) are principal.

2. How to prove factoriality. We have just seen that factoriality is a desirable property for a ring. On the other hand proving that a ring is factorial is rarely trivial, so it is useful to have at hand as many characterizations of factorial rings as possible.

As we have seen in Section 1, factoriality of A means that every nonzero element of A admits a decomposition as a product of irreducible ones, and that this decomposition is unique up to units. The existence of such a decomposition is usually easy to check; it follows from this "chain condition" for principal ideals (valid in any factorial ring):

(3) *Every strictly increasing sequence of principal ideals of A is finite*, which is itself equivalent to the "maximal condition":

(3′) *Every nonempty family of principal ideals of A admits a maximal element.*

For example (3) (or (3′)) holds when the ring A is noetherian, and most rings that are encountered in arithmetic or in geometry are noetherian. Furthermore, with proper caution, property (3) may pass to the direct limits. We henceforth assume that (3) holds.

As to *uniqueness*, things are not so easy. Unique factorization in a ring A implies that any irreducible element p of A enjoys the stronger property that

(4) *If p divides a product ab, then it divides a or b.*

Conversely, assuming (3), a well-known proof copied from elementary number-theory shows that (4) implies the uniqueness of the decomposition into irreducible factors. An element p which enjoys property (4) is called a *prime* element of A; this means that the principal ideal Ap is a prime ideal ($ab \in Ap \Rightarrow a \in Ap$ or $b \in Ap$), or, equivalently, that the factor ring A/Ap is a domain. As shown in elementary number-theory, property (4) is equivalent to

(4′) *Any two elements of A admit a greatest common divisor*, and also to:

(4″) *Any two elements of A admit a least common multiple.*

A rather handy form of (4″) is

(4‴) *The intersection of any two principal ideals of A is a principal ideal.*

If we deal with a *noetherian* domain A, it can be proved that every nonunit in A is contained in a prime ideal of height 1 (i.e., a prime ideal which is minimal among nonzero prime ideals). From this one easily deduces that a noetherian domain A is factorial iff

(5) *Every prime ideal of height* 1 *of A is principal.*

This condition has already been met at the end of Section 1, when we discussed the geometric meaning of factoriality.

More technical characterizations of factorial rings may be given in the framework of the theory of *Krull rings*, for which we refer to Bourbaki, "Algèbre Commutative," Chap. VII, "Diviseurs" [4]. Let us only say that the class of Krull rings contains the class of factorial rings and is more stable under various ring-theoretic operations. Furthermore, it is in general easy to test whether a given ring is a Krull ring or not. The problem is therefore to test whether a given Krull ring is factorial, and, if not, to measure its "nonfactoriality."

3. Properties stronger than factoriality. For proving that Z is factorial, one usually first proves that Z is *principal* (i.e., that every ideal of Z is principal). Then the chain condition (3) is very easy, and condition (4‴) is obvious. The example of a polynomial ring in several variables over a field shows that being principal is a stronger property than being factorial; thus it could seem to be dangerous to concentrate on this stronger property for proving factoriality. However, we have a reliable touchstone for telling us whether the danger exists or not. In fact the commutative algebraists have developed an extensive theory of the *dimension* of a ring, and many methods for computing the dimension of a ring are available. Moreover the principal rings are characterized as being the factorial rings of *dimension* 0 *or* 1. Thus the dimension of the ring A we are studying will tell us whether it is reasonable to attempt to prove that A is principal.

In most geometric cases, principality is proved by proving separately factoriality and one-dimensionality. But, in *algebraic number theory*, there are methods for proving directly that a ring is principal. For instance, let K be a number-field of finite degree n over the rationals, let A be the ring of algebraic integers of K, d the absolute discriminant of A, and $2r_2$ the number of nonreal conjugates of K in C. Then, by using Minkowski's theory of lattice points in convex sets, one can prove that every nonzero ideal $\mathfrak{A}$ of A may be written as $\mathfrak{A}=xb$, where x is an element of K^* and where b is an ideal in A for which

$$\operatorname{card}(A/b) < \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} (|d|^{1/2}). \tag{6}$$

Now the right hand side of (6) can be computed by standard methods, whereas the ideals b for which A/b has a given cardinal c are finite in number, and are easy to determine if c is not too large. Thus, if it happens that all the ideals b for which (6) is satisfied are principal, then the ring A is principal. The reader may apply the method to the ring $A=Z[i]$ of Gaussian integers (here $r=2$, $r_2=1$, $|d|=4$, whence the right hand side of (6) is <2, and (6) thus implies $b=A$); he may then feel that this is a very sophisticated method for proving that $Z[i]$ is principal! In fact the usual proof for $Z[i]$, as well as for Z or for a polynomial ring $k[X]$ over a field k, uses the fact that these rings are *euclidean*. Let us recall that an integral domain A is said to be euclidean if there exists a mapping $\phi: A\rightarrow N$ (the positive integers) such that, for every nonzero b in A, every class modulo Ab admits a representative r such that $\phi(r)<\phi(b)$ (i.e., every a in A may be written $a=bq+r$ with $\phi(r)<\phi(b)$). A euclidean ring A is principal for, given a nonzero ideal b in A, we choose a nonzero element x of b for which $\phi(x)$ is minimal, and see that x generates b. For this proof, it is not necessary to assume that ϕ takes its values in N; any well ordered set W would work as well. A mapping $\phi: A\rightarrow W$ satisfying the above property is called an algorithm on A. If we consider a given ring A and a large-enough well ordered set W (e.g., such that card $(W)>$card (A)), the theory of well ordered sets shows that every algorithm on A is isomorphic (in an obvious sense) with an algorithm with values in W. Furthermore, if $\phi_\alpha: A\rightarrow W$ is a family of algorithms on A, then $\phi=\inf_\alpha\phi_\alpha$ is also an algorithm, so that A (if euclidean) admits a smallest algorithm. If the residue fields of A are finite, this smallest algorithm ϕ_0 actually takes its values in N (the general case is still open). But it is not necessarily the usual algorithm: in the case of Z, $\phi_0(n)$ is the number of binary digits of the integer $|n|$ $(n\in Z)$; however, for polynomials in X over a field k, $\phi_0(P(X))$ is the degree of the polynomial $P(X)$.

Much work has been done by arithmeticians for determining the number fields for which the ring A of integers is euclidean; most of them studied the more restricted problem of finding out whether the usual "norm-function" (i.e., $\phi(x)=$card (A/Ax) for $x\neq 0$) is an algorithm or not. For imaginary quadratic fields, the five fields $Q(\sqrt{-d})$ for $d=1, 2, 3, 7, 11$ are the only ones for which the norm is an algorithm, and are also the only euclidean ones. But there are four principal noneuclidean rings of integers in imaginary quadratic fields for $d=19$, 47, 67 and 163 (the problematic existence of a fifth one has recently been disproved). As to real quadratic fields $Q(\sqrt{m})$ $(m>0)$, the list of those which are euclidean for the norm is known:

$$m = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73.$$

Many others are known to be principal, but we do not know whether their number is finite or not. Also we do not know whether some of them might not be euclidean for another algorithm than the norm; a bit of evidence induces the writer to think that $Q(\sqrt{14})$ deserves to be studied in this respect (see [5], [6]). Summarizing, one might say that the theory of euclidean rings has a quite different flavor from that of factoriality.

4. Nagata's Theorem. Masayoshi Nagata has proved a theorem which is very useful for showing that a ring is factorial. We recall that, if A is an integral domain with quotient field K and if S is a multiplicatively closed subset of $A(0 \notin S)$, then the fractions a/s with $a \in A$ and $s \in S$ form a subring of K, denoted by $S^{-1}A$, and called the *ring of quotients* of A with respect to S. Now suppose that A satisfies the finiteness condition (3) (see Section 2), that S is generated by *prime* elements (Section 2), and that $S^{-1}A$ is factorial; then Nagata's theorem states that A itself is *factorial*. If S is generated by a finite number of prime elements, one can dispense with condition (3). A very easy converse of Nagata's theorem is that any ring of quotients of a factorial ring is factorial.

Gauss's lemma about *polynomial rings* is an easy consequence of Nagata's theorem. In fact let R be a factorial ring, L its quotient field, and $S = R - \{0\}$. Since a prime element p of R remains prime in the polynomial ring $A = R[X]$ (for $A/pA = (R/pR)[X]$ is a domain), S is generated by prime elements of A. But $S^{-1}A = L[X]$ is a polynomial ring in one variable over a field, whence is euclidean and factorial. Hence $A = R[X]$ is factorial by Nagata. By induction the same holds for polynomial rings in several variables over a factorial ring.

Let us sketch three other *examples* of application of Nagata's theorem (complete proofs are left to the reader):

(a) Let k be an algebraically closed field of characteristic $\neq 2$, and $F(X_1, \cdots, X_n)$ a nondegenerate quadratic form over k, with $n \geqq 5$. Then $A = k[X_1, \cdots, X_n]/(F)$ is factorial. (By a change of variables, write $F = X_1X_2 + G(X_3, \cdots, X_n)$; denote by x_j the image of X_j in A; then x_1 is prime since G is irreducible (for $n \geqq 5$); taking $S = \{1, x_1, \cdots, x_1^j, \cdots\}$, we see that $S^{-1}A = k[x_1, x_3, \cdots, x_n][1/x_1]$ is factorial as a ring of quotients of a polynomial ring.)

(b) Let k be a field in which -1 is not a square, and $A = k[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ ("the ring of the 2-sphere"); then A is factorial. (Denote by x, y, z the images of X, Y, Z in A; then $x^2 + y^2 = (1+z)(1-z)$; take S generated by $1 - z$, which is prime; now $S^{-1}A$ is factorial as in (a).)

(c) Let k be a field and $A = k[X, Y, Z]/(X^r + Y^s + Z^t)$ where the exponents r, s, t are pairwise relatively prime; then A is factorial. (Denote by x, y, z the images of X, Y, Z in A, so that $z^t = -(x^r + y^s)$; suppose first that $t \equiv 1 \pmod{rs}$, i.e., $t = 1 + drs$; take S generated by z (which is prime), and set $x' = x/z^{ds}$, $y' = y/z^{dr}$; then $z = -(x'^r + y'^s)$ and $S^{-1}A = k[x', y'][1/z]$ is factorial; in the general case, one chooses an integer j such that $jt \equiv 1 \pmod{rs}$, and replaces z by some jth root w of z.)

5. Further Results. The theory of factorial rings is nowadays much more developed than has been sketched above. For example, in [2] of the bibliography, we find about 80 pages of lecture notes entirely devoted to factoriality with sizeable prerequisites from commutative and homological algebra; moreover these notes did not contain everything known on the subject when they were written (1963), and the theory has progressed since that time. We will thus briefly sketch some highlights of this theory, without defining some of the terms

we use; for detailed definitions, proofs and connected results, we refer the reader to the bibliography.

(1) *Power Series.* In Section 4 we have stated Gauss's lemma about polynomial rings. It is a particular instance of the "transfer" of some property from a ring A to the polynomial ring $A[X]$. Many similar transfers are known, and also transfers of properties from a ring A to the formal power series ring $A[[X]]$. Thus it was reasonable to conjecture that, if A is factorial, so is $A[[X]]$. This conjecture has been disproved (see [**7**]). In the first counter-examples given, the ground ring A was a noncomplete local ring, and taking formal power series over a noncomplete local ring could be deemed, by some mathematicians, to be an unnatural (or even immoral) operation. Doubts were settled very recently by P. Salmon [**13**], who constructed a complete local factorial ring A such that $A[[X]]$ is not factorial.

(2) *Regular Rings.* The notion of a regular ring is defined in commutative algebra; in the geometric case, it corresponds to the notion of a nonsingular variety. In 1957, M. Auslander and D. Buchsbaum proved, by homological methods, that any regular local ring is factorial. Their proof has been streamlined by I. Kaplansky [**1**], [**2**], and by N. Bourbaki [**4**].

On the other hand, if A is a regular and factorial ring (not necessarily local), then both $A[X]$ and $A[[X]]$ are factorial.

(3) *Galoisian going-down.* Let A be a factorial ring, and G a finite group of automorphisms of A; the elements of A which are invariant by G form a subring of A, traditionally denoted by A^G. Let A^* be the multiplicative group of units A. Then if the cohomology group $H^1(G, A^*)$ vanishes, the ring A^G is factorial ([**2**], [**3**], [**17**]).

Here the writer cannot resist giving an amusing example. We take for A a polynomial ring $A=k[X_1, \cdots, X_n]$ (k: a field, $n\geqq 5$), and for G the *alternating group* A_n, acting on A by permutations of the variables. Here the ring of invariants A^G is generated over k by the elementary symmetric functions $s_1, \cdots, s_n$ and by the "discriminant" $d=\Pi_{i<j}(X_i-X_j)$; it is known that $d^2=P(s_1, \cdots, s_n)$ where P is a polynomial over k. Furthermore $A^*=k^*$ is trivially operated by G, so that $H^1(G, A^*)=\mathrm{Hom}(G, k^*)$ (classical formula in the cohomology of groups). Since $G=A_n$ is a simple group ($r\geqq 5$) and since k^* is commutative, we have $\mathrm{Hom}(G, k^*)=0$ and A^G is factorial.

The same method has given an example of a factorial ring which is not a Macaulay ring [**18**]. Notice that P. Murthy has proved that a factorial Macaulay ring is necessarily a Gorenstein ring.

In characteristic $p\neq 0$, there is a parallel theory in which automorphisms are replaced by derivatives ([**2**], [**9**], [**16**]). As above the proofs of factoriality are partly computational, and (especially in characteristic 2) the complete performance of these computations is sometimes more accessible than in the case of automorphisms.

(4) *Complete Intersections.* A local ring A is called a "complete intersection" if it is isomorphic to some R/I, where R is a regular local ring and I an ideal

generated by a regular R-sequence (this means that I may be generated by $\dim(R)-\dim(R/I)$ elements). By using powerful methods of his theory of schemes (the latest version of algebraic geometry), A. Grothendieck proved that a complete intersection A, such that A_p is factorial whenever $\dim(A_p)\leqq 3$, is itself factorial ([19]). This generalizes older geometric theorems of F. Severi, S. Lefshetz and A. Andreotti. No purely ring-theoretic proof of Grothendieck's theorem is known.

(5) *Two-dimensional Factorial Rings.* We have already said that the factorial rings of dimension one are the principal rings; among them, the local ones are the discrete valuation rings and are considered as well known. In dimension 2, we have already seen a good number of examples of factorial rings: e.g., the rings of the surfaces $x^i+y^j+z^k=0$ (i, j, k pairwise relatively prime) and of the sphere $x^2+y^2+z^2-1=0$; localizing the first ones at the origin, we obtain many non-regular local factorial rings of dimension 2. These local rings are not complete and, moreover, the factoriality of their completions $C=K[[x, y, z]]$ was in doubt. First G. Scheja and D. Mumford proved that the complete ring C of the surface $x^2+y^3+z^5=0$ is factorial. Then P. Salmon, for the counterexample alluded to in (1), proved the same for the surface $x^2+y^3+tz^6=0$ over a field K of the form $K=k(t)$ with t transcendental over k.

In this last example the ground field K is not algebraically closed. Now E. Brieskorn, by using techniques from algebraic geometry, has proved that, among the complete two-dimensional local rings over an algebraically closed field K, only two are factorial: the regular ring $K[[X, Y]]$ (formal power series), and the ring $K[[x, y, z]]$ with $x^2+y^3+z^5=0$ (cf. [13]). It can be noted that the latter is the ring of invariants of an icosahedral group acting on the former [1].

A bibliography of factorial rings

An elementary exposition can be found in:

1. P. Samuel, Anneaux Factoriels (red. A. Micali), Bol. Soc. Mat., São Paulo, 1964.

More complete results in:

2. P. Samuel, Lectures on unique factorization domains, (notes by Pavman Murthy) Tata Institute for Fundamental Research lectures, No. 30, Bombay, 1964.

3. ———, Lectures in commutative algebra (notes by M. Bridger), mimeographed by Brandeis University, Waltham, Mass., 1964–65 (write to Brandeis).

For a treatment of factorial rings, in the framework of Krull rings, see:

4. N. Bourbaki, Algèbre Commutative, Chap. VII "Diviseurs," Hermann, Paris, 1966.

For the case of number-fields, see:

5. Hardy-Wright, An introduction to the theory of numbers, Clarendon, Oxford, 1960, and also the tables in

6. Borovič-Safarevič, Théorie des nombres, Gauthier-Villars, Paris, 1966. (German and English translations also available.)

Most results, up to 1964, about factorial rings are given in [1], [2], [3]. For the reader's convenience, we however quote:

7. P. Samuel, On unique factorization domains, Ill. J. Math., 5 (1961) 1–17.

8. ———, Sur les anneaux factoriels, Bull. SMF, 89 (1961) 155–173.

9. ———, Classes de diviseurs et dérivées logarithmiques, Topology, 3, Supp. 1 (1964) 81–96.

10. ———, Modules réflexifs et anneaux factoriels, In Colloque International de Clermont-Ferrand, ed. CNRS, Paris, 1965.

11. P. Samuel, Sur les séries formelles restreintes, C.R. Acad. Sci., Paris, 1962.

The ring of the surface $x^2+y^3+z^5=0$ is studied in

12. F. Klein, Lectures on the icosahedron, Dover, New York, 1956, Chap. 2, Sections 12 and 13.

13. E. Brieskorn, Local rings which are UFD's, (preprint, MIT, Oct. 1966), and in articles of G. Scheja (Math. Ann., 1965), and D. Mumford (Publ. I.H.E.S., 9 (1961)).

The first example of a complete local ring A for which $A[[t]]$ is not factorial was given in:

14. P. Salmon, Sulla non-factorialita . . . , Rend. Lincei, June 1966.

A further discussion of this example is in:

15. N. Zinn-Justin, Dérivations des corps et anneaux de caractéristique p, (Thèse Paris 1967); in print in Mémoires Soc. Math. France, 1967.

For the theory of the "purely inseparable going-down," see:

16. N. Hallier, Utilisation des groupes de cohomologie dans la théorie de la descente p-radicielle, C.R. Acad. Sci. Paris, 261 (1965) 3922–3924, and also [15]. (NB: Hallier is the maiden-name of Mrs. Zinn-Justin.)

For examples of "galoisian going-down," see:

17. M. J. Dumas, Sous anneaux d'invariants d'anneaux de polynômes, C.R. Acad. Sci. Paris, 260 (1965) 5655–5658.

18. M. J. Bertin, Sous groupes cycliques d'ordre $p^n \cdots$, C.R. Acad. Sci. Paris, April 1967. (NB: Dumas is the maiden-name of Mrs. Bertin.)

A proof of Grothendieck's theorem on the factoriality of some complete intersections is in

19. A. Grothendieck, Séminaire de Géometrie Algébrique 1961–1962, exposé XI, mimeographed by the Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 92-Bures sur Yvette. France.

SOME GENERALIZED "ISOMOMENT" EQUATIONS AND THEIR GENERAL SOLUTIONS

J. ACZEL, University of Waterloo, Ontario, Canada, and P. FISCHER,
Automatizálási Kutató Intézet, Budapest, Hungary

1. Introduction. In a previous paper [3] one of us has proved that *all real solutions of the* "isomoment" (terminology of S. Kotz [8]) *functional equation*

$$f\left(\sum_{k=1}^{n} x_k^m/n\right) = \sum_{k=1}^{n} f(x_k)^m/n \tag{1}$$

satisfied for all

$$x_k \geqq 0 \tag{2}$$

$(k=1, 2, \cdots, n)$ and for fixed integers $n>1$, $m>1$ *are continuous and given by*

$$f(x) = 0, \tag{3_t}$$

$$f(x) = 1, \qquad f(x) = x \tag{3_i}$$

in case of any integer $m>1$, *and* further

$$f(x) = -1, \qquad f(x) = -x \tag{3_o}$$

in case of odd $m>1$.

The question arises whether similar results hold for other (negative or noninteger) exponents m. For $m=0$, (1) says only $f(1)=1$ and any function with this property is a solution. For $m=1$, all functions of the form $f(x)=a(x)+b$ with arbitrary constant b and arbitrary solutions a of Cauchy's functional equation

$$a(x+y) = a(x) + a(y) \qquad (x \geqq 0, y \geqq 0) \tag{4}$$

satisfy (1) and only these (see [3] Remark 1 and together with other preliminaries, also section 2 here below). But (4) has noncontinuous solutions ([7], [2] and [4], also for $x>0$, $y>0$).

The remaining cases are those of negative and/or noninteger exponents m. In the cases where m is a negative integer, to be treated in section 3, $f(x)\neq 0$ should be supposed (which eliminates the trivial solution (3_t)) and, rather than (2),

$$x_k > 0 \qquad (k = 1, 2, \cdots, n) \tag{5}$$

should be taken as domain of the functional equation (1) (cf. [3] Remark 2).

In the cases of noninteger m, in order that (1) should make sense, either $f(x)>0$ might be supposed or $f(x_k)$ could be replaced by $|f(x_k)|$ on the right-hand side of (1):

$$f\left(\sum_{k=1}^{n} x_k^m/n\right) = \sum_{k=1}^{n} |f(x_k)|^m/n, \tag{6}$$

$$(n > 1; m \neq 0; f(x) \neq 0 \text{ if } m < 0; x_k > 0; x_k < 0; k = 1, 2, \cdots, n).$$

If $m>0$, also $x_k\geqq 0$, respectively $f(x)\geqq 0$ might be allowed. In both cases it is easy to see that f is continuous as will be shown in section 4, where in analogy to (6), also the functional equation

$$\left|f_1\left(\sum_{k=1}^{n} x_k^m/n\right)\right| = \sum_{k=1}^{n} |f_1(x_k)|^m/n, \tag{7}$$

$$(n > 1, m \neq 0; f(x) \neq 0 \text{ if } m < 0; x_k>0; k = 1, 2, \cdots, n)$$

will be completely solved, which has noncontinuous solutions too.

2. Preliminaries. A fundamental lemma. Both

$$f\left(\sum_{k=1}^{n} x_k^m/n\right) = \sum_{k=1}^{n} f(x_k)^m/n \qquad (n > 1; m \neq 0; f(x) \neq 0 \text{ if } m < 0; x_k > 0; k = 1, 2, \cdots, n) \tag{1'}$$

and (6) are particular cases of

$$f\left(\sum_{k=1}^{n} x_k^m/n\right) = \sum_{k=1}^{n} h(x_k) \qquad (n > 1, m \neq 0; x_k > 0, k = 1, 2, \cdots, n). \tag{8}$$

As mentioned before, we will confine ourselves to the domain

$$x_k > 0 \qquad (k = 1, 2, \cdots, n). \tag{5}$$

It is easy to join 0 to the domain in the cases $m>0$, but the solution for (5) is in general more complicated than for (2), because of the absence of the unit-element 0. We apply the following simplified version of the methods displayed in [5] and [1]:

After transforming equation (8), by writing

$$y_k = x_k^m/n > 0 \qquad (k = 1, 2, \cdots, n); \qquad g(y) = h((ny)^{1/m}), \tag{9}$$

into

$$f\left(\sum_{k=1}^{n} y_k\right) = \sum_{k=1}^{n} g(y_k), \tag{10}$$

we put into (10) $y_3 = y_4 = \cdots = y_n = 1$ and denote $f_0(y) = f(y+n-2) - (n-2)g(1)$ in order to get $f_0(y_1+y_2) = g(y_1) + g(y_2)$. Then

$$g(y_1 + y_2) + g(1) = f_0(y_1 + y_2 + 1) = g(y_1) + g(y_2 + 1)$$

or, with $a(y) = g(y+1) - g(1)$:

$$g(y_1) + a(y_2) = g(y_1 + y_2) = g(y_2) + a(y_1) \qquad \text{for all } y_k > 0 \ (k = 1, 2). \tag{11}$$

Putting, say, $y_2 = 1$, $c = g(1) - a(1)$ into (11), we get

$$g(y) = a(y) + c \tag{12}$$

and, again by (11),

$$a(y_1 + y_2) = a(y_1) + a(y_2) \qquad \text{for all } y_k > 0 \quad (k = 1, 2). \tag{13}$$

Equations (10), (12) and (13) give with $b = nc$ and $y_k = x/n$, $k = 1, 2, \cdots n$,

$$f(x) = f(n(x/n)) = ng(x/n) = na(x/n) + nc = a(x) + b,$$

so that (8), and in particular (1′) or (6), always imply

$$f(x) = a(x) + b \qquad \text{for all } x > 0 \tag{14}$$

with a satisfying (13). As all solutions, *bounded from one side on an interval* ([6], or on a set of positive measure [9]) *or with not everywhere dense graphs* ([7], [10]) of (13), are continuous and of the form $a(x) = ax$, we get *in all these cases* $f(x) = ax + b$.

On the other hand, (8) gives with $x_1 = x_2 = \cdots = x_n = x^{1/m}$

$$f(x) = nh(x^{1/m}) \qquad \text{for all } x > 0, \tag{15}$$

in particular, for (1′)

$$f(x^m) = f(x)^m \qquad (x > 0) \tag{16}$$

or with (14)

$$a(x^m) = (a(x) + b)^m - b \qquad (x > 0), \tag{17}$$

and for (6)

$$f(x) = |f(x^{1/m})|^m \qquad (x > 0). \tag{18}$$

Reciprocally, (13), (14) and (15) imply

$$\sum_{k=1}^{n} h(x_k) = \sum_{k=1}^{n} f(x_k^m)/n = \sum_{k=1}^{n} (a(x_k^m) + b)/n$$
$$= a\left(\sum_{k=1}^{n} x_k^m/n\right) + b = f\left(\sum_{k=1}^{n} x_k^m/n\right),$$

that is (8). Similarly (13), (14) and (16) or (17) imply (1′), and (13), (14) and (18) imply (6).

We summarize:

LEMMA: *The following equivalences hold:*
(19) ((13) *and* (14) *and* (15))⇔(8),
(20) ((13) *and* (14) *and* (16))⇔(1′), *and* ((13) *and* (14) *and* (17))⇔(1′),
(21) ((13) *and* (14) *and* (18)⇔(6).

Comparison with [3] shows:

COROLLARY 1. (3_t), (3_i) *and* (3_o) *are the only solutions of* (1′) *for integer* $m>1$ *also if the domain is* (5).

If in (1′) $m=1$ then (16) is trivially satisfied so that in this case we get only (14) with (13) as system equivalent to (1′) and, as (13) has noncontinuous solutions ([7], [4], [2]) so has (1′) in the case $m=1$.

3. Negative integer exponents. If in (1′) $f(x)\neq 0$, and $m<-1$ is a negative integer then by (20) we get (13), (14) and from (16)

$$f(x^{m^2}) = f(x^m)^m = f(x)^{m^2}. \tag{22}$$

On the other hand, again by (20), the system (13), (14) and (22) imply (1′) for *positive* integer $m^2>1$ in place of m and so (the squares of even or odd integers being even and odd, respectively) we can apply the above corollary and get (3_i) *and in the case of odd* $m\neq -1$ *also* (3_o) *as the only solutions of* (1) *for* $m<-1$ *integer and* $f(x)\neq 0$.

Only the case $m=-1$ remains unsettled among the negative integers. Here (1′) has by (20), (16), (17) the system consisting of

$$f(1/x) = 1/f(x) \quad \text{or} \quad a(1/x) = 1/(a(x)+b) - b \tag{23}$$

and (14), (13) as equivalent. These give

$$a\left(x+\frac{1}{x}\right) = a(x) + a\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) - 2b = f(x) + \frac{1}{f(x)} - 2b. \tag{24}$$

Now, on one hand, every $t\geqq 2$ can be represented as $t=x+(1/x)$ $(x>0)$, on the other hand the right-hand side of (24) is not smaller than $2-2b$ if $f(x)>0$ and not greater than $-2-2b$ if $f(x)<0$ while $f(x)=0$ was excluded, so that the solution of (13) omits the interval $(-2-2b,\ 2-2b)$ of length 4 on $[2,\ \infty)$ and thus cannot be noncontinuous ([7], [10]).

Thus $f(x)=ax+b$ and by (23) $a/x+b=1/(ax+b)$, and this gives the following solutions of (1) for $m=-1$

$$\begin{aligned} a &= 0,\ b = 1: & f(x) &= 1\\ b &= 0,\ a = 1: & f(x) &= x\\ a &= 0,\ b = -1: & f(x) &= -1\\ b &= 0,\ a = -1: & f(x) &= -x, \end{aligned}$$

and only these. So also for $m=-1$ *we have* (3_i) and (3_o) (all continuous) *as only solutions of* (1).

We can now unite the results of this section:

THEOREM 1. *All nonvanishing real solutions of the functional equation*

$$f\left(\sum_{k=1}^{m} x_k^m/n\right) = \sum_{k=1}^{n} f(x_k)^m/n \tag{1}$$

—supposed valid for all positive $x_k (k=1, 2, \cdots, n)$ and for fixed integers $m<0$, $n>1$—are continuous. The functions (3_i) and, in case of odd m, also (3_o) represent all solutions of this equation.

COROLLARY. *All solutions of the systems of functional equations*

$$f\left(\sum_{k=1}^{n} x_k/n\right) = \sum_{k=1}^{n} f(x_k)/n,\quad f(x^m) = f(x)^m\ (x_k > 0,\ x > 0,\ n < 1, f(x) \neq 0) \tag{25}$$

or

$$f(x_1 + x_2) = f(x_1) + f(x_2),\quad f(x^m) = f(x)^m \qquad (x_k > 0,\ x > 0, f(x) \neq 0) \tag{26}$$

(where m is a—positive or negative—integer constant, different from 0 *and from* 1*) are continuous and given by (3_i) and, if m is odd, also (3_o), or by $f(x)=x$, and, if m is odd, also $f(x)=-x$, respectively.*

Proof: (26) is the $b=0$ case of (13), (14) and (16), so the Lemma and Theorem 1 give the result. From (25)

$$f\left(\sum_{k=1}^{n} x_k^m/n\right) = \sum_{k=1}^{n} f(x_k^m)/n = \sum_{k=1}^{n} f(x_k)^m/n$$

follows, which is (1), so again Theorem 1 gives the result.

4. Noninteger exponents. If in (1)

$$f(x) > 0 \quad \text{or} \quad f(x) \geqq 0 \tag{27}$$

are supposed, then in (14) $a(x) \geqq -b$ for $x \in (0, \infty)$ and a is a solution of (13), bounded on an interval, thus continuous. Similarly, by the Lemma, (6) implies (18):

$$f(x) = |f(x^{1/m})|^m \geqq 0,$$

i.e. (27), and we can proceed as before. So in both cases $f(x) = ax + b \geqq 0$ for $x > 0$ and

$$ax^m + b = (ax + b)^m \qquad (x > 0; m \neq 0).$$

One sees (e.g. by differentiation) immediately that $a = 0$, $b = 1$ or $b = 0$, $a = 1$, (also $a = b = 0$ for $f(x) \geqq 0$) if $m \neq 1$, while a, b can be arbitrary in the case $m = 1$, but such that $ax + b \geqq 0$ for $x > 0$ which is the case iff $a \geqq 0$, $b \geqq 0$($a > 0$, $b \geqq 0$ iff $f(x) > 0$ has to hold). We have:

THEOREM 2. *Let $m \neq 1$ be a real constant. All nonnegative solutions of the equation* (1′) *and all solutions of* (6) *are given by $f(x) = x$, $f(x) = 1$, which are joined by $f(x) = 0$ in case of positive m. If $m = 1$, then all nonnegative (positive) solutions of* (1′) *and all solutions of* (6) *are given by $f(x) = ax + b$ with $a \geqq 0$, $b \geqq 0$ (or $a > 0$, $b \geqq 0$, respectively).*

In analogy to (6) we can consider also (7). With

$$f(x) = |f_1(x)| \geqq 0, \tag{28}$$

we get (1′), so by Theorem 2 we have for f (besides possibly $f(x) = 0$ for $m > 0$) $f(x) = 1$ or $f(x) = x$ if $m \neq 1$ and $f(x) = ax + b$ ($a \geqq 0$, $b \geqq 0$) if $m = 1$. By (28)

$$f_1(x) = e(x)f(x), \qquad |e(x)| = 1.$$

As the equation (7) contains only the absolute value of f_1, the function $e(x)$ can be arbitrary with absolute value equal to 1 on the positive reals; or

THEOREM 3. *The solutions of* (7) *are, besides $f_1(x) = 0$ for $m > 0$, $f_1(x) = e(x)$ and $f_1(x) = e(x)x$ if $m \neq 1$ and $f_1(x) = e(x)|ax + b|$ if $m = 1$ with arbitrary constants a, b and with an arbitrary function $e(x)$ satisfying $|e(x)| = 1$, and only these.*

Of course, e and thus f_1 might be discontinuous.

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References

1. J. Aczel, On a generalization of the functional equation of Pexider, Acad. Serbe Sci. Publ. Inst. Math., Beograd, 4 (18), (1964) 77–80.

2. ———, Lectures on Functional Equations and their Applications, Academic Press, New York-London, 1966.

3. ———, General solution of "isomoment" functional equations, this MONTHLY, 74 (1967) 1068–1071.

4. J. Aczel and P. Erdös, The nonexistence of a Hamel-Basis and the general solution of Cauchy's functional equation for non-negative numbers, Publ. Math. Debrecen, 12 (1965) 259–263.

5. J. Aczel and H. Kiesewetter, Über die Reduktion der Stufe bei einer Klasse von Funktionalgleichungen, Publ. Math. Debrecen, 8 (1958) 348–363.

6. G. Darboux, Sur le théorème fondamental de la géométrie projective, Math. Ann., 17 (1880) 55–61.
7. G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x+y)=f(x)+f(y)$, Math. Ann., 60 (1905) 459–462.
8. S. Kotz, On the solutions of some "isomoment" functional equations, this MONTHLY, 72 (1965) 1072–1075.
9. A. Ostrowski, Mathematische Miszellen XIV. Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen, Jber. Deutsch. Math.-Verein, 38 (1929) 54–62.
10. R. San Juan, An application of Diophantine approximation to the functional equation $f(x_1+x_2)=f(x_2)+f(x_2)$, (Spanish) Publ. Inst. Mat. Univ. Litoral 6 (1946) 221–224.

COMPACT-LIKE OPERATORS AND THE EBERLEIN THEOREM

KARL GUSTAFSON, University of Colorado, Boulder

In this paper we will prove the following related new results 1–7 below concerning operators of compact type (see definitions below). Using these results, we briefly summarize presently-known relations between these operator classes (Theorem 6) and also the characterization of certain normed linear spaces in terms of these operator class relationships (Theorem 7). We also provide partial answers to a question posed in [5] concerning how the operator type depends on its range space; it is in this connection that a framework of incomplete spaces is essential. Due to attention to organization, the use of definitions of sequential type, and the use of the Eberlein Theorem, our proofs are brief and elementary.

(Our setting is that of normed linear spaces; the results stated herein would extend somewhat into the class of locally convex topological vector spaces via definitions and arguments more topological in nature.)

1. *When the domain of the operator is reflexive, the weakly compact operators are exactly the weakly continuous ones* (see Theorem 1).
2. *A reflexive domain affirmatively answers the question of* [5] *for all types of compact-like operators* (see Corollary 2).
3. *The Eberlein* (*-Shmulyan*) *Theorem* [4] *extends to the class of all normed linear spaces* (see Theorem 4).
4. *When the domain is of Dunford-Pettis type and the range space is reflexive, the weakly compact operators are exactly the strictly singular operators* (see Theorem 5).
5. *All domains with weak convergence equivalent to strong convergence are of Dunford-Pettis type* (see Theorem 7).
6. *The composition AB, B weakly compact, A completely continuous, is compact* (see Theorem 8).
7. *Any semigroup T_t of operators such that T_t is both weakly compact and completely continuous* (*as in Brownian Motion*) *is compact* (see Corollary 9).

We now introduce some notation. Let $[\cdot]$ denote a class of linear operators $[T: X\rightarrow Y]$ with $D(T)=X$, a normed linear space, and with range $R(T)$ in another normed linear space Y. We assume X and Y (not necessarily complete) to be infinite dimensional; however, if not, the operator classes all collapse to $[F]$ and the theorems remain true. Let $\{x_n\}$ be a sequence in X, let $\rightarrow$ denote strong (normed) convergence, let $\rightharpoonup$ denote weak ($x'(x_n-x)\rightarrow 0\ \forall x'\in X'$, the dual space) convergence, and let $\subset$ denote inclusion (not necessarily strict). We consider the following operator classes.

[CC] = completely continuous: $x_n\rightharpoonup x\Rightarrow Tx_n\rightarrow Tx$.
[WC] = weakly continuous: $x_n\rightharpoonup x\Rightarrow Tx_n\rightharpoonup Tx$.
[C] = continuous: $x_n\rightarrow x\Rightarrow Tx_n\rightarrow Tx$.
[DC] = demicontinuous: $x_n\rightarrow x\Rightarrow Tx_n\rightharpoonup Tx$.
[Cl] = closed: $x_n\rightarrow x$, $Tx_n\rightarrow y\Rightarrow Tx_n\rightarrow Tx$.
[K] = compact: $\{x_n\}$ bounded$\Rightarrow$some $Tx_{n_k}\rightarrow y$.
[PK] = precompact: $\{x_n\}$ bounded$\Rightarrow$some $\{Tx_{n_k}\}$ is Cauchy.
[WK] = weakly compact: $\{x_n\}$ bounded$\Rightarrow$some $Tx_{n_k}\rightharpoonup y$.
[SS] = strictly singular: $T\in[C]$ and for X_0 any infinite dimensional subspace of X there exists an infinite dimensional subspace of X_0 on which $T\in[PK]$.

Of course, there are many interesting subclasses of [K], such as the nuclear operators, the finite range operators [F]: $T\in[C]$ and $R(T)$ finite dimensional, the $[C_p]$ trace classes when $X=Y=H$, a Hilbert space, and others. We should note at this time that when $X=Y=H$, all the above classes collapse to just [K] and [C].

We could also have considered subclasses of [Cl] such as

$$[\mathrm{WCl}]: x_n\rightharpoonup x,\ Tx_n\rightharpoonup y\Rightarrow Tx_n\rightharpoonup Tx, \qquad [\mathrm{CCl}]: x_n\rightharpoonup x,\ Tx_n\rightarrow y\Rightarrow Tx_n\rightarrow Tx,$$
$$[\mathrm{DCl}]: x_n\rightarrow x,\ Tx_n\rightharpoonup y\Rightarrow Tx_n\rightharpoonup Tx,$$

but because they are all exactly equal to [Cl] when X and Y are complete, by use of the closed graph theorem (as in Theorem 6 (6)), we did not include them in these comparisons. Similarly the class [DP]: $x_n\rightarrow x\Rightarrow Tx_n\rightarrow y$ and its variations [WDP], [CDP], [DDP] were not included, although there are some interesting relationships between these and the above operator classes (e.g., [CDP] = [CC] and [WDP] = [WC] for X reflexive).

We consider the following spaces, often as domains for the operators.

X is reflexive: the natural isometric injection of X into X'' is surjective.

X is D-P (Dunford-Pettis): for every complete Y, $[\mathrm{WK}]\subset[\mathrm{CC}]$.

X is very irreflexive: X contains no infinite dimensional reflexive subspaces.

X is $S=w$: $x_n\rightharpoonup x\Rightarrow x_n\rightarrow x$.

As is well known [4], among complete spaces the reflexive ones are exactly the locally sequentially weakly compact (l.s.w.k.) ones: every bounded sequence contains a weakly convergent subsequence. Theorem 4 will extend this to all normed linear spaces; the (l.s.w.k.) property is extremely useful when dealing with compact-like operators, as the following proofs will reveal.

It is shown in [3, see summarizing statement, p. 511] that most nonreflexive Banach spaces are D-P spaces, and the fact that the D-P spaces and the reflexive spaces are disjoint was observed and used in [8]. We do not introduce the subprojective and superprojective spaces of [11], although there are interesting questions concerning the exact overlapping of all of these and other classifications of normed linear spaces. For examples of the above spaces, see [2, 3, 6, 12].

The following result does not require Y complete.

THEOREM 1. *When X is reflexive,* $[\mathrm{WK}]=[\mathrm{C}]=[\mathrm{WC}]$.

Proof. Since always $[\mathrm{WK}]\subset[\mathrm{C}]\subset[\mathrm{WC}]$, e.g., see Theorem 6, it is sufficient to show that $[\mathrm{WC}]\subset[\mathrm{WK}]$. Let $\{x_n\}$ be bounded and $T\in[\mathrm{WC}]$; then since X is (l.s.w.k.), there exists a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ and $\{Tx_{n_k}\}$ converge weakly.

Concerning Theorem 1, it is well known (and immediate by (l.s.w.k.)) that $[\mathrm{K}]=[\mathrm{CC}]$ when X is reflexive. It has recently been brought to the author's attention that $[\mathrm{WK}]=[\mathrm{C}]$ is essentially shown in [9], (there the conditions are X only almost reflexive but Y weakly complete), and that $[\mathrm{WK}]=[\mathrm{C}]$ is implied in the results of [1].

Any function $T: X\to Y$ induces another function $T_0: X\to R(T)$ with $T_0x=Tx$. Although many operator classes retain their type (Corollary 2 below) under this range reduction, it is not necessarily true that $[\mathrm{K}]$ does, since $R(T)$ is not necessarily closed in Y. Recently in [5] a counterexample T: $c_0\to l_2$ was given and the question asked: when is $T_0\in[\mathrm{K}]$?

COROLLARY 2. $T\in[\cdot]\Rightarrow T_0\in[\cdot]$ *whenever* $[\cdot]=[\mathrm{CC}]$, $[\mathrm{WC}]$, $[\mathrm{C}]$, $[\mathrm{DC}]$, $[\mathrm{Cl}]$, $[\mathrm{PK}]$, *and* $[\mathrm{SS}]$ *always. X reflexive or $R(T)$ closed is sufficient for* $[\mathrm{K}]$ *and* $[\mathrm{WK}]$.

Proof. The first part follows from the definitions. Since when X is reflexive, $[\mathrm{K}]=[\mathrm{CC}]$ and $[\mathrm{WK}]=[\mathrm{WC}]$, the second part follows from the first part. The condition $R(T)$ closed is worth observing since criteria for a closed range exist (e.g., X and Y complete and a positive minimum modulus for T).

That X reflexive is sufficient for $T_0\in[\mathrm{K}]$ was shown by Whitley in [10], and $T_0\in[\mathrm{WK}]$ was recently shown in [1]. In the terminology of these recent papers, Corollary 2 shows that reflexive spaces are perfect and weakly perfect, and that all spaces are "perfect" for all of the other classes of compact-like operators.

Our results above, obtained independently of the above mentioned papers, were characterized by extremely short proofs utilizing the (l.s.w.k.) property of reflexive spaces. There immediately arises the possibility that the (l.s.w.k.) spaces are a larger class than the reflexive spaces; that this is not the case is important to note in this context of compact-like operators.

LEMMA 3. *A* (l.s.w.k.) *normed linear space is complete.*

Proof. Let $\{x_n\}$ be Cauchy in X. Then $\{x_n\}$ has a strong (and therefore

weak) limit z_1 in the completion Z of X; also $\{x_n\}$ has a subsequence converging weakly in X (and therefore in Z) to a z_2 in X. By the uniqueness of weak limits (applied to x_n in Z), $z_1=z_2$.

Since apparently this extension of the Eberlein Theorem is not explicitly given in the literature, we state it here.

THEOREM 4. *Among all normed linear spaces, the (l.s.w.k.) ones are exactly the reflexive ones.*

Clearly this result extends further, for example, to the (C.s.w.k.) ones, i.e., those where every Cauchy sequence contains a weakly convergent subsequence, and perhaps to less trivial extensions.

It would be interesting to have criteria for $T_0 \in [\mathrm{K}]$ for X among the D-P spaces, where all the counterexamples presently exist [**5**, **10**]; unfortunately the definition of D-P spaces involves complete Y. Note that if we had used a Banach space (e.g., Y complete) setting, the T_0 question could not have been approached in the above manner, since it is basically a question of how incomplete $Y_0=R(T)$ can be.

That X very irreflexive and Y reflexive implies $[\mathrm{SS}]=[\mathrm{WK}]=[\mathrm{C}]$ was shown in [**6**]; the next result extends this to the large class of all D-P spaces.

THEOREM 5. *If X is* D-P *and Y is reflexive, then* $[\mathrm{CC}]=[\mathrm{SS}]=[\mathrm{WK}]=[\mathrm{C}]$.

Proof. That $[\mathrm{WK}]=[\mathrm{C}]$ when Y is reflexive is immediate from $[\mathrm{WK}]\subset[\mathrm{C}]$ and Y(l.s.w.k.), and when X is complete we have the important result of [**11**] that $[\mathrm{WK}]\cap[\mathrm{CC}]\subset[\mathrm{SS}]$; these two relations combined with $[\mathrm{WK}]\subset[\mathrm{CC}]$ for X D-P yield the desired result.

It seems that no concise summary of relations between these compact-like operator classes is currently available; using the above results and known results, we now give such a summary, emphasizing domain properties and the five classes [K], [CC], [SS], [WK], and [C].

THEOREM 6. *For X and Y normed linear spaces, always*:

(1) $[\mathrm{K}]\subset[\mathrm{PK}]\subset[\mathrm{SS}]\subset[\mathrm{C}]$;

(2) $[\mathrm{K}]\subset[\mathrm{CC}]\subset[\mathrm{C}]=[\mathrm{WC}]=[\mathrm{DC}]\subset[\mathrm{Cl}]$;

(3) $[\mathrm{K}]\subset[\mathrm{WK}]\subset[\mathrm{C}]$.

When Y is complete:

(4) $[\mathrm{K}]=[\mathrm{PK}]$.

When X is complete:

(5) $[\mathrm{WK}]\cap[\mathrm{CC}]\subset[\mathrm{SS}]$.

When X and Y are complete:

(6) $[\mathrm{C}]=[\mathrm{WCl}]=[\mathrm{DCl}]=[\mathrm{CCl}]=[\mathrm{Cl}]$.

When Y is reflexive and X is complete:

(7) $[\mathrm{K}]\subset[\mathrm{CC}]\subset[\mathrm{SS}]\subset[\mathrm{WK}]=[\mathrm{C}]$.

When X is reflexive:

(8) $[\mathrm{K}]=[\mathrm{CC}]\subset[\mathrm{SS}]\subset[\mathrm{WK}]=[\mathrm{C}]$.

When X (complete) is D-P *and Y is complete*:

(9) $[\mathrm{K}] \subset [\mathrm{WK}] \subset [\mathrm{CC}] \cap [\mathrm{SS}] \subset [\mathrm{C}]$.

When X (complete) is either D-P *or very irreflexive, and Y is reflexive*:

(10) $[\mathrm{K}] \subset [\mathrm{CC}] \subset [\mathrm{SS}] = [\mathrm{WK}] = [\mathrm{C}]$.

When Y is $S=w$:

(11) $[\mathrm{K}] = [\mathrm{WK}] \subset [\mathrm{SS}] \subset [\mathrm{CC}] = [\mathrm{C}]$.

Proof. Because our proofs of some of the minor results included may be new, we now indicate proofs of everything in Theorem 6 that is not immediate from the definitions. In (2), that $[\mathrm{K}] \subset [\mathrm{CC}]$ follows from the boundedness of weak limits and $[\mathrm{K}] \subset [\mathrm{C}]$, that $[\mathrm{DC}] \subset [\mathrm{Cl}]$ is by the uniqueness of weak limits, and that $[\mathrm{C}] = [\mathrm{WC}] = [\mathrm{DC}]$ follows from $D(T') = Y' \Leftrightarrow T \in [\mathrm{C}]$, (see [**7**]). (That $[\mathrm{C}] = [\mathrm{WC}]$ is sometimes shown to follow from the fact that a normed linear space is bornological.) In (3) ,that $[\mathrm{WK}] \subset [\mathrm{C}]$ follows from the boundedness of weak limits (sometimes this is unnecessarily required in the definition of $[\mathrm{WK}]$). The important relation (5) originally appeared in [**11**] and is extended to incomplete Y in [**7**]. Since always $[\mathrm{C}] \subset [\mathrm{WCl}] \subset [\mathrm{CCl}] \cap [\mathrm{DCl}] \subset [\mathrm{Cl}]$, equality (6) is the closed graph theorem applied to (2). The equality in (7) is by the (l.s.w.k.) of Y, and the rest then follows from (5). Relation (8) has been shown above from the (l.s.w.k.) of X. Relation (9) follows from (5) and the fact that X is D-P. The D-P part of (10), with also $[\mathrm{CC}] = [\mathrm{SS}]$, was shown above, and the irreflexive part was shown in [**6**]. When either X is $S=w$ or Y is $S=w$, many operator classes coincide, and (11) is included only to emphasize this fact and that completeness is not essential in (5).

We refer the reader to the references concerning which inclusions above have been shown to be (in general) strict by construction of examples.

We recall that: (i) although most types of normed linear spaces are distinguished in terms of some inherent structure, the D-P spaces were defined by means of the behavior of two of their compact-like operator classes; (ii) always $I \in [\mathrm{C}]$, $I \in [\mathrm{K}]$ iff X is finite dimensional, and $I \notin [\mathrm{SS}]$ for any infinite dimensional X. In the following statement we extend this characterization of X in terms of the behavior of I and the compact-like operator classes.

THEOREM 7. *Let X be a normed linear space. Then the following three properties are equivalent*:

(1a) *X is reflexive*;

(1b) $[\mathrm{WK}] = [\mathrm{WC}]$ *for every Y*;

(1c) $I \in [\mathrm{WK}]$.

Similarly, the following two properties are equivalent:

(2a) *X is* D-P;

(2b) $[\mathrm{WK}] \subset [\mathrm{CC}] \cap [\mathrm{SS}]$ *for every complete Y*; *and then* (2c) $I \notin [\mathrm{WK}]$.

Similarly, the following three properties are equivalent:

(3a) *X is $S=w$*;

(3b) $[\mathrm{CC}] = [\mathrm{C}]$ *for every Y*;

(3c) $I \in [\mathrm{CC}]$;

and then (3d) *X is both* D-P *and very irreflexive.*

Proof. (1a)$\Rightarrow$(1b) by Theorem 1, (1b)$\Rightarrow$(1c) clearly, and (1c) implies X(l.s.w.k.), which, by Theorem 4, implies X reflexive. Of course, many other conditions equivalent to X reflexive are known (e.g., see [2], p. 56). (2a)$\Rightarrow$(2b) by (9) of Theorem 6, with the converse by definition, and (2c) restates the disjointness of the D-P and reflexive spaces. (3a), (3b), and (3c) are obvious, and the last part of (3d) was shown in [11]. That X is D-P follows from (3b) and $[\mathrm{WK}]\subset[\mathrm{C}]$.

We conclude by noting the properties of the compact-like operator classes under composition. Let $B: X\rightarrow Y$, $A: Y\rightarrow Z$, X, Y, and Z normed linear spaces, let $A\in[A]$ where $[A]$ denotes one of the five classes [K], [CC], [SS], [WK], and [C], similarly $B\in[B]$. Let $A\cdot B$ denote the composition, so that there are 25 possibilities to consider. (Concerning sums, that $[A]+[A]\subset[A]$ for each of the five classes is easily seen; e.g., see [7] for a demonstration of this for [K] and [SS].)

THEOREM 8. *$A\cdot B\subset[A]\cap[B]$ always. In particular,* $[\mathrm{CC}]\cdot[\mathrm{WK}]=[\mathrm{K}]$.

Proof. The first part follows from the fact that left or right composition with [C] preserves the other four classes; see [7] for a demonstration of this for [SS], the other three classes being easily verified. Thus the composition always strengthens the compactness by preserving both original compactness properties.

The second part follows from the implications $\|x_n\|<M\Rightarrow Bx_{n_k}\rightarrow y\Rightarrow ABx_{n_k}\rightarrow Ay$. This is an extension of the known result (see [3]) that $[\mathrm{WK}]\cdot[\mathrm{WK}]=[\mathrm{K}]$ when Y is a D-P space, which is clear from Theorem 8 since then $A\in[\mathrm{WK}]\subset[\mathrm{CC}]$. Among the 25 compositions, this is the only (general) strengthening by the composition to a compactness property stronger than both of the original ones. Of course, if one imposes various combinations of reflexive and D-P conditions on X, Y, and Z, one obtains reductions due to Theorem 6.

However, the second part of Theorem 8 is of interest for the following reason. Letting P_t denote a probability transition of Markov type (i.e., possessing the Chapman-Kolmogorov property), then P_t generates a (contraction) semigroup $T_t=\int_S P_t$ (see [12]), and under certain probabilistic assumptions on (each) P_t (e.g., strongly Fellerian in the strict sense), it can be shown that (each) $T_t\in[\mathrm{K}]$, $t>0$. However, one can extend this result to a larger (less restrictive) set of probabilistic assumptions by any of the following (somewhat equivalent) procedures: (i) letting P_t be only such that $T_t\in[\mathrm{WK}]$; (ii) letting T_t possess certain averaging properties (e.g., $T_t(B(S))\subset C(S)$); (iii) letting $R(T_t$(bounded sequences)) possess certain pointwise convergence properties (e.g., see [3, p. 265, Corollary 4]). Then $T_t\in[\mathrm{WK}]$, and since (usually) $D(T_t)$ is of D-P type, $T_t=T_{\frac{1}{2}t}\cdot T_{\frac{1}{2}t}\in[\mathrm{K}]$ by Theorem 8. These observations (to some extent probably known, but apparently not in the literature) evolved in a discussion of Brownian Motion with R. Cairoli.

By Theorem 8 we therefore have the following generalization of this phenomenon to arbitrary semigroups.

COROLLARY 9. *Let T_t be any semigroup of bounded linear operators on a normed linear space X such that $T_t \in [\mathrm{WK}] \cap [\mathrm{CC}]$; then $T_t \in [\mathrm{K}]$.*

Note that by [11] (i.e., (5) of Theorem 6), a strong result in itself, in general one has only $[\mathrm{WK}] \cap [\mathrm{CC}] \subset [\mathrm{SS}]$.

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References

1. D. R. Arterburn, Weak perfect compactness and generalized adjoints, dissertation, N. Mexico State University, 1965.
2. M. M. Day, Normed linear spaces, Springer-Verlag, Berlin, 1958.
3. N. Dunford and J. Schwartz, Linear operators I, Interscience, New York, 1963.
4. W. F. Eberlein, Weak compactness in Banach spaces, Proc. Nat. Acad. Sci. U.S.A., 33 (1947) 51–53.
5. Seymour Goldberg and E. O. Thorp, The range as range space for compact operators, J. Reine Angew. Math., 211 (1962) 113–115.
6. Seymour Goldberg and E. O. Thorp, On some questions concerning strictly singular operators, Proc. Amer. Math. Soc., 14 (1963) 334–336.
7. Seymour Goldberg, Unbounded linear operators, McGraw-Hill, New York, 1966.
8. A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type C(K), Canadian J. Math., 5 (1953) 129–173.
9. E. Lacey and R. J. Whitley, Conditions under which all the bounded linear maps are compact, Math. Ann., 158 (1965) 1–5.
10. E. O. Thorp, The range as range space for compact operators II, J. Reine Angew. Math., 217 (1965) 69–78.
11. R. J. Whitley, Strictly singular operators and their conjugates, Trans. Amer. Math. Soc., 113 (1964) 252–261.
12. K. Yosida, Functional analysis, Springer-Verlag, Berlin, 1965.

SERIES WHOSE TERMS ARE OBTAINED BY ITERATION OF A FUNCTION

GEORGE BRAUER, University of Minnesota

Let $f(x)$ be a real-valued function defined on an interval I_a: $[0, a]$ satisfying the conditions

(1) f is continuous on I_a,

(2) $f(0)=0$, $0 \leqq f(x) < x$, $0 < x \leqq a$,

and let the series $\sum_{n=0}^{\infty} u_n$ be formed from the function f by defining $u_0 = x$, $u_{n+1} = f(u_n)$, for $n \geqq 0$. In [2] Fort and Schuster showed that if the function f satisfies the additional conditions:

(3) f is differentiable in I_a,

(4) there exists a positive constant c such that $f'(x) \geqq c$ in I_a,

(5) if $0 < x_1 < x_2 < b$, then $f(x_1)/x_1 > f(x_2)/x_2 > 0$,

then, for each point $x=u_0$ in I_a, the series $\sum_{n=0}^{\infty} u_n$ converges or diverges according as the integral $\int_0^a t/(t-f(t))dt$ converges or diverges. In [1] I studied the convergence of series $\sum u_n$ obtained from functions f satisfying only (1) and (2). In this note I shall extend the results in [1], answering a question raised in [1], and make some remarks about the Dirichlet series $\sum_{n=0}^{\infty} u_n^s$. Throughout the paper f will be assumed to satisfy (1) and (2). As in [1], each point u_0 in I_a such that the series $\sum_{n=0}^{\infty} u_n$ converges, will be called a point of convergence for f; the set of points of convergence will be designated as the set of convergence.

THEOREM 1. *If f satisfies a Lipschitz condition in some interval $I_b(0<b\leqq a)$, as well as conditions* (1) *and* (2) *on I_a, and the integral $\int_0^a t/(t-f(t))dt$ converges, then all points of I_a are points of convergence for f.*

Proof. Let u_0 be a point of I_a. If $u_n=0$ for some n then u_0 is clearly a point of convergence, in fact the series $\sum u_n$ contains only finitely many nonzero terms. If $u_n\neq 0$ for any value of n, $u_{n-1}-u_n>0$ for all n. Conditions (1) and (2) ensure that $\{u_n\}$ is a null sequence; therefore there is a number u_0 such that $u_n\in I_b$ for $n\geqq N_0$. If z is a point in the interval $[u_n, u_{n-1}]$ then, if L denotes a Lipschitz constant for f so that $x-f(x)$ has $L+1$ as a Lipschitz constant, we have

$$0 \leqq z-f(z) \leqq u_n - f(u_n) + (L+1)(u_n - z) \leqq (L+2)(u_n - u_{n+1}),$$

and therefore

$$(L+2)z/(z-f(z)) \geqq z/(u_{n-1}-u_n) \geqq u_n/(u_{n-1}-u_n)$$

for all $z\in[u_n, u_{n-1}]$. Consequently

$$\sum_{n>N_0} u_n = \sum_{n>N_0} u_n/(u_{n-1}-u_n)(u_{n-1}-u_n)$$
$$\leqq (L+2)\sum_{n>N_0}\int_{u_n}^{u_{n-1}} \frac{(t)dt}{t-f(t)} \leqq (L+2)\int_0^a \frac{t}{t-f(t)}\,dt.$$

The conclusion follows.

We note that the above inequality is a comparison of the area under the graph of the step function

$$y = u_n/(u_{n-1}-u_n) \qquad u_n \leqq x \leqq u_{n-1}$$

and the area under the curve $y=x/(x-f(x))$.

THEOREM 2. *If f satisfies a Lipschitz condition in some interval $I_b(0<b\leqq a)$ as well as conditions* (1) *and* (2) *on I_a, and there is a point of divergence u_0 such that $u_n/u_{n-1}\to 1$ then zero is an isolated point of convergence.*

Proof. Since $u_n/u_{n-1}\to 1$, there is, for each $\epsilon>0$, a number n_0 such that $u_n\geqq u_{n-1}/(1+\epsilon)$ or $u_n\geqq(u_{n-1}-u_n)/\epsilon$ for $n\geqq n_0$. Also since $\{u_n\}$ is a null sequence n may be chosen so that $u_{n-1}\in I_b$ for $n\geqq n_0$. If v_0 is a point of convergence, there must exist arbitrarily large n such that $v_n\leqq\epsilon u_n$. Suppose that n is chosen greater

than n_0 and so that $v_n \leqq \epsilon u_n$ ($\epsilon < 1$). The proof of Theorem 3 of [1] shows that if for each $\delta > 0$ the interval $(0, \delta)$ contains points of convergence, then there is a point w in (u_{n-1}, u_n) such that $f(w) = v_n$; moreover since $w < u_{n-1}$, $w \in I_b$. But

$$f(u_{n-1}) - f(w) \geqq (1 - \epsilon)u_n \geqq (1 - \epsilon)(u_{n-1} - u_n)/\epsilon \geqq (1 - \epsilon)(u_{n-1} - w)/\epsilon.$$

Since ϵ is an arbitrary positive number, the above inequality contradicts the hypothesis that f satisfies a Lipschitz condition in I_b. Thus the theorem is proved.

We note that if u_n vanishes for some n then the Dirichlet series $\sum u_n^s$ converges for all s, while if u_n never vanishes then this series has some nonnegative number as its abscissa of convergence.

THEOREM 3. *If $\sigma_c \geqq 0$, then the set of points $x = u_0$ such that the series* (6) *has σ_c as its abscissa of convergence is of type $F_{\sigma\delta}$.*

Proof. Let

$$U = \{u_0 \mid \textstyle\sum u_n^\sigma \text{ converges for } \sigma > \sigma_c, \text{ diverges for } \sigma < \sigma_c\},$$

$$D_\sigma = \{u_0 \mid \textstyle\sum u_n^\sigma \text{ diverges}\},$$

$$C_\sigma = \{u_0 \mid \textstyle\sum u_n^\sigma \text{ converges}\}.$$

A point u_0 lies in U if and only if it lies in each set C_σ for $\sigma > \sigma_c$ and in each set D_σ for $\sigma < \sigma_c$. Let $\{\sigma^{(r)}\}$ denote a decreasing sequence of numbers converging to σ_c; let $\{\sigma'^{(r)}\}$ denote an increasing sequence of numbers converging to σ_c. We have

$$U = \bigcap C_{\sigma^{(r)}} \bigcap D_{\sigma'^{(r)}}.$$

By the method used in the proof of Theorem 2 the set of points u_0 such that $\sum u_n^\sigma$ converges is of type F_σ, that is, each set $C_{\sigma^{(r)}}$ is of type F_σ while each set $D_{\sigma^{(r')}}$ is of type G_δ. Thus U is of type $F_{\sigma\delta}$.

The set of points u_0 such that the Dirichlet series $\sum u_n^s$ converges for all s is of type F_σ since it is the union of the sets

$$F_n = \{u_0 \mid u_n = 0\}, \; n = 0, 1, \cdots.$$

THEOREM 4. *Let σ_c be a nonnegative number and let M denote the set of points u_0 in I_a such that the Dirichlet series $\sum u_n^s$ has σ_c as its abscissa of convergence. Let $s' = \sigma_c + i\tau'$ be a point on the line $\sigma = \sigma_c$ and let α be a complex number. Let E denote the set of points v_0 in M such that the function $G(s) = \sum_{n=0}^\infty v_n^s$ tends to α as s tends to s' along the segment $\tau = \tau'$, $\sigma < \sigma_c$. The set E is of type $G_{\delta\sigma\delta}$.*

Proof. Let $s^{(r)} = \sigma^r + i\tau'$ be a sequence of points such that $\{\sigma^{(r)}\}$ is a decreasing sequence of numbers tending to σ_c.

LEMMA. *If $\rho > 0$ and r_0 is a natural number, then the set of points z_0 in M such that $\left| \sum_{n=0}^\infty z_n^{s^{(r)}} - \alpha \right| < \rho$ for $r > r_0$ is of type G_δ.*

The proof of this lemma is similar to that of Theorem 2 of [1]. For each

positive number ϵ and each natural number r_0 let $E(\epsilon, r_0)$ denote the set of points v_0 in M such that $|G(s^{(r)})-\alpha|<\epsilon$ for $r>r_0$; by the lemma the set $E(\epsilon, r_0)$ is of type G_δ for fixed ϵ and r_0. We have $E=\bigcap_j\bigcup_{r_0}E(1/j, r_0)$, and thus E is a set of type $G_{\delta\sigma\delta}$.

Schuster and Fort [2] proved that if f satisfies condition (1)–(5), then the Dirichlet series $\sum u_n^s$ converges or diverges at $s=\sigma$ for all points $x=u_0$ in I_a according as the integral $\int_0^a t^\sigma/(t-f(t))dt$ converges or diverges. If we adapt the proof of Theorem 1 in [1], we can show if f is monotone near the origin, then for all points $x=u_0$ in I_a, the series $\sum_{n=0}^\infty u_n^s$ have the same abscissa of convergence.

THEOREM 5. *If in addition to satisfying* (1) *and* (2), *the function f is monotone in some interval I_b $0<b\leqq a$ and u_0 is a point of I_a such that the series $\sum_{n=0}^\infty u_n^\sigma(u_n/u_{n+1}-1)$ converges uniformly when σ is bounded away from zero, the functions $\sum u_n^s$ and $\sum v_n^s$ have the same singularities in the half plane $\sigma>0$.*

LEMMA. *If $0<B<A$, then for each real number β*

$$|A^{\sigma+i\beta}-B^{\sigma+i\beta}| \leqq |A^\sigma - B^\sigma| + |\beta|(A/B-1).$$

Proof. We have

$$\begin{aligned} |A^{\sigma+i\beta}-B^{\sigma+i\beta}| &\leqq |A^\sigma(A^{i\beta}-B^{i\beta})+(A^\sigma-B^\sigma)B^{i\beta}| \\ &\leqq A^\sigma|A^{i\beta}-B^{i\beta}| + A^\sigma - B^\sigma \\ &\leqq A^\sigma - B^\sigma + A^\sigma|\exp(i\beta\log A)-\exp(i\beta\log B)|. \end{aligned}$$

If we now use the fact that $|e^{i\theta}-e^{i\phi}|\leqq|\theta-\phi|$ for all real numbers θ and ϕ, we see that the above expression is no greater than $A^\sigma-B^\sigma+A^\sigma|\beta|[\log A-\log B]$ $=A^\sigma-B^\sigma+A^\sigma|\beta|(\log A/B)$. But for all $x\geqq 0$ $\log(1+x)\leqq x$, and therefore

$$\log(A/B)\leqq A/B-1.$$

This completes the proof of the lemma.

To prove the theorem we note that since $\{u_n\}$ is a null sequence there is a natural number N_0 such that $u_n\in I_b$ when $n\geqq N_0$. Let $v_0\in I_a$; there is no loss in generality in taking v_0 so that $v_{N_0}\in(u_{N_0+N}, u_{N_0})$ and hence $v_n\in(u_{n+1}, u_n)$ for $n\geqq N_0$. We will show that the series $\sum_{n=0}^\infty u_n^s-v_n^s$ converges uniformly in each infinite rectangle $R=\sigma\geqq\eta$, $|\tau|\leqq T$, where η and T are positive constants, I_n such a rectangle, we have by the preceding lemma for $n>N_0$

$$|u_n^s-v_n^s| \leqq u_n^\sigma - v_n^\sigma + |\tau|u_n^\sigma(u_n/v_n-1) \leqq u_n^\sigma - u_{n+1}^\sigma + |\tau|u_n^\sigma(u_n/u_{n+1}-1),$$

since $u_{n+1}<v_n<u_n$. The series $\sum u_n^\sigma-u_{n+1}^\sigma$, $\sum u_n^\sigma(u_n/u_{n+1}-1)$ converge uniformly when σ is bounded away from 0. Hence the series $\sum u_n^s-v_n^s$ converges uniformly in R and therefore it represents an analytic function for $\sigma>0$. Consequently the singularities of the function $\sum u_n^s$ are the same as those of the function $\sum_{n=0}^\infty v_n^s$.

Let μ_δ denote the measure of the intersection of the set of divergence with the

interval $(0, \delta)$. The theorem of Fort and Schuster quoted above led me to conjecture in [1] that $\lim_{\delta\to 0} \mu_\delta/\delta=0$ if and only if $\int_0^a t/(t-f(t))dt<\infty$. The following shows that this conjecture is false. For each integer m let

(a) $f(x)=4mx/(4m+1)\ 1/4m-1/40m(4m+1)\leqq x\leqq 1/4m$,

(b) between each pair of points $1/4m-1/40m(4m+1)-1/(4m+1)^8$ and $1/4m-1/40m(4m+1)$ in I_a we define f as the linear function which takes the value 0 at $x=1/4m-1/40m(4m+1)-1/(4m+1)^8$ and the value $1/(4m+1)-1/[10(4m+1)^2]$ at $x=1/4m-1/40m(4m+1)$,

(c) between each pair of points $1/4m$ and $1/4m+1/(4m+1)^8$ we define f as the linear function which takes the value $1/(4m+1)$ at $x=1/4m$ and the value 0 at $x=1/4m+1/(4m+1)^8$,

(d) for all other points x we define $f(x)=0$.

The inequality

$$1/4(m+1)+1/(4m+5)^8 \leqq 1/4m-1/40m(4m+1)-1/(4m+1)^8$$

which may be verified for $m\geqq 1$ ensures that the intervals of definition do not overlap so that $f(x)$ is well defined. On each interval $\mathscr{I}_m$: $[1/4m-1/40m(4m+1)]$, $x/(x-f(x))\geqq 4m+1$ so that

$$\int_{\mathscr{I}_m} t/(t-f(t))dt \geqq 1/4m,$$

and thus

$$\int_0^a t/(t-f(t))dt \geqq (1/40)\sum 1/m = \infty.$$

On the other hand, each interval $\mathscr{I}_m$ is mapped by f on the interval $[1/(4m+1)-1/10(4m+1)^2,\ 1/4m+1)]$, $m=1, 2, \cdots$ and since

$$1/(4m+1) \leqq 1/4m - 1/40m(4m+1)^2 - 1(4m+1)^8,$$
$$1/(4m+1) - 1/10(4m+1)^2 \geqq 1/(4[m+1]) + 1/[4(m+1)+1]^8$$

for $m>1$, the second iterate of f maps all points in the intervals $\mathscr{I}_m$ on 0. Hence all points of divergence must lie in one of the intervals $[1/4m-1/40m(4m+1)-1/(4m+1)^8,\ 1/4m-1/40m(4m+1)]$ or $[1/4m,\ 1/(4m+1)^8]$. If $\delta>0$, then

$$\mu_\delta \leqq 2\sum_m (4m+1)^{-8} = O(\delta^7)$$

the sum being taken over all m greater than $[1/4\delta]$. Thus $\mu_\delta/\delta\to 0$.

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References

1. G. Brauer, Sets of convergence for series defined by iteration, Canad. Math. Bull., 9 (1966) 83–87.

2. M. K. Fort and Seymour Schuster, Convergence of series whose terms are defined recursively, this MONTHLY, 71 (1964) 994–998.

QUADRATIC POLYNOMIALS WITH THE SAME RESIDUES

K. S. WILLIAMS, Carleton University, Ottawa

It is the purpose of this note to prove the following:

THEOREM. *Two quadratic polynomials* $a_1x_1^2+b_1x_1+c_1$ *and* $a_2x_2^2+b_2x_2+c_2$ *with integral coefficients have the same residues modulo every prime* $p>3$ *not dividing* a_1a_2, *if and only if they are related by a nonsingular rational linear transformation, that is to say, if and only if there exist rationals* r *and* s *with* $r\neq 0$ *such that*

$$a_1(rx+s)^2+b_1(rx+s)+c_1\equiv a_2x^2+b_2x+c_2.$$

For example, this theorem tells us that $12x_1^2+14x_1+1$ and $75x_2^2+55x_2+7$ have the same residues (mod p) for every prime $p\geqq 7$, since

$$12(\tfrac{5}{2}x+\tfrac{1}{3})^2+14(\tfrac{5}{2}x+\tfrac{1}{3})+1\equiv 75x^2+55x+7.$$

We note that there is no *integral* transformation relating $12x_1^2+14x_1+1$ and $75x_2^2+55x_2+7$. In any particular case it is easy to decide whether the two quadratics also have the same residues (mod p) for $p=2$ or 3 or any $p\mid a_1a_2$. In the above example they do for $p=2$, 3 but not for $p=5$.

We begin by calculating the number N_p ($p>2$) of common residues (mod p) of the two quadratic polynomials $a_1x_1^2+b_1x_1+c_1$ and $a_2x_2^2+b_2x_2+c_2$ (a_1, $a_2\not\equiv 0$ (mod p)), that is, the number of integers r satisfying $0\leqq r\leqq p-1$ for which both the congruences

$$a_1x_1^2+b_1x_1+c_1\equiv r,\qquad a_2x_2^2+b_2x_2+c_2\equiv r$$

are soluble. We denote the number of solutions x_i ($i=1,2$) of $a_ix_i^2+b_ix_i+c_i\equiv r$ by $N_i(r)$ so that

$$N_p=\sum_{r=0}^{p-1}1,\qquad N_i(r)>0\qquad (i=1,2).$$

Let $d_i=b_i^2-4a_ic_i$ ($i=1,2$) then we have

$$N_i(r)=1+\left(\frac{d_i+4a_ir}{p}\right)\qquad (i=1,2)$$

and so $N_p=\sum_{i,j=0}^{1}N_{ij}$, where

$$N_{ij}=\sum_{r=0}^{p-1}1,\qquad \left(\frac{d_1+4a_1r}{p}\right)=i\qquad \left(\frac{d_2+4a_2r}{p}\right)=j.$$

We now evaluate each N_{ij} ($i,j=0,1$) in turn. For convenience we set $e=a_1d_2-a_2d_1$. Then

$$N_{00} = \sum_{r=0}^{p-1} 1 = \begin{cases} 1, & \text{if } e \equiv 0, \\ 0, & \text{if } e \not\equiv 0, \end{cases}$$
$$r \equiv -4^{-1}a_1^{-1}d_1,$$
$$r \equiv -4^{-1}a_2^{-1}d_2;$$

that is $N_{00} = 1-(e^2/p)$. Also

$$N_{01} = \sum_{r=0}^{p-1} 1, \qquad r \equiv -4^{-1}a_1^{-1}d_1, \qquad \left(\frac{d_2+4a_2r}{p}\right) = 1,$$
$$= \begin{cases} 1, & \text{if } \left(\dfrac{a_1e}{p}\right) = 1, \\ 0, & \text{if } \left(\dfrac{a_1e}{p}\right) = 0 \quad \text{or} \quad -1, \end{cases}$$

that is

$$N_{01} = \frac{1}{2}\left\{\left(\frac{a_1e}{p}\right) + \left(\frac{e^2}{p}\right)\right\}.$$

Similarly $N_{10} = \frac{1}{2}\{(-a_2e/p)+(e^2/p)\}$. Finally

$$N_{11} = \sum_{r=0}^{p-1} 1, \qquad \left(\frac{d_1+4a_1r}{p}\right) = \left(\frac{d_2+4a_2r}{p}\right) = 1$$
$$= \frac{1}{4}\sum_{r=0}^{p-1}\left\{1+\left(\frac{d_1+4a_1r}{p}\right)\right\}\left\{1+\left(\frac{d_2+4a_2r}{p}\right)\right\}$$
$$r \not\equiv -4^{-1}a_1^{-1}d_1 \quad \text{or} \quad -4^{-1}a_2^{-1}d_2$$
$$= \frac{1}{4}\left[\sum_{r=0}^{p-1}\left\{1+\left(\frac{d_1+4a_1r}{p}\right)\right\}\left\{1+\left(\frac{d_2+4a_2r}{p}\right)\right\}\right.$$
$$\left. - \left\{1+\left(\frac{a_1e}{p}\right)\right\} - \left\{1+\left(\frac{-a_2e}{p}\right)\right\} + \left\{1-\left(\frac{e^2}{p}\right)\right\}\right]$$
$$= \frac{1}{4}\left[p + \sum_{r=0}^{p-1}\left(\frac{16a_1a_2r^2+4(a_1d_2+a_2d_1)r+d_1d_2}{p}\right)\right.$$
$$\left. -1-\left(\frac{a_1e}{p}\right)-\left(\frac{-a_2e}{p}\right)-\left(\frac{e^2}{p}\right)\right]$$
$$= \frac{1}{4}\left[p+\left\{p-p\left(\frac{e^2}{p}\right)-1\right\}\left(\frac{a_1a_2}{p}\right)-1-\left(\frac{a_1e}{p}\right)-\left(\frac{-a_2e}{p}\right)-\left(\frac{e^2}{p}\right)\right].$$

Hence we have proved

LEMMA 1. *For* $p>2$, $p\nmid a_1a_2$

$$N_p = \frac{1}{4}\left[\left\{1 + \left(\frac{a_1a_2}{p}\right) - \left(\frac{a_1a_2e^2}{p}\right)\right\}p + \left\{3 - \left(\frac{e^2}{p}\right) + \left(\frac{a_1e}{p}\right) + \left(\frac{-a_2e}{p}\right) - \left(\frac{a_1a_2}{p}\right)\right\}\right].$$

This gives the following table of values of N_p $(p>2)$.

TABLE

						$p\equiv 1$ (mod 4)	$p\equiv 3$ (mod 4)		$p\equiv 1$ (mod 4)	$p\equiv 3$ (mod 4)
$\left(\frac{a_1}{p}\right)$	$\left(\frac{a_2}{p}\right)$	$\left(\frac{e}{p}\right)$	$\left(\frac{a_1a_2}{p}\right)$	$\left(\frac{e^2}{p}\right)$	$\left(\frac{a_1e}{p}\right)$	$\left(\frac{-a_2e}{p}\right)$	$\left(\frac{-a_2e}{p}\right)$	$\left(\frac{a_1a_2e^2}{p}\right)$	N_p	N_p
1	1	1	1	1	1	1	−1	1	$\frac{1}{4}(p+3)$	$\frac{1}{4}(p+1)$
1	1	0	1	0	0	0	0	0	$\frac{1}{2}(p+1)$	$\frac{1}{2}(p+1)$
1	1	−1	1	1	−1	−1	1	1	$\frac{1}{4}(p-1)$	$\frac{1}{4}(p+1)$
1	−1	1	−1	1	1	−1	1	−1	$\frac{1}{4}(p+3)$	$\frac{1}{4}(p+5)$
1	−1	0	−1	0	0	0	0	0	1	1
1	−1	−1	−1	1	−1	1	−1	−1	$\frac{1}{4}(p+3)$	$\frac{1}{4}(p+1)$
−1	1	1	−1	1	−1	1	−1	−1	$\frac{1}{4}(p+3)$	$\frac{1}{4}(p+1)$
−1	1	0	−1	0	0	0	0	0	1	1
−1	1	−1	−1	1	1	−1	1	−1	$\frac{1}{4}(p+3)$	$\frac{1}{4}(p+5)$
−1	−1	1	1	1	−1	−1	1	1	$\frac{1}{4}(p-1)$	$\frac{1}{4}(p+1)$
−1	−1	0	1	0	0	0	0	0	$\frac{1}{2}(p+1)$	$\frac{1}{2}(p+1)$
−1	−1	−1	1	1	1	1	−1	1	$\frac{1}{4}(p+3)$	$\frac{1}{4}(p+1)$

LEMMA 2. *If $p>3$, the quadratics $a_1x_1^2+b_1x_1+c_1$ and $a_2x_2^2+b_2x_2+c_2$ have exactly the same residues* (mod p), *if and only if* $(a_1a_2/p)=+1$ *and* $e\equiv 0$ (mod p).

Proof. This is immediate from the table as the quadratics $a_1x_1^2+b_1x_1+c_1$ and $a_2x_2^2+b_2x_2+c_2$ have exactly the same residues if and only if $N_p=\frac{1}{2}(p+1)$. (Recall that the number of residues (mod p) of a quadratic polynomial ax^2+bx+c (a, b, c, integers, $a\not\equiv 0$ (mod p) $(p>2)$ is $\frac{1}{2}(p+1)$.)

Our last lemma is based upon an idea contained in a paper of H. Salié [1].

LEMMA 3. *For any prime q, there exists an integer $l\equiv 1$* (mod 4) *and* $\not\equiv 0$ (mod q) *such that if p is a prime* $\equiv l$ (mod $4q$) *then* $(q/p)=-1$.

Proof. If $q=2$ take $l=5$ as $(2/p)=-1$ for primes $p\equiv 5$ (mod 8). We may therefore suppose that $q>2$. Let

$$L = \{l \mid l = 1, 5, 9, 13, \cdots, 4q-3\}.$$

The number of integers in L is just q. They are distinct (mod q) for if l_1, $l_2\in L$

with $l_1 \equiv l_2 \pmod{q}$ then as $q>2$ and $l_1 \equiv l_2 \equiv 1 \pmod 4$ we have $l_1 \equiv l_2 \pmod{4q}$ i.e. $l_1 = l_2$. Hence the residues of the integers in $L \pmod q$ form a complete residue set $(\text{mod } q)$. Let n denote the least positive quadratic nonresidue $(\text{mod } q)$ and choose $l \in L$ such that $l \equiv n \pmod q$. Then $l \equiv 1 \pmod 4$, $l \not\equiv 0 \pmod q$ and if p is a prime $\equiv l \pmod{4q}$ (so that in particular $p \equiv 1 \pmod 4$) we have by the law of quadratic reciprocity

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)(-1)^{[(p-1)(q-1)]/4} = -1,$$

as required.

We are now in a position to prove the theorem.

Proof of Theorem. From Lemma 2, the quadratics $a_1x_1^2+b_1x_1+c_1$ and $a_2x_2^2+b_2x_2+c_2$ have the same residues modulo every prime p strictly greater than 3 and not dividing a_1 or a_2 if and only if

$$\left(\frac{a_1a_2}{p}\right) = +1 \quad \text{and} \quad e \equiv 0 \pmod p$$

for these primes. Now $e \equiv 0 \pmod p$ for any infinity of primes p if and only if $e=0$ i.e. if and only if

$$a_1d_2 = a_2d_1. \tag{1}$$

Clearly if a_1a_2 is a square then $(a_1a_2/p) = +1$ for all $p \nmid a_1a_2$. We now show conversely that if $(a_1a_2/p) = +1$ for all $p>3$ not dividing a_1a_2 then a_1a_2 is a square. Suppose that it is not. Then it can be expressed as

$$\pm\, p_1p_2 \cdots p_sk^2 \quad \text{or} \quad -k^2,$$

where $p_1, \cdots, p_s$ are $s \geqq 1$ distinct primes. We deal with the case when a_1a_2 is positive first. To obtain the necessary contradiction it suffices to show the existence of a prime $p > 3a_1a_2$ such that $(a_1a_2/p) = -1$. We do this by showing the existence of such a prime p with $(p_1/p) = \cdots = (p_{s-1}/p) = +1$ and $(p_s/p) = -1$. Let l_1 denote the integer l given by Lemma 3 with $q = p_s$. We now define an integer l_2 as follows: if $s=1$ take $l_2 = l_1$ and if $s>1$ choose l_2 such that

$$l_2 \equiv 1 \pmod{4p_1 \cdots p_{s-1}}$$
$$l_2 \equiv a_1 \pmod{4p_s}.$$

This is possible as $4 = (4p_1 \cdots p_{s-1}, 4p_s) \mid l_1 - 1$. Obviously $(l_2, 4p_1 \cdots p_s) = 1$. By Dirichlet's theorem there exists an infinity of primes $\equiv l_2 \pmod{4p_1 \cdots p_s}$. Let p denote the least such $> 3a_1a_2$. Then by Lemma 3 $(p_s/p) = -1$. Also

$$\left(\frac{p_i}{p}\right) = \left(\frac{p}{p_i}\right)(-1)^{[(p-1)(p_i-1)]/4} = \left(\frac{l_2}{p_i}\right) = \left(\frac{1}{p_i}\right) = 1,$$

for $i = 1, 2, \cdots, s-1$ as required.

We now deal with the case when a_1a_2 is negative. Suppose firstly that $a_1a_2 = -k^2$. We show this is impossible. By Dirichlet's theorem there are an

infinity of primes $\equiv 3 \pmod 4$. Take p to be the least such one $> -3a_1a_2$. Then $p>3$ and $p \nmid a_1a_2$ and moreover

$$\left(\frac{a_1a_2}{p}\right)=\left(\frac{-k^2}{p}\right)=\left(\frac{-1}{p}\right)=-1,$$

which is a contradiction. Thus if a_1a_2 is negative it must be of the form $-p_1p_2 \cdots p_sk^2$, where $p_1, \cdots, p_s$ are $s \geqq 1$ distinct primes. As in the case when a_1a_2 was assumed to be positive we can find a prime $p > -3a_1a_2$ for which $(-a_1a_2/p) = -1$. This prime is $\equiv 1 \pmod 4$ so $(-a_1a_2/p) = (a_1a_2/p)$, completing the proof that a_1a_2 must be a square.

Now let $a_1a_2 = a^2$ and set $r = a/a_1$, $s = (b_2 - b_1r)/(2a_1r)$, so that both r and s are rational. Then $a_1r^2 = a^2/a_1 = a_2$,

$$2a_1rs + b_1r = (b_2 - b_1r) + b_1r = b_2$$

and

$$\begin{aligned} a_1s^2 + b_1s + c_1 &= \frac{1}{4a_1r^2}\left\{(b_2 - b_1r)^2 + 2b_1r(b_2 - b_1r) + 4a_1c_1r^2\right\} \\ &= \frac{1}{4a_1r^2}\left\{b_2^2 - d_1r^2\right\} = \frac{1}{4a_2}\left\{b_2^2 - \frac{a_2d_1}{a_1}\right\} \\ &\frac{1}{4a_2}\left\{b_2^2 - d_2\right\} = c_2 \qquad \text{(from (1))} \end{aligned}$$

giving

$$a_1(rx+s)^2 + b_1(rx+s) + c_1 \equiv a_2x^2 + b_2x + c_2$$

as required.

Reference

1. H. Salié, Über den kleinsten positiven quadratischen Nichtrest nach einer Primzahl, Math. Nachr., 3 (1949/50) 7–8.

MATHEMATICAL NOTES

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

THE CONCEPT OF A TORSION MODULE

V. DLAB, Institute of Advanced Studies, Australian National University, Canberra

The concept of a torsion group (or of a torsion element) in the theory of abelian groups (in what follows, we shall use the word group to mean always additive abelian group) is very simple: G is torsion if all elements of G have

finite order; or, the order (annihilator) ideal of any element is nonzero. Every torsion group then enjoys the following "radical" properties (cf. [**3**]):

(t_1) Every subgroup of a torsion group is torsion.

(t_2) Every group G possesses a torsion subgroup $T(G)$ which contains all torsion subgroups of G.

(t_3 $T(G/T(G))=0$.

The properties (t_1), (t_2), (t_3) follow as a simple consequence of the following obvious basic properties of the family $\mathfrak{I}$ of all nonzero ideals of the ring Z of all integers:

(i_0) If $I\in\mathfrak{I}$, then $I: n=\{x \mid x\in Z \text{ and } xn\in I\}\in\mathfrak{I}$ for every $n\in Z$.

(i_1) If $I_1\subseteq I_2$ and $I_1\in\mathfrak{I}$, then $I_2\in\mathfrak{I}$.

(i_2) If $I_1\in\mathfrak{I}$ and $I_2\in\mathfrak{I}$, then $I_1\cap I_2\in\mathfrak{I}$.

(i_3) If $I_2: n\in\mathfrak{I}$ for all n from $I_1\in\mathfrak{I}$, then $I_2\in\mathfrak{I}$. The properties (i_0), (i_1), (i_2) ensure the validity of (t_1) and (t_2) and together with (i_3), the validity of (t_3).

However, in a general (associative) ring R with unity, the family $\mathfrak{I}^R$ of all nonzero left ideals may no longer satisfy the above conditions. $\mathfrak{I}^R$ always possesses (i_1), but the validity of (i_0) and (i_2) for $\mathfrak{I}^R$ is equivalent to the fact that the zero-ideal $\{0\}$ is irreducible (i.e. that any two nonzero left ideals intersect nontrivially); and, furthermore, $\mathfrak{I}^R$ satisfies (i_0)–(i_3) if and only if R is an Ore domain, i.e. R has no zero-divisors and $\{0\}$ is irreducible.

Hence, in the theory of R-modules (left unital modules over R), the concept of a torsion R-module—as an R-module all nonzero elements of which are of nonzero orders—will be satisfactory if and only if R is an Ore domain.

As a consequence of this very limiting restriction on R, many authors have attempted to generalize the concept of torsion in order to be applicable to more general situations. A natural way of doing this is to exploit some suitable characteristic property of the family of all nonzero ideals of Z. One of the ideas appearing frequently is to use the property of being essential.

A left ideal L of R is said to be *essential* (in R) if it intersects with any other nonzero left ideal of R nontrivially.

Indeed, all nonzero ideals of Z are essential. Moreover, one feels it rather natural to require an element of a "big," viz. essential, order to be torsion.

Now, the family $\mathcal{E}^R$ of all essential ideals of a ring R satisfies (i_0), (i_1), (i_2), but not, in general, (i_3). Hence, we face the task of constructing a new family $\mathfrak{I}^R_\square$ from the family $\mathcal{E}^R$ in such a way that $\mathfrak{I}^R_\square$ will retain the properties (i_0), (i_1), (i_2) and will, in addition, possess (i_3).

Here, we like to describe two (formally opposite) solutions of this problem, i.e. two different ways of defining torsion R-modules. The first way insists that every element of essential order be torsion; thus, we have to enlarge $\mathcal{E}^R$. The other way stresses the fact that only elements of essential order can be torsion, i.e. we have to reduce $\mathcal{E}^R$.

(*) Denote by $\mathfrak{I}^R_*$ the family of all maxi ideals of R. A left ideal L of R is said to be *maxi* if, for any $\rho\in R\backslash L$, there is $\sigma\in R$ such that $L:\sigma\rho$ is a proper essential ideal of R. Clearly, $\mathfrak{I}^R_*\supseteq\mathcal{E}^R$. Moreover, $\mathfrak{I}^R_*$ satisfies (i_0)–(i_3) and thus yields a

definition of a $*$-torsion R-module (see [2] and [4]).

(o) Denote by $\mathfrak{J}_0^R$ the family of all strong ideals of R. A left ideal L of R is said to be *strong* if, for any $\rho\in R\backslash L$, there is no $\sigma\neq 0$ of R such that $(L:\rho)\sigma=\{0\}$. Clearly, $\mathfrak{J}_0^R\subseteq\mathcal{E}^R$. Again, $\mathfrak{J}_0^R$ satisfies (i_0)–(i_3) and yields a definition of a o-torsion R-module (see [3] and [5]).

These two definitions coincide for example in the case of an Ore domain R; for, then $\mathfrak{J}_*^R=\mathfrak{J}^R=\mathfrak{J}_0^R$ $(=\mathcal{E}^R)$. In fact, we can prove that the following four assertions are equivalent (cf. [2]):

(i) $\mathfrak{J}_*^R=\mathfrak{J}_0^R \quad (=\mathcal{E}^R)$.

(ii) $\mathfrak{J}_*^R=\mathcal{E}^R$.

(iii) $\mathfrak{J}_0^R=\mathcal{E}^R$.

(iv) The singular ideal $S(R)$ of R equals to $\{0\}$; here, $S(R)$ is the ideal of all elements of R whose left annihilators are essential in R.

Indeed, it is obvious that $\mathfrak{J}_0^R=\mathcal{E}^R$ implies immediately $S(R)=\{0\}$ (and vice versa). But then, since for a nonessential maxi ideal L there is an element $\rho\in R\backslash L$ such that $L:\rho$ is the left annihilator of ρ, necessarily $\mathfrak{J}_*^R=\mathcal{E}^R$. Hence, we have proved the two implications (iii)$\rightarrow$(iv)$\rightarrow$(i). In order to complete the proof we are going to show that (ii)$\rightarrow$(iv). Thus, let $S(R)\neq\{0\}$. Consider a maximal (left) ideal L satisfying $L\cap S(R)=\{0\}$ and denote by K the ideal generated by $S(R)$ and L; evidently, K is essential in R. Hence, for any $\rho\in R\backslash L$ there exists $\sigma\in R$ such that $L:\sigma\rho\neq R$ is essential in R. Here, σ can be taken as the unity of R provided that $K:\rho\subseteq L:\rho$ and as an element satisfying $\sigma\rho\in K\backslash L$ otherwise. Thus, $L\in\mathfrak{J}_*^R\backslash\mathcal{E}^R$, i.e. $\mathfrak{J}_*^R\neq\mathcal{E}^R$, as required.

Here we refrain from advocating one or the other concept; we feel that both have their merits. However, we would like to conclude this article with a few remarks.

First, let us mention that, for any ring $R(\neq\{0\})$, $\mathfrak{J}_0^R$ never coincides with the family $\mathfrak{L}^R$ of all left ideals of R; this results in the fact that there always exist nontrivial o-torsion-free R-modules. However, $\mathfrak{J}_0^R$ can consist only of the single element R. On the other hand, the family $\mathfrak{J}_*^R$ can take on either of these extremes $\mathfrak{L}^R$ or $\{R\}$. Apart from the case when

$$\mathfrak{J}_*^R=\mathfrak{J}_0^R=\{R\} \qquad (\text{i.e. } \mathcal{E}^R=\{R\}), \tag{1}$$

we can have

$$\mathfrak{J}_*^R=\mathfrak{L}^R, \qquad \mathfrak{J}_0^R\neq\{R\}, \tag{2}$$

$$\{R\}\neq\mathfrak{J}_*^R\neq\mathfrak{L}^R, \qquad \mathfrak{J}_0^R=\{R\}, \tag{3}$$

and, even,

$$\mathfrak{J}_*^R=\mathfrak{L}^R, \qquad \mathfrak{J}_0^R=\{R\} \qquad (\text{and } \{R\}\neq\mathcal{E}^R\neq\mathfrak{J}^R). \tag{4}$$

Thus, any nonzero module over a ring R satisfying (4) will be at the same time $*$-torsion and o-torsion-free. This seemingly disturbing situation is not a result of an improper choice of the definitions of $*$- and o-torsion, but should be, as a matter of fact, expected; simply, because there are rings R (see the examples of R_1 and R_4 below) which, apart from $\mathfrak{L}^R$ and $\{R\}$, possess no other families of left ideals satisfying (t_1), (t_2), (t_3). In order to present some simple examples of rings satisfying (1)–(4), take

R_1 to be a complete matrix ring over an arbitrary division ring,

R_2 to be the ring of all pairs (x, y) of integers with component-wise addition and the multiplication defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2 + y_1x_2),$$

R_3 to be the ring of all triples (x, y, z) of elements from the field Z_p (Z modulo a prime p) with component-wise addition and the multiplication defined by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1x_2, x_1y_2 + y_1x_2, x_1z_2 + z_1x_2 + z_1z_2),$$

R_4 to be the ring of all triplets (x, y, z) of elements from Z_p with component-wise addition and the multiplication

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1x_2, x_1y_2 + y_1x_2, x_1z_2 + z_1x_2).$$

Finally, let us remark that a certain equivalence can be defined on the families of ideals of R under which $\mathcal{E}^R$ and $\mathfrak{I}_*^R$ are always equivalent; moreover, $\mathfrak{I}_*^R$ contains any other element of its equivalence class (see [2]). Neither of these statements is true for $\mathfrak{I}_0^R$ (see [3]). In this sense, $\mathfrak{I}_*^R$ is a natural close extension of $\mathcal{E}^R$. One consequence to mention here is that any maximal independent set of elements of essential orders is a maximal independent set of $*$-torsion elements; and the cardinalities of such maximal independent sets define torsion rank of a module (see [1]).

Note added in proof. The referee has kindly drawn the author's attention to a related paper of S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc., 121 (1966) 223–235.

References

1. V. Dlab, The concept of rank and some related questions in the theory of modules, Comment. Math. Univ. Carolinae, 8 (1967) 39–47. To appear in full in Fund. Math.
2. ———, Distinguished submodules, J. Austral. Math. Soc., to appear.
3. ———, Distinguished sets of ideals of a ring, Czechoslovak Math. J., to appear.
4. A. W. Goldie, Torsion-free modules and rings, J. of Algebra, 1 (1964) 268–287.
5. J. P. Jans, Some aspects of torsion, Pacific J. Math., 15 (1965) 1249–1259.

ON COEFFICIENT IDENTITIES FOR CYCLOTOMIC POLYNOMIALS $F_{pq}(x)$

DAVID ZEITLIN, Minneapolis, Minnesota

1. **Introduction.** The cyclotomic polynomial $F_{pq}(x)$ is defined by

$$(1.1) \qquad (1 - x^p)(1 - x^q)F_{pq}(x) = (1 - x^{pq})(1 - x),$$

where p and q denote two distinct odd primes. Bang [1] and Migotti [2] have shown that the coefficients of $F_{pq}(x)$ are ± 1 or 0. Recently, Sister Marion Beiter [3] proved that if $F_{pq}(x)=\sum_{n=0}^{d} C_n x^n$, where $d=\phi(pq)=(p-1)(q-1)$, then $C_n=(-1)^\delta$ if $n=\alpha q+\beta p+\delta$ in exactly one way and $C_n=0$ otherwise, where α, β are nonnegative integers and $\delta=0$, 1. She also determined the middle coefficient C_n, where $n=d/2$, as $(-1)^{k-1}$, where k is the least positive solution of the congruence $qx\equiv 1 \pmod p$.

Carlitz [4] has recently proved the following result:

Let $\theta_0(pq)$ denote the number of terms with positive coefficients in $F_{pq}(x)$. Take $q>p$ and define u by means of $qu\equiv -1 \pmod p$, $(0<u<p)$. Then we have

$$\theta_0(pq) = (p-u)(uq+1)/p. \tag{1.2}$$

An application of this result is given in Lemmas 4 and 5.

We will derive some new identities for the coefficients of cyclotomic polynomials.

2. Coefficient relations for $F_{pq}(x)$. Let $d=\phi(pq)$ and write $F_{pq}(x)=\sum_{n=0}^{d} C_n x^n$. Since $F_{pq}(x)=x^d F_{pq}(1/x)$ (see (1.1)), we obtain the symmetric property

$$C_n = C_{d-n} \qquad (n = 0, 1, \cdots, d). \tag{2.1}$$

For p and q both distinct odd primes, we note that $F_{pq}(x^2)=F_{pq}(x)F_{pq}(-x)$, which gives

$$C_k = \sum_{j=0}^{2k} (-1)^j C_j C_{2k-j} \qquad (k = 0, 1, \cdots, d), \tag{2.2}$$

$$\sum_{j=0}^{2k-1} (-1)^j C_j C_{2k-1-j} = 0 \qquad (k = 1, 2, \cdots, d). \tag{2.3}$$

Using (2.1), we obtain from (2.3) for $k=d/2$

$$\sum_{j=0}^{d-1} (-1)^j C_j C_{j+1} = 0. \tag{2.4}$$

Since $F_{pq}(1)=F_{pq}(-1)=1$ (as well as $F_{pq}(i)=F_{pq}(-i)=1$), we conclude that

$$\sum_{k=0}^{d/2} C_{2k} = 1, \qquad \sum_{k=1}^{d/2} C_{2k-1} = 0. \tag{2.5}$$

Setting $k=(d/2)-j$, we may transform A into B, where

$$A = \sum_{k=(d/4)+1}^{d/2} C_{2k} = \sum_{j=0}^{(d/4)-1} C_{d-2j} = \sum_{j=0}^{(d/4)-1} C_{2j} = B.$$

Since (see (2.5)) $B+C_{(d/2)}+A=1$, we obtain

LEMMA 1.

$$C_{(d/2)} = 1 - 2 \sum_{k=0}^{(d/4)-1} C_{2k}. \tag{2.6}$$

Since $C_{(d/2)} = \pm 1$, we obtain from (2.6)

LEMMA 2.

$$\sum_{k=0}^{(d/4)-1} C_{2k} = \begin{cases} 0 & \text{for } C_{(d/2)} = 1, \\ 1 & \text{for } C_{(d/2)} = -1. \end{cases} \tag{2.7}$$

Noting that

$$\sum_{j=0}^{(d/2)-1} (-1)^j C_j^2 = \sum_{j=(d/2)+1}^{d} (-1)^j C_j^2 = H,$$

we obtain from (2.2) for $k = d/2$ (using (2.1))

$$C_{(d/2)} = \sum_{j=0}^{d} (-1)^j C_j^2 = H + C_{(d/2)}^2 + H.$$

Thus,

$$C_{(d/2)} - C_{(d/2)}^2 = 2H, \tag{2.8}$$

and since $C_{(d/2)} = \pm 1$, we obtain from (2.8)

LEMMA 3.

$$\sum_{j=0}^{(d/2)-1} (-1)^j C_j^2 = \begin{cases} 0 & \text{for } C_{(d/2)} = 1, \\ -1 & \text{for } C_{(d/2)} = -1. \end{cases} \tag{2.9}$$

Our (2.7) and (2.9) are useful check formulas for the evaluation of C_j in $F_{pq}(x)$.

The results of Carlitz for $\theta_0(pq)$(see (1.2)) can now be applied to evaluate $C_{(d/2)}$. To see this, let $\theta_1(pq)$ denote the number of terms with negative coefficients in $F_{pq}(x)$. From $F_{pq}(1) = 1$ we obtain $\theta_0(pq) = \theta_1(pq) + 1$. Since the coefficients of $F_{pq}(x)$ are symmetric about $C_{(d/2)}$ (see (2.1)), there must be as many positive coefficients below $C_{(d/2)}$ as above it (and the same is true for the negative coefficients). Thus, we have

LEMMA 4.
(2.10) *If* $\theta_0(pq)$ *is odd, then* $C_{(d/2)} = 1$.
(2.11) *If* $\theta_0(pq)$ *is even, then* $C_{(d/2)} = -1$.

If we apply (2.10) and (2.11) to the results in [4], we obtain

LEMMA 5.

$$C_{(d/2)} = 1 \quad \textit{for } (p, q) = (3, 3k+1); \tag{2.12}$$

$$C_{(d/2)} = -1 \quad \textit{for } (p, q) = (3, 3k+2); \tag{2.13}$$

$$(2.14)\qquad C_{(d/2)} = 1 \quad \textit{for } (p, q) = (5, 5k + 1), (5, 5k + 2);$$

$$(2.15)\qquad C_{(d/2)} = -1 \quad \textit{for } (p, q) = (5, 5k + 3), (5, 5k + 4),$$

where $d=\phi(pq)=(p-1)(q-1)$. *For the special case* $p=2$, *we have* $d=q-1$ *and* $C_{(d/2)}=(-1)^{d/2}$.

We note (see (1.1)) that $F_{pq}(x^r)$ is a factor of $1-x^{rpq}$ and that $F_{pq}(-x^r)$ is a factor of $1+x^{rpq}$, where $r=1, 2, \cdots$. If we define $f(x)=(1+x)/(2\sqrt{x})$, then it is readily verified that

$$(2.16)\qquad 1 - x^{rpq} = (1 - x)x^{(rpq-1)/2}U_{rpq-1}(f(x)),$$

$$(2.17)\qquad 1 + x^{rpq} = 2x^{rpq/2}T_{rpq}(f(x)),$$

where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively. Thus, we have

LEMMA 6. *Let* $f(x)=(1+x)/(2\sqrt{x})$. *Then, for* $r=1, 2, \cdots$,

$$(2.18)\qquad F_{pq}(x^r) \textit{ is a factor of } x^{(rpq-1)/2}U_{rpq-1}(f(x));$$

$$(2.19)\qquad F_{pq}(-x^r) \textit{ is a factor of } 2x^{rpq/2}T_{rpq}(f(x)).$$

Added in proof. Equating coefficients of x^n in (1.1) gives $C_0=C_d=1$, $C_1=C_{d-1}=-1$,

$$(3.1)\qquad C_n - C_{n-q} - C_{n-p} + C_{n-p-q} = 0 \qquad (n = 2, 3, \cdots, pq - 1),$$

where $C_{-n}=C_{d+n}=0$, $n=1, 2, 3, \cdots$. If $q>p$, then $C_2=C_3=\cdots=C_{p-1}=0$; $C_p=1$; $C_{p+1}=-1$; $C_{p+s}=C_{p-q+s}$, $s=2, 3, \cdots, p-1$; $C_{q+n}=C_{q-p+n}$, $n=2, 3, \cdots, p$; and $C_{p+q+k}=C_{p+k}+C_{q+k}$, $k=2, 3, \cdots, p-1$. Using (3.1) and (2.1), we obtain

$$(3.2)\qquad C_{(d/2)} = C_{(d/2)+q} + C_{(d/2)+p} - C_{(d/2)+p+q}.$$

For odd primes, $p<q<r$, the coefficients of $F_{pqr}(x)$, where $d=\phi(pqr)$, satisfies relations (2.1), (2.2), $\cdots$, (2.6), and (2.8). Noting (2.6), we now have

LEMMA 7. $C_{(d/2)}$ *is an odd integer.*

We omit the details of the proof for the following lemmas:

LEMMA 8. *For polynomials* $F_{pq}(x)$, *where* $d=\phi(pq)$, *we have*

$$(3.3)\qquad \sum_{k=0}^{d} k^2C_k = d(d + pq + 1)/6, \qquad \sum_{k=0}^{d} (-1)^k k^2C_k = d(pq + 1)/2;$$

$$(3.4)\qquad \sum_{k=0}^{d} k^3C_k = d^2(pq + 1)/4, \qquad \sum_{k=0}^{d} (-1)^k k^3C_k = d^2(3pq + 3 - d)/4.$$

LEMMA 9. *For* $F_{pqr}(x)$, *where* $d=\phi(pqr)$ *and* $M=rpq+p+q+r$, *we have*

$$(3.5)\qquad \sum_{k=0}^{d} k^2C_k = d(d + M)/6, \qquad \sum_{k=0}^{d} (-1)^k k^2C_k = dM/2;$$

(3.6) $\sum_{k=0}^{d} k^3C_k = d^2M/4, \qquad \sum_{k=0}^{d} (-1)^k k^3 C_k = d^2(3M-d)/4.$

LEMMA 10. *For $F_{pq}(x)$, $F_{pqr}(x)$, $F_{pqrs}(x)$, $\cdots$, we have, for the proper value of d,*

(3.7) $\sum_{k=0}^{d} kC_k = \sum_{k=0}^{d} (-1)^k kC_k = d/2, \qquad 2\sum_{k=0}^{d} (-1)^k kC_k^2 = dC_{(d/2)}.$

References

1. A. S. Bang, Om Ligningen $\phi_n(x)=0$, Nyt Tidskrift for Mathematik, 6 (1895, part B) 6–12.
2. A. Migotti, Zur Theorie der Kreisteilungsgleichung, Akad. Wiss., (2) 87 (1883) 7–14.
3. Sister Marion Beiter, The midterm coefficient of the cyclotomic polynomial $F_{pq}(x)$, this MONTHLY, 71 (1964) 769–770.
4. L. Carlitz, The number of terms in the cyclotomic polynomial $F_{pq}(x)$, this MONTHLY, 73 (1966) 979–981.

A PROOF OF SYLVESTER'S THEOREM ON COLLINEAR POINTS

V. C. WILLIAMS, Portland State College

The object of this paper is to present a new proof of the theorem below, a theorem conjectured in 1893 by J. J. Sylvester [5]. This theorem was first proved by T. Gallai after 1930. Since then, a number of proofs have been given, using a variety of methods. Some of these are to be found in [1, pp. 27, 28], [2, p. 30], [3, pp. 65, 181], [4, pp. 42, 57].

THEOREM. *Let S be a finite set of points of the Euclidean plane. If on the line through any two distinct points of S, there is a third point of S, then all the points of S lie on one line.*

Proof. The proof is indirect. We suppose that the points of S are not collinear. Then there is at least one nondegenerate triangle with vertices in S; there are at most finitely many. Thus there is at least one triangle $P_0P_2P_4$ which has an angle α larger than or equal to any angle in any of the rest of the triangles (Fig. 1).

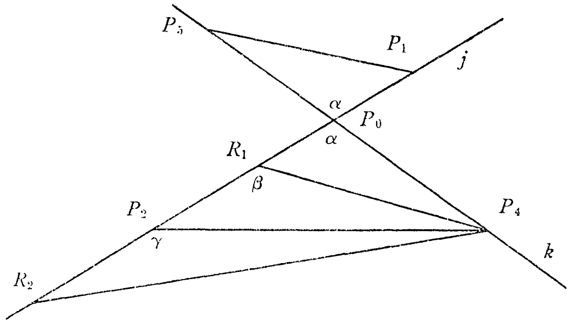

FIG. 1

In Figure 1, the only point of S on line j and below P_0 is P_2, for if there were other points such as R_1 or R_2 we would have $\beta > \alpha$ or $\gamma > \alpha$ respectively. However, there is a third point P_1 of S on j, which must lie above P_0. By the same reasoning, there is a point P_5 of S on k and above P_0.

There is no point of S on line m (through P_2 and P_4) and to the left of P_2, for if R were such a point, then $\beta > \alpha$ (Fig. 2).

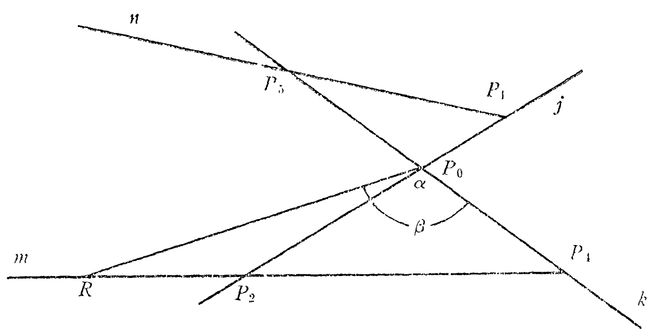

FIG. 2

Similarly, there is no point of S on m and to the right of P_4; thus a third point P_6 in S and on m must lie between P_2 and P_4. By the same reasoning a third point P_3 in S and on n must lie between P_1 and P_5 (Fig. 3).

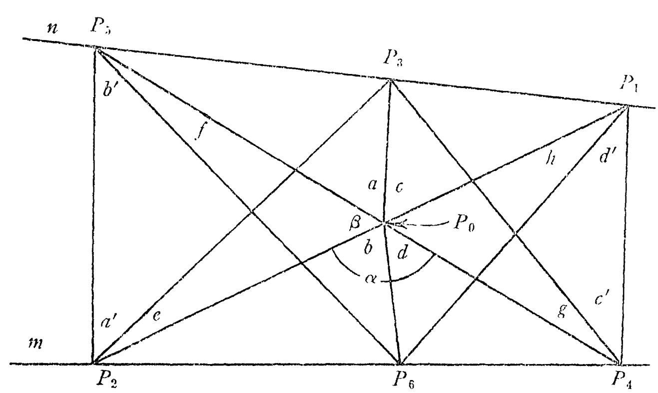

FIG. 3

In Figure 3, we observe that

$$\beta + a' + b' + e + f = \pi,$$
$$\beta + c' + d' + g + h = \pi,$$
$$2\beta + a + b + c + d = 2\pi;$$

thus

$$2\beta + a' + b' + c' + d' + e + f + g + h = 2\beta + a + b + c + d;$$

and $a'+b'+c'+d' < a+b+c+d$. Therefore at least one of these inequalities must hold:

$$a' < a, b' < b, c' < c, \text{ or } d' < d.$$

Without loss of generality, we may suppose that $a' < a$. Let P be the point on the segment P_2P_3 such that the angle PP_0P_5 equals a', as in Figure 4. Since $a' < a$, $P \neq P_3$.

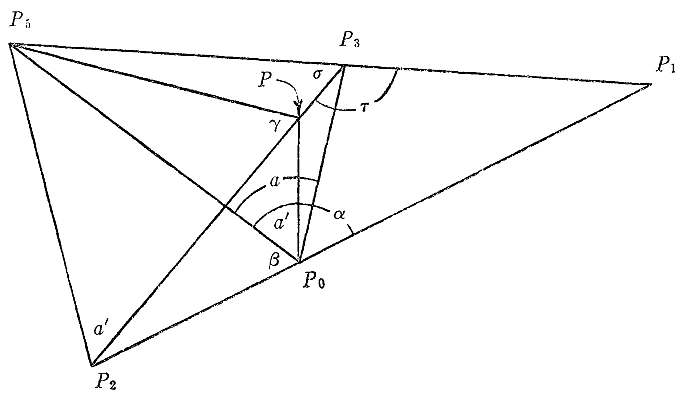

FIG. 4

It is an interesting and elementary exercise in similar triangles to show that in Figure 4, the angles β and γ are equal. Then we have $\beta = \gamma > \sigma$; hence $\tau > \alpha$. The triangle $P_1P_2P_3$ has the angle τ larger than the angle α in triangle $P_0P_2P_4$, a contradiction. The theorem is established.

One application of Sylvester's theorem is pedagogical. It is not possible to draw a picture of a 7-point, 7-line projective geometry without bending at least one line.

The author is indebted to the referee for a suggestion resulting in extensive simplification of the author's first proof.

References

1. H. S. M. Coxeter, A problem of collinear points, this MONTHLY, 55 (1948) 26–28.
2. ———, The Real Projective Plane, 2nd Ed., Cambridge, 1955.
3. ———, Introduction to Geometry, Wiley, New York, 1961.
4. H. Hadwiger, H. Debrunner and V. Klee, Combinatorial Geometry in the Plane, Holt, New York, 1964.
5. J. J. Sylvester, Mathematical Question 11851, Educational Times, 59 (1893) 98.
6. H. Zassenhaus, What is an angle?, this MONTHLY, 61 (1954) 369–378.

OSCILLATORY PROPERTIES OF CERTAIN NONLINEAR EQUATIONS

G. DiAntonio, Indiana University of Pennsylvania

The purpose of this note is to reveal the oscillatory behavior of solutions of the equations

$$x'' - xx'^2 + q^2x^3 = 0, \tag{1}$$

$$x'' + xx'^2 - q^2x^3 = 0, \qquad x' = dx/dt. \tag{2}$$

In response to a question of W. R. Utz [1], we will show that solutions of (1) valid for all large t may be either oscillatory or nonoscillatory and that solutions of (2) are all nonoscillatory. The solution $x\equiv 0$ is excluded as oscillatory.

The methods employed, which make use of an alternate (x, c) plane, thus extending the methods of Utz, may also be used to treat the equations

$$x'' + xx'^2 + q^2x^3 = 0 \tag{3}$$

$$x'' - xx'^2 - q^2x^3 = 0. \tag{4}$$

To discover the nature of solutions of (1), set $v=x'$ to secure

$$\frac{dv}{dx} - xv = -q^2x^3v^{-1}. \tag{5}$$

Equation (5) is a Bernoulli equation and, setting $v^2=z$, we have

$$v^2(x) = z(x) = e^{x^2}q^2\int_{x_0}^{x} u^2e^{-u^2}(-2u)du + ke^{x}.$$

Two integrations by parts yield

$$v^2 = z = q^2(1+x^2) + ce^{x^2}, \tag{6}$$

wherein $c=\exp(-x_0^2[v^2(x_0)-q^2(x_0^2+1)]$.

The nature of the curve (6) depends upon the value of c. Let

$$F(x) = q^2(1+x^2) + ce^{x^2}$$

so that we are interested in the curve $v^2=F(x)$. By sketching the curve $F(x)=0$, that is, $c=-q^2(1+x^2)e^{-x^2}$ in the (x, c) plane, it is seen that the curve is inverted bell-shaped, asymptotic to the x-axis, lies in the lower half plane, symmetric about the c axis, and having $(0, -q^2)$ as absolute minimum. The following cases now arise.

(a) If $c<-q^2$, then $v^2=F(x)<0$ for all x and so there is no real solution.

(b) If $c=-q^2$, then $v=x=0$ is the only point on the curve $v^2=F(x)$. That is, there is only the trivial solution $x\equiv 0$.

(c) If $-q^2<c<0$, then $v^2=F(x)$ is a simple closed curve symmetric with respect to both the x and v axes. From the graph of the curve $F(x)=0$ in the x, c-plane one sees that $v^2=F(x)$ cuts the x-axis exactly twice and cuts the v-axis at the two points $v=\pm\sqrt{(q^2+c)}$. For such values of c any solution of (1) valid for all large t is oscillatory.

(d) If $c \geqq 0$, then $F(x) > 0$ for all x and so the curve $v^2 = F(x)$ does not cut the x-axis. That is, no solution is oscillatory.

We summarize these possibilities in the following theorem.

THEOREM 1. *Let* $c = \exp(-x_0^2)[x_0'^2 - q^2(x_0^2+1)]$. *A solution of* (1) *with initial conditions* $x = x_0$, $x' = x_0'$ *and valid for all large* t *is oscillatory if* $-q^2 < c < 0$ *and nonoscillatory if* $c \geqq 0$.

The analysis of equation (2) is similar. The equation corresponding to (6) is

$$v^2 = z = q^2(x^2 - 1) + ce^{-x^2}. \tag{7}$$

The curve $v^2 = F(x) = 0$ for equation (7) may be described as follows. It is similar to a parabola symmetric about the c axis, opening downward with intercepts at $(\pm 1, 0)$, and having $(0, q^2)$ as highest point.

For no value of c (except the trivial case $c = q^2$) is (7) a closed curve and so we have the following theorem.

THEOREM 2. *No solution of equation* (2) *is oscillatory.*

A similar analysis of (3) and (4) reveals, as was shown by Utz, that all solutions of (3), $x \neq 0$, valid for all large t are oscillatory while all of the solutions of (4) are nonoscillatory.

The author wishes to thank the referee for helpful suggestions.

Reference

1. W. R. Utz, The behavior of solutions of the equations $x'' \pm xx'^2 \pm x^3 = 0$, this MONTHLY, 74 (1967) 420–423.

ARITHMETIC FUNCTIONS AND DISTRIBUTIVITY

M. V. SUBBARAO, University of Alberta

1. Introduction. In a recent paper Lambek [2, Theorem 2] proved the equivalent of the following theorem:

Let f, g, h, k be completely multiplicative functions, then

$$(f \circ g)(h \circ k) = fh \circ fk \circ gh \circ gk \circ w, \tag{1.1}$$

where $w(n) = f(\sqrt{n})\, g(\sqrt{n})\, h(\sqrt{n})\, k(\sqrt{n})\, \mu(\sqrt{n})$ if n is a square and $w(n) = 0$ otherwise and $\mu(n)$ is the Moebius function. Here and throughout what follows "$\circ$" denotes Dirichlet convolution. A function f is called multiplicative (completely multiplicative) provided $f(1) = 1$ and $f(mn) = f(m)\, f(n)$ for all coprime integers m and n (for all positive integers m and n).

In this note, we give a very short proof of (1.1) different from Lambek's and then extend (1.1) to the case of triple product $(f \circ g)\,(h \circ k)\,(u \circ v)$, where all the functions $f, \cdots, v$ involved are completely multiplicative. As a corollary we obtain the identity:

$$(1.2)\quad \sum \sigma_\alpha(n)\sigma_\beta(n)\sigma_\gamma(n)/n^s$$
$$= \zeta(s)\zeta(s-\alpha)\zeta(s-\beta)\zeta(s-\gamma)\zeta(s-\alpha-\beta)\zeta(s-\beta-\gamma)\zeta(s-\gamma-\alpha)$$
$$\cdot\zeta(s-\alpha-\beta-\gamma)\theta(s),$$

where $\zeta(s)$ is the Riemann Zeta function, $\sigma_\alpha(n)$ represents the sum of the αth powers of the divisors of n, and $\theta(s)=\sum F(n)/n^s$, $F(n)$ being a multiplicative function of n defined, for arbitrary prime p, by $F(1)=1$; $F(p^m)=0$ for $m=1$ or 5 or $m>6$;

$$F(p^4) = p^{\alpha+\beta+\gamma}F(p^2)$$
$$= -p^{\alpha+\beta+\gamma}(p^{\alpha+\beta+\gamma}(p^\alpha+p^\beta+p^\gamma+3)+p^{\beta+\gamma}+p^{\gamma+\alpha}+p^{\alpha+\beta})$$

and

$$F(p^3) = p^{\alpha+\beta+\gamma}(p^\alpha+1)(p^\beta+1)(p^\gamma+1);\quad F(p^6) = -p^{3(\alpha+\beta+\gamma)}.$$

On lettering $\gamma\to-\infty$, (1.2) reduces to a well-known result of Ramanujan [3].

Later in this note we consider the case of unitary convolution and show that the distribution law analogous to (1.1) holds 'exactly' without an 'error term' like w in (1.1). In fact, if f, g, h, k are arbitrary *multiplicative* functions, then

$$(1.3)\qquad (f\cdot g)(h\cdot k) = fh\cdot fk\cdot gh\cdot gk,$$

where '$\cdot$' denotes the unitary convolution operation defined by

$$(1.4)\qquad (f\cdot g)(n) = \sum_{\substack{d\mid n\\(d,n/d)=1}} f(d)g(n/d).$$

If S denotes the set of all multiplicative functions, and X denotes the natural product of two such functions, the ring $(S,\ \cdot,\ X)$ has some interesting properties which will be dealt with elsewhere.

2. Dirichlet convolution and distributivity. To prove (1.1) set $f(p)=a_p$, $g(p)=b_p$, $h(p)=c_p$, $k(p)=d_p$, $p^{-s}=x_p$ where p is an arbitrary prime. Then using formal Dirichlet series we have

$$(2.1)\qquad \sum (f\circ g)(n)/n^s = \prod_p (1-a_px_p)^{-1}(1-b_px_p)^{-1},$$

where throughout the paper, $\sum$ denotes summation over all positive integers n and $\prod_p$ denotes product over all primes p. Thus,

$$(2.2)\quad \sum (f\circ g)(h\circ k)(n)/n^s = \prod_p \left\{1+\sum \frac{a_p^{n+1}-b_p^{n+1}}{a_p-b_p}\right\}\left\{\frac{c_p^{n+1}-d_p^{n+1}}{c_p-d_p}\right\}x_p^n.$$

(If $b_p=a_p$ in the above, $[a_p^{n+1}-b_p^{n+1}]/[a_p-b_p]$ is to be taken as its limiting value $(n+1)a_p^n$ as $b_p\to a_p$; and similarly for $[c_p^{n+1}-d_p^{n+1}]/[c_p-d_p]$.) The result (1.1) now follows immediately from the elementary identity

$$(2.3)\qquad 1+\sum \frac{a^{n+1}-b^{n+1}}{a-b}\,\frac{c^{n+1}-d^{n+1}}{c-d}\,x^n=\frac{1-abcdx^2}{(1-acx)(1-adx)(1-bcx)(1-bdx)}$$

on noticing that $\sum w(n)/n^s=\prod_p (1-abcdx^2)$.

We shall next extend the result (1.1) to the triple product $(f\circ g)(h\circ k)(u\circ v)$, where all the functions $f, g, \cdots, v$ are completely multiplicative. Set, as above, $f(p)=a_p, \cdots, k(p)=d_p, u(p)=i_p, v(p)=j_p, p^{-s}=x_p$, and use the identity

$$(2.4)\qquad \begin{aligned} &1+\sum\left\{\frac{a^{n+1}-b^{n+1}}{a-b}\right\}\left\{\frac{c^{n+1}-d^{n+1}}{c-d}\right\}\left\{\frac{i^{n+1}-j^{n+1}}{i-j}\right\}x^n \\ &=\frac{1-rx^2+2abcdij(a+b)(c+d)(i+j)x^3-abcdijrx^4+(abcdij)^3x^6}{(1-acix)(1-acjx)(1-adix)(1-adjx)(1-bcix)(1-bcjx)(1-bdix)(1-bdjx)}\end{aligned}$$

where

$$(2.5)\qquad r=(a^2+b^2)cdij+(c^2+d^2)abij+(i^2+j^2)abcd+3abcdij.$$

We thus obtain the following

THEOREM 1. *For arbitrary completely multiplicative functions f, g, h, k, u, v, we have*

(2.6) $(f\circ g)(h\circ k)(u\circ v)=fhu\circ fhv\circ fku\circ fkv\circ ghu\circ ghv\circ gku\circ gkv\circ t$, *where $t(n)$ is a multiplicative function defined for arbitrary primes p as follows:*

$$t(p^m)=\begin{cases} 1 & \text{if } m=0;\\ 0 & \text{if } m=1 \text{ or } 5 \text{ or } m>6;\\ -r & \text{if } m=2;\\ 2abcdij(a+b)(c+d)(i+j) & \text{if } m=3;\\ -abcdijr & \text{if } m=4;\\ (abcdij)^3 & \text{if } m=6,\end{cases}$$

where r is given by (2.5).

Setting $f(n)=n^\alpha$, $h(n)=n^\beta$; $j(n)=n^\gamma$; $g(n)=k(n)\equiv 1$, we obtain the result (1.2) stated earlier.

One can easily see that the general situation is as follows: If $f_{i1}, f_{i2} (i=1, \cdots, r)$ are all completely multiplicative, then the natural product of the r functions $f_{i1}\circ f_{i2}$ $(i=1,\cdots,r)$ equals the Dirichlet convolute of the 2^r functions $f_{1j}f_{2j}\cdots f_{rj}$ $(j=1, 2)$ and a certain multiplicative function $t_r(n)$ which vanishes whenever the canonical form of n has a prime occurring to an exponent <2 or $>2^r-2$.

3. Unitary convolution and distributivity. We shall first note the

LEMMA. *The arithmetic function f satisfies $f(g\cdot h)=fg\cdot fh$ for all arithmetic functions g and h if and only if h is multiplicative.*

Here '·' denotes the unitary convolution defined in (1.4). The 'if' part of the proof is quite trivial. For the 'only if' part, given any integer M, define

$$\delta_M(n) = \begin{cases} 1 & \text{if } n = M, \\ 0 & \text{otherwise.} \end{cases}$$

Then, evaluating $f(\delta_M \cdot \delta_N) = f_M \cdot f_N$ at $n = MN$, one obtains $f(MN) = f(M)f(N)$ as required.

THEOREM 2. *For any multiplicative functions f, g, h and k, we have*

$$(f \cdot g)(h \cdot k) = fh \cdot fk \cdot gh \cdot gk.$$

Proof. Since f and g are multiplicative, so also is $f \cdot g$. Application of the lemma yields

$$(f \cdot g)(h \cdot k) = [(f \cdot g)h] \cdot [(f \cdot g)k].$$

Applying the lemma again, the right member in the above relation becomes $(fh \cdot gh) \cdot (fk \cdot gk)$, thus proving the theorem.

4. Concluding remarks. It is well known (see, for example, [1]) that if a_n and b_n are sequences, each of which satisfies a linear recurrence relation with constant coefficients, the sequence $a_n b_n$ satisfies a similar relation. Moreover, if a_n is generated by the function

$$A(x) = \sum a_n x^n = \frac{p(x)}{(1 - \theta_1 x)^{e_1} \cdots (1 - \theta_k x)^{e_k}},$$

where the θ_i are distinct numbers and $p(x)$ a polynomial in x of degree $< e_1 + e_2 + \cdots + e_k = N$ and if b_n is generated by $B(x)$, then the sequence $a_n b_n$ is generated by

$$C(x) = \sum a_n b_n x^n = \sum_{j=1}^{k} \frac{1}{(e_j - 1)!} \frac{\partial^{e_j - 1}}{\partial s^{e_j - 1}} \cdot \left\{ \frac{s^{N-1} p(x/s) B(s)}{(s - \theta_1 x)^{e_1} \cdots (s - \theta_{j-1} x)^{e_{j-1}} (s - \theta_{j+1} x)^{e_{j+1}} \cdots} \right\}_{s = \theta_j}$$

Using this method one easily obtains (2.3) and (2.4) and many others. It can be used for example to obtain an identity for $\sum \sigma_\alpha(n)\sigma_\beta(n)\sigma_\gamma(n)\sigma_\delta(n)/n^s$ similar to (1.2). The details are left to the interested reader. One can similarly obtain identities involving other arithmetic functions also which satisfy a linear recurrence relation with constant coefficients. As simple illustrations we might mention:

$$(4.1) \qquad \sum \sigma(n)\phi(n)/n^s = \zeta(s-1)\zeta(s-2) \prod_p (1 - (p+1)p^{-s} + p^2 p^{-2s}); \quad s > 2;$$

and

$$(4.2) \qquad \sum \overset{*}{\sigma}_\alpha(n) \overset{*}{\sigma}_\beta(n)/n^s = \zeta(s)\zeta(s-\alpha)\zeta(s-\beta)\zeta(s-\alpha-\beta)F(s),$$

where

$$F(s) = \prod_p \{1 + (p^\alpha + p^\beta + p^{2\alpha+\beta} + p^{\alpha+2\beta} + 2p^{\alpha+\beta})p^{-2s} + p^{\alpha+\beta}(1 + p^\alpha)(1 + p^\beta)p^{-3s} - 3p^{2\alpha+2\beta}p^{-4s}\}.$$

Here $\phi(n)$ denotes the Euler totient; $\sigma(n) = \sigma_1(n)$; and

$$\overset{*}{\sigma}_\alpha(n) = \sum_{\substack{d \mid n \\ (d,n/d)=1}} d^\alpha = \text{the sum of the } \alpha\text{th powers of the unitary divisors of } n.$$

References

1. D. A. Klarner, A ring of sequences generated by rational functions, this MONTHLY, 74 (1967) 813–816.
2. J. Lambek, Arithmetic functions and distributivity, this MONTHLY, 73 (1966) 969–973.
3. S. Ramanujan, Some formulae in the analytic theory of numbers, Messenger of Math., 45 (1916) 81–84 (or, collected Papers of S Ramanujan, Chelsea, New York, 1962, pp. 133–135).

BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the MONTHLY. *Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.*

THE ARITHMETIC FUNCTION $\tau_{k,r}(n)$

R. SIVARAMAKRISHNAN, Government Engineering College, Trichur, India

1. We define $\tau_{k,r}(n)$ as the number of ways of expressing n as the product of k factors each of which is an rth power including unity ($r \geqq 1$), regarding factorizations as distinct, according to the order of the factors. The special case for $r=1$ is $\tau_k(n)$ (vide [1]).

From the definition it can be easily shown that

(1.1) $\tau_{k,r}(n)$ is multiplicative in n

$$\tau_{k,r}(n) = \begin{cases} \tau_k(n^{1/r}), & \text{if } n \text{ is an } r\text{th power} \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

2. Using (1.1) and (1.2), we prove the following theorems:

$$\sum_{d \mid n} \tau_{k+1,r}(d)\phi_r\left(\frac{n}{d}\right) = \sum_{d \mid n} d\tau_{k,r}\left(\frac{n}{d}\right), \tag{2.1}$$

where $\phi_r(n)$ is the extension of Euler's ϕ-function due to V. L. Klee [2].

(2.2) $$\text{If } n = \prod_{i=1}^{\nu} p_i^{m_i}, \quad \text{where} \quad (m_i, k) = 1, \quad \text{then}$$
$$\tau_{k,r}(n) \equiv 0 \pmod{k^r}.$$

(2.3) $$\text{If } n \text{ is a } k\text{th power, } \tau_{k,1}(n) \equiv \tau_{k-1,1}(n) \pmod{k}.$$

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Mail Address: Dept. of Mathematics, Government Engineering College, Trichur—9, India.

References

1. M. G. Beumer, The arithmetic function $\tau_k(n)$, this MONTHLY, 69 (1962) 777–781.
2. V. L. Klee, A generalization of Euler's ϕ-function, this MONTHLY, 55(1948) 358–359.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

A DEVELOPMENT OF THE JORDAN CURVE THEOREM AND THE SCHOENFLIES THEOREM FOR POLYGONS

S. B. JACKSON, University of Maryland and University of Liverpool

In giving an axiomatic development of plane geometry, the separation properties play an important role. The simpler ones, such as the separation of the plane into two convex sets by a line and the separation of the plane into two connected sets by an angle, are usually comparatively painless, being either adopted as axioms or readily deduced from the axioms. To obtain such a result as the Jordan Curve Theorem for polygons, however, is much more troublesome and laborious. Yet this result is essential in many general arguments about polygons. The axiom system adopted by Moise in his *Elementary Geometry from an Advanced Standpoint* and due originally to G. D. Birkhoff is now quite widely employed. It seems appropriate therefore to present a development, based on this axiom system, of properties of polygons including both the Jordan Theorem for polygons and the Schoenflies Theorem for polygons. It is to be noted, however, that no use is made of the parallel axiom, so the proofs are valid for both the Euclidean and hyperbolic planes.

The features which distinguish the chosen axiom system from, say, the Hilbert axiom system do not equally affect all parts of the discussion. Hence substantial parts of the development are similar to other presentations of these topics. In this connection, note for example the work of Lennes [3], Courant and Robbins [2] and Bagemihl's translation of Knopp [1, pp. 15–19] in which the section noted is actually the work of the translator.

Making use of the Plane Separation Axiom [4], it is easy to define the interior of a triangle. Specifically, the interior of ΔABC is the intersection of the three open half planes consisting of the A side of line BC, the B side of line AC, and the C side of line AB. Since, by the axiom, half planes are convex, it follows that the interior of a triangle is a convex set as is the union of a triangle and its interior. As a result of this convexity and of Pasch's "Axiom" which is here an easy consequence of the Plane Separation Axiom, it is a simple matter to verify that the interior of ΔABC consists precisely of those points P of the plane for which there exists some ray $\overrightarrow{PQ}$ which does not contain a vertex of ΔABC and meets the triangle exactly once.

LEMMA 1. *Let $\mathcal{S}$ and $\mathcal{T}$ be closed polygons, not necessarily simple, of m and n sides respectively ($m, n \geqq 3$) such that neither polygon contains a vertex of the other. If a point P belongs to q sides of $\mathcal{S}$ and r sides of $\mathcal{T}$, it is said to be an intersection of multiplicity qr. Then the total number, p, of intersections, counted according to multiplicity, is an even number.*

Proof. Note that the number p is merely the number of times a side of $\mathcal{S}$ meets a side of $\mathcal{T}$. The proof will be by induction on m. If $m=3$ then $\mathcal{S}$ is a triangle. By the definition above of the interior, and hence exterior, of a triangle, it follows readily that, in tracing $\mathcal{T}$, whenever a side of $\mathcal{T}$ meets a side of $\mathcal{S}$ there is a crossing from interior to exterior or conversely. Since $\mathcal{T}$ is closed it must cross $\mathcal{S}$ the same number of times in each direction, so the total number of crossings is even. This is precisely the number p of intersections counted according to multiplicity. Hence the lemma holds for the case $m=3$.

Assume now that the lemma holds for $m \leqq k$ and consider the case $m=k+1$. Let the $k+1$ vertices of $\mathcal{S}$ be $P_0, P_1, \cdots, P_k$. If necessary, let $\mathcal{T}$ be distorted slightly so that none of its vertices is collinear with two vertices of $\mathcal{S}$. This can be done without changing the number p. Draw segment $\overline{P_0P_2}$ and consider the polygons $\mathcal{S}'$ with vertices $P_0P_1P_2$ and $\mathcal{S}''$ with vertices $P_0P_2 \cdots P_k$. Let s' be the number of intersections of $\mathcal{T}$ with segments $\overline{P_0P_1}$ and $\overline{P_1P_2}$, let s'' be the number of intersections of $\mathcal{T}$ with the remaining sides of $\mathcal{S}$, let r be the number of intersections of $\mathcal{T}$ with segment $\overline{P_0P_2}$, in each case counting the intersections according to multiplicity. The number of intersections of $\mathcal{T}$ with $\mathcal{S}'$ is therefore $s'+r$ while the number of intersections with $\mathcal{S}''$ is $s''+r$. Since both $\mathcal{S}'$ and $\mathcal{S}''$ have no more than k sides, it follows from the induction assumption that $s'+r$ and $s''+r$ are both even. Their sum, $s'+s''+2r$, is therefore even, whence $s'+s''$ is even. This completes the proof since $p=s'+s''$.

LEMMA 2. *If P and Q are points exterior to a circle, there is a polygonal path from P to Q which is wholly exterior to the circle.*

The proof of this lemma is easy and will be omitted. To say that a point is exterior to a circle means simply that its distance from the center exceeds the length of a radius.

LEMMA 3. *Let $\mathcal{P}$ be a closed polygon, not necessarily simple, W a point not on $\mathcal{P}$ and $\overrightarrow{WA}$ and $\overrightarrow{WB}$ rays from W not containing any vertices of $\mathcal{P}$. Then the numbers of intersections of $\mathcal{P}$ with $\overrightarrow{WA}$ and $\overrightarrow{WB}$ are either both odd or both even, intersections being counted according to multiplicity.*

Proof. The set of distances from W to points of $\mathcal{P}$ is clearly bounded. Hence there exists a circle with center W such that $\mathcal{P}$ is interior to the circle. Let A' and B' be points of $\overrightarrow{WA}$ and $\overrightarrow{WB}$ respectively which are exterior to this circle. By Lemma 2, A' and B' can be joined by a polygonal path exterior to the circle and hence containing no points of $\mathcal{P}$. If $\mathcal{S}$ is the polygon consisting of this polygonal path together with segments $\overline{WA'}$ and $\overline{WB'}$, the intersections of $\mathcal{P}$ and $\mathcal{S}$ are precisely the intersections of $\mathcal{P}$ with rays $\overrightarrow{WA}$ and $\overrightarrow{WB}$. By Lemma 1, the total number of these intersections, counted according to multiplicity, is even, so the numbers for $\overrightarrow{WA}$ and $\overrightarrow{WB}$ are either both even or both odd as was to be shown.

By Lemma 3, if the point W does not lie on the closed polygon $\mathcal{P}$ then either all rays from W not containing vertices of $\mathcal{P}$ meet $\mathcal{P}$ in an even number of points or all meet $\mathcal{P}$ in an odd number of points. We distinguish these cases by saying that W has even parity or odd parity with respect to $\mathcal{P}$.

LEMMA 4. *Let A and B be points not on a closed polygon $\mathcal{P}$, not necessarily simple, and consider a polygonal path from A to B which contains no vertex of $\mathcal{P}$ and has no vertex on $\mathcal{P}$. If this path meets $\mathcal{P}$ in q points, then A and B have the same or different parities according as q is even or odd.*

Proof. Consider a circle such that all points of $\mathcal{P}$ are interior to it. Let $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ be any rays from A and B not containing vertices of $\mathcal{P}$, and let A' and B' be chosen outside the circle. There is, by Lemma 2, a polygonal path from B' to A' exterior to the circle and hence not meeting $\mathcal{P}$. Let $\mathcal{S}$ be the closed polygon consisting of this path from B' to A', the given path from A to B, and the segments $\overline{AA'}$ and $\overline{BB'}$. If t_A and t_B are the numbers of intersections of $\mathcal{P}$ with rays $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$, respectively, we see at once that $\mathcal{P}$ and $\mathcal{S}$ have t_A+t_B+q intersections, namely t_A on $\overline{AA'}$, t_B on $\overline{BB'}$, and q on the given path. By Lemma 1, t_A+t_B+q is even. Hence if q is even then t_A+t_B is even so A and B have the same parity, while if q is odd then t_A+t_B is odd so A and B have different parities.

It has been noted that the points not on a closed polygon $\mathcal{P}$ are divided into two sets, the set of points with even parity and the set of points with odd parity. We denote these sets by S_e and S_o respectively.

If A is a point not on the closed polygon $\mathcal{P}$, consider the set of points which can be joined to A by a polygonal path not meeting $\mathcal{P}$. This set may be called the component for point A. It follows at once from the definition that if the component for point A and the component for point B have a point C in common, then the sets are the same. For there is a polygonal path from B to C not meeting $\mathcal{P}$ and one from C to A not meeting $\mathcal{P}$, and the combination gives a path from B to A. Hence the points P which can be joined to A can also be

joined to B and conversely. Thus all points not on $\mathcal{P}$ are partitioned into a certain number of disjoint sets called components. Moreover, by Lemma 4, all points in any one component have the same parity, so any component is a subset of S_e or of S_o. Thus we may speak of odd and even components.

The lemmas above have applied to any closed polygon. For the work below, however, it is necessary to consider specifically simple closed polygons.

LEMMA 5. *If A is any point of a simple closed polygon $\mathcal{P}$, there is a circular region with center A which contains points of exactly two components, one odd and one even.*

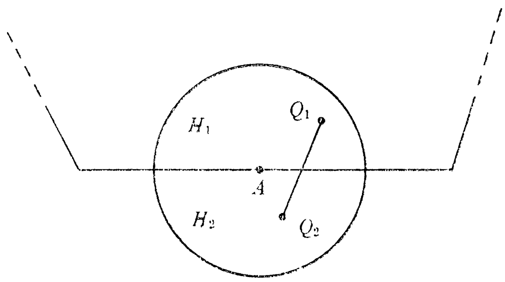

FIG. 1

Proof. Consider first the case when A is not a vertex of $\mathcal{P}$. In this case there is a circle about A whose interior contains no points of $\mathcal{P}$ except on the side containing A (Fig. 1). The points of this circular region not on $\mathcal{P}$ consist of the intersections H_1 and H_2 of the circular region with the two half planes determined by the side of $\mathcal{P}$ through A. Since both a circular region and a half plane are convex, H_1 and H_2 are convex. Hence any two points of H_1 or H_2 can be joined by a segment not meeting $\mathcal{P}$. Hence all points of H_1 are in the same component and similarly for H_2. But the segment joining a point Q_1 of H_1 and Q_2 of H_2 meets $\mathcal{P}$ exactly once since Q_1 and Q_2 are in opposite half planes. Hence, by Lemma 4, Q_1 and Q_2 have different parities and must lie in different components. Thus the circular region contains points of exactly two components, one odd and one even.

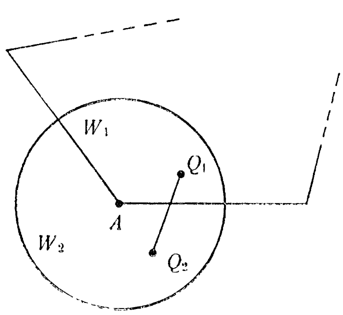

FIG. 2

The case when A is a vertex of $\mathcal{P}$ is similar. In this case a circle can be found about A whose interior contains only points of $\mathcal{P}$ lying on the sides through A (Fig. 2). The points of the circular region not on $\mathcal{P}$ then consist of the set W_1 which is the intersection of the region with the interior of the angle having vertex A and the set W_2 which is the intersection of the region with the exterior of this angle. Since the interior of an angle is convex (being the intersection of two half planes) all points of W_1 are in the same component just as in the former case. The exterior of an angle is not convex so W_2 need not be convex. However, it is easy to show that any two points of W_2 can be joined by a polygonal path not meeting $\mathcal{P}$, so again all points of W_2 are in the same component. As before, there is a segment joining some point of W_1 to a point of W_2 and meeting $\mathcal{P}$ in a single point. Hence, again by Lemma 4, these points have different parities, so one component is odd and one is even. This completes the proof.

The process of Lemma 5 associates with every point A of a simple closed polygon $\mathcal{P}$ a unique pair of components, C_1 and C_2. Note that all points of $\mathcal{P}$ within the circular regions of Lemma 5 must be associated with the same pair C_1 and C_2, for about any such point can be found a circular region wholly contained within that about A and hence containing points only from components C_1 and C_2. That is, all points of $\mathcal{P}$ close enough to A are associated with the same pair of components. We shall now prove that *all* points of $\mathcal{P}$ are associated with the same pair of components.

LEMMA 6. *All points of a simple closed polygon $\mathcal{P}$ are associated with the same pair of components.*

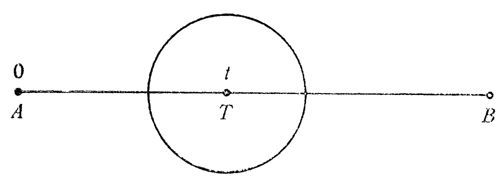

FIG. 3

Proof. It is sufficient to show that all points of any side of $\mathcal{P}$ have this property. Let $\overline{AB}$ be any side of $\mathcal{P}$ and let A be associated with components C_1 and C_2. Let a coordinate system be introduced on $\overline{AB}$ according to the Ruler Axiom such that the coordinate of A is zero and the coordinate of B positive (Fig. 3). Assume that there are points of $\overline{AB}$ not associated with C_1 and C_2. Let t be the greatest lower bound of the coordinates of such points, and let T be the point with coordinate t. By the remark above, all points of $\overline{AB}$ close enough to A are associated with C_1 and C_2 so $T \neq A$ and $t > 0$. By the same remark, all points of $\mathcal{P}$ close enough to T are associated with the same components. This is impossible since, by definition of T, any interval about T contains points not associated with C_1 and C_2 and also points of $\overline{AT}$ which are associated with C_1 and C_2. Hence the assumption that there are points of $\overline{AB}$ not associated with this pair of components is false and the lemma is proved.

It is to be noted that the proof of Lemma 6 made use of a coordinate system, since this is the machinery provided by the Ruler Axiom. The essential idea, however, is that of Dedekind's Axiom and an analogous argument without the use of coordinates could be given if this axiom were available.

THEOREM A. (Jordan Curve Theorem for Polygons) *If $\mathcal{P}$ is a simple closed polygon, the set of points of the plane not on $\mathcal{P}$ is the union of two disjoint sets such that two points of the same set may be joined by a polygonal path not meeting $\mathcal{P}$, while a polygonal path joining points in different sets must meet $\mathcal{P}$. These sets are the sets S_o and S_e of points having odd and even parity respectively.*

Proof. The set of points not on $\mathcal{P}$ is already known to be the union of the disjoint sets S_o and S_e. It remains to show that these sets satisfy the stated conditions. By Lemma 6, the points of $\mathcal{P}$ are all associated with a pair of components C_1 and C_2 and, by Lemma 5, one of these components is of odd parity and hence a subset of S_o while the other is of even parity and a subset of S_e. For definiteness suppose $C_1 \subset S_o$ and $C_2 \subset S_e$. Let P be any point of S_o and consider a ray from P which intersects $\mathcal{P}$. Let Q be the first intersection of this ray with $\mathcal{P}$ so that $\overline{PQ}$ contains no point of $\mathcal{P}$ except Q. Any two points of $\overline{PQ} - \{Q\}$ belong to the same component since they are joined by a polygonal path not meeting $\mathcal{P}$. But all points of this set close enough to Q must belong to C_1 or C_2 by Lemma 5. Hence P belongs to C_1 or C_2. Since P has odd parity it cannot belong to C_2 whence it follows that $P \in C_1$. This argument shows that $P \in S_o$ implies $P \in C_1$, whence $S_o \subset C_1$. Since it was already known that $C_1 \subset S_o$, it follows that $C_1 = S_o$. An exactly similar argument shows that $C_2 = S_e$. Thus S_o and S_e are components and any two points of either set may be joined by a polygonal path not meeting $\mathcal{P}$. But since S_o and S_e have different parity it follows by Lemma 4 that every polygonal path joining them must meet $\mathcal{P}$. This completes the proof.

The sets S_o and S_e are called, respectively, the interior and the exterior of $\mathcal{P}$. If a point P is outside a circle containing $\mathcal{P}$ it is trivial that there exist rays from P not meeting $\mathcal{P}$, so all sufficiently distant points belong to the exterior, S_e, which is in accord with our intuitive use of the term exterior. Indeed, through P there exist lines not meeting $\mathcal{P}$ so there are lines wholly in the exterior of a polygon $\mathcal{P}$. On the other hand, if any line contains an interior point of $\mathcal{P}$, i.e., a point of odd parity, then the line consists of two rays each of which must meet $\mathcal{P}$. Hence there can be no lines which lie wholly in the interior of a polygon.

For the case when $\mathcal{P}$ is a triangle, a definition had already been given of the terms interior and exterior in terms of certain half planes. As noted just before Lemma 1, however, the interior and exterior thus defined are precisely the sets S_o and S_e. Hence there is no ambiguity in the two definitions.

By the use of Theorem A, it is not difficult to obtain the following result, the proof of which will be omitted.

LEMMA 7. *If two points of a simple closed polygon $\mathcal{P}$ are joined by a simple polygonal path $\mathcal{S}$ which is (except for the endpoints) interior to $\mathcal{P}$, then $\mathcal{S}$ separates*

the interior of $\mathcal{P}$ into two disjoint regions which are the interiors of the two simple closed polygons thus formed.

It is now possible to deduce results concerning decompositions of the interior of a simple closed polygon.

LEMMA 8. *If $\mathcal{P}$ is a simple closed polygon with n sides $(n>3)$, then there exists a pair of vertices P_i, P_j such that the segment $\overline{P_iP_j}$ is, except for its endpoints, interior to $\mathcal{P}$.*

An outline of the proof of this lemma may be given as follows. Let the vertices of $\mathcal{P}$ be $P_1, P_2, \cdots, P_n$ and consider the vertex P_1. If there is a vertex P_k such that segment $\overline{P_1P_k}$ is, except for endpoints, interior to $\mathcal{P}$, then the lemma holds. Consider therefore the contrary case when no such vertex P_k exists. As in Lemma 5, consider a circle with center P_1 such that it meets $\mathcal{P}$ only at points of sides $\overline{P_1P_2}$ and $\overline{P_1P_n}$. By Lemma 5 the two circular sectors into which this circular region is divided by $\angle P_2P_1P_n$ are in different components, and by Theorem A, one of these is interior to the polygon $\mathcal{P}$. Let T (Fig. 4) be any point of this interior sector. Since T is interior to $\mathcal{P}$ it has odd parity so the ray from T away from P_1 meets $\mathcal{P}$. Let Q be the first such intersection so that segment $\overline{P_1Q}$ is, except for endpoints, interior to $\mathcal{P}$. If T varies in the circular sector interior to $\mathcal{P}$, point Q varies, but always on the same side $\mathcal{S}$ of $\mathcal{P}$ since otherwise, for some position of T, Q would be a vertex which contradicts our assumption. Hence as T approaches segment $\overline{P_1P_2}$ (or $\overline{P_1P_n}$) the point Q approaches the intersection L (or M) of $\mathcal{S}$ with ray $\overrightarrow{P_1P_2}$ (or $\overrightarrow{P_1P_n}$).

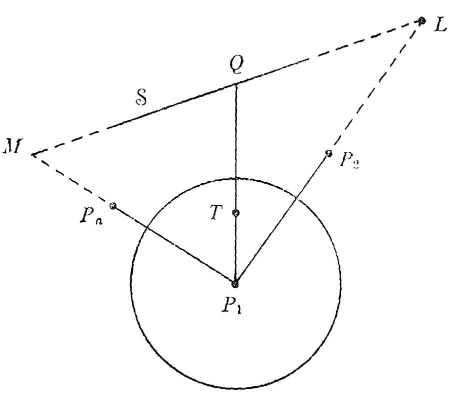

FIG. 4

Note that the argument given establishes the existence of the points L and M and hence of ΔP_1LM. Because, by the Plane Separation Axiom, the union of a triangle and its interior is convex, it follows readily that the interior points of segments $\overline{P_1Q}$ for Q between L and M constitute the interior of ΔP_1LM. Since these points are, by their definition, interior to $\mathcal{P}$, the interior of ΔP_1LM is a

subset of the interior of $\mathcal{P}$. (This actually implies that the measure of the interior angle of the polygon $\mathcal{P}$ at P_1 is less than π, though this fact is not needed to complete the proof.) Since $\mathcal{P}$ is simple, L and M cannot be interior points of $\overline{P_1P_2}$ or $\overline{P_1P_n}$ respectively. Moreover, we cannot have both $L=P_2$ and $M=P_n$ since this would make $\mathcal{P}$ a triangle contradicting the assumption $n>3$. By the convexity already noted of the union of ΔP_1LM and its interior, the segment $\overline{P_2P_n}$ is, except for endpoints, interior to ΔP_1LM and hence interior to $\mathcal{P}$. It is therefore the segment $\overline{P_iP_j}$ required by the lemma.

A formal proof of this lemma from the axioms would require an argument like that of Lemma 6 using a coordinate system on $\mathcal{S}$ to show that the points Q do indeed all belong to the same side of $\mathcal{P}$ and that all points between L and M are actually attained. It is an interesting corollary of the discussion that, of any two consecutive vertices of $\mathcal{P}$, a segment satisfying the required conditions can be drawn from at least one of them. An alternative proof of this lemma, also depending on a continuity argument, is given by Bagemihl as noted above in [1, pp. 15–19].

THEOREM B. *If $\mathcal{P}$ is a simple closed polygon of n sides ($n \geqq 3$), the closed region bounded by $\mathcal{P}$ can be decomposed into exactly $n-2$ nonoverlapping closed triangular regions such that the vertices of each triangle are also vertices of $\mathcal{P}$.*

Proof. The proof is by induction on n. For the case $n=3$ $\mathcal{P}$ is a triangle and the lemma holds trivially since $n-2=3-2=1$.

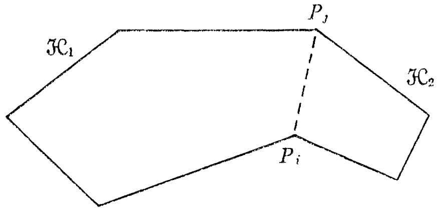

FIG. 5

Assume now the lemma holds for $n \leqq k$ and consider the case $n=k+1$. Let $\overline{P_iP_j}$ be the segment guaranteed by Lemma 8. This produces two simple closed polygons, $\mathcal{H}_1$ and $\mathcal{H}_2$ (Fig. 5), and by Lemma 7 the closed region bounded by $\mathcal{P}$ is the union of those bounded by $\mathcal{H}_1$ and $\mathcal{H}_2$ and these regions do not overlap, i.e., have no interior points in common. Let $\mathcal{H}_i$ ($i=1, 2$) contain r_i sides of $\mathcal{P}$, whence $\mathcal{H}_i$ is a polygon of r_i+1 sides and $r_1+r_2=k+1$. Since the number of sides of each $\mathcal{H}_i$ does not exceed k, it follows by the induction assumption that the region bounded by $\mathcal{H}_i$ can be decomposed into exactly $(r_i+1)-2=r_i-1$ triangular regions. Hence the region bounded by $\mathcal{P}$ can be decomposed into nonoverlapping triangular regions whose number is

$$(r_1 - 1) + (r_2 - 1) = (r_1 + r_2) - 2 = (k + 1) - 2.$$

Thus the lemma holds for the case $n=k+1$ and the induction is complete.

LEMMA 9. *If $\mathcal{P}$ is a simple closed polygon with n sides and if $n>3$, there exist two nonadjacent vertices, P_j and P_k of $\mathcal{P}$ such that the segments $\overline{P_{j-1}P_{j+1}}$ and $\overline{P_{k-1}P_{k+1}}$ are, except for endpoints, interior to $\mathcal{P}$.*

Proof. Consider such a decomposition as is guaranteed by Theorem B. Every side of $\mathcal{P}$ is a side of a unique triangle in the decomposition. But there are $n-2$ triangles and n sides. Hence at least two of the triangles contain two sides of $\mathcal{P}$. The common vertices of these pairs of sides are the required vertices in the lemma. This completes the proof.

THEOREM C. (Schoenflies Theorem for Polygons) *If $\mathcal{P}$ is a simple closed polygon, there is a homeomorphism of the plane onto itself taking $\mathcal{P}$ and its interior onto a circle and its interior.*

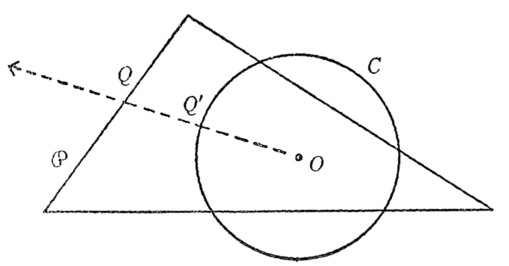

FIG. 6

Proof. The proof is by induction on the number of sides, n, of the polygon. If $n=3$ then $\mathcal{P}$ is a triangle. Let C be a circle which, for convenience, we take with center O interior to $\mathcal{P}$ (Fig. 6). Let Q be any point of $\mathcal{P}$, and let Q' be the unique point where ray $\overrightarrow{OQ}$ meets C. If P is any point of ray $\overrightarrow{OQ}$, let it be transformed into point P', also on $\overrightarrow{OQ}$, such that $OP'=(OQ'/OQ)OP$. This is clearly a homeomorphism of the plane onto itself. It carries points Q of $\mathcal{P}$ to points Q' of C and transforms segment $\overline{OQ}$ onto segment $\overline{OQ}'$. That is, it takes $\mathcal{P}$ and its interior onto C and its interior. Hence it is the required transformation and the theorem holds when $n=3$.

Assume now that the theorem holds for $n \leqq k$ and consider the case $n=k+1$. Let A be one of the vertices guaranteed by Lemma 9 such that segment $\overline{BC}$ (Fig. 7) joining the two adjacent vertices is interior to $\mathcal{P}$. Draw the line joining A to a point D of segment $\overline{BC}$ and on this line choose points O and H as shown such that A and D lie between O and H with O interior to $\mathcal{P}$ and H exterior. These points can be chosen such that quadrilateral $OBHC$ has no points in common with $\mathcal{P}$ except B and C. We shall define a homeomorphism of the plane which is the identity except for points of the region bounded by $OBHC$.

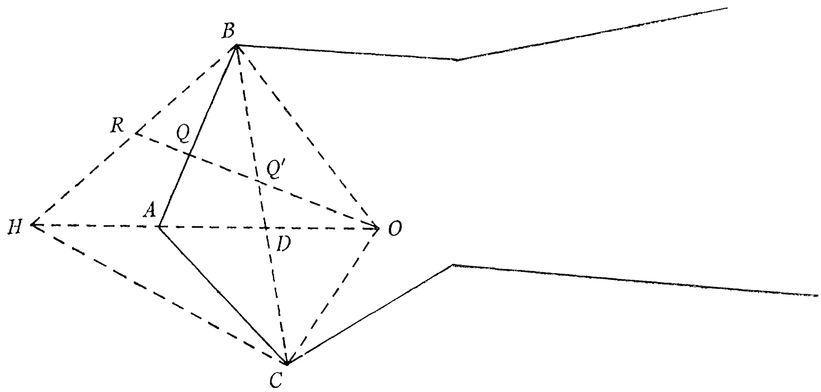

Fig. 7

Let Q be any point of $\overline{AB}$ ($\overline{AC}$) and let Q' be the point where ray $\overrightarrow{OQ}$ meets $\overline{BD}$ ($\overline{DC}$), and R the point where this ray meets $\overline{HB}$ ($\overline{HC}$). The segment $\overline{OR}$ is mapped onto itself by mapping $\overline{OQ}$ linearly on $\overline{OQ'}$ and $\overline{QR}$ linearly on $\overline{Q'R}$. Specifically, if $P \in \overline{OQ}$ then its image point P' is such that $OP' = (OQ'/OQ)OP$ and if $P \in \overline{QR}$ then P' is such that $P'R = (Q'R/QR)PR$. This is readily seen to be a homeomorphism of $OBHC$ and its interior onto itself which is the identity map on the quadrilateral. For example, if $P \in \overline{OB}$ then $Q = Q' = B$ so that $OP' = (OQ'/OQ)OP = OP$ and $P = P'$. The mapping which is defined above for $OBHC$ and its interior and is the identity elsewhere is therefore a homeomorphism of the plane onto itself. Moreover, it maps $\overline{BA}$ and $\overline{AC}$ onto $\overline{BD}$ and $\overline{DC}$ and maps interior points of triangle ABC into interior points of triangle OBC. Consider now the simple k-sided polygon $\mathcal{P}'$ consisting of $\overline{BC}$ and the sides of $\mathcal{P}$ other than $\overline{AB}$ and $\overline{AC}$. Since triangle OBC is interior to $\mathcal{P}'$, this homeomorphism carries $\mathcal{P}$ and its interior onto $\mathcal{P}'$ and its interior. But by the induction assumption there is a homeomorphism of the plane onto itself taking $\mathcal{P}'$ and its interior onto a circle and its interior. The product of these homeomorphisms is the required homeomorphism for $\mathcal{P}$. Hence the theorem holds for the case $n = k+1$ and the inductive proof is complete.

References

1. K. Knopp, Theory of Functions, Part 1, translated from the 5th German edition by F. Bagemihl, Dover, New York, 1945.

2. R. Courant, and H. Robbins, What is Mathematics?, Oxford University Press, New York, 1941.

3. N. J. Lennes, Theorems on the simple finite polygon and polyhedron, Amer. J. Math., 33 (1911) 37–62.

4. E. Moise, Elementary Geometry from an Advanced Standpoint, Addison-Wesley, Reading, Mass., 1963.

3. D. Pedoe, A Geometric Introduction to Linear Algebra, Wiley, New York, 1963, pp. 126–128.
4. D. E. Faddeev and I. S. Sominskii, Problems in Higher Algebra, Translated by J. L. Brenner, Freeman, San Francisco, 1965, p. 84.
5. P. Kahn, Introduction to Linear Algebra, Harper and Row, New York, 1967, pp. 269–270

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS

COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706

THE EFFECT OF REMEDIAL INSTRUCTION ON MATHEMATICAL ACHIEVEMENT

ESTHER OTTLEY, Howard University

This study was originally designed to investigate and to interpret the effectiveness of the program in remedial mathematics for college students at Howard University in Washington, D. C. It has contributed to the design of college remedial programs in mathematics.

The investigation was confined to 2475 freshmen students who enrolled at the University in September 1961, or, in September 1962, and the following subproblems were found to be closely identified with the main problem: (1) to measure the improvement shown by freshmen students of deficient mathematics background after they have undergone a period of remediation; (2) to compare or contrast the subsequent performance of remedial students in their first precalculus three-hour course in mathematics with that of nonremedial students; (3) to determine the areas of greatest weakness in fundamental mathematics at the college entrance level and the areas in which most improvement is shown after remediation; (4) to identify the critical factors which are responsible for the improvement; and (5) to investigate the effectiveness of remediation at various levels of scholastic ability.

The findings and recommendations have led to the construction of a revitalized program for remedial students in mathematics at Howard University and it is yielding highly satisfactory if not surprising results.

Techniques and procedure. The analytical survey method of research was used in determining the significant gains in mathematical achievement as evidenced by the retest scores of remedial students on the mathematics placement test; and an equivalent-group design was used in comparing the subsequent performance of remedial students with that of other freshmen students in their first precalculus course.

A graph of the pretest and retest score distribution for remedial students on

the mathematics placement test according to different ability levels gave evidence of the effectiveness of remediation at various ability levels; and a test and textbook analysis used in conjunction with a questionnaire identified areas of fundamental mathematics in which there was greatest improvement as well as areas of greatest weakness.

Results. It was shown that significant improvement was made by remedial students, in general, after a period of remediation. The quality of improvement was directly correlated with the ability level of the student and the areas of most improvement were, for the most part, areas of emphasis of the textbook and areas which do not involve much problem solving, nor abstract thinking. There was a decided lack of knowledge of fundamental concepts. *The Cooperative Mathematics Pretest for College Students* which was the mathematics placement test did not measure, nor did the course content include units on appropriate modern concepts for the high school curriculum. It was also shown that the subsequent performance of remedial students in pre-calculus classes was not significantly different from that of nonremedial students, and the performance of both groups was low.

Conclusions and present practice. It was concluded that entering freshmen are generally weak in their mathematical preparation and only 40 per cent of remedial students could be approved to enter precalculus courses after the first semester of remedial work. It was highly possible that much of the learning was incidental and might have been accomplished by virtue of self-directed effort and application. Directed teaching stands to make a valuable contribution by treating diagnosed weaknesses and by giving special attention to the 44 per cent of remedial students with ability level below 290 who are still deficient after a semester of remedial work. *The School and College Ability Test* (SCAT) was used as the instrument to measure the ability rating.

Fewer than one-fourth of remedial students with ability levels at 300 and higher were automatically positioned into passing the retest examination with little expenditure of effort. It was this group which showed the most improvement, and which could profit from placement in an accelerated or enrichment section.

The low performance in pre-calculus courses of students with high ability scores and seemingly adequate mathematical backgrounds might be attributed to one or more of several causes. These causes might include heavy work-study program, poor application, lack of diligence, and poor orientation to college level work.

The recommendations were based largely on the conclusions as gathered from the statistical analysis of scores and from the voluntary comments on the questionnaire:

1. A mathematics placement test serves as a valuable instrument of discrimination. A test which closely reflects the changing high school curriculum and the modern trend in mathematics and which has since been adopted at Howard University is the College Entrance Examination Board Achievement Test in mathematics.

2. The materials of the course should reflect the new look in mathematics and should provide some enrichment as well as understandings and appreciations. The materials cannot be limited to the use of a single text and supplementary materials must be used regularly. Appropriate programmed material, and review work in specific areas have been introduced to supplement a new choice of text.

3. It is to be expected that the admission of students of such a wide range of ability will precipitate large enrollments in remedial classes each semester. The results of experiments in large-group instruction, supplemented by individualized help through the medium of tutorial assistance and the facilities of a mathematics laboratory (with library) has led to the adoption of large-group classes in remedial classes. The groups have ranged in size from 150–250 students and approximately 70 per cent of the enrolled students have successfully scored above the national norm on The Cooperative Achievement Tests in Arithmetic, Structure of the Number System, Algebra II, and Plane Geometry in two consecutive semesters of testing. Follow-up work is being done, and the project continues to be handled as an experiment.

4. Teachers should be assigned to the program on the basis of their training and interest and there should be adequate staff and equipment to insure frequent evaluation of progress and technique. The use of qualified graduate assistants and undergraduate readers has provided a good working staff together with the assigned teacher. This permits individual help, regular written assignments, and more drill work. The use of the overhead projector and voice amplification is almost indispensable to the management of the large group in mathematics. The contemplated film program will provide some of the reinforcement and enrichment which is needed.

5. Qualifying examinations are now being offered three times during the semester and provide incentive and motivation to work and to remedy basic defects. A high proficiency standard, if required of all remedial students, would insure that their subsequent performance in succeeding mathematics courses would be improved.

It is hoped that this brief glance at the activities of one institution in the area of mathematics remediation will help to direct in some fruitful way the continuing programs in remedial mathematics at the college level which have come to be part of college instructional programs everywhere.

AN OKLAHOMAN REPORTS

G. K. Goff, Oklahoma State University

In October 1962, CUPM sponsored the Oklahoma Conference on the Training of Teachers of Elementary School Mathematics. This was the first of ten such conferences throughout the nation. At our conference in Oklahoma City we adopted a timetable [1] for meeting CUPM recommendations for training teachers at Level I. This timetable stipulated September 1967, as the target date for fulfilling the recommendations.

At the fall meeting of the Oklahoma Council of Teachers of Mathematics a report on the progress made toward accomplishing the goals set forth at the 1962 conference was requested by the Grade 12 through University Section. A questionnaire was prepared and sent to the eighteen colleges and universities in Oklahoma that recommend certification of elementary teachers. Fourteen of these institutions responded, and the results are summarized in the table which follows.

SEMESTER HOUR REQUIREMENTS AND ELECTIVES IN MATHEMATICS FOR OKLAHOMA COLLEGES AND UNIVERSITIES

Elementary	*Universities*					*Colleges*								
Required Mathematics	3	3	6	6	12	6	6	6	10	6	6	6	6	6
Required Math Education	3	2	2			3				3		2	4	
Electives**		9	6			9	6	6		12			6	
Secondary														
Total Hours of Math Required	28	28	31	34	30	35	30	37	39	36	32	34	33	28
Min. No. of hrs Above Calculus	12	16*	21	18	9	18	12	18	23	18	14	15	20	18
No. of hrs. Prescribed	9	13*	15	15	9	3	6	15	17	18	6	6	20	18
Math. Educ. Required						3		2			2	2		
No. of Elective 3 hr. Courses***	1 of 2	1 of 3	2	1 of 2	0	5 of 9	2	1	2 of 4	0	2 of 5	3 of 6	0	0

Each column represents a university or college in Oklahoma.
All numerals represent semester hour credits except in the bottom row.
* Includes Philosophy 103.
** Mathematics courses specifically for elementary teachers.
*** 1 of 2 indicates one course is to be selected from 2 prescribed electives.

The questionnaire also asked for the requirements for certification of secondary teachers, and these results are also included in the table.

Reference

1. CUPM Report, Ten Conferences on the Training of Teachers of Elementary School Mathematics, Number 7, February 1963.

ON THE DITCHLEY CONFERENCE AND CURRICULUM REFORM

PETER HILTON, Cornell University, Co-Chairman, Cambridge Conference on School Mathematics

I was pleased to see the report by my friend and colleague, Earle Lomon, on the Ditchley Conference in the December issue of the Monthly (E. L. Lomon, The Ditchley Conference on School Mathematics of Two Countries, this MONTHLY, 74 (1967) 1251) I am very much in agreement with Lomon's assessment of the priorities for future work on curricular reform and development; but I do have one significant disagreement with hm and feel it should be ventilated since it refers to a point which Lomon regards as essential.

I refer to the wide divergence of attitudes among those active in curriculum development in this country; Lomon attests to a similar divergence in Britain and asserts that the meeting at Ditchley clearly revealed this divergence. I

speak only for myself in saying that I detected no such divergence, but I might claim that my experience at first hand of the situation in both countries reinforces my belief that no such comparable divergence exists in Britain.

At the secondary level the movement for curricular reform in Britain is dominated by the School Mathematics Project (SMP). Of the 12 participants in the Ditchley Conference on the British side 5 are actively associated with SMP and 5 others have no special affiliations. Moreover there are strong similarities of educational background among those first five which are shared by most of the independents. Thus it is only to be expected that a certain homogeneity of outlook would be revealed by the conference and it was one of my most vivid impressions from the conference that this homogeneity was even more conspicuous than I had expected. On the other hand, the American contingent was, as Professor Lomon states, very varied in its special interests and approaches and it was a particularly revealing—and perceptive—observational aside of Professor Thwaites that the Americans were clearly benefiting from their meeting in Britain to discover each others' attitudes and to find out the current state of the various curricular projects represented by the U. S. participants.

I do not take up your time just to record impressions, but to draw attention to certain important differences of procedure in the two countries which I believe to be important and which bring with them advantages and disadvantages. The homogeneity of SMP, to be brief, confers efficiency on its operations at some cost in diversity, and daring of experiment. Moreover, the (deliberate) dominance of highly qualified and highly talented 'public school' teachers in SMP ensures the teachability of the material produced by SMP and brings to SMP a highly desirable continuity of operation; again, however, the price, compared with similar reform activities in this country, is a tendency to accept current educational, and examining, structures —and to maintain a fairly constant attitude towards educational values. Here, in this country, it is, I believe, a great strength of the reform movement in mathematical education that it is supported, very actively, by a number of university mathematicians (and even by university scientists, as evidenced by Professor Lomon's invaluable contributions). On the other hand, although such mathematicians can bring fresh ideas to bear on educational problems, they are, almost all, amateurs in the field of education. I use this description advisedly; for they are amateurs in the sense that the study of problems of mathematical education is not their profession, but they are also amateurs in the practical sense that they are able to give only a small proportion of their time to such problems. I believe we in the C.C.S.M. would benefit enormously by being able to involve high school (and elementary school) teachers of mathematics of the calibre of Messrs. Cundy, Quadling and Tammadge in our operations; but I also believe that it would be splendid if Professors Atiyah, Lighthill and Coulson—and others like them—were more closely involved in mathematical education in Britain than is indicated by their presence 'on Sunday evening' at the Ditchley Conference.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, Maine 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before March 31, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed, stamped postcards.

E 2125. *Proposed by F. A. Butter, Jr., California State College at Long Beach*

For every fixed real number x, the sequence $\{u_n(x)\}$, where

$$u_n(x) = (1 + n^{-1})^{n+x} \qquad (n = 1, 2, \cdots),$$

converges to e as $n \to \infty$. Determine numbers a, b, $a<b$, such that $\{u_n(x)\}$ is monotone increasing for $x \in (-\infty, a)$ and monotone decreasing for $x \in [b, \infty)$. For $x \in (a, b)$ discuss the variation of $u_n(x)$ with n, and estimate bounds. (The cases $x=0$, 1 have been widely treated.)

E 2126. *Proposed by Erwin Just, Bronx Community College, New York*

If n is an integer greater than 2, and k is a nonzero integer, prove that $\sum_{r=0}^{n} (rk+1)x^{n-r}$ has no integral zeros.

E 2127. *Proposed by John Wilker, University of Toronto*

Let A, B, C and D be any four points in Euclidean 3-space. If every sphere or plane through A and B meets every sphere or plane through C and D, what can be said about the points?

E 2128. *Proposed by Jovan Vukmirović, Belgrade, Yugoslavia*

In the triangle ABC let M be any point on side BC. Prove that

$$(AM - AC)\cdot BC \leqq (AB - AC)\cdot MC.$$

E 2129. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha, and H. E. Chrestenson, University of Guana and Reed College*

Let J_k denote the integers modulo k. We seek three nonzero elements x, y, z of J_k such that $x^n + y^n = z^n$ for all positive integers n. Show that if k is not a power

of a prime, then J_k contains such a triple of elements. For what other values of k do these triples exist?

E 2130. *Proposed by J. M. Quoniam, Saint-Etienne, France*

Show how to construct a regular octahedron whose vertices lie on six parallel planes. Under what conditions is the problem possible?

E 2131. *Proposed by R. A. Struble, North Carolina State University at Raleigh*

If $f: R^n \to R^n$ is bijective and maps connected sets onto connected sets and nonconnected sets onto nonconnected sets, then f is a homeomorphism.

E 2132. *Proposed by Gregory Wulczyn, Bucknell University*

Let r, s be positive integers, $(r, s)=1$, rs not a square. Then there are infinitely many pairs of triangular numbers, $T_a=\frac{1}{2}a(a+1)$, $T_b=\frac{1}{2}b(b+1)$ such that $T_a/T_b=s/r$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Number of Elements in a Set of m-tuples

E 1985 [1967, 589]. *Proposed by E. O. Buchman, University of California, Los Angeles*

The functions $t(m)$ are defined recursively by $t(0)=1$, $t(m+1)=\sum_{i=0}^{m}\binom{m}{i}t(i)$, $i=1, 2, \cdots$. Show that $t(m)$ is the number of elements in the set $T_m=\{(a_1, \cdots, a_m): a_i \text{ is a positive integer, } a_1=1,\ a_i \leqq 1+\max_{k<i}a_k\}$. See also E 1785 [1966, 672].

I. *Solution by Douglas Lind, University of Virginia.* We obtain a slightly more detailed description of T_m. Let $T_{m,k}=\{(a_1, \cdots, a_m)\in T_m: \max_{i\leq m}a_i=k\}$, and denote the number of elements in a set S by $|S|$. Now each element in $T_{m+1,k}$ comes either from one in $T_{m,k}$ by appending a "1," "2," $\cdots$, or "k," or from one in $T_{m,k-1}$ by appending a "k". The elements produced in this manner are distinct, so that

$$|T_{m+1,k}| = k|T_{m,k}| + |T_{m,k-1}|.$$

Since $|T_{1,1}|=1$, $|T_{1,k}|=0$ for $k>1$, the numbers $|T_{m,k}|$ have the same initial values and obey the same recurrence relations as the Stirling numbers of the second kind $S(m, k)$ given by

$$S(m, k) = \frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}r^m.$$

Hence $|T_{m,k}|=S(m, k)$, and $|T_m|=\sum_{k=1}^{m}|T_{m,k}|=\sum_{k=1}^{m}S(m, k)=B_m$, where B_m is the mth Bell number. The Bell numbers are generated by

$$\exp(e^x - 1) = \sum_{m=0}^{\infty} B_m x^m/m!, \tag{1}$$

and appear as the solution to the difference equation in E 461. [1941, 701]. It is well known and easy to show from (1) that

$$B_{m+1} = \sum_{i=0}^{m} \binom{m}{i} B_i. \tag{2}$$

Since $B_0 = t(0) = 1$, and both obey (2), we see

$$|T_m| = B_m = t(m),$$

the desired conclusion.

II. *Solution by D. P. Roselle, University of Maryland.* The numbers $t(m)$ are the Bell (or exponential) numbers and satisfy $t(m) = \sum_{k=1}^{m} S(m, k)$, with $S(m, k)$ the Stirling number of the second kind or, for our purposes, the number of partitions of the set $X = \{x_1, \cdots, x_m\}$ into exactly k nonempty subsets.

Now put $T_m(k) = \{(a_1, \cdots, a_m) \in T_m : \max_{1 \leq i \leq m} a_i = k\}$ and for each $\alpha \in T_m(k)$, define the partition $P = [B_1, \cdots, B_k]$ of X by $x_j \in B_i$ if and only if $a_j = i$. Clearly this is a 1-1 mapping of $T_m(k)$ onto the set of partitions of X into exactly k nonempty subsets. Consequently $|T_m(k)| = S(m, k)$ and thus

$$t(m) = \sum S(m, k) = \sum |T_m(k)| = |T(m)|.$$

Also solved by L. J. Burton, C. A. Church, Jr., J. A. Ewell, N. J. Fine, M. G. Greening (Australia), C. B. A. Peck, Stanton Philipp, and the proposer.

N Riflemen

E 2003 [1967, 720]. *Proposed by Robert Abilock, Brandeis University*

N riflemen are distributed at random points on a plane. At a signal, each one shoots at and kills his nearest neighbor. What is the expected number of riflemen who are left alive?

Comment by Michael Goldberg, Washington, D. C. This problem is not properly posed when the region is not limited in size or shape. The answer for one finite region of given shape is not the same for another region of the same area but of a different shape. When the regions are increased while retaining the same shape, the two different answers remain unchanged. Hence, if both regions are increased without limit, the two answers are not the same.

For $n = 3$, the expected number of survivors is one. But for larger n the problem seems too difficult to be considered as an elementary problem.

The one-dimensional version of the problem with the riflemen distributed at random, uniformly along a linear segment, may be a more appropriate problem. For this problem it is not difficult to determine the expected number of riflemen left for small values of n.

Always an Integer

E 2004 [1967, 720]. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Show that the following quotient is integral for all integers $n > 0$ and $m > 1$,

$$\left\{\prod_{i=0}^{n-1}(m^n - m^i)\right\} \Big/ n!.$$

I. *Solution by N. J. Fine, Pennsylvania State University.* The numerator may be written as $m^{n(n-1)/2}(m-1)(m^2-1)\cdots(m^n-1)$. Let p be any prime $\leqq n$. Then p divides $n!$ exactly α times, where $\alpha = \sum_{j\geqq 1}[n/p^j]$. If $p|m$, then p divides the numerator $n(n-1)/2$ times, and

$$\alpha < \sum_{j\geqq 1}(n/p^j) = n/(p-1) \leqq n \leqq n(n-1)/2,$$

provided that $n>2$, as we may assume, since the cases $n=1$ and $n=2$ are trivial. If p does not divide m, then $m^{s(p-1)}-1$ is divisible by p for all positive integers s. The number of multiples of $(p-1)$ up to n is $\beta=[n/(p-1)]$, so the numerator is divisible by p^β. Using the trivial inequality $[x+y]\geqq[x]+[y]$, we have

$$\beta = \left[\frac{n}{p-1}\right] = \left[\sum_{j\geqq 1}\frac{n}{p^j}\right] \geqq \sum_{j\geqq 1}\left[\frac{n}{p^j}\right] = \alpha.$$

This completes the proof.

II. *Comment by Andrzej Mąkowski, Warsaw, Poland.* If m is a power of a prime, the given expression is the number of linearly independent sets of n vectors in n-dimensional linear space over $GF(m)$. Indeed, one vector may be chosen in m^n-1 ways, the second in m^n-m ways, etc. The denominator $n!$ is introduced to cancel the sequence.

Also solved by I. K. Abruob, Charles Bahne, Jr., L. J. Burton, D. Ž. Djorković, J. A. Ewell, M. G. Greening (Australia), Donald Jeffords, Douglas Lind, Helen M. Marston, Dale Peterson, Simeon Reich (Israel), and the proposer.

G. Wulczyn notes that the problem also appears as Aufgabe 539 (proposed by L. Carlitz) in Elemente der Mathematik, Nov. 1966, with solution by A. Bager in the same journal, Nov. 1967, p. 107.

Carlitz notes that the result appears as a special case in his paper, *Note on a paper of Laksov*, Math. Scand., 19 (1966), 38–40.

A Sum of Multiples of Given Primes

E 2005 [1967, 720]. *Proposed by W. A. McWorter, Ohio State University*

Let $p_1, \cdots, p_t$ be distinct primes and n a positive integer, and let $k=p_1p_2\cdots p_t$. Show that there exist nonnegative integers $a_1, \cdots, a_t$ such that

$$\sum_{i=1}^{t} a_i p_i = \binom{kn-1}{k-1} - 1.$$

Solution by Stanley Rabinowitz, Far Rockaway, N. Y. Claim: If M is any integer greater than or equal to k, then there exist nonnegative integers $a_1, \cdots, a_t$ such that

$$\sum_{i=1}^{t} a_i p_i = M, \qquad (t > 1).$$

Proof: The case $t=2$ is proved in problem E 1967 [1968, 675]. If it is true for t primes, then $k=p_1p_2 \cdots p_t$ is a linear combination of the p's with nonnegative coefficients. But k and p_{t+1} are relatively prime, so any number $\geqq kp_{t+1}$ is also such a linear combination of the $(t+1)$ p's. Hence by induction our claim is true for all $t \geqq 2$.

If $n>1$, $\binom{kn-1}{k-1}-1 \geqq kn-1 \geqq k$, so the theorem is true for $t>1$ by the above. If $t=1$, we have modulo p,

$$\binom{pn-1}{p-1} \equiv (pn-1)(pn-2)\cdots(pn-p+1)/(p-1)!$$
$$\equiv (1)(2)\cdots(p-1)/(p-1)! \equiv 1.$$

Hence

$$\binom{pn-1}{p-1} - 1 = ap.$$

Also solved by D. Ž. Djoković, J. A. Ewell, M. G. Greening (Australia), Donald Jeffords, Helen M. Marston, and the proposer.

A Locus Problem

E 2006 [1967, 861]. *Proposed by A. S. Howard, Eastern Kentucky State College*

Given two fixed points A, B in the Euclidean plane, let C be free to move on a circle in the plane with A as center. Find the locus of P the point of intersection of BC with the internal bisector of angle A of the triangle ABC.

Solution by C. F. Pinzka, University of Cincinnati. Since $PC/BP=AC/AB$ we have

$$\frac{BP}{BC} = \frac{BP}{BP+PC} = \frac{AB}{AB+AC}, \quad \text{a constant,}$$

it is clear that the locus is homothetic to the circle A about center B. Thus the locus is a circle symmetric about AB, passing through A, and with radius $(AB)(AC)/(AB+AC)$.

Also solved by sixty-eight other readers.

Cross-cancellative Semigroup

E 2007 [1967, 861]. *Proposed by W. A. Donnell, North Texas State University*

Show that a cross-cancellative (i.e., $ab=bc$ implies $a=c$) semigroup S is a cancellative semigroup.

I. *Solution by Peter Kornya, University of British Columbia.* From $(ab)a = a(ba)$ cross-cancellation gives $ab=ba$. Hence S is commutative and the result follows.

II. *Solution by D. E. Penney, University of Georgia.* More can be proved from less. Assume that S is only a groupoid satisfying the weak associative law $(xy)x = x(yx)$ as well as the cross-cancellation law

$$ab = bc \qquad \text{implies } a = c.$$

Then $xy = a \Rightarrow y(xy) = ya \Rightarrow (yx)y = ya \Rightarrow yx = a \Rightarrow xy = yx$, so that S is commutative. Consequently $ab = ac \Rightarrow ba = ac \Rightarrow b = c$, so that S is cancellative.

If $S = [0, 1]$ and $xy = \frac{1}{2}(x+y)$, we obtain a groupoid satisfying the weak associative and cross-cancellative laws (and in addition, $x^2 = x$ for all x), but which is not a semigroup. Thus the weakened hypothesis is strictly weaker.

Also solved by one hundred and seventy-four other readers.

Convergent Sequences

E 2008 [1967, 861]. *Proposed by Murali Rao, University of British Columbia*

Let $\{a_n\}$ be a sequence of positive numbers such that $\sum a_n$ converges. Find a necessary and sufficient condition for the existence of a sequence of positive numbers $\{b_n\}$ such that $\sum a_n/b_n$ and $\sum b_n$ both converge.

Solution by James R. Kuttler and Nathan Rubinstein, The Johns Hopkins University, Applied Physics Laboratory. A necessary and sufficient condition that given a sequence $\{a_n\}$ of positive numbers, there exists a sequence $\{b_n\}$ of positive numbers such that the sums $\sum a_n/b_n$ and $\sum b_n$ converge is that $\sum a_n^{1/2}$ converges. Necessity follows from Schwarz's inequality:

$$\sum a_n^{1/2} \leqq \left(\sum a_n/b_n\right)^{1/2}\left(\sum b_n\right)^{1/2}.$$

Sufficiency is clear: let $b_n = a_n^{1/2}$.

Also solved by Einar Andresen (Norway), D. F. Dawson, D. Ž. Djoković, N. J. Fine, B. G. Klein, E. S. Langford, Neil Felsinger, Dan Marcus, R. D. Meredith, Henrik Meyer (Denmark), M. E. Muldoon, D. N. Page, C. B. A. Peck, O. E. Stanaitis, L. E. Ward, and the proposer.

Editorial Note. A more interesting problem would be to find useful necessary and sufficient conditions on the sequence $\{b_n\}$ to have the desired properties, rather than just the existence of a suitable $\{b_n\}$ (assuming, of course, that $\sum a_n^{1/2} < \infty$.) Some separate sufficient conditions are:

$$\overline{\lim}\; b_n/\sqrt{a_n} < \infty, \qquad \underline{\lim}\; b_n/\sqrt{a_n} > 0.$$

A Conditionally Convergent Series

E 2009 [1967, 861]. *Proposed by R. A. Avelsgaard and R. E. Stockton, Bemidji State College, Minnesota*

It is well known, as an example of Dirichlet's test, that the series $\sum_{n=2}^{\infty} \sin n/\log n$ converges. Does it converge absolutely?

I. *Solution by N. J. Fine, Pennsylvania State University.* Since $\sin x$ and $\sin(x+1)$ are never zero simultaneously, the function $|\sin x| + |\sin(x+1)|$ is always positive. Because it is continuous and periodic, there is a positive b such that $|\sin x| + |\sin(x+1)| \geqq b\,(x \in R)$. Hence

$$\sum_{n=2}^{\infty} \frac{|\sin n|}{\log n} = \sum_{k=1}^{\infty}\left(\frac{|\sin 2k|}{\log 2k} + \frac{|\sin(2k+1)|}{\log(2k+1)}\right)$$
$$\geqq \sum_{k=1}^{\infty} \frac{|\sin 2k| + |\sin(2k+1)|}{\log(2k+1)} \geqq b \sum_{k=1}^{\infty} \frac{1}{\log(2k+1)}.$$

Thus the series does not converge absolutely.

II. *Solution by M. E. Muldoon, York University, Toronto.* No. Suppose that the given series converges absolutely. Then, a theorem of Fatou (see, e.g., A. Zygmund, *Trigonometric Series* I (1959), Theorem 1.6, p. 232) would imply the convergence of $\sum_{n=2}^{\infty} (1/\log n)$, so we have a contradiction.

Also solved by Einar Andresen (Norway), D. A. Brannan, Peter Bundschuh (Germany), M. S. Demos, R. C. Entringer, Bengt Fornberg (Sweden), David Hoitsma, Peter Kornya, Joseph Lehner, Norman Miller, D. N. Page, C. B. A. Peck, J. R. Purdy, R. E. Shafer, Sidney Spital, O. E. Stanaitis, Kao Hwa Sze, Dimitrios Vathis (Greece), L. E. Ward, Steven Weintraub, and the proposers.

Editorial Note. An equivalent problem of showing that $\sum \sin n\theta/n \log n$ diverges absolutely ($\theta \neq k\pi$) occurs in Bary, *Treatise on Trigonometric Series*, Vol. 1, p. 90, footnote. See also Th. 1, p. 87.

Always an Even Integer

E 2010 [1967, 861]. *Proposed by D. R. Rao, Osmania University, Hyderabad, India*

Show that $[(m-1)!/m]$ is even for all $m>4$. Brackets indicate the greatest integer function.

Solution by C. V. Heuer, Concordia College. If m is an odd prime, by Wilson's theorem $N=\{(m-1)!+1\}/m$ is an integer. N must be odd since its numerator is odd. Therefore $[(m-1)!/m]=N-1$ is even.

If $m(>4)$ is not prime it is known that $m|(m-1)!$. Say $md=(m-1)!$. Here d must be even since for $k=[\frac{1}{2}(m+1)]$, 2^k is a divisor of $(m-1)!$ while 2^k does not divide m. Hence $[(m-1)!/m]=[d]=d$ is even.

Also solved by fifty-nine other readers.

Editorial Note. Through an oversight this problem was published also in the Canadian Math. Bull. as Proposal 113. A solution by A. Makowksi (Poland) appears on p. 122, vol. 10 (1967) no. 1.

A Functional Equation

E 2011 [1967, 861]. *Proposed by R. C. Entringer, University of New Mexico*

Solve the functional equation $f(x)=f(x/(1-x))$, $x\neq 1$, subject to the condition that f is continuous at $x=0$.

I. *Solution by Anders Bager, Hjørring, Denmark.* As $-1/n\to 0$ as $n\to\infty$, the functional equation and continuity at 0 give:

$$f(-1) = f(-\tfrac{1}{2}) = f(-\tfrac{1}{3}) = \cdots = f(0).$$

Let $t \notin \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \cdots\}$. Then backward application of the functional equation gives

$$f(t) = f\left(\frac{t}{t+1}\right) = f\left(\frac{t}{2t+1}\right) = f\left(\frac{t}{3t+1}\right) = \cdots.$$

As $\lim_{n\to\infty} t/(nt+1) = 0$, we get $f(t) = f(0)$. The only solutions then are of the form $f(x) = k$, where k is any constant.

II. *Solution by Emil Grosswald, University of New Hampshire.* Define $g(x) = f(1/x)$; then $g(x) = g(x-1)$ so that $g(x)$ is periodic with period one. Conversely, if $g(x)$ is periodic of period one and $f(x) = g(1/x)$, then $f(x) = g(1/x) = g(1/x-1) = g((1-x)/x) = g(1/[x/(1-x)]) = f(x/(1-x))$. Consequently, without the requirement of continuity at the origin, $f(x) = g(1/x)$, where $g(x)$ is an arbitrary periodic function, of period one. Imposing now the condition that f be continuous at the origin, g has to be continuous as $x \to \infty$, i.e., there exists a constant A such that for every $\epsilon > 0$, $|g(x) - A| < \epsilon$, provided that $x \geqq x_0(\epsilon)$; in particular, for all x_2, x_3 satisfying $x_0 < x_1 < x_2 < x_3 < x_1 + 1$, $|g(x_2) - g(x_3)| < 2\epsilon$. It follows that $g(x)$ approaches a constant along a full interval of length one. By periodicity this means that $g(x)$ itself reduces to a constant; hence so does f.

Also solved by forty-nine other readers.

Diophantine Equation with Homogeneous Members

E 2012 [1967, 861]. *Proposed by R. S. Kulkarni, Harvard University*

Let $f(x_1, \cdots, x_r)$, $g(y_1, \cdots, y_s)$ be two homogeneous polynomials with integral coefficients, and suppose the degree of f is relatively prime to the degree of g. Show that there exist infinitely many integral solutions of the equation $f = g$.

Solution by Dan Marcus, Rutgers—The State University. More generally, infinitely many solutions exist with the x_i in the preassigned ratio $u_1 : u_2 : \cdots : u_r$ (not all zero), and the y_j in the ratio $v_1 : v_2 : \cdots : v_s$ (not all zero).

Put $U = f(u_1, u_2, \cdots, u_r)$, $V = g(v_1, v_2, \cdots, v_s)$; let m and n be the degrees of f and g, respectively. Then positive integers α and β exist such that $\alpha m - \beta n = 1$. $f = g$ is satisfied, for each integer t, by

$$x_i = xt^n u_i, \qquad y_j = yt^m v_j,$$

where x and y are integers, not both zero, satisfying

$$Ux^m = Vy^n.$$

If either U or V is zero, it is clear we can find a nontrivial solution. Otherwise, take

$$x = (U^{\beta n - 1} V)^\alpha, \qquad y = (U^{\beta n} V)^\beta.$$

Also solved by Einar Andresen (Norway), L. Carlitz, N. J. Fine, Donald Jeffords, D. C. B. Marsh, R. D. Meredith, J. R. Purdy, Hugo Sun, and the proposer.

The result has been obtained earlier by A. A. Aucoin. See Bull. Amer. Math. Soc., (1939) 330; also (1942) 933.

An Identity Involving All Selections from n Integers

E 2013 [1967, 861]. *Proposed by Walter Noll, Carnegie-Mellon University*

m, n being positive integers, prove the identity

$$S_{n,m} = \sum_{r=1}^{n} (-1)^{n-r} \sum (x_{i_1} + \cdots + x_{i_r})^m = \begin{cases} n!x_1x_2 \cdots x_n & \text{if } m = n \\ 0 & \text{if } m < n, \end{cases}$$

where the second summation is taken over all possible selections $i_1, \cdots, i_r$ of r of the first n natural numbers.

I. *Solution by J. W. Moon, University of Alberta.* Consider a set of n different boxes such that the ith box is subdivided into x_i different compartments, for $i=1, 2, \cdots, n$. Let $S_{n,m}$ denote the number of ways of placing m different marbles in the compartments of these boxes so that every box contains at least one marble. It is obvious that $S_{m,n}=n!x_1 \cdots x_n$ if $m=n$ and 0 if $m<n$. The expression for $S_{n,m}$ as a double sum follows immediately from the method of inclusion and exclusion.

Notice that when $x_1= \cdots =x_n=1$, the identity reduces to the well-known formula

$$\Delta^n 0^m = \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} r^m = \begin{cases} n! & \text{if } m = n \\ 0 & \text{if } m < n. \end{cases}$$

II. *Solution by L. Carlitz, Duke University.* It follows from

$$S_{n,m} = \sum_{r=0}^{n} (-1)^{n-r} \sum (x_{i_1} + \cdots + x_{i_r})^m$$

that

$$\sum_{m=0}^{\infty} S_{n,m} \frac{z^m}{m!} = \sum_{r=0}^{n} (-1)^{n-r} \sum \exp[(x_{i_1} + \cdots + x_{i_r})z]$$
$$= \prod_{j=1}^{n} (e^{x_j z} - 1).$$

Hence

$$S_{n,m} = \begin{cases} 0 & (0 \leqq m < n) \\ n!x_1x_2 \cdots x_n & (m = n). \end{cases}$$

Moreover

$$S_{n,n+1} = \tfrac{1}{2}(n+1)!x_1x_2 \cdots x_n(x_1 + x_2 + \cdots + x_n),$$

$$S_{n,n+2} = \tfrac{1}{6}(n+2)!x_1x_2 \cdots x_n(x_1^2 + x_2^2 + \cdots + x_n^2)$$
$$+ \tfrac{1}{4}(n+2)!x_1x_2 \cdots x_n(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n),$$

and so on.

Note that for $x_1 = \cdots = x_n = 1$, $S_{n,m}/n!$ reduces to the Stirling number of the second kind $S(m, n)$.

Also solved by Einar Andresen (Norway), M. T. L. Bizley (England), D. Ž. Djoković, J. A. Ewell, N. J. Fine, M. G. Greening (Australia), Donald Jeffords, E. S. Langford, R. D. Meredith, C. B. A. Peck, J. R. Purdy, Stephen Spindler, Sidney Spital, D. P. Sumner, Oswald Wyler, and the proposer.

Fine also notes that $S_{n,m}$ is the sum of all the terms in the multinomial expansion of $(x_1+x_2+\cdots+x_n)^m$ which involve all the variables.

A. Makowski observes that the problem was proposed by A. Pelezynski as problem 68 in Wiadomosci Matematyczme, 7 (1964) p. 245. No solution has been published.

n Distinct Prime Divisors

E 2014 [1967, 861]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York City*

Show that $2^{2^n}-1$ has at least n distinct prime divisors.

Solution by Donna May, Student, Wake Forest University. Noting that the result is obvious for $n=1$, we assume that $2^{2^{k-1}}-1$ has at least $k-1$ distinct prime divisors. Now

$$2^{2^k} - 1 = (2^{2^{k-1}} - 1)(2^{2^{k-1}} + 1).$$

No prime can divide both the odd numbers on the right since they differ by 2. Hence $2^{2^k}-1$ is divisible by at least $(k-1)+1$ different prime factors and a proof by induction is complete.

Also solved by one hundred and twenty-nine other readers.

Properties of a Certain Matrix

E 2015 [1967, 1005]. *Proposed by W. A. McWorter, Ohio State University*

Let $\ll$ be an anti-reflexive partial order on the set $\{1, \cdots, n\}$. Let A be an $n\times n$ matrix where the (i, j)-entry is 1 or 0 according as $i \ll j$ or not. (i) Prove that there exists a permutation matrix P such that PAP^{-1} is upper triangular. (ii) Show that $A=0$ if A is idempotent.

Solution by George Alberts, Laurel, Maryland. (i) Let π be a permutation of $\{1, \cdots, n\}$ which extends the partial order. In other words, $i \ll j$ implies $\pi(i) < \pi(j)$ in the natural order of the integers. Let P be the permutation matrix corresponding to π. Then PAP^{-1} has the same entries as A, but the rows and columns of A appear permuted according to π. Since π extends $\ll$, PAP^{-1} is upper triangular, by the assumed anti-symmetry of $\ll$.

(ii) Suppose A is idempotent. Then so is PAP^{-1}, which is upper triangular, with all zeros on the main diagonal by the anti-reflexivity of $\ll$. Such a matrix is

nilpotent. PAP^{-1}, being both nilpotent and idempotent, is the zero matrix. It follows that $A=0$.

Also solved by C. Q. Artino, Anders Bager (Denmark), M. G. Greening (Australia), Peter Kornya, and the proposer.

Integral Solutions of a System

E 2016 [1967, 1005]. *Proposed by J. M. Katri, Baroda, India*

Let Δ_r be the triangular number $\frac{1}{2}r(r+1)$. Solve in integers the system:

$$\Delta_a + \Delta_b = \Delta_c + \Delta_d, \qquad x^2 + y^2 = z^2 + w^2,$$
$$\Delta_{a+x} + \Delta_{b+y} = \Delta_{c+z} + \Delta_{d+w}.$$

Solution by D. C. B Marsh, Colorado School of Mines. By setting $A=2a+1$, $B=2b+1$, $C=2c+1$, $D=2d+1$ and using the first pair of equations to eliminate all squared terms from the third, one obtains the equivalent system:

$$A^2 + B^2 = C^2 + D^2, \qquad x^2 + y^2 = z^2 + w^2, \qquad Ax + By = Cz + Dw.$$

The second equation has known parametric solutions given by

$$x = \tfrac{1}{2}(rs + pq), \qquad y = \tfrac{1}{2}(qs - pr), \qquad z = \tfrac{1}{2}(rs - pq), \qquad w = \tfrac{1}{2}(qs + pr)$$

for integers p, q, r, s satisfying $rs+pq\equiv qs+pr\equiv 0 \pmod 2$.

The first equation is satisfied by

$A=\frac{1}{2}(ij+gh)$, $B=\frac{1}{2}(hj-gi)$, $C=\frac{1}{2}(ij-gh)$, $D=\frac{1}{2}(hj+gi)$ for integers g, h, i, j

satisfying (so that A, B, C, D are odd)

$$ij \pm gh \equiv hj \pm gi \equiv 2 \pmod 4),$$

Substitution of these expressions into the third equation reduces it to

$$(gs - jp)(hr - iq) = 0.$$

We may generate integral solutions of the original system by choosing for g, j any two integers not both multiples of 4, then choosing any integers s, p such that $s/p=j/g$. There then exist solutions h, i, q, r to the congruences above. Choose any such solutions and thence calculate a, b, c, d, x, y, z, w.

Also solved by J. W. Baldwin, Michael Goldberg, L. Kuipers, Norman Miller, D. N. Page, J. R. Purdy, Simeon Reich (Israel), Gregory Wulczyn, and C. C. Yalavigi (India).

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before May 31, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5629. *Proposed by M. W. Hudgins, University of Iowa*

Does there exist in the Euclidean plane a family of disjoint open arcs con-

taining, for each point P distinct from the origin O, one joining P to O?

5630. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove the inequality $1+r^{\lambda}\leqq(1+r^2)^{\lambda/2}(0\leqq r\leqq 1;\ \lambda\geqq 2)$, and determine positive numbers a, b, c such that the inequality $1+r^a\leqq(1+r^b)^c$ $(0\leqq r\leqq 1)$ holds true.

5631. *Proposed by A. D. Ziebur, State University of New York at Binghamton*

Suppose f is continuous in a neighborhood of the point $(0, c)$ in R^2, and is such that $f(t, x)$ is nondecreasing in x. Define the *Picard transformation* P by means of the equation

$$P\phi(t) = c + \int_0^t f(s, \phi(s))ds.$$

Show that if $P^2\phi(t)=\phi(t)$ for each t in some interval $[0, a)$, then $P\phi(t)=\phi(t)$ for $t\in[0, a)$.

5632. *Proposed by D. K. Cohoon, W. Lafayette, Indiana*

Show that the line E^1 is the only one of the Euclidean spaces E^n $(n\geqq 1)$ which does not admit a fixed-point-free homeomorphism f with the property that the iterate f^m has fixed points for some positive integer m.

5633. *Proposed by George Shapiro, Harvard University*

Let $G(x, y)$ designate the free group on two generators x and y, and let $G(x)$ designate the subgroup generated by x. Does there exist a sub-semigroup S of $G(x, y)$ such that the set-theoretic union $S\cup S^{-1}=G(x, y)$ and $S\cap S^{-1}=G(x)$, where $S^{-1}=\{x^{-1}: x\in S\}$?

5634. *Proposed by H. E. Thomas, Jr., University of Michigan*

Find all integral solutions (n, r) of $\sum_{i=1}^{n} i=\sum_{i=1}^{r} i^2$.

5635. *Proposed by Otto Plaat, University of San Francisco*

Let S be a metric space such that (1) closed spheres in S are compact, and (2) given points p, q in S there is an isometry T of S such that $T(p)=q$. Prove that every isometry of S (into S) is onto S. (This yields a non-algebraic proof that the isometries of E^n are onto.)

SOLUTIONS OF ADVANCED PROBLEMS

Subdividing a Rectangle into Triangles

5479 [1967, 329]. *Proposed by Fred Richman and John Thomas, New Mexico State University*

Let N be an odd integer. Can a rectangle be dissected into N nonoverlapping triangles, all having the same area?

See John Thomas's paper, *A Dissection Problem*, Mathematics Magazine, 41(1968)187–190, which gives a partial solution. No other solutions have been submitted.

Rational Triangles

5499 [1967, 599]. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur, and J. S. Biggerstaff, Portland, Oregon*

(1) Are there rational numbers a, b such that no rational numbers c, k exist for which there is a triangle having sides a, b, c and area k?

(2) Prove or disprove the conjecture: for every positive rational number k there exist positive rational a, b, c such that a triangle with sides a, b, c has area k (e.g., triangle with sides 3/2, 5/3, 17/6 has area 1).

Solution (1) *by D. D. Ang, Saigon University, and D. E. Daykin and T. K. Sheng, University of Malaya.* We show that there is no triangle ABC with sides a, b, c and area k all rational, having $a=1$, $b=2$. If there were such a triangle then angles A, B, C would have rational sines and cosines. Hence $r=\tan \frac{1}{2}A = \sin A/(1+\cos A)$ would be rational, and $\sin A = 2r/(1+r^2)$. Similarly $\sin B = 2s/(1+s^2)$ for rational s, and by the sine rule

$$s(1 + r^2) - 2r(1 + s^2) = 0.$$

Solving this quadratic equation for r, we find that no nonzero rational solutions r, s exist because it is known that

$$s^4 + s^2 + 1 = t^2$$

has no nonzero solutions. [See Dickson, *History of the Theory of Numbers*, v. 2, pp. 636–638.]

Also solved by N. J. Fine and S. L. Segal.

Editorial Notes. Fine, in a paper on *Rational Triangles* (forthcoming issue of this MONTHLY) offers a collection of numbers (a, b) for which there are no rational triangles. Segal offers additional values (a, b) by letting $b=2$ and selecting a an integer for which the equation

$$u^4 + (2 - a^2)u^2v^2 + v^4 = t^2$$

has no integral solutions u, v, t; for example $a=5$ (cf. Pocklington, Proc. Cambridge Phil. Soc., (1914) 108–120.)

In regard to part (2), the fact that for every rational k there is a rational triangle of area k is presented by Fine in his paper. Segal points out that the earliest study of rational triangles seems to be due to Brahmagupta (7th century A.D.). See Dickson's *History*, v. 2, p. 191, and Fine's paper referred to above.

The Truncated Exponential Series

5541 [1967, 1269]. *Proposed by Robert Breusch, Amherst College*

Find

$$\lim_{m\to\infty} \frac{m!}{m^m}\left(2\sum_{k=0}^{m} \frac{m^k}{k!} - e^m\right).$$

Solution by H. W. Gould, West Virginia University. Ramanujan posed the relation

$$(1) \qquad \sum_{k=0}^{n-1} \frac{n^k}{k!} + \frac{n^n}{n!}\theta_n = \frac{1}{2}e^n, \quad \text{where} \quad \frac{1}{3} < \theta_n < \frac{1}{2}$$

as a problem in the Journal of the Indian Mathematical Society in 1911 [12], and later gave a refinement in a letter to Hardy dated Jan. 16, 1913. The refinement shows that in fact lim θ_n as $n \to \infty$ is 1/3.

We may rewrite (1) in the form

$$\frac{n!}{n^n}\left(2\sum_{k=0}^{n}\frac{n^k}{k!} - e^n\right) = 2(1-\theta_n),$$

from which it is then evident that the solution of 5541 is 4/3.

Many proofs and variations of (1) are known. Ramanujan's result was stated and used in the solution of Problem E 1583 [1964, 208]. Ramanujan's relation can be found in Copson's *Complex Variables* [6], and Copson gave a variation [7] by proving that

$$\sum_{k=0}^{n-1}(-1)^k\frac{n^k}{k!} + (-1)^n\frac{n^n}{n!}\phi_n = e^{-n}, \qquad \text{with} \quad \lim_{n\to\infty}\phi_n = \tfrac{1}{2}.$$

Of special interest are the proofs of (1) and its extensions as given by Buckholtz [1], Carlitz [2], Castagnetto [3], Cheng [4], Karamata [9], Szegö [14], and Watson [15]. It is considered worthwhile to take this opportunity to offer an extended list of references dealing with Ramanujan's remarkable relation.

1. James D. Buckholtz, Concerning an approximation of Copson, Proc. Amer. Math. Soc., 14 (1963) 564–568.

2. L. Carlitz, The coefficients in an asymptotic expansion, Proc. Amer. Math. Soc., 16 (1965) 248–252.

3. Louis Castagnetto, Dimostrazione elementare di un teorema del Tricomi, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8) 39 (1965) 413–417.

4. Tseng Tung Cheng, The normal approximation to the Poisson distribution and a proof of a conjecture of Ramanujan, Bull. Amer. Math. Soc., 55 (1949) 396–401.

5. S. Chowla and F. C. Auluck, An approximation connected with exp x, Math. Student, 8 (1940) 75–77.

6. E. T. Copson, Theory of Functions of a Complex Variable, Oxford, 1935. Note p. 230, prob. 18–19.

7. ———, An approximation connected with e^{-n}, Proc. Edinburgh Math. Soc., (2) 3 (1932–33) 201–206.

8. B. V. Gnedenko, The Theory of Probability, Chelsea, New York, 1962. Note p. 302, prob. 3.

9. J. Karamata, Sur quelques problèmes posés par Ramanujan, J. Indian Math. Soc., 24 (1961) 343–365. Note p. 323, prob. 294.

10. U. Ph. Lely and F. Schuh, Bewijs der formule

$$\lim_{x=\infty} e^{-x}\left(\frac{x^x}{x!}+\frac{x^{x+1}}{(x+1)!}+\frac{x^{x+2}}{(x+2)!}+\cdots+\right)=\frac{1}{2},$$

Christiaan Huygens, 1 (1921–22) 163–165.

11. Beppo Levi, Approximations to $n!$ for large values of n, Applications, Math. Notae, 3 (1943) 148–154. MR 5 (1944) 60.

12. S. Ramanujan, Problem 294, J. Indian Math. Soc., 3 (1911) 128; 4 (1912) 151–152.

13. ———, Collected Papers of, Cambridge, 1927. Note pp. 323–324.

14. G. Szegö, Über einige von S. Ramanujan gestellte Aufgaben, J. London Math. Soc., 3 (1928) 225–232.

15. G. N. Watson, Theorems stated by Ramanujan (V): Approximations connected with e^x, Proc. London Math. Soc., (2) 29 (1928) 293–308.

16. Problem E 1583, this MONTHLY, 70 (1963), 437; 71 (1964), 208–209.

17. Problem 5054, this MONTHLY, 69 (1962) 926; 70 (1963) 906–907. Note references cited here also.

18. Problem P 29, Canad. Math. Bull., 3 (1960) 78; 297–298.

Also solved by M. G. Beumer (Netherlands), D. Borwein, L. Carlitz, D. A. Hejhal, E. S. Langford, J. G. Mauldon (England), M. E. Muldoon, C. B. A. Peck, Henry Ricardo, G. L. Richardson, Tetsundo Sekiguchi, R. E. Shafer, Michael Skalsky, Sidney Spital, F. W. Steutel (Netherlands), J. H. van Lint (Netherlands), R. S. C. Wong, and the proposer.

The Degree of Products of Algebraic Numbers

5542 [1967, 1269]. *Proposed by Robert Breusch, Amherst College*

The first edition of Herstein, *Topics in Algebra* contains (p. 174) the following problem: "If $a, b \in K$ are algebraic over F of degree m and n respectively, and if m and n are relatively prime, prove that ab is algebraic over F of degree mn." Here K is an extension field over the field F. The problem has been deleted from the third edition because, as stated, it is incorrect.

Find counterexamples.

Solution by R. L. Walter, Florida State University. Let F be the field of rational numbers. Consider the cyclotomic polynomial $\phi p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$, p an odd prime. Then if a is a primitive pth root of unity, we have $\phi p(a) = 0$. Since $\phi p(x)$ is irreducible over F, the degree of a is $p-1$. Also let $b = 2^{1/p}$; then the degree of b is p. We have both a, b algebraic and $(p, p-1) = 1$; but $(ab)^p = 2$. That is, ab is a root of $x^p - 2$; hence the degree of ab is not $p(p-1)$.

Also solved by Sami Beraha & Randal Miller, P. T. Bhattacharya (India), A. K. Charnow, L. D. Crowson, D. Ž. Djoković, H. M. Edgar, Octavio Garcia R. (Mexico), D. A. Hejhal, John Kieffer, Peter Kornya, Kenneth Kramer, Douglas Lind, Andrzej Makowski (Poland), J. G. Mauldon (England), Albert Pollatchek, Z. Z. Uoiea, W. C. Waterhouse, A. K. Wayman, Jr., Charles Wells, Linda E. Wells, William Worobey, and the proposer.

The proposer notes that the proposition is correct if F is the field of rationals and if the product ab is replaced by the sum $a+b$.

Lifting Endomorphisms

5543 [1967, 1269]. *Proposed by U. Güntzer, Göttingen, and J. J. Hirschfelder, University of Notre Dame*

Let A be a commutative ring with 1, M an A-module, L a submodule of M. A homomorphism $f: M \to M$ with $f(L) \subset L$ induces a homomorphism $f': M/L \to M/L$. Can every endomorphism of M/L be obtained in this way?

Solution by J. S. Golan, The Hebrew University, Jerusalem, Israel. No. Let $R = Z$, the ring of integers. Let $M = Z \oplus Z_2$, $L = 2Z \oplus 0$. Then M/L is isomorphic to $Z_2 \oplus Z_2$. Define $f': M/L \to M/L$ by $f'(x, t) = (t, 0)$. Then f' is an endomorphism of M/L which cannot possibly come from one of M since all endomorphisms of M must take first coordinates into first coordinates and second coordinates into second coordinates.

Also solved by D. Ž. Djoković, Douglas Lind, Ka Menehune, W. R. Scott, W. C. Waterhouse, Charles Wells, and the proposer.

An Exponential Sum

5544 [1967, 1270]. *Proposed by D. R. Hayes, Oak Ridge National Laboratory*
Consider the exponential sum

$$S(n) = \frac{1}{2^n} \sum_{x=0}^{2^p-1} e\left(\frac{x^3 + 2x^2}{2^n}\right),$$

where $e(u) = e^{2\pi i u}$. Show that for $1 \leqq n \leqq 8$, the value of $S(n)$ is given by

n	1	2, 3, 4	5	6	7	8
$S(n)$	0	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1-i}{8}$	0

Show that for $n > 8$, the value of $S(n)$ is given by

$$2^{-\frac{1}{2}(n-1)}[(1 + i) + (1 - i)\cdot e(a_n)],$$

where a_n is determined as follows: Expand $2^{-n}(32/27)$ as a 2-adic number

$$2^{-n}\left(\frac{32}{27}\right) = 2^{5-n} + 2^{6-n} + 2^{9-n} + \cdots.$$

Then a_n is the rational number which is the sum of the negative powers of 2 which appear in this expansion.

Solution by Oswald Wyler, Carnegie-Mellon University. By inspection $S(0) = 1$ and $S(1) = 0$. By a simple computation

$$\begin{aligned} &2^{-n}[(x + 2^{n-k})^3 + 2(x + 2^{n-k})^2] \\ &\qquad = 2^{-n}(x^3 + 2x^2) + 2^{-k}(3x^2 + 4x) + 2^{n-2k}(3x + 2 + 2^{n-k}). \end{aligned} \tag{1}$$

For $k = 1$ and $n \geqq 2$, the last term at the right is an integer, and the middle term $\equiv \frac{1}{2}$ (mod 1) for odd x. Thus the terms with odd x cancel out two by two in the sum for $S(n)$, $n \geqq 2$.

If $n \leqq 4$, then $2^{-n}(x^3+2x^2)$ is an integer for all even x. It follows that $S(2) = S(3) = S(4) = \frac{1}{2}$.

If $n \geqq 5$ and $k = 3$ in (1), then the last term on the right side is an integer for even x, and the middle term $\equiv \frac{1}{2}$ (mod 1) if $x \equiv 2$ (mod 4). Thus the terms with $x \equiv 2$ (mod 4) cancel out two by two in the sum for $S(n)$, and only the terms with $x \equiv 0$ (mod 4) remain. Putting $x = 4y$ and $n = m+5$, we have $2^{-n}(x^3+2x^2) = 2^{-m}(2y^3+y^2)$. Thus the terms of $S(m+5)$ depend only on the residue class of y (mod 2^m), and we have

$$S(m + 5) = 2^{-(m+2)} \sum_{y=1}^{2^m} e(2^{-m}(2y^3 + y^2)). \tag{2}$$

$S(5)=\frac{1}{4}$ and $S(6)=0$ follow from (2) by inspection.

Let now $m\geqq 2k$, $k\geqq 1$. By a simple computation,

$$(3)\quad \begin{aligned}2^{-m}[2(y+2^{m-k}u)^3+(y+2^{m-k}u)^2] &= 2^{-m}(2y^3+y^2)\\ &+2^{1-k}(3y^2+y)u+2^{m-2k}u^2(6y+2^{m-k+1}u+1),\end{aligned}$$

where the last term at the right is an integer. If $2^{1-k}(3y^2+y)=c$, and if $f_m(y)=e(2^{-m}(2y^3+y^2))$, then it follows from (3) that

$$\sum_{u=1}^{2^k} f_m(y+2^{m-k}u)=f_m(y)\sum_{u=1}^{2^k} e(cu).$$

Since 2^kc is an integer, the sum at the right is 2^k if c is an integer, and 0 if c is not an integer. Thus

$$(4)\qquad S(m+5)=2^{-(m-k+2)}\sum e(2^{-m}(2y^3+y^2)),$$

with summation over all y such that $0\leqq y<2^{m-k}$ and $3y^2+y\equiv 0 \pmod{2^{k-1}}$.

Since y and $3y+1$ are relatively prime, the congruence $3y^2+y\equiv 0\pmod{2^{k-1}}$ is satisfied iff $y\equiv 0 \pmod{2^{k-1}}$ or $3y+1\equiv 0\pmod{2^{k-1}}$. We write $y=z+2^{k-1}v$, where z is one of the two solutions of the congruence $3z^2+z\equiv 0 \pmod{2^{m-k}}$, and v is determined mod 2^{m-2k+1}. We have

$$(5)\quad \begin{aligned}2^{-m}(2y^3+y^2) &= 2^{-m}(2z^3+z^2)+2^{-(m-k)}(3z^2+z)v\\ &+2^{-(m-2k+2)}v^2(1+6z+2^kv),\end{aligned}$$

where the second term at the right is an integer.

If m is even, we put $k=\frac{1}{2}m$ and use (5) with $v=0$ and $v=1$. Adding the resulting two terms of (4), we get

$$(6')\qquad 2^{-(k+2)}f_m(z)[1+e(2^{-2}(1+6z+2^k))].$$

If m is odd, we put $k=\frac{1}{2}(m-1)$, and we use (5) with $v=0, 1, 2, -1$. Adding the resulting terms of $S(m+5)$, we obtain

$$(6'')\qquad 2^{-(k+3)}f_m(z)e(2^{-3}(1+6z))[e(2^{k-3})+e(-2^{k-3})].$$

We note that the terms with $v=0$ and $v=2$ cancel in $(6'')$.

If $m=2$ or $m=3$, $k=1$, then we obtain all terms in (4) from $(6')$ or $(6'')$ with $z=0$. It follows that $S(7)=2^{-3}(1-i)$ and $S(8)=0$.

For $k\geqq 2$, the two terms in the bracket of $(6'')$ are equal. We take out a factor $e(2^{-2})=i$ in $(6')$ and a factor $e(2^{-3})=2^{-1/2}(1+i)$in $(6'')$. Then the two expressions become

$$(7')\qquad 2^{-(2+\frac{1}{2}m)}f_m(z)(1+ie(\tfrac{1}{2}z));$$

$$(7'')\qquad 2^{-(2+\frac{1}{2}m)}e(2^{k-3})f_m(z)(1+i)e(3z/4).$$

$S(m+5)$ is the sum of two such terms, one for each solution of $3z^2+z\equiv 0$

(mod 2^{m-k}). These solutions are $z=0$ and $3z+1\equiv 0 \pmod{2^{m-k}}$. Thus

$$(8) \qquad S(n) = 2^{-\frac{1}{2}(n-1)}[(1+i)+(1-i)e(a_n)],$$

where again $n=m+5$, and $a_n=2^{-m}(2z^3+z^2)$ for $3z+1\equiv 0 \pmod{2^{m-k}}$.

Formula (8) is valid for all $n\geqq 9$ except $n=10$. For $n=10$, i.e., $m=5$ and $k=2$, we have $e(2^{k-3})=-1$ in (7″), so that $S(10)$ is the negative of the value given by (8). Thus a correction in the statement of the problem becomes necessary.

We evaluate a_n as follows. If $3z+1\equiv 0 \pmod{2^{m-k}}$, with $2k\leqq m$, then $27(2z^3+z^2)\equiv 1 \pmod{2^m}$. Thus $a_n=2^{-m}c_n=2^{5-n}c_n$ for an integer c_n such that $27c_n\equiv 1 \pmod{2^{n-5}}$. One way to obtain this integer c_n is to break off the 2-adic development of $1/27$ after $n-5$ places, or that of $32/27$ after n places, counting 2-adic units as the first place.

Also solved by L. Carlitz.

A Functional Equation

5545 [1967, 1270]. *Proposed by E. A. Maier, L. C. Eggan and Howard Gage, University of Oregon*

Let f be a real-valued continuous function defined on the real numbers such that

$$f(x+y) = f(x) + f(y) + kxy.$$

Prove that $f(x)=(k/2)x^2+cx$, where $c=f(1)-k/2$.

Solution by Charles Riley, Keene State College, N. H. Let $g(x)=f(x)-kx^2/2$. Then $g(x+y)=g(x)+g(y)$ for each x, y, so that $g(x)=g(1)x$ by the hypothesis on $f(x)$. Thus $f(x)-kx^2/2=(f(1)-k/2)x$ for each real x.

Also solved by Robert Abrams, J. Aczel, Arnold Adelberg, A. S. Adikesavan (India), Frank Alden, D. R. Anderson, T. M. Apostol, Bernard August, Anders Bager (Denmark), Robert Baumel, D. W. T. Bean, Stephen Berman, W. J. Blundon, R. D. Boswell, Jr., M. G. Boyce, J. L. Brenner, J. L. Brown, Jr., Roxanne M. Byrne, Benedict Carlat, L. Carlitz, P. R. Chernoff, F. A. Chimenti & J. S. Wasileski, Donald Cohen, Ted Cullen, D. Ž. Djoković, W. G. Dotson, Jr., E. S. Eby, R. J. Egbert, R. C. Entringer, J. L. Ercolano & F. G. Gustavson, G. J. Eterovich, William Fox, M. F. Friedell, Octavio Garcia R. (Mexico), John Gill, Stuart Goff, Robert Goldstein (England), W. R. Gordon, M. G. Greening (Australia), Charles Hanna, D. A. Hejhal, Robert Heller, Dennis Henkel, A. S. B. Holland, R. A. Horn, John Horvath, W. H. Jones, P. L. Kannappan, C. F. Kaun, B. G. Klein, A. M. Krall, Kenneth Kramer, Peter Krauchthaler (Switzerland), J. R. Kuttler & V. G. Sigillito, H. E. Lahmann (Germany), E. S. Langford, D. J. Leeming, O. P. Lossers (Netherlands), J. G. Mauldon (England), Renate McLaughlin, D. E. Myers, C. A. Oster, P. J. Ownes (England), C. B. A. Peck, D. E. Penney, G. W. Petrie, David Promiscow, L. R. Pujara, Gosula N. Reddy, L. Naga Muni Reddy (India), Simeon Reich (Israel), Zenon Reynarowych, G. H. Ryder, David Ryeburn, John Schroeter, B. L. R. Shawyer, Marlow Sholander, Harry Siller, Michael Skalsky, Al Somayajulu, P. K. Subramanian, D. P. Sumner, Hugo Sun, Chang Sung-shen (Taiwan), W. W. Symes, D. H. Voelker, Albert Wehrly, R. J. Weinacht, Roger Weitzenkamp, J. E. Wetzel, Albert White, J. C. Williams, M. Z. Williams & C. D. Meyer, Oswald Wyler, David Zeitlin, and the proposers.

Editorial Note. Several solvers generalized by considering the functional equation $f(x+y) = f(x)+f(y)+g(x, y)$. If $p(x)$ is a polynomial and $g(x, y)$ may be expressed as $p(x+y)-p(x)-p(y)$, Goldstein proves the solution to be of the form $cx+p(x)$. If $g(x, y)$ is expressible as $kg(x)g(y)$, Maier, Eggan and Gage express the solution $f(x)$ as $f(1)x+k[g(1)G(x)-G(1)g(x)]$, where $G(x) = \int_0^x g(t)dt$.

If f has a continuous second derivative, Voelker shows that $g(x, y)$ may be expressed as

$$\int_b^y \int_a^x q(u+v)dudv$$

and derives the solution

$$f(x) = \int_b^x \int_a^\xi q(u)dud\xi + (x-b)f'(a) + f(b).$$

A Convergent Integral with Large Integrand

5546 [1967, 1270]. *Proposed by T. S. Shores, University of Kansas*

Prove or disprove: If f is a nonnegative real-valued measurable function on the reals such that for any $a<b$ and $R>0$,

$$m\{(a, b) \cap \{x \mid f(x) \geqq R\}\} > 0,$$

then

$$\int_{-\infty}^{\infty} f(x)dx = +\infty.$$

Solution by Dennis Henkel, Milwaukee, Wisconsin. The statement is not true. Let $\{r_n\}$ be an enumeration of the rationals and define

$$f_k(x) = \begin{cases} (x-r_k)^{-1/2} & \text{if } x \in (r_k, r_k+1) \\ 0 & \text{if } x \notin (r_k, r_k+1). \end{cases}$$

Write $f = \sum_{n=1}^{\infty} f_n/2^n$. By the Lebesgue monotone convergence theorem, we see that $\int_{-\infty}^{\infty} f(x)dx = 2/3$. However, for any $a<b$, $R>0$, there is some rational r in (a, b), and an open interval I contained in (a, b) with r as left end point, such that $f(x) > R$ for all $x \in I$.

Also solved by P. R. Chernoff & W. C. Waterhouse, R. A. Christiansen, Edmund Deaton, Z. Ditzian & A. Meir, D. Ž. Djoković, G. J. Foschini, D. A. Hejhal, B. G. Klein, O .P. Lossers (Netherlands), J. G. Mauldon (England), Frank Meyer, Harry Miller, J. C. Morgan II, Tassos Nakassis (Greece), Hugh Noland, Paul Odlyzko, P. J. Owens, N. T. Peck (England), P. M. Perdew, Z. R. Pop-Stojanovic, A. C. Segal, J. J. Uhl, Z. Z. Uoiea, Linda E. Wells, Oswald Wyler, and the proposer.

A Number-theoretic Identity

5547 [1967, 1271]. *Proposed by Anders Bager, Hjørring, Denmark*

Let $\rho(n)$ (resp. $\nu(n)$) be the number of prime divisors of the positive integer n counted with (resp. without) multiplicity. Prove that $\sum_{d|n} (-1)^{\rho(d)} 2^{\nu(n/d)} = 1$ for all n.

I. *Solution by Douglas Lind, University of Virginia.* We generalize the result by showing that for any integer t,

$$h(n) = \sum_{d\mid n} t^{\rho(d)}(1 - t)^{\nu(n/d)} = 1$$

for all n. Using the convention $0^0 = 1$, the functions $f(n) = t^{\rho(n)}$, $g(n) = (1-t)^{\nu(n)}$ are easily shown to be multiplicative, implying that the Dirichlet convolution $h(n)$ is also. If $n = p^k$, where p is a prime, then

$$h(p^k) = t^k + \sum_{j=0}^{k-1} t^j(1 - t) = t^k + 1 - t^k = 1,$$

and hence $h(n) = 1$ for all n.

II. *Solution by Henry Ricardo, Yeshiva University and Manhattan College.* We use the generating functions

$$(1) \qquad \sum_{n=1}^{\infty} (-1)^{\rho(n)}/n^s, \qquad\qquad (2) \qquad \sum_{n=1}^{\infty} 2^{\nu(n)}/n^s,$$

where s is real and >1. Then $F(s)$, equal to the product of series (1) and (2), has coefficients

$$a_n = \sum_{d\mid n} (-1)^{\rho(d)} 2^{\nu(n/d)}.$$

Since series (1) is equal to $\zeta(2s)/\zeta(s)$, and series (2) is equal to $\zeta^2(s)/\zeta(2s)$ we see that $F(s) = \zeta(s)$. (See Hardy & Wright, *An Introduction to the Theory of Numbers*, fourth edition (1960), Theorems 300 and 301, where $\lambda(n) = (-1)^{\rho(n)}$ and $\omega(n) = \nu(n)$.) This implies that, for all $n \geqq 1$,

$$a_n = \sum_{d\mid n} (-1)^{\rho(d)} 2^{\nu(n/d)} = 1.$$

Also solved by Arnold Adelberg, R. G. Buschman, P. R. Chernoff & W. C. Waterhouse, R. C. Entringer, M. A. Ettrick, R. D. Fray, Ray Glenn, Myron Goldberg, M. G. Greening (Australia), J. Hanumanthachari (India), Heiko Harborth (Germany), D. A. Hejhal, John Kieffer, Peter Krauchthaler (Switzerland), E. S. Langford, P. A. Lindstrom, Andrzej Makowski (Poland), Dan Marcus, Simeon Reich (Israel), Jonathan Ryshpan, R. Sivaramakrishnan (India), David A. Smith, Al Somayajulu, Sidney Spital, E. W. Trost (Switzerland), Emanuel Vegh, C. S. Venkataraman (India), Rogert Weitzenkamp, Charles Wells, K. S. Williams, Oswald Wyler, and the proposer.

Makowski uses E 1951 [1968, 544] to obtain the result.

Sivaramakrishnan proves that if $g(n)$ is multiplicative (requiring $(m, n) = 1$) and $f(n)$ is completely multiplicative $[f(m)f(n) = f(mn)$, all $m, n]$, and if $g(p) = g(p^\alpha)$, $\alpha > 1$, then

$$\sum_{d\mid n} f(d)g(n/d) = 1$$

if and only if $g(p^m) = 1 - f(p)$ for all primes p, $m \geqq 1$, $f(p) \neq 1$.

Quartic Residues of $p\equiv 1$ (mod 4)

5548 [1967, 1271]. *Proposed by E. J. Barbeau, The University of Western Ontario*

Let p be a prime, congruent to 1 (mod 4). Show that $\frac{1}{4}(p-1)$ and -4 are quartic residues (mod p), and that the fourth roots of $\frac{1}{4}(p-1)$ consist of two pairs of consecutive integers.

Solution by Iain Cummins, Edinburgh, Scotland. If $p\equiv 1$ (mod 4), there is an integer a such that $a^2\equiv -1 \pmod p$. Then $(a\pm 1)^2=a^2\pm 2a+1\equiv \pm 2a \pmod p$ and $(a\pm 1)^4\equiv 4a^2\equiv -4 \pmod p$, i.e., -4 is a quartic residue.

Since p is odd, there is an integer c such that $2c\equiv 1 \pmod p$. Then $16[c(a\pm 1)]^4 \equiv -4\equiv 4(p-1) \pmod p$, showing that $\frac{1}{4}(p-1)$ is a quartic residue.

Since $c(a+1)$ and $c(a-1)$ differ by $2c\equiv 1 \pmod p$, these two fourth roots of $\frac{1}{4}(p-1)$ may be taken as consecutive integers b, $b+1$. Then $p-b-1$, $p-b$ is also a pair of consecutive fourth roots. The pairs do not overlap unless one of the roots, say $b=\frac{1}{2}(p-1)$, but then $(p-1)^4\equiv 1\equiv -4 \pmod p$ which can hold only for $p=5$, whereby 1, 2, 3, 4 are all fourth roots of $\frac{1}{4}(p-1)$, and so the result holds also in this case.

Also solved by Arnold Adelberg, M. G. Beumer (Netherlands), W. J. Blundon, L. Carlitz, P. A. Catlin, Mannis Charosh, J. H. E. Cohn (England), M. G. Greening (Australia), A. J. Keeping, John Kieffer, Kenneth Kramer, D. C. Kurtz, H. F. Mattson, Jr., Bob Prielipp, E. J. F. Primrose (England), Simeon Reich (Israel), Jonathan Ryshpan, E. W. Trost (Switzerland), Emanuel Vegh, Roger Weitzenkamp, Charles Wells, K. S. Williams, Gregory Wulczyn, Oswald Wyler, and the proposer.

Beumer remarks that the problem is mentioned in Dickson, *History of the Theory of Numbers*, V.1, p. 386, reference 49a.

Splitting Quotient Groups

5549 [1967, 1271]. *Proposed by Steve Ligh, University of Houston*

Let G be an additively written torsion free abelian group. H is any subgroup of G. If the automorphism γ of G/H that takes x onto $-x$ can be lifted to an automorphism α of G that is the identity on H, then G/H splits.

Solution by R. B. Hardin, Jr., and E. W. Poluianov, Wayne State University. In terms of a diagram we know

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & G \arrow[d, "\beta"] \\ G/H \arrow[r, "\gamma"] & G/H \end{tikzcd}$$

is commutative, where β is the natural map. Let $T/H=(G/H)_t$, the torsion part of G/H. We have on one hand

$$\gamma(t+H) = -t+H, \qquad \forall t\in T,$$

and on the other hand, as T/H is torsion, there exists an n such that $nt \in H$. This implies that $\alpha(nt) = nt$, or $n(\alpha(t) - (t)) = 0$. The torsion freeness of G gives $\alpha(t) = t$. These two results imply that

$$t + H = \beta(t) = \beta(\alpha(t)) = \beta\alpha(t) = \gamma\beta(t) = \gamma(t + H) = -t + H.$$

As $t+H$ was arbitrary, we have $2(G/H)_t = 0$, or $(G/H)_t$ is a bounded (by 2) subgroup of G/H. The torsion subgroup of an abelian group is always pure, and any pure subgroup of bounded order is a summand. Therefore the torsion subgroup of G/H is a summand, i.e., G/H splits.

Also solved by Kenneth Kramer, Ka Menehune, and the proposer.

With his solution, Menehune offers references to relevant theorems in I. Kaplansky, *Infinite Abelian Groups:* p. 14(e) and p. 18, Theorem 7.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. All unsigned material is by the editor. Correspondence about Reviews will be welcome.

Beginning with the January 1969 issue, film reviews will be edited by Seymour Schuster, Carleton College, Northfield, MN 55057. All correspondence concerning films should be sent to him.

Mathematical Logic. By Joseph R. Shoenfield. Addison-Wesley, Reading, Mass., 1967. vii+344 pp. \$12.75.

This book is a landmark in the history of its subject: the first which is and looks like a genuine graduate text for mathematicians. Fittingly it has appeared just as logic has become popular in the mathematical community. The book probably contains everything that was suitable for textbook presentation by, say, 1966. The sections on recursion theory, including the theory of hyperarithmetic functions, and on set theory fill a particularly urgent need; nothing in the published literature is comparably compact and complete.

The author successfully avoids several familiar defects of earlier compendia on mathematical logic; no rambling historical introductions, no examples from "everyday reasoning" of matters which are more incisively illustrated by mathematical applications, no verbal tricks which obscure issues (like super-clever advertisements which prevent "sponsor identification"). He goes out of his way to pick out the principal theorems, often giving them a carefully chosen name, and arranges them in a beautifully economical, and sometimes novel, order.

The technical mathematical portions are very readable. However, some of the usually short, introductory passages and some philosophical comments on results seem weak. Worse, the author rarely reexamines, at the end of a chapter, his introductory material in the light of the technical development. (This is

reminiscent, at least to this reviewer, of "physical" passages in old-fashioned books on differential equations written by so called applied mathematicians: they didn't improve the mathematical exposition and they certainly didn't convey any idea of rigorous physical reasoning.) There are not enough careful distinctions where they are needed most: in the informal analysis of basic concepts needed, (if one attempts such an analysis at all). For instance, some comment is needed to distinguish the use of "visualize" on page 3 in connection with finitary methods from that on page 239 in the justification of the axioms of set theory. When analyzing the axiom of choice, quite properly, in terms of definability (p. 259), a word of warning on the notion of definability used is called for: there are purely logical theorems of the form $\exists x A x$ such that no set-theoretically definable x_0 can be shown to satisfy $A x_0$. A warning is needed because once he has got to page 259, the reader is almost bound to think of set-theoretic definability! The general idea of the justification (p. 306) of inaccessibles in terms of generalized inductive definitions or fixed points of operations, is surely correct; but it is spoiled by omitting mention of the essential property of the cumulative type structure involved, namely continuity at limit ordinals; after all one does not postulate an ordinal α which is closed under the operation: if x is a set of ordinals $<\alpha$, so is $x \cup \{x\}$! Finally, the reviewer is convinced that a brief model theoretic and proof theoretic treatment, in the author's masterly fashion, of some simple topic such as propositional calculus or equational systems, would have conveyed (his own ideas of) the nature of mathematical logic much better than the disjointed and inarticulate remarks in the first few pages of Chapter I, which was supposed to do this job.

For the record, it should be said that the reviewer has heard the reactions of many students, both here and abroad, to this book. They unanimously regard it, not only as the most useful, but as the best and most agreeable text in the field. This applies not only to the technical parts, but also, for good or ill, to the informal explanations.

G. KREISEL, University of California, Los Angeles

Topology—An introduction with applications to topological groups. By George McCarty. McGraw-Hill, New York, 1967. vii+263 pp. $8.95. (Telegraphic Review, Jan. 1968).

The author provides a concise outline of his book in the preface:

"This text introduces the student to that part of geometry which is generally labeled 'topology.' It will give him that familiarity with elementary point set topology, including an easy acquaintance with the line and the plane, which has become prerequisite to most graduate programs in mathematics. Nevertheless, it is not a collection of such topics; rather, it early employs the language of point set topology to define and discuss topological groups. These geometric objects in turn motivate a further discussion of set-theoretic topology and of its applications in functions spaces. An introduction to homotopy and the fundamental group then brings the student's new theoretical knowledge to bear on very concrete problems: the calculation of the fundamental group of circle and a proof

of the fundamental theorem of algebra. Finally, the abstract development is brought to a satisfying fruition with the classification of topological groups by equivalence under local isomorphism."

As this outline promises, there is a surprising amount of substantial mathematics in these 263 pages. The author's approach might be said to be "pre-categorical" in nature, there being much emphasis upon the algebraic properties of morphisms (a term he uses regularly), commutative diagrams, etc. This means that the book admirably meets the topology requirements for the pregraduate training of research mathematicians as recommended by CUPM.

Of course, the book places heavy demands upon the reader. No less could be expected of a book which starts at the very beginning and, without missing much in between, achieves the meaningful mention of fiber bundles, H-spaces and Lie groups! The exercises and problems are an essential part of the book, which is as it should be for the expected audience. And the student will be richly rewarded for his efforts. In short, then, this fine book meets its stated objectives very well indeed.

J. G. HOCKING, Michigan State University

Laplace Transform Theory. By M. G. Smith. Van Nostrand, New York, 1966. x+123 pp. $6.00 (cloth) $3.25 (paper).

The author presents a concise development of the theory and application of the Laplace transform. Beginning with a discussion of the D-operator, the author brings out its analogy to the Laplace transform but indicates limitations which do not carry over to the Laplace transform. In Chapters 2 and 3, the foundation of the subject is developed with the definition of the Laplace transform in terms of the Riemann integral, and with the derivation of its basic elementary properties. An appeal to the Fourier Integral Theorem, proved in the Appendix, yields the inverse integral considered in Chapter 4, and leads, as well, to the uniqueness theorem for the Laplace transform. The convolution integral is derived in Chapter 5. Throughout Chapters 2–5, applications of the Laplace transform are given to solutions of ordinary linear differential equations with constant coefficients. More general applications to ordinary linear differential equations with variable coefficients, to linear partial differential equations, to integral equations, and to linear difference equations are considered in Chapters 6–9. The final chapter of the book deals with asymptotic formulae relating a function and its transform. Tables of operational formulae and of common Laplace transforms are included in Appendices.

The book is written in a clear, formal style. Each chapter begins with a brief discussion of the topic considered. This is followed by illustrative examples, worked out in detail, and finally, by a number of exercises for which answers are given.

For those with a working knowledge of complex function theory, the book should serve as a source for a compact, though not rigorous, introduction to the Laplace Transform.

DEBORAH TEPPER HAIMO, University of Missouri, St. Louis

Introduction to Linear Programming, with Applications. By William R. Smythe, Jr. and Lynwood A. Johnson. Prentice-Hall, Englewood Cliffs, N. J., 1966. viii+221 pp. $9.95.

This text is intended for readers having little or no college mathematics. It seems admirably suited to this intention. The writing is lucid, the material is well motivated, proofs are rigorous without being obscure, and the exercises furnish both drill and further insights.

The brief first chapter presents graphical two-variable examples. Chapter two is a model introduction to linear algebra, and concludes with a presentation in tableau form of the pivotal exchange method for a change of basis. The reader, then, is well prepared for the simplex method presented with clarity and illustrated by many examples in chapter three. Degeneracy is discussed and a nonconstructive proof (involving a limit of a sequence) of the termination of the simplex method is given. Chapter four on integer programming proves the elegant theorems on network flows and applies these to transportation problems. Examples in the last chapter illustrate a wide range of applications to industrial planning.

Only brief mention (two pages) is made of duality, although the fundamental duality theorem is proved. The omission of matrix games and the inclusion of only three references for further reading may disappoint some potential users.

W. F. TYNDALL, Franklin and Marshall College

A short introduction to numerical analysis. By M. V. Wilkes. Cambridge University Press, 1966. 77 pp. $4.75.

This 77-page book covers a satisfactory amount of material, and has some exercises to which a student may devote himself for better understanding. Among the subjects treated are Newton-Raphson iteration, quadrature formulas, numerical integration of ordinary, and in one case even partial, differential equations, finite difference methods, and solution of simultaneous linear equations. The treatment of some subjects is necessarily cursory, not to say superficial, but there are no errors of commission in the book. On the other hand, rigorous statements about convergence are lacking, and so is systematic reference to sources. The book might well be used in some survey courses on numerical methods.

JOEL BRENNER, Stanford Research Institute, University of British Columbia, and University of Arizona

Elementary Partial Differential Equations. By Paul W. Berg and James L. McGregor. Holden Day, San Francisco, 1966. ix+421 pp. $11.75.

This is a very carefully written text. So true is this that a trivial printer's error came as a shock on page 229. Theorems are stated precisely and proved rigorously. It is quite true, as the authors state, that only the calculus and a course in ordinary differential equations are prerequisites, but the student requires a considerable degree of sophistication to absorb the full content.

The book begins with two chapters covering definitions, classical elementary

techniques, constant coefficient partial differential equations of the first two orders, and the heat equation carefully derived from physical considerations. It continues with a broad coverage including eigenvalue expansions, asymptotic solutions, Fourier series (an outstanding feature of the book), theorems on existence, uniqueness, and representation, the wave equation derived from physical considerations, problems on infinite and semi-infinite intervals, the Fourier integral, the Parseval equation for Fourier transforms, the convolution theorem, heat flow in two and three dimensions and the vibrating rectangular membrane, the Dirichlet problem, and Poisson's integral formula for the disc. The final chapter gives a solid treatment of Bessel functions of order n. Special attention is given to eigenvalue problems and eigenfunction expansions connected with Bessel's equation of order zero. The problems are considered in polar and cylindrical coordinates, and the usual relations connecting Bessel functions are derived.

The exercises at the end of each chapter are extensions of the discussion. None is trivial, and they reflect impressively careful preparation. At the end of the book it is pleasing to see not only answers for selected problems, but hints for their solution. This is a valuable aid for the independent reader. One of many excellent features is the continuing review of and presentation of elementary theorems and material needed at a given point.

The text is suitable for a one semester course in partial differential equations at the undergraduate level or at the first year graduate level. It can serve both as an introduction to the subject and also as a bridge to the more advanced material. It is a needed and substantial addition to the existing literature. The authors are to be commended for their work.

M. J. HELLMAN, Long Island University

Geometry Revisited. H. S. M. Coxeter and S. L. Greitzer. New Mathematical Library 19. Random House, New York, 1967. School Edition from Singer. xiv, +193 pp.

This attractive book is designed to give pleasure and mathematical instruction to those who have already studied some Euclidean geometry. The proofs are detailed and clear, the subject matter eclectic, as one would expect, and there are exercises and hints for solution.

The reviewer is glad to note that despite recent diatribes against a course on Euclidean geometry "based on the nine-point circle and Stewart's theorem," these are both discussed by the authors. The preface is slightly on the defensive: "Geometry still possesses all those virtues that the educators ascribed to it a generation ago." The best answer to those who would get rid of geometry completely is the publication of good books on geometry, and the book under review is a good book.

Some of the topics the authors consider are presented in a new light. For instance Chapter 5, "An Introduction to Inversive Geometry" shows that a circle can be defined as the locus of a point which has certain separation properties, and this is invariant under inversion. The same chapter also introduces the

notion of "inversive distance," a recent discovery of the senior author. There are many delightful things in the book. The proofs, as one would expect, are very elegant.

DANIEL PEDOE, University of Minnesota

Boundary Value Problems. By F. D. Gakhov. Translated from Russian by I. N. Sneddon. Addison-Wesley, Reading, Mass., and Pergamon Press, London, 1966. xix+561 pp. $22.50.

This book offers an excellent account of the treatment of elliptic boundary value problems and singular integral equations through use of Cauchy type integrals. In addition to a highly readable style of mathematical writing, the author has included numerous historical references throughout. A one year course in complex variable theory is sufficient to understand the book.

The basic theorems relating to integrals (single and multiple) of the Cauchy type are developed in chapter 1 (84 pp). Of particular interest are the properties of Cauchy integrals that involve functions which satisfy Hölder conditions. The familiar and important Sokhotski or Plemelj formulas are developed. The level of rigor has been reduced to produce a readable account. The reader seeking a sounder treatment would profit from reading "Simplified Treatment of Integrals of Cauchy Type" by N. Levinson (SIAM Reviews, Vol. 7, No. 4 (1965)). These results are first applied to Riemann type boundary problems. The familiar theorems of potential theory as they relate to the Dirichlet and Neumann problems are obtained. An account of singular integral equations is given followed by a treatment of Hilbert type problems and singular integral equations with a Hilbert kernel. Relations between the Riemann and Hilbert problems are then obtained. Examples are provided throughout to illustrate how the methods apply in different situations (interior problems, half-plane problems, etc).

The last half of the book examines generalizations of the Riemann and Hilbert problems including the case in which the functions entering the boundary conditions exhibit singularities. Because of the broad coverage, this book should prove to be a valuable addition to mathematical literature.

L. R. BRAGG, Oakland University

A survey of texts for a one year course in real analysis at the advanced undergraduate or beginning graduate level.

In 1965 CUPM published the following booklets: (A) Preparation for Graduate Study in Mathematics (B) Pregraduate Preparation of Research Mathematicians. An outline for a Real Analysis course in (A) contains the following topics: (a) Real Numbers, (b) Complex Numbers, (c) Set Theory, (d) Metric Spaces, (e) Euclidean Spaces, (f) Continuity, (g) Differentiation, (h) Riemann-Stieltjes Integral, (i) Series of Numbers, (j) Series of Functions, (k) Series Expansions. The Real Analysis course in (B) includes: (a) Set Theory (b) Real Numbers (c) Metric Spaces (d) Topological Spaces (e) Normed Linear Spaces (f) Lebesgue Integrals and Measures (g) Differentiation (h) Applications to Classical Analysis (i) Integration on Groups (j) Measure Spaces (k) Banach

Algebras (1) Spectral Resolution of Self-Adjoint Operators. More detailed course descriptions are given in the booklets obtainable by writing to CUPM, P.O. Box 1024, Berkeley, California 94701.

The following seem suitable for course (A):

R. G. Bartle. *The Elements of Real Analysis*. Wiley, 1964. Highly recommended. Also a good introduction to Functional Analysis.

Tom M. Apostol. *Mathematical Analysis*. Addison-Wesley, 1957. Includes complex analysis, vector analysis, Fourier series, and usual advanced calculus topics; enough for three semesters, but real analysis can be covered in a year's course. Recommended.

(a) M. E. Munroe. *Introductory Real Analysis*. Addison-Wesley, 1965. (b) W. H. Fleming. Functions of Several Variables, Addison-Wesley, 1965. Six chapters of (a) and all of (b) would make an excellent combination for a year's course.

Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 1953.

The following books are possible texts or supplementary material for course (B). Items (i), (j), (k), (l) are missing from all of these texts. No claim is made of completeness in selection.

Asplund and Bungart. *A First Course in Integration*. Holt, Rinehart and Winston, 1966. Suitable even though items (c), (d), and (e) are not treated formally. A supplementary text could be T. H. Hildebrandt, *Introduction to the Theory of Integration*. Academic Press, 1963. The latter seems a bit advanced for this level.

R. G. Bartle. *The Elements of Integration*. Wiley, 1966. A compact treatment of Lebesgue Integration. May be used as a text with the first two-thirds of the same author's book on Elements of Real Analysis.

Casper Goffman. *Real Functions*. Prindle, Weber and Schmidt, 1953. Recommended for supplementary reading. Does not treat items (c), (d), and (e) as such.

J. Dieudonné. *Foundations of Modern Analysis*. Academic Press, 1960. Recommended. It does not treat the Lebesgue Integral, but may easily be supplemented by Goffman above. The chapter on analytic functions may be omitted.

Richard R. Goldberg. *Methods of Real Analysis*. Blaisdell, 1964. Highly recommended. It does not treat metric linear spaces as such but contains an exposition of $C[a, b]$, L^2 and Fourier Series.

Heider and Simpson. *Theoretical Analysis*. Saunders Company, 1967. Very highly recommended. Seems to have been written especially for this course! Contains an excellent treatment of all the topics listed above except (i) and (k). It is also an excellent introduction to Functional Analysis.

S. T. Hu. *Elements of Real Analysis*. Holden-Day, 1967. Highly recommended. It treats all topics except (i), (j), (k), and (l) in the usual "Hu" style. It is also excellent for self-study.

McShane and Botts. *Real Analysis*. Van Nostrand, 1959. Recommended especially for beginning graduate students. The treatment of L^p spaces is very condensed. Excellent for supplementary reading.

John F. Randolph. *Basic Real and Abstract Analysis*. Academic, 1968. No detailed treatment of metric or metric linear spaces, but E^m is treated as a metric space and spaces l^2 and L^2 are discussed.

H. L. Royden. *Real Analysis*. Macmillan, 1968. Highly recommended as a text or for supplementary reading. Also an excellent introduction to Functional Analysis. Includes more material than needed for a first year course. Allows selection to emphasize various topics. The approach is topological but accessible for a first course.

B. Sz-Nagy. *Introduction to Real Functions and Orthogonal Expansions*. Oxford, 1965. Recommended highly for supplementary reading. Contains no formal treatment of metric spaces, topological spaces or metric linear spaces. The approach is classical. Contains applications to L^p Spaces, Fourier Series, Orthogonal Sequences of Functions and Summability Theory.

Angus E. Taylor. *General Theory of Functions and Integration*. Blaisdell, 1965. Recommended for supplementary reading. It seems more suitable for a bone fide Function Theory Course.

G. S. GILL, Brigham Young University

FILMS

Volumes of Shells. By George F. Leger. Calculus Film Project of the MAA under direction of H. M. MacNeille. Available (rent or buy) from Modern Learning Aids in the U. S. and Canada. 8 min., 16 mm., color.

This eight minute color film discusses the representation of the volume of a solid of revolution as a definite integral, approximating the given solid by a finite union of cylindrical shells. The presentation is well conceived and should be accessible to the average calculus student. Although the same material could be presented just as clearly by a talented lecturer working at the blackboard, the film should be a useful supplement to the standard calculus course, either for showing in the classroom or (even better) for individual viewing outside the classroom on the compact equipment available for this purpose. The film should probably be regarded as supplementary to the lecturer's presentation of the material, rather than intended to replace it. A series of such films when so used should be very beneficial in stimulating the students' interest in calculus, by adding variety to the subject and making it less formidably abstract. They might also serve to improve the quality of the lecturer's presentation, by providing both him and the students a standard of comparison.

ERRETT BISHOP, Univ. of California, La Jolla

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)—18 (second graduate year). A boldface star (★) marks a notable book of general interest.

Algebra

Matrix Theory. By J. N. Franklin (Calif. Inst. of Tech.). Prentice-Hall, Englewood Cliffs, N.J. 1968. $10.95. "A rigorous but practical approach to matrix theory for

modern mathematicians, engineers, and scientists who expect to use digital computers." Topics are determinants, linear equations, differential equations, eigenvalues, eigenvectors and canonical forms, Jordan canonical form, variational principles and perturbation theory, numerical methods (100 pages). There is a slip on page 171 where the author connects vector norms and convex bodies. The omission of the condition that the convex body be centrally symmetric makes the following Theorem 4 incorrect. T (15).

Idealtheorie. By Wolfgang Krull. 2nd rev. ed. Springer-Verlag, New York, 1968. ix+160 pp. $7.00. Corrections and three supplements (to sections 15, 17, and 35). P.

Basic Algebraic Systems. An Introduction to Abstract Algebra. By Richard Laatsch (Miami Univ. Oxford, Ohio). McGraw-Hill, New York, 1968. xii+224 pp. $7.95. Topics are fundamental concepts (41 pages), groups, subgroups and homomorphisms, two-operation systems, polynomials, topics in group theory, vector spaces, Boolean Algebra and mathematical proof. T (14–15).

Elementary Linear Algebra. By Marvin Marcus and Henryk Minc (Univ. of California, Santa Barbara). Macmillan, New York, 1968. xv+267 pp. $8.95. Designed for a short course at the freshman-sophomore level, and therefore a candidate for a part of the modern elementary calculus sequence. Topics include determinants, characteristic roots, and quadratic functions (forms). T (13–14).

Introduction to the Theory of Abstract Algebras. By Richard S. Pierce (Univ. of Washington). Holt, Rinehart and Winston, New York, 1968. viii+147 pp. $5.50. "Abstract algebra" is the "universal algebra" of recent books with that title by P. M Cohn and G. Gratzer. Chapters are on set theory, basic concepts, subdirect decompositions, direct decompositions, free algebras, varieties of algebras. There is a bibliography and symbol list. T (16–17), S, P, L.

Combinatorial Identities. By John Riordan (Bell Telephone Lab. Murray Hill, N.J.). Wiley, New York, 1968. xii+256 pp. $15.00. An effort to bring order into what seems to be the intrinsically chaotic subject of identities involving binomial coefficients and related quantities. P, L.

Monotone Processes of Convex and Concave Type. By R. Tyrrell Rockafellar. Memoirs of the AMS, Num. 77. AMS, Providence, R.I., 1967. $1.60 (paper). Convex analogues of linear transformations. P.

Linear and Matrix Algebra. By Bernard Vinograde (Iowa State Univ.). Heath, Boston, Mass., 1967. ix+252 pp. $6.95 (cloth), $3.95 (paper). Presupposes only elementary analytic geometry and no abstract algebra. Beginning with intuitive motivations, the author adopts linear transformations as the main idea, but "a good deal of emphasis is given to matrix algebra because it is exceptionally adaptable to the needs of the physical and social sciences." T (13–15).

Analysis

Hilbert transformation, gebrochene Integration und Differentiation. By Paul L. Butzer and Walter Trebels (both of Rhein.-Westf. Techn. Hochschule, Aachen). Westdeutscher Verlag, Köln and Opladen, 1968. 81 pp. 70 DM. An extensive account of fractional integration and differentiation of the real line. The price of nearly 20¢ a page will undoubtedly win top honors for 1968!.

Metric Spaces. By E. T. Copson (Univ. of St. Andrews). Cambridge Tracts in Mathematics and Mathematical Physics No. 57. Cambridge Univ. Press, New York, 1968. 143 pp. $5.00. The author aims to "provide a more leisurely treatment of metric

spaces" than is usually found in texts on functional analysis and thereby to provide a sound basis for courses in that area. T (15–16), S, P, L.

Introduction to Analysis. By Edward Gaughan (New Mexico State Univ.). Brooks/Cole, Belmont, California., 1968. viii+310 pp. $8.95. The goals are rigor and "an accurate intuitive feeling for analysis" via the usual topics of one dimensional calculus. T (16).

Integral Equations. By Guido Hoheisel (Univ. of Cologne). Translated by A. Mary Tropper (Univ. of London). Nelson, London, 1967. 96 pp. $5.50. " . . . a survey of the main results in the theory of non-singular linear integral equations." T (16).

Total Positivity. Vol. I. By Samuel Karlin (Stanford Univ.). Stanford University Press, Calif., 1968. xi+576 pp. $17.50. An impressive volume, the first of two on this concept which plays an important role in various domains of mathematics, statistics, mechanics, and economics. Here the concentration is on the analytic structure of totally positive functions. Volume II will elaborate the oscillation characteristics of the resolvent kernel and of eigenfunctions associated with integral operators induced by totally positive kernels. There are historical comments and a bibliography. P, L.

The Operator of Translation Along the Trajectories of Differential Equations. By M. A. Krasnosel'skii. Trans. Math. Mon. Vol. 19. AMS., Providence, R.I., 1968. vi+294 pp. "15.40. " . . . certain problems in theory of periodic solutions of nonautonomous systems of ordinary differential equations with right-hand members that are periodic with respect to time." P.

Analysis I. By Serge Lang (Columbia Univ.). Addison-Wesley, Reading, Mass., 1968. xi+460 pp. $10.75. Part one reviews calculus for those who have not had the assumed two year course. Parts two through five cover convergence (normed vector spaces, limits, compactness, series, integrals in one variable), applications to the integral (approximation with convolutions), Fourier series, improper integrals, Fourier integral, calculus in vector spaces (including ordinary differential equations), and multiple integration (and differential forms). The author has tried to steer a course between the old computational advanced calculus and extreme abstraction. T (15).

Advanced Calculus. By Lynn H. Loomis and Shlomo Sternberg (both of Harvard Univ.). Addison-Wesley, Reading, Mass., 1968. 580 pp. $13.75. For an honors course to follow calculus of one variable and some linear algebra. This book deals with calculus in normed vector spaces and then with calculus of differentiable manifolds. The last three chapters are on exterior calculus, potential theory and classical mechanics. The symbol-word ratio is high. T (14–15, honors).

Linear Difference Equations. By Kenneth S. Miller (Columbia University, Electronics Research Lab. Now the Riverside Research Institute). Benjamin, New York, 1968. x+105 pp. $9.50 (cloth), $4.95 (paper). The author has "attempted to present the subject matter of linear difference equations in such a manner, that consistent with providing a unified picture, the overlap of content with recent texts is minimized." Some of the material is new. There are no exercises. The price of 5¢ a page for hard cover and 5¢ a page for paperbacks suggests that the publisher is trying for some sort of record. S, P.

Introduction to Complex Analysis. Revised Edition, By Zeev Nehari (Carnegie Inst. of Tech.). Allyn and Bacon, Boston, Mass., 1968. xi+272 pp. $7.95. The original edition of 1961 has been revised by minor improvements, new exercises, a more general version of Cauchy's theorem, and a proof of the Riemann mapping theorem. T (15–16).

Boundary Value Problems of Mathematical Physics. Vol. II. By Ivar Stakgold (Northwestern Univ.). Macmillan, New York, 1968. viii+408 pp. $13.95. Volume 1 was reviewed telegraphically in June 1967. This volume is built on the first one and deals with distributions, generalized solutions, potential theory, equations of revolutions, variational and related methods. There are appendices on spherical harmonics and asymptotic expansions. T (16), S, P.

Applications

The Philosophy of Quantum Mechanics. By D. I. Blokhintsev. Reidel, Dordrecht, Holland, and Humanities Press (distributor in U.S. and Canada), New York, 1968. vii+132 pp. $10.50. Mostly mathematics and physics, but some quite interesting philosophy. The Russian title (1965) was *Basic Questions of Quantum Mechanics*. P.

Language, Mathematics, and Linguistics. By Charles F. Hockett (Cornell Univ. and the Rand Corp.). Mouton, The Hague, 1967. 243 pp. 21 DG (paper). A mathematical introduction followed by an account of the author's own work. T.

Mathematical Economics. By Kelvin Lancaster (Columbia Univ.). Macmillan, New York, 1968. xii+411 pp. $10.95. Designed for a graduate course in mathematical economics and as a reference work for economists. The main parts of the book are optimizing theory, static economic models, dynamic economic models, and mathematical tools (including foundations, linear algebra, convexity, matrices, mappings, differential and difference equations, and calculus of variations). T, S, P.

Operational Research. French-English, English-French Vocabulary. Edited by A. L. Oliver. American-Elsevier, New York, 1968. x+151 pp. vi+148 pp. $9.75. The vocabulary covers probability and statistics, logic, programming, graphs, scheduling, flow, decision theory, game theory, queueing theory, stock control, reliability and maintenance, econometrics and military operational research. L.

Mathematical Linguistics in the Soviet Union. By Ferenc Papp. (Univ. of Debrecen). Mouton, The Hague, 1966. 165 pp. 22 DG (paper). In addition to chapters on structuralism, statistical modelling, structural modelling, machine translation, other applications of mathematical linguistics and mathematical linguistics in higher education, there are fifty pages on history and a twenty-eight page bibliography. S, P, L.

Computers, etc.

Die Programmierer. Eliten der Automation. By Karl Bednarik. Fischer Bücherei, Frankfurt, 1967. 140 pp. 95¢ (paper). A popular discussion of computers, programming, and information handling. P.

Computer Studies in the Humanities and Verbal Behavior. A quarterly published by Mouton and Co., P.O. Box 1132, The Hague, Netherlands. $10.00 per volume. There is a very broad board of editors including P. R. Halmos and H. P. Edmundson. The first number contains a bibliography on computer art.

Programming, an Introduction to Computer Languages and Techniques. By Ward Douglas Maurer (Dept. of Elec. Engineering, Univ. of Calif., Berkeley). Holden-Day, San Francisco, 1968. xiv+306 pp. $10.50. This book is intended for students who already have some knowledge of specific programming languages. It concentrates on general principles and techniques or programming in algebraic and machine language. There is no bibliography, but the book is intended as a reference for programming terminology, and the index contains over 1,000 terms. T (15–17), P, L.

Education

Objectives for Mathematics Learning. Some Ideas for the Teacher. By Shmuel M. Avital and Sara J. Shettleworth (both of The Ontario Institute for Studies in Education). OISE, Toronto, 1968. 57 pp. $1.50. Although directed primarily to precollege mathematics this pamphlet contains some ideas relevant to all mathematics teaching. TT.

Mathematics Teaching Area Examination. By Morris Bramson (Martin Van Buren High Sch., New York City). Arco, New York, 1968. 140 pp. $5.50 (cloth), $3.95 (paper). The main content consists of six sample tests similar to the National Teachers Examination in Mathematics of the Educational Testing Service. The NTE examinations are designed for teachers who have completed their training and are about to seek positions. They cover elementary mathematics through calculus, and many of the questions blend mathematical and pedagogical judgment. They reflect throughout the "new mathematics." TT.

CUPM Newsletter. CUPM, P.O.B. 1024, Berkeley, California 94701. May be obtained without charge. Number 2 (May, 1968) includes a history and survey of CUPM activities.

Sixth Report of the International Clearinghouse on Science and Mathematics Curricular Developments. Edited under the direction of J. David Lockard. A joint project of the American Association for the Advancement of Science and Science Teaching Center, University of Maryland. Single copies free (25¢ postage) from the editor, Science Teaching Center, University of Maryland, College Park, Maryland, 20742. xlvii +441 pp. The previous reports were reviewed here in May and December 1967. The clearinghouse was established in 1962 and its first report was released in 1963. Like previous reports, this gives a great deal of well organized and indexed information about curricular projects and development. The CUPM is described on pages 424–425. S, L.

General

Science, Numbers, and I. By Isaac Asimov (Boston Univ.). Doubleday, Garden City, 1968. xii+240 pp. $4.95. A little mathematics and lots of Asimov. Suitable present for juveniles.

English-German Technical and Engineering Dictionary. By Louis De Vries (Iowa State Univ.), and Theo. M. Hermann. McGraw-Hill, New York, 1967. 1154 pp. $27.50. Mathematical coverage is inadequate. For example an appropriate German equivalent of "lattice" does not appear.

Basic Concepts of Mathematics and Logic. By Michael C. Gemignani (SUNY at Buffalo). Addison-Wesley, Reading, Massachusetts, 1968. 280 pp. $7.95. "A first look at mathematics at the college level" with strong emphasis on logic and set theory, for either terminal or continuing students. T (13).

James and James. Mathematics Dictionary. Third Edition. Van Nostrand, Princeton, New Jersey, 1968. vii+517 pp. $17.50 (Multilingual edition). This new edition edited by R. C. James and E. F. Beckenbach, has the same contributors and translators as the previous edition. It has been revised and augmented by about 800 new terms in order to bring the total to approximately 8000. By slightly increasing the page size, the publisher has accomplished this in fewer pages. Birth and death dates of some mathematicians have been added. The dictionary remains the only serious one in English reaching through the most common terminology of undergraduate mathematics. The multilingual edition contains vocabularies in French, German, Russian

and Spanish. It is too bad that there is not also an Italian vocabulary, which would be more useful than the Spanish. T, L.

Kleine Enzyklopädie. Mathematik. VEB Verlag Enzyklopädie, Leipzig, 1967. 837 pp. 28 MDN. Part I, 322 pages, on pre-college mathematics: Part II, 303 pages, on college level analysis, geometry probability, statistics and applications; Part III, 91 pages, containing brief introductions to special fields (set theory, abstract algebra, functional analysis, information theory, etc); numerical tables and detailed index. Although primarily elementary in character and quite weak in modern algebra and topology, this little encyclopedia has much valuable information and is attractively printed with the use of many bright colors. It includes nearly 1000 illustrations, most of them in color, and a collection of 72 plates giving a panorama of mathematical personalities, education, applications, instruments, and documents. There should be as fine a product in English, not a translation but a somewhat similar work with more adequate treatment of modern mathematics. P, L.

Modern Elementary Mathematics: A Programmed Introduction. By Robert M. Todd (Boston Univ.) and Cecil W. McDermott (Hendrix College). Allyn and Bacon, Boston, Mass., 1967. xii+292 pp. $3.95 (paper). Sets, sentences, relations, numbers, numeration systems and arithmetic, real numbers. S, TT.

Mathematics. The Alphabet of Science. By Margaret F. Willerding (San Diego State College) and Ruth A. Hayward (General Dynamics, Convair Div.). Wiley, New York, 1968. xii+285 pp. $6.95. For a course for nonmathematicians. Topics: logic, divisibility, primes, congruences, Pythagorean theorem, groups and fields, probability and statistics, finite geometry, matrices, computers and numeration, computer language. Historical comments and some portraits. T (13).

Geometry and Topology

Thirteen Papers on Group Theory, Algebraic Geometry and Algebraic Topology. AMS Translations, Series 2, Vol. 66. AMS, Providence, R.I., 1968. iv+272 pp. $13.80. P.

Twelve Geometric Essays. By H. S. M. Coxeter. Southern Illinois University Press, Carbondale, and Fefer and Simons, London. xi+274 pp. $7.00. Photo-offset reprint of twelve papers (1935–1965) with a charming preface describing the origin of each. The essays cover a number of topics in geometry and applications. Several supplement the author's *Regular Polytopes* (1948, 1963).

CUPM Geometry Conference. Proceedings. Part 1: Convexity and Applications. Based on Lectures by Branko Grünbaum and Victor Klee. Part II: Geometry in other Subjects. Based on Lectures by A. M. Gleason and Norman Steenrod. Part III: Geometric Transformation Groups and other topics. Based on lectures by H. S. M. Coxeter, H. Busemann, G. Culler, P. Hammer, P. J. Kelly, and W. Prenowitz. Limited number of free copies obtainable from CUPM, P.O. Box 1024, Berkeley, California 94701.

Foundations of Euclidean and Non-Euclidean Geometry. By Ellery B. Golos (Ohio Univ.). Holt, Rinehart and Winston, New York, 1968. xiii+225 pp. $7.95. This is not a book on the foundations of Geometry, but an introduction "in keeping with the spirit of Euclid, and . . . the modern developments in axiomatic mathematics." The approach is axiomatic and synthetic. The emphasis is on method rather than information. The author's purpose is to " . . . blend several aspects of mathematical thought, namely, intuition, creative thinking, abstraction, rigorous deductions, and the excitement at discovery." The three parts are an analysis of axiomatic systems, an axiomatic

development of elementary geometry, and non-Euclidean Geometry. TT, T (for an introduction to axiomatic synthetic geometry), S.

A Second Introduction to Analytic Geometry. By G. Hochschild (Univ. of California, Berkeley). Holden-Day, San Francisco, 1968. 63 pp. $3.50 (paper). This is "an examination of the basic geometrical features of Euclidean three-space from the point of view of rigorous mathematics," designed to help the college student relate the topics treated in the elementary courses to the more rigorous formulations to come. S.

Methods of Algebraic Geometry. By W. V. D. Hodge (Pembroke College, Cambridge) and D. Pedoe (Univ. of Minnesota). Volume I. Book I: Algebraic Preliminaries. Book II: Projective Space. Volume II. Book III: General Theory of Algebraic Varieties in Projective Space. Book IV: Quadrics and Grassmann Varieties. Cambridge Univ. Press, New York, 1968. viii+440 pp., xi+394. $2.95 each (paper). A welcome paperback reprint of the well known work first published in 1947 and reprinted in 1953. There is no mention of book five which made up a third volume of the original work and is not included here. T, S, P, L.

Introduction to Differential Geometry and Riemannian Geometry. By Erwin Kreyszig. Translated from the German edition (Akademische Verlagsgesellschaft, Leipzig 1967.) Mathematical Expositions no. 16. University of Toronto Press, Toronto, 1968. xii+370 pp. $10.95. An extension of the earlier book Differential Geometry, this deals with curves and surfaces in three dimensional Euclidean space and n-dimensional Riemannian geometry. The last chapter is on hypersurfaces and the last section on principal directions of a tensor and principal curvatures. T (15–16).

Topics from Inversive Geometry. By Albert E. Meder, Jr. Houghton Mifflin, Boston, Mass., 1967. iii+59 pp. $1.20 (paper). Although apparently designed for high school students, this pamphlet would be enlightening to many college students. S, L.

Space Time and Relativity. By Rolf Nevanlinna (Academy of Finland). Translated from the German by Gordon Reece. Addison-Wesley, Reading, Mass., 1968. xii+158 pp. $6.00 (cloth), $3.50 (paper). A philosophical, historical high level popularization based on lectures at the Universities of Helsinki and Zürich to students of all faculties. This book has been translated and augmented by some exercises for possible use as a textbook as well as for general reading. It presupposes no university level mathematics but touches on fundamental issues. The first sentence is "Geometry is the study of certain invariant time-independent forms and properties of space." T (13), S, P, L.

Foundation of Euclidean and Non-Euclidean Geometries According to F. Klein. By L. Redei (Jozsef Attila Univ. Szeged, Hungary). Pergamon, New York, 1968. x+400 pp. $19.00. Intl. Ser. Monog. in Pure and Appl. Math. No. 97. The author fills in gaps and extends Klein's treatment, confining himself to the foundations. P, L.

Stereograms. By Donald W. Stove. Editorial Adviser A. E. Meder, Jr., Houghton Mifflin, New York, 1966. iii+44 pp. $1.60. Although written for high school enrichment, this pamphlet is of interest to college students. It includes simple instructions for making stereograms, numerous examples and a viewer. S, P.

Some Properties Related to Compactness. By J. Van der Slot. Mathematical Centre Tracts, No. 19, Mathematisch Centrum, Amsterdam, 1968. 56 pp. ". . . a systematic study of a few topological generalizations of compactness in Hausdorff spaces," with special interest in obtaining "a characterization of real-compactness in which we do not use explicitly the special properties of real-valued continuous functions." P.

Differential Topology. First Steps. Andrew Wallace (Univ. of Pennsylvania). Benjamin, New York, 1968. xi+130 pp. \$9.50 (cloth), \$3.95 (paper). Intended to answer the question "What is Differential Topology about?" by someone familiar with advanced calculus, differential equations, and linear transformation of quadratic form but without previous knowledge of topology. Topics are topological spaces, differential manifolds, submanifolds, tangent spaces and critical points, critical and noncritical levels, spherical modifications, two-dimensional manifolds. The final chapter is called "Second Steps" and is followed by references and guide to further reading. T, S, P, L.

Analytic Geometry. By William Wernick (City Coll. of New York). Silver Burdett, Morristown, N.J., 1968. 294 pp. \$4.72. Coordinates and vectors in one and two dimensions, including curve sketching and conic sections. T (12–13).

Probability and Statistics

Introduction to Statistics. By Frank W. Carlborg (Northern Illinois Univ.). Scott Foresman, Glenview, Illinois, 1968. 301 pp. \$8.75. Addressed primarily to scientists and businessmen and assuming no previous knowledge of probability or statistics, the book gives the necessary preliminaries, but concentrates on multiple regression and analysis of variance. Only 10% of the material presupposes calculus and may be omitted. T.

Applied Statistical Decision Theory. By Howard Raiffa and Robert Schlaifer. MIT Press, Cambridge, Mass., 1968. xxviii+356 pp. \$3.95 (paper). A paperback reprint of the 1961 treatise on applied Bayesian statistical methods. There are no problems sets. S, P.

Internationale Tagung über mathematische Statistik und ihre Anwendungen. Berlin, Mai 6–9, 1966. Edited by K. Schroder. (Abhandlg. Dtsch. Akad. Wiss. Berlin, 1967, 4.) Akademie-Verlag, Berlin, 1968. 206 pp. 32.50 DM. Thirty five papers by thirty seven authors. P.

Statistics for Economists. By Wieslaw Sadowski. Translated from the Polish by J. Stadler. Pergamon, New York, 1968. vii+323 pp. \$10.00. In spite of the title this seems to be a fairly standard introduction presupposing calculus. It does not contain applications to economics. T (15).

Probability and Statistics. By Stephen S. Willoughby (New York Univ.). Silver Burdett, Morristown, N.J., 1968. 224 pp. \$4.40. An elementary but up to date treatment ending with hypothesis testing and game theory. The printing is unusually attractive. There are historical notes and portraits of Cardano, Feller, Kolmogorov, Chebyshev, Jakob Bernoulli, Fisher and von Neumann. T (12–13).

NOTABLE PAPERS

If you know of a recent paper that should be mentioned here, drop us a note with complete information (author in full, title in full, name of serial in full, date of serial, volume, page number) and a brief indication of content.

SMSG: The First Decade. By E. G. Begle in The *Mathematics Teacher*, March 1968, 239–245.

Mathematics—Practical and Impractical. By David Hawkins (Univ. of Colorado) in *Mathematics Teaching* (A quarterly published by the Association of Teachers of Mathematics, Vine Street Chambers, Nelson, Lancashire, England), Winter 1967. Discussion and examples of abstraction, applicable at all levels.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Hunter College: Dr. F. J. Weyl, Special Assistant to President, National Academy of Sciences, has been appointed Dean of Sciences and Mathematics, and Professor of Mathematics; Dr. Barry Cherkas, Catholic University of America and Applied Physics Laboratory, Johns Hopkins University, has been appointed Assistant Professor.

Herbert H. Lehman College: Associate Professor Seymour Hayden, Clark University, has been appointed Professor; Assistant Professor Joseph Lewittes, Yeshiva University, has been appointed Associate Professor; Associate Professor Annita Tuller has been promoted to Professor.

Dr. W. E. Hartnett, Parke Mathematical Laboratories, Inc., has been appointed Professor at State University College at Plattsburgh, New York.

Professor R. E. Horton, Dean of Los Angeles City College, has been appointed President of Los Angeles Valley College.

Associate Professor J. W. Kenelly, Clemson University, has been appointed Professor and Chairman of the Mathematics Department at Louisiana State University, New Orleans.

Associate Professor D. E. Koontz, Elizabethtown College, has been appointed Chairman of the Mathematics Department.

Dr. J. H. Manheim, Carnegie Fellow at the Center for the Study of Higher Education at the University of Michigan, has been appointed Dean of the College of Liberal Arts and Sciences, Bradley University.

Colonel C. G. Meehan, USA Retired, Assistant Professor at Grove City College, has been promoted to Associate Professor.

Dr. Catherine M. Murphy, Catholic University, has been appointed Assistant Professor at Purdue University.

Dr. R. A. Welker, Planning Research Corporation, Los Angeles, has accepted a position as Project Scientist with Booz, Allen Applied Research Inc.

Professor Emeritus N. A. Court, University of Oklahoma, died on July 20, 1968. He was a Charter Member of the Association.

Dr. J. A. Jensen, University of Wyoming, died on June 22, 1968. He was a member of the Association for eight years.

FELLOWSHIP AND RESEARCH OPPORTUNITIES IN MATHEMATICS

The Division of Mathematical Sciences of the National Research Council calls attention to a number of fellowships and other support for research in the mathematical sciences at both the predoctoral and postdoctoral levels to be awarded during the year 1968–69. Copies of the complete announcement are available from: Division of Mathematical Sciences, National Research Council, 2101 Constitution Avenue, N. W., Washington, D. C. 20418.

MATHEMATICAL SPECTRUM—A MAGAZINE OF CONTEMPORARY MATHEMATICS

The *Mathematical Spectrum* is a new magazine which addresses itself primarily to high school and undergraduate students. It will be published by the Oxford University Press on behalf of the Applied Probability Trust, a nonprofit making body established

by Trust Deed for the encouragement of study and research in the mathematical sciences. Its first two issues will be published during the academic year 1968–1969, with an annual subscription price of $1.20 (block subscriptions of 5 or more: $1.00 each). For further information, please write to Oxford University Press, 200 Madison Avenue, New York, N. Y. 10016.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-NINTH SUMMER MEETING OF THE ASSOCIATION

The Forty-ninth Summer Meeting of the Mathematical Association of America was held at the University of Wisconsin, Madison, Wisconsin, from Monday, August 26, to Wednesday, August 28, 1968, in conjunction with meetings of the American Mathematical Society, the Institute of Mathematical Statistics, the Society for Industrial and Applied Mathematics, and the Pi Mu Epsilon Fraternity. There were registered 2029 persons, including 891 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning, and on Wednesday afternoon. All sessions were held in the Union Theater of the Wisconsin Memorial Union at the University of Wisconsin. Presiding officers at the three Earle Raymond Hedrick Lectures were President E. E. Moise, First Vice-President Victor Klee, and Second Vice-President R. J. Walker; at the lecture by Professor Lefschetz, Professor D. C. Spencer; at the lecture by Dr. Engel, President Moise; at the lecture by Mr. Johntz, Professor C. O. Oakley; at the lecture on Tuesday by Professor Gehring, Professor L. V. Ahlfors; and at the Report from CUPM on Wednesday afternoon, Professor M. W. Pownall. The seventeenth series of Earle Raymond Hedrick Lectures was delivered by Professor Hyman Bass of Columbia University. The Program Committee consisted of E. R. Fadell, Chairman; J. G. Harvey, Morris Marden, P. E. Miles, and J. B. Rosser.

FIRST SESSION OF THE ASSOCIATION

Welcome on behalf of the University by Vice-President R. L. Clodius of the University of Wisconsin, Madison, and on behalf of the department of mathematics by Professor R. H. Bing.

The Earle Raymond Hedrick Lectures: *Algebraic K-Theory*, Lecture I, *Stability Theorems in Topology and in Algebra*, by Professor Hyman Bass, Columbia University.

The topological treatment of linear algebra in terms of vector bundles was briefly described. Then it was shown how some of this topological theory can be translated into a purely algebraic setting. This leads to various conjectures, and a few theorems, centered mainly around "stability" phenomena.

The Early Development of Algebraic Geometry, by Professor Solomon Lefschetz, Princeton University and Brown University.

This paper discussed the development of algebraic geometry, mostly of plane curves and as a chapter in the theory of algebraic functions of a single complex variable from Legendre and Abel to Riemann, Noether, Weierstrass with underscoring of the fundamental influence of Cauchy's theory of analytic functions.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, *Grothendieck Groups and Whitehead Groups*, by Professor Bass.

Topological K-theory suggests the introduction of the Grothendieck and Whitehead groups of a ring, and some of the desired formal properties of these groups were established, in analogy with their topological counterparts.

On Systematic Use of Applications in Mathematics Teaching, by Dr. Arthur Engel, Stuttgart, Germany.

By a systematic use of applications, the level, efficiency, and usefulness of mathematics teaching can be vastly increased. This is shown by outlining a course for grades 5 to 7. The aim of the course is that *the students learn thinking in statistical terms.*

First, students are familiarized with generators of random digits. Then they are confronted with many typical probabilistic problems. The random digits are used to solve the problems by simulation. By this students acquire the intuitive background of probability. Soon they realize that some of the problems can also be solved by reasoning. From this point on a significant part of elementary arithmetic, fractions, and algebra are absorbed into probability. Students acquire the necessary skill in these areas by solving probabilistic problems from many areas, like reliability, communication theory, hypothesis testing, random walks, Markov chains, etc.

Mathematicians Teach Disadvantaged Children on Daily Basis, by Mr. William Johntz, Director, Project S.E.E.D. (Special Elementary Education for the Disadvantaged), University of California, Berkeley.

The speaker described a five-year old project in which persons with graduate mathematical training, including Ph.D.'s, daily teach abstract conceptually oriented algebra to disadvantaged students (K-6) from the California ghettos using an open-ended provocative questioning process in which the students are led to discover mathematical concepts and structure for themselves. Even disadvantaged children with extremely low achievement and I.Q. scores demonstrate (arm waving) enthusiasm and competence partly because abstract mathematics is free of prior failure experiences and handicapping home influences. Their success raises the student's self image and consequently his motivation. The mathematicians are, thereby, turned on.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, *Congruence Subgroups and Reciprocity Laws*, by Professor Bass.

The calculation of Whitehead groups for rings of algebraic integers leads naturally to the "congruence subgroup problem," whose solution in terms of reciprocity laws was described.

Business Meeting of the Association; presentation of Lester R. Ford Awards and presentation of the honorary title of Executive Director Emeritus of the Association to Professor H. M. Gehman.

Quasiconformal Mappings, by Professor F. W. Gehring, University of Michigan.

This lecture was concerned with elementary properties of quasiconformal mappings. Various definitions for plane quasiconformal mappings were discussed first. Then a convergence theorem was proved illustrating the use of extremal length. Finally some of the problems in higher dimensions were mentioned.

FOURTH SESSION OF THE ASSOCIATION

Report from CUPM

Survey of Current CUPM Activities, by Professor R. P. Boas, Chairman of CUPM, Northwestern University.

The speaker outlined the current activities of CUPM and some of the Committee's plans for the future. The most significant current activities are covered by the addresses by Professors Rosenberg and Finkbeiner that follow. Recent ceilings imposed on the allowable expenditures under NSF grants may mean that most of the planned activities will have to be postponed for a year or more.

The Illinois Section's Commission on the Mathematics Preparation of Teachers of Elementary School Mathematics (CMPTESM): A Second Step Toward Program Improvement by Mr. C. L. Greeno, Illinois Institute of Technology.

The Illinois Section of the Association recently initiated a novel program for assisting to improve undergraduate programs in mathematics. At present, the approach appears to be successful and very promising as a regional activity to complement the efforts of CUPM. In the pilot effort, the Illinois Section's activities have been conducted primarily by CMPTESM. Although the Commission's efforts are limited to Illinois and to only one aspect of the undergraduate programs, it seems likely that the orientation and approach used could be readily adapted to the needs of other states with respect to each of the programs already treated by CUPM. This report reviewed the general features of CMPTESM's program, especially those that might be suggestive to other states. Its value hinges largely around philosophic orientations that have not been formally established. Consequently, this paper was offered not as an official statement of either the Illinois Section or its CMPTESM, but as reflecting observations made by the author as an individual while serving as a member of the Commission.

Mathematics in the Two-Year College for the University Parallel Student, by Professor Alex Rosenberg, Cornell University.

The curriculum in mathematics for prospective transfer students in Two-Year colleges, recommended by CUPM, was discussed. The basic course sequences were outlined. These consist of Calculus Preparatory Work, Calculus, Linear Algebra, Probability and Statistics, and courses for the training of elementary school teachers. In addition to these, some optional courses in Numerical Analysis, Finite Mathematics, Differential Equations, etc., which could be offered if students and staff are available, were described. There also was a discussion about the implementation of the recommended program, and questions of articulation were touched on briefly.

Preparation for Teaching College Mathematics, by Professor D. T. Finkbeiner II, Kenyon College.

Currently a majority of the persons who teach mathematics in the colleges and universities of the United States do not hold a Ph.D. degree in a mathematical science. Although the output of Ph.D.'s is rising, increasing demands for higher quality and quantity of undergraduate instruction in mathematics underline the importance of the following question: "What is the best way to arrange the early part of the graduate program in mathematics to provide background for effective college teaching, without retarding the student's progress toward the Ph.D.?" An answer is proposed in the forthcoming CUPM report *A Beginning Graduate Program in Mathematics,* described in this talk.

Introducing Applications into the Undergraduate Mathematics Curriculum, by Professor R. P. Boas, Chairman of CUPM, Northwestern University.

The speaker, substituting for Professor Donsker, described Donsker's plan for encouraging the writing of undergraduate textbooks in several branches of applied mathematics. A series of such books is planned by a commercial publisher; it is hoped that, when they are available, departments of mathematics will be able to offer courses in applied mathematics of the kind recommended by CUPM. The speaker also urged that introductory calculus should be presented in the spirit of applied mathematics rather than as an abstract mathematical theory.

Each of the above presentations was followed by questions and discussion from the audience.

SPECIAL SESSIONS OF THE ASSOCIATION

a. Planning and Implementation of New Mathematics Facilities

A session on the Planning and Implementation of New Mathematics Facilities was held on Monday at 7:00 P.M. in Room 102 of Van Vleck Hall. It was organized by Profes-

sor J. W. Givens, Director, Applied Mathematics Division, Argonne National Laboratory. Professor J. S. Frame of Michigan State University presided.

The Office of Education Science Facilities Program was discussed by Mr. O. E. Stamberg, Chief of the Graduate Facilities Branch, Bureau of Higher Education, Office of Education, HEW. The Graduate Science Facilities Program at the National Science Foundation was discussed by Mr. L. O. Herwig, Staff Associate with the Institutional Relations Division of the NSF. The Conference Board of the Mathematical Sciences publication on "Buildings and Facilities for the Mathematical Sciences" by J. Sutherland Frame (with John W. McLeod, F.A.I.A.) was discussed by Dr. T. A. Botts, Executive Director of CBMS. Copies of this reference volume were available for inspection and order.

This was followed by a panel discussion by members of the panel and questions from the audience. The discussion was led by Professor Frame. Members of the panel, in addition to the previous speakers, were Professor Elizabeth Scott, University of California, Berkeley, Professor D. E. Myers, University of Arizona, and Mr. Kent Peters, partner in Peters and Martinson, architectural firm, member of American Institute of Architects and designer of Van Vleck Hall.

b. Film Showings

Film showings were held in Room 6210 of the Social Science Buidling. This included a showing of all the films produced by the CEM Calculus Film Project, beginning on Sunday at 7:00 P.M. with some introductory remarks by Professor H. M. MacNeille, Project Director. The showing of films on Sunday ended at 9:40 P.M.

On Monday evening the following films were shown:

7:00–8:00 P.M. FIXED POINTS: A Lecture by Solomon Lefschetz (A CEM Individual Lectures film in color).

8:10–8:50 P.M. CHALLENGING CONJECTURES: A Lecture by R. H. Bing (A CEM Individual Lectures Film in b & w).

On Tuesday evening the following films were shown:

Films of the NCTM Series: Mathematics for Elementary School Teachers (in color)

7:00–7:27 P.M. Development of Our Decimal Numeration System.

7:30–7:54 P.M. Multiplication Algorithms and the Distributive Property.

8:00–8:27 P.M. The Whole Number System—Key Ideas.

This was followed by a showing of the films of the Calculus Project not already shown on Sunday evening. The showing of films terminated at 9:45 P.M.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Sunday at 10:00 A.M. in the Lake Shore Room of the Wisconsin Center at the University of Wisconsin, with thirty-four members present. Among the items of business transacted were the following:

The Board elected Professor P. D. Lax as an additional Associate Editor of the MONTHLY. The Board elected the following as Associate Editors of the MATHEMATICS MAGAZINE for the five-year term beginning January 1, 1969: Professors H. W. Eves, Raoul Hailpern, R. E. Horton, E. A. Maier, Hans Sagan, and S. T. Sanders.

The Board elected Professor E. F. Wilde of Beloit College as Governor from the Wisconsin Section to fill the unexpired part of the term ending June 30, 1969, of Professor C. J. Vanderlin, who has moved from the Wisconsin Section for a period of two years.

The Board voted to invite Professor E. A. Bishop of the University of California, San Diego, to deliver the eighteenth series of Earl Raymond Hedrick Lectures at the 1969 Summer Meeting.

The Board voted to increase the amounts of the prizes to high ranking individuals and teams in the William Lowell Putnam Mathematical Competition, to award plaques

to each of the top ten teams, and certificates to each of the top ten or honorable mention individuals, and to honor the First Place Team at a dinner.

The Board instructed the Secretary to submit to the membership at the business meeting next January an amendment to the By-Laws providing that the Board of Governors shall establish the annual dues for ordinary membership (see page 1053 of this MONTHLY).

The Board passed a resolution of thanks to the General Electric Company for its contribution of $2000 toward printing and distributing the brochure YOU'LL NEED MATH.

The Board approved the following schedule of future meetings of the Association: New Orleans, Louisiana, January 25–27, 1969; University of Oregon, August 25–27, 1969; Miami, Florida, January 24–26, 1970; University of Wyoming, August 24–26, 1970; Atlantic City, New Jersey, January, 1971.

The Executive Director reported the membership of the Association as 17,723 individual members, an increase of 127 since the corresponding date last year, 3 corporate members, and 244 academic members.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President Moise presiding.

The fourth set of Lester R. Ford Awards were presented by President Moise to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE in 1967. The Awards, in the amount of $100 each, were presented for six articles (for further details on these Awards, see the August–September issue of this MONTHLY, page 827).

The title of Executive Director Emeritus of the Association was conferred upon Professor Harry M. Gehman, effective August 1, 1968. The citation which appears on pages 1052–3 of this MONTHLY was prepared by Professor E. A. Cameron and read by President Moise. After a brief response by Professor Gehman in which he expressed his gratitude, the audience paid a standing tribute to him.

President Moise announced that the film LET US TEACH GUESSING: A DEMONSTRATION WITH GEORGE POLYA had received the blue ribbon award in the "Math and Physics" film category of the Educational Film Library Association's 10th Annual American Film Festival held in New York City, May 28–June 1. The blue ribbon award is given to only one film in this category. A blue ribbon medal and accompanying certificate were presented in person to Professor R. A. Rosenbaum, Project Director of the Individual Lectures Project, and in absentia to Professor George Polya and Dr. A. N. Feldzamen, Executive Director of the Project.

President Moise also announced that the film INFINITE ACRES was nominated for final showing at the same festival. This film by Professor Melvin Henriksen was produced by the Calculus Film Project, whose Director was Professor H. M. MacNeille.

The Secretary then reported on some of the actions taken by the Board on Sunday. He announced that the Committee on Assistance to Developing Colleges had started an employment register for these colleges. Letters were sent to all developing colleges asking them to list the openings, and announcements were placed in the MONTHLY and the NOTICES of the AMS asking people interested in helping developing colleges to send their names to the Committee. The lists were then exchanged so that contacts could be made from both directions. The lists were compiled at the mathematics department at Indiana University and met with unexpected interest from both sides. A special table was set up at the summer meeting in Madison to continue this recruiting.

The Secretary also reported that the Committee on Institutes, in a letter to Dr. Peter Muirhead, Associate Commissioner, Bureau of Higher Education, Office of Education, Washington, D.C., had suggested the possibility that the Office of Education might pro-

vide funds for summer institutes in urban areas for the training of junior college teachers in the mathematical sciences—these institutes to be run by qualified universities in these areas. In this experiment, the Committee is particularly anxious to reach the areas of little opportunity in these cities where it would seem that the teaching personnel would be less trained as well. In his reply, Dr. Muirhead encouraged the Committee's "efforts and interest in the training of junior college teachers, particularly the summer institutes in urban areas." The Secretary emphasized that the deadline for proposals for such institutes to be held in the summer of 1970 is expected to be as early as July 1, 1969. He announced that several such institutes are planned for the summer of 1969.

The Secretary reported the receipt of several grants from the National Science Foundation including one for support of the Individual Lectures Film Project for the period April 1, 1968 to March 31, 1970. He also reported receipt of an additional amount of $4,700 from the same anonymous donor who had previously contributed funds to the Association to support certain activities of the Committee on Educational Media which were not provided for in the NSF grant.

The Secretary announced that copies of the third edition of the GUIDEBOOK TO DEPARTMENTS IN THE MATHEMATICAL SCIENCES IN THE UNITED STATES AND CANADA, published in 1968, are now available from the Washington Office of the Association for 50 cents per copy. This book provides in summary form information about the location, size, staff, library facilities, course offerings, and special features of departments in the mathematical sciences in 1400 four-year colleges and universities in the United States and Canada.

The Secretary then introduced the Association's new Executive Director, Dr. A. B. Willcox, and announced that the Association's new central office is expected to be ready for occupancy in October of this year.

The Secretary acknowledged the splendid job done by the local Committee on Arrangements in providing so thoughtfully for the welfare of the participants at the meeting and for the smooth operation of all of its aspects. He introduced Professor R. H. Bing, Chairman of the Committee, who coordinated so effectively the work of the Committee, and singled out for special commendation Professor J. B. Rosser, who supervised so carefully the arrangements for the SIAM beer party and the picnic, and Professor R. A. Brualdi, Publicity Director of the meeting, for his thoughtful and successful handling of the publicity.

MEETING OF SECTION OFFICERS

The meeting of representatives of the Sections was held on Monday evening in the Lake Shore Room of the Wisconsin Center. Professor L. E. Mehlenbacher, Chairman of Committee on Sections, presided. Fifty-eight persons were present, representing twenty-seven of the twenty-eight Sections of the Association.

President Moise emphasized the vital role played by the Sections of the MAA. He noted that not more than one-tenth of the membership is able to visit one of the two national meetings held each year, so that the Sections have to play a large part in giving effective service to the individual members of the Association.

Dr. Willcox, the newly appointed Executive Director of the Association, urged the officers of the Sections to visit the new central office of the Association at 1225 Connecticut Avenue, N.W., Washington, D.C., halfway between the Mayflower Hotel and DuPont Circle, to make suggestions on how the Association might better serve its membership and to ask for information. He emphasized that he is particularly anxious to maintain and even increase the contacts between the Sections and the central office of the Association and, for this purpose, plans to visit the Sections whenever possible. He also announced his desire to expand and broaden the membership of the Association.

Professor Nura D. Turner reported on the results of the participation of the Upstate

New York MAA Contest Section in the Fourth British Mathematical Olympiad held on May 20, 1968 in London. An article will appear in an early issue of the MONTHLY regarding this Olympiad.

Professor C. T. Salkind, Chairman of the Committee on High School Contests, spoke on the first year of the changed format of the MAA Annual High School Mathematics Contest. The Committee was concerned with two important objectives in experimenting with the changed format, namely, maintaining discrimination in the higher scores and raising the lower scores to reduce the feeling of "frustration." He presented statistics showing success on both points.

Professor Abraham Schwartz, Governor of the New York Metropolitan Section, gave a report on the Mathematics Speakers Bureau of the Metropolitan New York Section. During the last academic year, 25 speakers had given 58 talks, compared to 48 speakers giving 167 talks the previous year when the program was supported by NSF. At least part of the decrease in participation, however, was due to the teachers' strike in New York City at the beginning of the fall term.

Professor H. M. Bacon, Chairman of the Committee on Secondary School Lecturers, reported on the work of that Committee. He summarized the responses received in reply to the questionnaires sent to all Section secretaries concerning programs of visiting lecturers to secondary schools which are or have been in effect within the Sections. The Committee is encouraged by the reports of lecturer programs sponsored by some of the Sections. These efforts deserve commendation and whatever support they can be given. At present, this must be principally moral support, but the Committee is optimistic enough to hope that support in more tangible form may eventually be developed.

Professor R. D. Boswell, Jr., Chairman of the Committee on Visiting Lecturers, reported on the work of that Committee. During 1967–68, 226 institutions were visited and the lecturers spent 304 days of lecturing. He reported that many members of the Committee and many lecturers feel that a two-day visit is much more effective than a one-day visit in that the lecturer gets to know the institution better and can be more effective in conferences. For 1968–69, the Committee has 77 lecturers lined up with 19 of them new to the program or the region in which they will be lecturing. Thirty-two of the lecturers are new to the program or have served only one year previously.

Professor Dorothy L. Bernstein, Chairman of the Committee to Study the Reorganization of Association Publications, reported on the work of that Committee. The recommendations of this Committee were accepted with some modifications by the Board of Governors at the January meeting in San Francisco. The recommendations included a strong emphasis on expository articles on contemporary mathematics in both the MONTHLY and the MATHEMATICS MAGAZINE, survey articles, book reviews, interestingly written pedagogical articles, no "minor research" papers, an attempt to make the MONTHLY a periodical of importance to anyone teaching collegiate mathematics and the MAGAZINE of importance to anyone with general mathematical interests, with a view to making the MAGAZINE also an official journal in the future.

Professor E. A. Nordhaus reported on the new procedures in the Michigan Mathematics Prize Contest. The examination is given in two parts, scheduled about a month apart. Part I is a multiple-choice examination, which is machine graded, and serves as a qualifying examination for Part II, which consists of five more difficult problems designed to test the mathematical ingenuity of the students. Each part takes 100 minutes to administer. To stimulate interest in mathematics throughout the school year, it is planned to inaugurate statewide team competition similar to that employed in the William Lowell Putnam Collegiate Competition. More flexibility regarding qualifiers for Part II will be introduced as a result of a rather curious observation that many students who performed rather poorly on the multiple-choice examination, and as a result barely qualified for Part II, did surprisingly well on the more difficult part of the examination.

Professor Arnold Wendt reported on the work of the Commission on the Mathematical Preparation of Teachers of Elementary School Mathematics of the Illinois Section.

This Commission has been charged by the Section to consider the question: "What mathematical background is necessary for the fully-qualified teacher of mathematics in the elementary school?" Its activities are supported financially by the Section. It is hoped that ultimately the Commission will be able to produce specific course objectives and concomitant course outlines for the guidance of those concerned with the mathematical competence of teachers of elementary school mathematics.

Dr. R. G. Long, Project Director of the Individual Lectures Film Project, reported on the work of that Project. The films to be produced by this Project are designed for junior and senior mathematics majors in the U.S. colleges and universities. They will, in every case, fit within the one-hour periods ordinarily available for film showings, and each will be accompanied by a printed manual which goes further than restating the content of the film. Rather than being films on classroom material which is already readily available, they will be on topics similar to those used by mathematics clubs: interesting new developments, applications of mathematics which have not yet assumed a standard position in courses, expositions that reveal connections between subjects familiar to the audience, etc.

Professor Mehlenbacher reminded the Section Officers of the existence of a fund available to the Committee on Sections for worthwhile projects of the Sections. During the past year, three requests for support of projects had been received. The total amount annually available to the Committee is $500, but it has never been spent.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. Two sets of Colloquium Lectures, each consisting of four lectures, were presented. Professor D. C. Spencer of Stanford University delivered one set on "Overdetermined Systems of Partial Differential Equations" on Tuesday at 1:30 P.M., on Wednesday at 8:30 A.M. and on Thursday and Friday at 10:00 A.M. Professor J. W. Milnor of Princeton University and the University of California, Los Angeles, delivered the other set on "Uses of the Fundamental Group" on Tuesday at 2:45 P.M., on Wednesday at 9:30 A.M., and on Thursday and Friday at 9:00 A.M. Invited addresses were given by Professor W. C. Hsiang of Yale University on "Nonsimply Connected Differential Topology" on Thursday at 1:45 P.M., by Professor P. A. Griffith of the University of Houston and Princeton University on "Some Transcendental Problems in Algebraic Geometry" on Thursday at 3:00 P.M., and by Professor V. W. Guillemin of the Massachusetts Institute of Technology on "Recent Developments in the Theory of Pseudogroups" on Friday at 1:45 P.M.

The Institute of Mathematical Statistics met from Tuesday through Friday. Professor Herman Chernoff of Stanford University gave the Wald Lectures on the subject "Continuous Time Stopping Problems and Optimal Stochastic Control" at the following times and with these individual titles: "Stopping Problems and the Heat Equation" on Tuesday at 4:00 P.M., "Sequential Analysis and the One-Armed Bandit Problem" on Wednesday at 3:00 P.M., and "A Stochastic Control Problem and the Two-Armed Bandit Problem" on Thursday at 3:00 P.M.

The Society for Industrial and Applied Mathematics presented the John von Neumann Lecture on Wednesday at 8:00 P.M. It was delivered by Professor P. D. Lax of the Courant Institute of Mathematical Sciences, New York University, on "Nonlinear Partial Differential Equations."

The Pi Mu Epsilon Fraternity held sessions for contributed papers on Tuesday at 3:15 P.M. and on Wednesday at 10:40 A.M. in Room 139 of Van Vleck Hall. A banquet was held Tuesday at 6:00 P.M. in the East Dining Room of the Wisconsin Center. At this banquet, Professor G. S. Young of Tulane University spoke on "Pure and Applied Mathematics." A Dutch-treat breakfast meeting for Pi Mu Epsilon members was held on Wednesday at 8:00 A.M. in the Plaza Room of the Wisconsin Memorial Union.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements consisted of R. H. Bing, Chairman, H. L. Alder, P. T. Bateman, R. A. Brualdi, Sister Diane Drufenbrock, J. V. Finch, Simon Hellerstein, M. I. Knopp, Morris Marden, Mrs. Vera Nohel, J. B. Rosser, George Roussas, Mrs. Mary Ellen Rudin, Mrs. Jeanne Smith, M. B. Smith, Jr., L. F. Wahlstrom, G. L. Walker.

Registration headquarters were located in the lobby of the Wisconsin Center. Dormitory rooms and cafeteria facilities were provided by the University of Wisconsin. The Mathematical Sciences Employment Register was maintained in Room 226 and the second floor Lake Lounge of the Wisconsin Center, and book exhibits and exhibits of educational media were displayed in the Wisconsin Center.

A picnic was held on Wednesday at 5:15 P.M. at the Athletic Field. SIAM conducted a beer party on Wednesday at 9:00 P.M. in the Wisconsin Memorial Union cafeteria. A bus excursion to Spring Green left on Wednesday at 9:30 A.M. returning at 4:00 P.M. and included a guided visit to the Frank Lloyd Wright School of Architecture and buildings of local sandstone, stucco, and natural wood combinations designed by Wright. Conducted nature walks of the Wisconsin Arboretum were available.

HENRY L. ALDER, *Secretary*

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting at the University of Wisconsin, Madison, on August 25, 1968, elected to membership the fourteenth set of applicants for academic membership (for election of the other thirteen sets, see the April and November issues for 1962–68). Approval for election was given to the following 5 applicants for academic membership:

California State College at Los Angeles, Los Angeles, California
Grambling College, Grambling, Louisiana
University of Missouri at Kansas City, Kansas City, Missouri
Northern Illinois University, DeKalb, Illinois
Richard Bland College, Petersburg, Virginia

HENRY L. ALDER, *Secretary*

HONORARY TITLE CONFERRED UPON PROFESSOR H. M. GEHMAN

At the business meeting of the Association held on August 27, 1968 at the University of Wisconsin, Madison, the title of Executive Director Emeritus was conferred upon Professor Harry M. Gehman, effective August 1, 1968. The following citation, prepared by Professor E. A. Cameron, was read by President Moise.

"Harry Merrill Gehman, by a recent vote of the Board of Governors, has been given the title of Executive Director Emeritus. Professor Gehman received his Ph.D. degree at the University of Pennsylvania, was National Research Council Fellow at the University of Texas, Assistant Professor at Yale, Professor and Department Head at the State University of New York at Buffalo, organizer and Department Head at Shrivenham University, U.S. Army, England, and a vigorous researcher in topology in the early years of that subject. These facts alone indicate a full and rich career in mathematics, but Harry Gehman's greatest contribution to our discipline has been his dedicated work for the Mathematical Association of America. He has served successively as the first Chairman of the Upper New York Section, member of the Association's Finance Committee, Secretary-Treasurer, Treasurer, Treasurer and Executive Director, and most recently as Executive Director. In addition, he has been a member of numerous committees, ex officio and appointive.

Since he assumed the office of Secretary-Treasurer, the membership has increased

manyfold. It now takes three men to carry out the duties which he alone performed so smoothly. Over the years he has wisely managed the financial affairs of the Association, efficiently coordinated the work of its committees, and diligently supervised its editing and publishing.

He brought to our organization a personal touch. At national meetings Harry was always on hand to give you a hearty greeting before you joined the line to register. He would not hesitate to write a delinquent member a friendly letter in longhand suggesting that it would be a fine thing for him to pay his back dues. If, as has been said, an institution is the lengthened shadow of a man, the Association could be said to have that relation to Harry Gehman. It was most appropriate that he received in 1966 the Award for Distinguished Service to Mathematics.

For nearly a quarter of a century, as other officers came and went, Harry Gehman served with devotion, wisdom and never failing good humor. He has left upon the Association a lasting imprint and to us, his colleagues, an inspiration and obligation to carry on the work he so well pursued. With deep gratitude for his long and distinguished service, by vote of the Board of Governors, we are proud to confer upon him the honorary title Executive Director Emeritus."

HENRY L. ALDER, *Secretary*

PROPOSED AMENDMENT TO THE BY-LAWS OF THE MAA

At the meeting of the Board of Governors held on August 25, 1968, in Madison, Wisconsin, the Secretary was instructed to submit to a vote of the membership an amendment to Article VII of the By-Laws which will give to the Board of Governors authority to set the dues of ordinary members. (It already has the authority to set the dues of institutional members.) This amendment is proposed since the impending move of the Association's office from Buffalo to Washington makes it much more difficult to predict expenses in the near future, thus making it desirable to have a procedure available allowing a change in dues of ordinary members on shorter notice than is possible under the existing By-Laws.

In accordance with these instructions of the Board, a motion will be made at the business meeting of the Association to be held in New Orleans on Sunday, January 26, to amend Article VII, Section 2, to read as follows:

"The Board shall establish the annual dues and privileges of membership for ordinary and institutional members. The dues of ordinary members shall include a subscription to the official journal."

Article VII, Section 3, shall then be deleted and Sections 4 through 6 renumbered as 3 through 5.

HENRY L. ALDER, *Secretary*

MATHEMATICAL SCIENCES EMPLOYMENT REGISTER

The Tulane Room in the Jung Hotel in New Orleans, Louisiana, will be the location of the Mathematical Sciences Employment Register during the annual meeting. The Employment Register will be open for *four* days, January 24 through January 27, 1969, from 9:00 A.M. to 5:00 P.M.

The Employment Register is sponsored by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics for the purpose of establishing communication between mathematical scientists available for employment and employers with positions to fill. As part of the service, interviews between applicants and employers are arranged.

Registration for the Employment Register is *separate and apart* from meeting registration, and it is, therefore, most important that both applicants and employers sign in at

the Employment Register desk as early as they can on Friday morning. No appointments will be scheduled for Friday, however A separate visual index will be maintained for Employment Register use only. *Appointments will be scheduled only for people who have actually signed in at the Register.* Requests for appointments can be submitted on any or all of the days the Employment Register is open.

There is no charge for registration except when the late registration fee of $5.00 is applicable. Provision will be made for anonymity of applicants upon payment of $5.00 to defray the cost involved in handling such a listing. Applicants and employers who wish to be listed should write to the Mathematical Sciences Employment Register, Post Office Box 6248, Providence, Rhode Island 02904, for applicant qualification forms or position description forms. These forms must be completed and returned to the Employment Register not later than December 15, 1968, in order to be included in the January lists.

Those forms which arrive too late to be included in the printed lists are taken to the meeting where they may be seen by applicants and/or employers who are interested in them. The printed lists will be mailed to subscribers during the first week in January. Lists can be ordered from the Employment Register office in Providence. They will also be available at the meeting.

A subscription to the lists, which includes three issues (January, May, and August) of both the applicants list and the positions list is available for $25.00 a year; the individual issues of both lists may be purchased in January, May, and August for $12.50. A subscription to the applicants list alone or single copies of that list are not available. Copies of the positions list only may be purchased for $3.00. Checks should be made payable to the American Mathematical Society and sent to the address given above.

RETIRED MATHEMATICIANS

The List of Retired Mathematicians Available for Employment will once again be published in January 1969 and will be distributed to subscribers to the Employment Register lists when the January issue is mailed. Besides being available to subscribers, the list is available on request from the Employment Register office. Copies will also be available at the annual meeting in New Orleans, Louisiana, January 23–27, 1969. Retired mathematicians who are interested in being included in the list may either request a form from the Employment Register office or send the following information: name, date of birth, highest degree earned and where it was obtained, most recent employment, present address, date available, references, preference foi academic or industrial employment, and geographic location preferred. The deadline for receipt of either the completed form or the above information is January 1, 1969.

SUMMER EMPLOYMENT OPPORTUNITIES

The 1969 List of Opportunities for Summer Employment for Mathematical Scientists and College Mathematics Students will be available in January 1969 and will be sent to subscribers to the Employment Register lists. Copies will also be available on request from the Employment Register office and during the annual meeting in New Orleans. There is no charge for the list. Academic institutions, industrial concerns, and government agencies that have summer openings and would welcome applications from mathematicians and students of mathematics may request forms for listing from the Employment Register office. The deadline for receipt of forms is January 1, 1969.

NOVEMBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the MAA was held at South Dakota State University in Brookings, South Dakota, on November 11, 1967. There were 93

persons in attendance, including 56 members of the Association. Professor J. E. Richards of South Dakota State University presided over the morning session; Professor Roy Dowling of the University of Manitoba presided over the afternoon session. The sessions were held in the Agricultural Engineering Hall.

After a welcome address by Dr. H. M. Briggs, President of South Dakota State University, the following program was presented:

1. *Some primitive problems in computer pattern recognition*, by Patricia Milic, South Dakota State University. (Introduced by M. F. Bryn, South Dakota State University.)

2. *Strongly continuous functions do have some applications*, by Gerald Heuer, Concordia College.

3. *Some examples of generalized Green's functions*, by W. S. Loud, University of Minnesota.

4. *An interesting approach to simultaneous differential equations*, by J. W. Mentele, South Dakota State University

5. *On solving difference equations with a formula for A^n*, by R. B. Kirchner, Carleton College.

6. *Distributions: differentiating nondifferentiable functions*, by Ian Richards, University of Minnesota (Invited speaker).

7. *On the existence of subgroups having a prescribed set of coset representatives*, by C. V. Heuer, Concordia College.

8. *The equivalence of arbitrary solids by piecewise congruence of their finite decompositions*, by R. J. Bitts, Arapahoe Junior College.

9. *Effects of precipitation on extended outdoor activities*, by R. P. Covert, South Dakota State University. (Introduced by M. F. Bryn, South Dakota State University.)

WALBERT KALINOWSKI, *Secretary-Treasurer*

MARCH MEETING OF THE KANSAS SECTION

The fifty-third annual spring meeting of the Kansas Section of the MAA was held at Marymount College, Salina, on March 23, 1968.

Sister Mary Paul presided at a joint morning meeting with the Kansas Association of Teachers of Mathematics at which time Professor Carolyn Eisele, Hunter College, presented a paper entitled "Mathematics is One of the Humanities."

A short business meeting was held and the following officers were elected: J. J. Brewer, Wichita State University, Chairman; D. L. Bruyr, Kansas State Teachers College, Vice-Chairman; L. J. Dixon, Kansas State University, Secretary-Treasurer.

Note was made of the death of a long-time member, Professor R. G. Sanger. The top six from Kansas in the Putnam competition were recognized and given awards.

At the afternoon session of the section the following papers were presented:

1. *Charles S. Peirce and the Mathematics of the Nineteenth Century*, by Carolyn Eisele, Hunter College.

2. *The CUPM Report, Qualifications for a College Faculty in Mathematics*, by John Jewett, Oklahoma State University.

L. J. DIXON, *Secretary-Treasurer*

MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the MAA was held on Saturday, March 23, 1968 at Grand Valley State College, Allendale, Michigan, in conjunction with the meeting of the Michigan Academy of Arts and Letters. Professor Beauregard Stubble-

field of Oakland University presided at the sessions. There were 44 members registered and with the late comers a total of 55 attended the meetings. A sudden snow storm was responsible for the small attendance and cancellation of more than half of the program.

Professor Stubblefield presided at the business luncheon meeting. The Secretary-Treasurer's report was given by Professor Y. Alavi on behalf of Professor Lester Serier.

The Nominating Committee, chaired by Professor James McKay of Oakland University, presented the nominees for the Chairman, Vice-Chairman, and the Secretary-Treasurer. The following officers were elected: Professor E. A. Nordhaus, Michigan State University, Chairman; Professor A. B. Clarke, Western Michigan University, Vice-Chairman; Professor H. T. Slaby, Wayne State University, Secretary-Treasurer.

Professor McKay presented on behalf of the Nominating Committee the following candidates for election as Governor of the Michigan Section: Professor L. M. Kelly and Professor N. D. Kazarinoff.

Chairman Stubblefield spoke on the interest of the Association in closer activities with junior and other two-year colleges. It was moved, seconded and passed that the section Chairman appoint a committee to constitute the addition of a second Vice-Chairman post in the constitution for the purpose of coordinating and unifying activities of the Michigan Section with junior colleges and other two-year institutions.

The following papers were presented:

1. *The orderly occurrence of prime numbers*, by G. P. Loweke, Wayne State University.

2. *Asymptotic solutions of a linear differential equation*, by P. F. Hsieh, Western Michigan University.

3. *Panel discussion:* A. B. Clarke, Western Michigan University; C. B. Stortz, Central Michigan University.

4. *The prediction problem: a case history in applied mathematics*, by H. A. Smith, Oakland University.

5. *On a generalization of a Fermat theorem*, by S. Leja, Western Michigan University.

L. H. SERIER, *Secretary-Treasurer*

APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at Old Dominion College in Norfolk, Virginia, on April 27, 1968. Professor Avron Douglis, Chairman of the Section, presided over the 83 in attendance.

During the business meeting, the High School Contest and the Putnam Competition were discussed, as well as the possibility of giving awards to regional winners. The following officers were elected: Chairman, G. N. Trytten; Vice-Chairmen, W. P. Reid and A. K. Aziz; Secretary, Denny Gulick; Treasurer, Joseph Milkman.

Dr. W. P. Reid gave one of the two main invited addresses, "The Practice of Applied Mathematics," pinchhitting for the ailing Dr. W. J. Youden.

Following a luncheon and welcome by the President of Old Dominion College, Dr. L. W. Webb, Jr., the second main invited address was given by Dr. W. J. Schneider, of Syracuse University, entitled "Mathematical Legends: Separating Wheat from Chaff."

The following short papers were presented:

1. *Another derivation of Green's matrix*, by A. L. Deal, Virginia Military Institute, Lexington.

2. *A characterization of the additive group of real numbers*, by Eleanor G. Dawley Jones, Virginia State College, Norfolk.

3. *Fourier analysis of spatial waves*, by R. L. Stallard, IBM, Wheaton.

4. *Sets of polynomials orthogonal simultaneously on four ellipses*, by Ruth Goodman, Westinghouse Electric Corp., Baltimore, Maryland.

5. *Mathematical analysis of wave motion in a doubly connected inhomogeneous region*, by Y. C. Lim, Westinghouse Electric Corp., Baltimore.

6. *A note on parentheses*, by S. R. Herron and E. F. Storm, University of Virginia, Charlottesville.

7. *A common form for special solutions of the radial heat equation*, by R. A. Kowalski, Westinghouse Defense and Space Center, Baltimore.

8. *Compactness and certain subclasses of Schlicht functions*, by R. A. Whiteman, IIT Research Institute, Annapolis.

D. GULICK, *Secretary*

APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at New Mexico State University, Las Cruces, on April 12–13, 1968, with ninety-two persons in attendance, including sixty-two members of the Association. At this meeting, records were broken in both attendance and the number of papers presented since the section was founded in 1938, when the first meeting was held at the very same institution. Professor R. J. Wisner, Chairman of the section, presided.

A banquet was held on the evening of April 12. Those present included Academic Vice President of the host institution, Dr. William O'Donnell, as the guest of honor; Dr. J. W. Jewett, Chairman of the Department of Mathematics, Oklahoma State University, was the guest speaker. The title of his talk was "CUPM Report: Qualifications For a College Faculty in Mathematics."

The following officers were elected: Chairman, Professor A. B. Gray, Jr., Northern Arizona University; Vice-Chairman, Professor James Nymann, University of Texas at El Paso; Secretary-Treasurer, Professor S. T. Kao, University of New Mexico, (reelected).

During the business meeting, Professor A. P. Hillman of UNM, the Governor of this section, presented a one-year Association Membership award to: Mr. R. L. Mercer, Mr. R. W. Mercer, both students of UNM, and Mr. C. G. Ullery, student of the University of Arizona, who were winners in the 28th annual William L. Putnam Intercollegiate Mathematical Competition.

The following papers were presented:

1. *A characterization of a certain class of matrices over a field*, by Beryl M. Green, Eastern New Mexico University.

2. *A theorem on finite groups*, by Louis Solomon, New Mexico State University.

3. *Some ideals in the polynomial near-ring* ($Z[x]$, $+0$), by Donna Doi, University of Arizona.

4. *A characterization of absolute continuity*, by C. L. DeVito, University of Arizona.

5. *A careful calculation*, by J. D. Thomas, Los Alamos Scientific Laboratory and New Mexico State University.

6. *A step-function property and its application to convolution*, by R. M. McGehee, New Mexico Institute of Mining Technology.

7. *An algorithm for finding the initial segments of solutions to infinite-play two-armed-bandit problems*, by S. J. Jakowitz, University of Arizona.

8. *An approach to the absolute optimal stopping rules*, by P. H. Randolph, New Mexico State University.

9. *Calculation of Bessel functions by digital computers*, by R. D. Halbegewachs, Sandia Corporation.

10. *The function concept*, by R. M. Conkling, New Mexico Highlands University.

11. *The range of the perturbation parameter for which a periodic solution of a second order equation exists in the regular case*, by Mohammed Bahauddin, New Mexico State University.

12. *An argument for geometry; the vibrating drum*, by J. A. Nickel, The Dikewood Corporation.

13. *Further generalization of the Droz-Farney Theorem*, by S. T. Kao, University of New Mexico.

14. *Unique representation of primitive factors of 2^n-1, n odd, in certain quadratic forms*, by Edgar Karst, University of Arizona.

15. *Some abstractions on maps and colorings*, by Stephen Wilson, Northern Arizona University.

16. *Continuing education facts and fallacies*, by M. M. Slayter, Sandia Corporation.

17. *Language and logic*, by Anna S. Henriques, College of Santa Fe.

18. *The lonesome saboteur*, by Lewis Springer, Bernard Dunn and P. H. Randolph, of Braddock, Dunn and McDonald. (Delivered by P. H. Randolph.)

19. *A philosophy of modern mathematics*, by Philip Hosford, New Mexico State University.

20. *Fun and games with freshmen honor students*, by W. M. Kreuger, New Mexico State University.

21. *Some comparisons of common topologies on linear spaces*, by James Forster, New Mexico Institute of Mining and Technology.

The meeting was concluded by an Invited Address by Dr. J. W. Jewett with the title: "CBMS Survey of Mathematical Sciences."

S. T. KAO, *Secretary-Treasurer*

MAY MEETING OF THE NEW JERSEY SECTION

The third joint meeting of the New Jersey Section of the MAA and the Association of Mathematics Teachers of New Jersey was held at Rider College on May 4, 1968. Professors Joshua Barlaz and J. J. Kinsella presided at the meeting. About one hundred and thirty persons attended the meeting, including approximately forty members of MAA.

There was no business meeting.

During the morning session the following papers were presented:

1. *How to teach how to choose*, by I. L. Battin Sr., Trenton State College (by invitation).

2. *The differential calculus—what should it be about?*, by R. M. Cohn, Rutgers—The State University (by invitation)

The following talk was given at the afternoon session:

Euclid's Algorithm—since Euclid, by P. M. Cohn, University of London (by invitation).

F. A. VARRICHIO, *Secretary*

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The Spring Meeting of the Upper New York State Section of the MAA was held at Hamilton College on May 11, 1968. Professor D. W. Hall, SUNY at Binghamton, presided at the morning session and Professor F. R. Olson presided at the afternoon session.

At the business meeting, the following officers were elected: Chairman, Professor F. R. Olson, State University College at Fredonia; Vice-Chairman, Professor John Perry, Wells College; Secretary-Treasurer, Professor Paul Schaefer, State University College at Geneseo.

Professor Hall announced that Mr. Michael Bosquet, University of Montreal, was awarded First Prize in the Undergraduate Paper Contest. Professor Nura D. Turner, SUNY at Albany, reported on the High School Mathematics Contest.

The following papers were presented during the morning session:

1. *Iterated radicals*, by C. S. Ogilvy, Hamilton College.

2. *Some characterizations of near-fields*, by C. J. Maxson, State University College at Fredonia.

3. *Direct calculation of matrix functions by idempotent matrices*, by V. Lovass-Nagy and D. L. Powers, Clarkson College of Technology.

4. *Do it with points and lines*, by F. Buekenhout, SUNY at Buffalo.

During the afternoon session, the following paper, the first annual Harry M. Gehman Invited Lecture, was presented:

The algebra of rectangular matrices, by M. F. Smiley, SUNY at Albany.

P. SCHAEFER, *Secretary-Treasurer*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The Annual Meeting of the Pacific Northwest Section of the MAA was held at Reed College, Portland, Oregon on June 14 and 15, 1968 in conjunction with the Six-Hundred-Fifty-Seventh Meeting of AMS and the annual meeting of the Northwest Section of SIAM. One-hundred-fifty-five persons were in attendance.

At the business meeting the following officers were elected: Chairman, Ronald Harrop, Simon Fraser University; First Vice-Chairman, Lloyd Montzingo, Seattle Pacific College; Second Vice-Chairman, Norman Barton, Vancouver City College; Secretary-Treasurer, E. A. Maier, University of Oregon. The section voted to award cash prizes of \$30, \$20, and \$10 and a membership in the MAA to those students who ranked first, second and third, respectively, among participants in the Putnam Competition from institutions in the Section.

The program of the MAA portion of the meeting was as follows:

1. *Arithmetic in Community Colleges*, by Joseph Hashisaki, Western Washington State College (by invitation).

2. *CUPM recommendations for qualifications of college teachers and for a university parallel mathematics curriculum for two-year colleges*, by Alex Rosenberg, Cornell University.

3. *Existence theorems for convex surfaces*, by W. J. Firey, Oregon State University (by invitation).

4. *Special topics in linear algebra*, by Thomas Hungerford, University of Washington (by invitation).

5. *Optimal instruction: Fact or Fantasy?*, by Melvin Torseth, Everett Community College (by invitation).

E. A. MAIER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fifty-Second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 26, 1969.

FLORIDA, Florida Atlantic University, Boca Raton, March 21–22, 1969.

ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.

INDIANA

IOWA, University of Northern Iowa, Cedar Falls, April 18, 1969.

KANSAS, Wichita State University, Wichita, March 29, 1969.

KENTUCKY, Morehead State University, Morehead, Spring 1969.

LOUISIANA-MISSISSIPPI, New Orleans, January 25–27, 1969.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1969.

MINNESOTA

MISSOURI, St. Louis University, St. Louis, April 26, 1969.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 25–26, 1969.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of Santa Clara, Santa Clara, February 8, 1969.

OHIO

OKLAHOMA-ARKANSAS, Arkansas State University, Jonesboro, March 21–22, 1969.

PACIFIC NORTHWEST, University of Oregon, Eugene, August 1969.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, May 9–10, 1969.

SOUTHEASTERN, Winthrop College, Rock Hill, South Carolina, March 28–29, 1969.

SOUTHERN CALIFORNIA, California State College at Fullerton, March 15, 1969.

SOUTHWESTERN, Northern Arizona University, Flagstaff, Spring 1969.

TEXAS, Texarkana College, Texarkana, April 18–19, 1969.

UPPER NEW YORK STATE

WISCONSIN, Oshkosh, May 2–3, 1969.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, New Orleans, Louisiana, January 23–26, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Statler-Hilton Hotel, Washington, D. C., May 7–9, 1969.

ASSOCIATION FOR SYMBOLIC LOGIC, New Orleans, Louisiana, January 22–23, 1969.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, New Orleans, Louisiana, January 25–26, 1969.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Brown Palace Hotel, Denver, June 17–20, 1969.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 10–12, 1969.

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THE AMERICAN MATHEMATICAL MONTHLY

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(The following abbreviations are used: AL—Algebra and Theory of Numbers; AN—Analysis; GT—Geometry and Topology; LF—Logic and Foundations; ME—Mathematical Education; OA—Other Applications; PS—Probability and Statistics; PM—Professional Matters.)

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GEOMETRY AND TOPOLOGY

LOGIC AND FOUNDATIONS

PROBABILITY AND STATISTICS

OTHER APPLICATIONS

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PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

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THE AMERICAN MATHEMATICAL MONTHLY

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THE AMERICAN MATHEMATICAL MONTHLY

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NOTICE OF CHANGE OF ADDRESS

Effective November 1, 1968, the address of the business office of the Association is:

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GEOMETRIC ASPECTS OF DIOPHANTINE EQUATIONS INVOLVING EQUAL SUMS OF LIKE POWERS

L. J. LANDER, Aerospace Corporation

1.0. Introduction. Solutions to Diophantine equations of the form

$$x_1^k + x_2^k + \cdots + x_n^k = y_1^k + y_2^k + \cdots + y_n^k \tag{1}$$

have been found by a variety of algebraic and geometric methods. In recent years it has become possible to obtain numerical solutions on digital computers [1], [2] by generating, sorting and comparing sums of integer powers. When for a particular choice of k and n there occur numerous solutions to (1) in small integers, the existence of a parametric solution of low degree is suggested. Numerical solutions may also suggest side conditions (such as $\sum x_i = \sum y_i$) whose imposition results in an algebraic simplification. The computer can be extremely useful as a tool in finding parametric solutions, both in performing rational arithmetic on polynomials and in numerically verifying algebraic identities. In this paper three specific examples of (1) are considered. In each case one or more parametric solutions is developed through a geometric approach and some numerical characteristics of solutions are mentioned.

The fundamental geometric methods are simple and have long been known [3]. More recently B. Segre [4], [5] has performed general studies of cubic and quartic Diophantine equations. If we consider (1) as the equation of a rational curve or surface K in Cartesian space, there is a correspondence between solutions to (1) and rational points on K. When (1) has obvious trivial solutions, initial rational points on K are located. If geometric procedures are found which locate new rational points on K given existing rational points, then new solutions to (1) may result.

Suppose K is a rational plane curve of the third degree and that a line L intersects K in two rational points P_1, P_2. Then L must intersect K again in a third rational point P. For, the equation of L is rational, and the abscissae of P, P_1, P_2 are obtained by solving the equations for K and L simultaneously. This is equivalent to finding the three roots of a rational cubic equation, and since two of the roots are known to be rational, so is the third. Thus given P_1, P_2 on K, a new rational point P is located. It may happen that P coincides with P_1 or P_2, or that P lies at infinity, in which case no new solution results.

Alternatively, if P_1 is a rational point on K, take L to be the tangent to K at P_1. L has a rational equation and double contact with K at P_1, and therefore a third rational intersection with K which may provide a new solution. Similarly, if K has a rational asymptote A and P_1 is a rational point on K, take L as the line through P_1 parallel to A. Then L has contact with K at P_1 and at infinity, and a third rational contact with K which may give a new solution.

Next suppose that K is a rational surface of the fourth degree in Cartesian 3-space. If K contains three rational points lying on a line L, then L intersects K again in a fourth rational point. Let K contain a rational line R and a rational

point P, and let the plane tangent to K at P intersect R in Q. Then the line PQ has two rational contacts with K at P and one at Q, and so a fourth rational contact with K, which may provide a new solution.

The foregoing geometric procedures suffice for the examples which follow; a more complete discussion is given by Segre in the papers referenced. The approach followed in solving (1) is to reduce it to an equation which represents a curve or surface K having a geometric property of the type discussed. After finding initial rational points on K, based on the trivial solutions of (1) we infer new solutions by applying the geometric transformation. The three equations considered here have $(k, n) = (4, 2)$, $(5, 4)$ and $(5, 3)$, respectively.

2.0. Equal sums of two fourth powers. Euler [6], [7] gave two parametric solutions of the Diophantine equation

$$A^4 + B^4 = C^4 + D^4 \tag{2}$$

but the general solution is presently unknown. We give a geometric derivation of Euler's solutions and apply the geometric methods to obtain a new parametric solution, several new particular solutions and two complex parametric solutions. Related work by other investigators is also discussed. In a previous study [8], [1] the 46 smallest *primitive* solutions of (2) (those in which any factor common to A, B, C, D has been removed) were found by generating and sorting integers of the form a^4+b^4 on a digital computer. These primitive solutions will be referred to as $\sigma_1, \sigma_2, \cdots$, ranked in order of increasing magnitude of the common sum A^4+B^4. We say $\sigma = (a, b, c, d)$ is a solution if $A = a$, $B = b$, $C = c$, $D = d$ satisfies (2). For example,

$$\sigma_1 = (158, 59, 134, 133), \qquad \sigma_7 = (1203, 76, 1176, 653),$$
$$\sigma_{14} = (3494, 1623, 3351, 2338), \qquad \sigma_{39} = (12231, 2903, 10381, 10203),$$
$$\sigma_{46} = (15109, 581, 14723, 8461).$$

2.1. Euler's Algebraic Solutions. Euler treated (2) by setting

$$A = p + q, \quad B = r - s, \quad C = p - q, \quad D = r + s \quad \text{or} \tag{3.1}$$
$$2p = A + C, \quad 2q = A - C, \quad 2r = D + B, \quad 2s = D - B \tag{3.2}$$

which reduces (2) to

$$pq(p^2 + q^2) = rs(r^2 + s^2). \tag{4}$$

To each set of four positive integers A, B, C, D satisfying (2) there correspond 64 sets of integers (p, q, r, s) satisfying (4) obtained by using (3.2) and generating trivially different sets such as (p, q, s, r), (q, p, r, s), $(p, -q, r, -s)$. A quite distinct solution of (4) is obtained by interchanging C and D in (3.1):

$$A = p_1 + q_1 \quad B = r_1 - s_1 \quad C = r_1 + s_1 \quad D = p_1 - q_1 \tag{5.1}$$
$$2p_1 = A + D \quad 2q_1 = A - D \quad 2r_1 = C + B \quad 2s_1 = C - B \tag{5.2}$$

and there are similarly 64 sets such as (p_1, q_1, r_1, s_1) which satisfy (4). For σ_1 we

have $(p, q, r, s) = (146, 12, 96, 37)$ and

$$(2p_1, 2q_1, 2r_1, 2s_1) = (291, 25, 193, 75).$$

The paired solutions to (4) are related by the equations

$$\begin{aligned} 2p_1 &= p+q+r+s & 2p &= p_1+q_1+r_1+s_1 \\ 2q_1 &= p+q-r-s & 2q &= p_1+q_1-r_1-s_1 \\ 2r_1 &= p-q+r-s & 2r &= p_1-q_1+r_1-s_1 \\ 2s_1 &= p-q-r+s & 2s &= p_1-q_1-r_1+s_1. \end{aligned} \tag{6}$$

On substituting $z=q/s$, $v=(rs/pq)-1$, (4) becomes

$$\begin{aligned} (z^2-1)^2 + v(3z^4-4z^2+1) + v^2(3z^4-6z^2) + v^3(z^4-4z^2) - v^4z^2 \\ = [(s/p)(z^2-v-1)]^2. \end{aligned} \tag{7}$$

Euler made the quartic function of v on the left-hand side of (7) a square in two ways. First it was equated to $[(z^2-1)+fv+gv^2]^2$ and f and g were chosen to make the coefficients of v and v^2 in the resulting equation vanish. A rational solution of v as a function of z results. Euler [6] did not give the final algebraic result explicitly but did give the numerical solutions (2061283, 1584749, 2219449, 555617) and σ_{39} for $z=2$ and $z=3$, respectively. By completing the polynomial calculation we find a solution of the 13th degree:

$$\begin{aligned} p &= 3(z^2-1)^2(9z^8-44z^6+190z^4+100z^2+1) \\ q &= z(z^{12}-214z^{10}-2481z^8-2804z^6-2481z^4-214z^2+1) \\ r &= 3z(z^2-1)^2(z^8+100z^6+190z^4-44z^2+9) \\ s &= z^{12}-214z^{10}-2481z^8-2804z^6-2481z^4-214z^2+1. \end{aligned} \tag{8}$$

These polynomials do not seem to have been previously recorded. In the second approach the quartic function in (7) was equated to $[(1+dv)(z^2-v-1)]^2$ and upon removing the common factor (z^2-v-1) there results

$$v^3(z^2+d^2) + v^2(3z^2-d^2z^2+d^2+2d) + v(3z^2-2dz^2+2d+1) = 0$$

which is solved rationally by choosing d so that the coefficient of v vanishes. This leads to the well-known solution [7] of the 7th degree:

$$\begin{aligned} p &= 2(4z^6+z^4+10z^2+1) & r &= 2z(z^6+10z^4+z^2+4) \\ q &= -z(z^2+1)(z^4-18z^2+1) & s &= -(z^2+1)(z^4-18z^2+1) \end{aligned} \tag{9}$$

which produces $\sigma_1, \sigma_7, \sigma_{14}$, for $z=3, 2, 5$. In addition to the two parametric solutions, Euler [9] found the particular solution $\sigma_5 = (542, 103, 514, 359)$ by making special assumptions.

2.2. Geometric interpretation. If we set

$$x = p/s \qquad y = r/s \qquad z = q/s \tag{10}$$

then (4) becomes

$$y^3 + y = x^3z + xz^3. \tag{11}$$

For a fixed rational z we may interpret (11) as the Cartesian equation of a cubic curve K in the xy plane, symmetric with respect to the origin and passing through the rational point P with coordinates $x=1$, $y=z$. Each rational point on K corresponds to a solution of (4) through the relations $p=xs$, $q=zs$, $r=ys$ for arbitrary s. The point P gives the trivial solution $p=s$, $q=r$. The tangent at $P_0(x_0, y_0)$ has slope $t=(z^3+3x_0^2z)/(3y_0^2+1)$ and intersects K again in $P_1(x_1\, y_1)$, where $x_1=-2x_0-[3t^2(y_0-tx_0)/(t^3-z)]$, $y_1=t(x_1-x_0)+y_0$. By setting $x_0=1$, $y_0=z$ we find

$$x_1 = \frac{-2(4z^6 + z^4 + 10z^2 + 1)}{(z^2 + 1)(z^4 - 18z^2 + 1)} \qquad y_1 = \frac{-2z(z^6 + 10z^4 + z^2 + 4)}{(z^2 + 1)(z^4 - 18z^2 + 1)}$$

and if $s=-(z^2+1)(z^4-18z^2+1)$, the solution (9) results.

If $z=m^3$, K has the rational asymptote $y=mx$, and a line through $P_0(x_0, y_0)$, parallel to this asymptote, must intersect K again in a rational point $P_2(x_2, y_2)$. We find

$$x_2 = [1 + (y_0 - mx_0)^2]/3m^2, \qquad y_2 = m(x_2 - x_0) + y_0 \quad \text{and if} \quad x_0 = 1,$$

$$y_0 = z, \quad \text{then } x_2 = (m^6 - 2m^4 + m^2 + 1)/3m^2, \; y_2 = m(m^6 + m^4 - 2m^2 + 1)/3m^2.$$

Setting $s=3m^2$ there results the following solution to (4):

$$\begin{aligned} p &= m^6 - 2m^4 + m^2 + 1 & r &= m(m^6 + m^4 - 2m^2 + 1) \\ q &= 3m^5 & s &= 3m^2. \end{aligned} \tag{12}$$

This solution was given by A. Gerardin [10] and an algebraic derivation of it due to P. S. Dyer [11] can be found also in Hardy and Wright [12]. The solutions to (2) produced by (12) are in fact the same as those given by (9). The explanation is that the rational transformation $m=(z+1)/(z-1)$ carries (p, q, r, s) of (12) into polynomials proportional to $(-s_1, r_1, p_1, -q_1)$ obtained by applying the transformation (6) to (p, q, r, s) of (9). If we reverse this procedure for Euler's first solution, applying (6) to (8) and then substituting $z=(m+1)/(m-1)$ we get the following solution which is simpler than, but equivalent to (8):

$$\begin{aligned} p &= m(m^{12} + 2m^{10} - 3m^8 + 7m^6 - 12m^4 + 5m^2 + 1) \\ q &= m^{12} - 4m^{10} + 3m^8 + 4m^6 - 6m^4 - m^2 + 1 \\ r &= m(m^{12} - m^{10} - 6m^8 + 4m^6 + 3m^4 - 4m^2 + 1) \\ s &= m^{12} + 5m^{10} - 12m^8 + 7m^6 - 3m^4 + 2m^2 + 1. \end{aligned} \tag{13}$$

This solution can also be found geometrically, since the line joining two rational points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ of K intersects K in a third rational point $P_3(x_3, y_3)$. The slope of the line is $t=(y_2-y_1)/(x_2-x_1)$ and $x_3=-[x_1+x_2+3t^2(y_1-tx_1)/(t^3-z)]$, $y_3=t(x_3-x_1)+y_1$. Given any initial solution (p_0, q_0, r_0, s_0) of (4) take

$x_1=1$, $y_1=z=q_0/s_0$, $x_2=p_0/s_0$, $y_2=r_0/s_0$. The slope t is then $(r_0-q_0)/(p_0-s_0)$. If we take (p_0, q_0, r_0, s_0) to be the polynomials (s, r, q, p) of (12), then $t=m$ and the new rational point (x_3, y_3) gives a solution equivalent to (13).

A new solution of the 19th degree is obtained by taking $(p_0, q_0, r_0, s_0)=(r, s, q, p)$ from (12), and $x_1=-1$, $y_1=-z=-q_0/s_0$, $x_2=p_0/s_0$, $y_2=r_0/s_0$ which leads to:

$$\begin{aligned}
p &= m^{19}-m^{18}-3m^{17}-3m^{16}+21m^{15}-9m^{14}-44m^{13}+74m^{12}\\
&\quad -39m^{11}-21m^{10}+84m^9-132m^8+115m^7-73m^6+45m^5\\
&\quad -21m^4+12m^3-6m^2+m-1,\\
q &= 3m^2(m^{12}-4m^{10}+18m^9-36m^8+27m^7+m^6-9m^5+9m^4\\
&\quad -9m^3+5m^2+1),\\
r &= 3m^2(m^{15}-3m^{14}-m^{13}+11m^{12}-12m^{11}+4m^{10}+10m^9-30m^8\\
&\quad +39m^7-37m^6+41m^5-33m^4+16m^3-8m^2+3m-1),\\
s &= (m^6-2m^4+m^2+1)(m^{12}-4m^{10}+18m^9-36m^8+27m^7+m^6\\
&\quad -9m^5+9m^4-9m^3+5m^2+1).
\end{aligned} \tag{14}$$

These polynomials give σ_1 for $m=-1$ and new solutions such as

$$\begin{aligned}
(A, B, C, D) &= (134413, 34813, 114613, 111637),\\
& (1057167, 552059, 1054067, 545991)
\end{aligned}$$

for $m=2$, -2. The particular result for $m=2$ was found by E. Fauquembergue [**13**] using another algebraic method.

B. Segre [**5**] has given a general geometric treatment of fourth order Diophantine equations which correspond to quartic surfaces containing rational lines. In his paper two geometric transformations are introduced which for (2) lead to Euler's solution (9) and more generally to a discontinuous infinity of rational parametric solutions, not given explicitly.

2.3. Complex solutions. The rational complex points $x=\pm iz$, $y=i$ lie on K and the tangent at each of these points has a second rational intersection with K, giving the following solutions to (4) of the 5th degree in Gaussian integers:

$$\begin{aligned}
p &= iz(z^4-2) & r &= i(-2z^4+1)\\
q &= z(z^4+1) & s &= z^4+1
\end{aligned} \tag{15.1}$$

$$\begin{aligned}
p &= -iz(z^4+2) & r &= -i(2z^4+1)\\
q &= z(z^4-1) & s &= z^4-1.
\end{aligned} \tag{15.2}$$

2.4. Particular solutions. By starting with any of the polynomial solutions already given and employing the geometric techniques discussed here, further parametric solutions may be obtained. Whether or not such procedures eventually yield all rational solutions to (2) is not apparent. The primitives σ_2

$=(239, 7, 227, 157)$ and $\sigma_3=(292, 193, 257, 256)$ found by A. Werebrusow **[14]**, **[15]** are not produced by Euler's formulas (nor are any of the other computer-derived solutions except $\sigma_1, \sigma_7, \sigma_{14}, \sigma_{39}$) but nevertheless are geometrically derivable one from the other. By taking $z=2/25$ in (11) the two rational points $x=233/75$, $y=82/75$ and $x=-274/225$, $y=-32/225$ on K correspond to σ_2, σ_3 and the line joining these points intersects K in the trivial point $x=-1$, $y=-2/25$. A number of similar relationships were found to hold among certain sets of the smallest known primitives, specifically

$$(\sigma_2, \sigma_{18}),\ (\sigma_3, \sigma_{11}),\ (\sigma_4, \sigma_{13}),\ (\sigma_4, \sigma_{22}),\ (\sigma_6, \sigma_{45}),\ (\sigma_{16}, \sigma_{40}),\ (\sigma_{19}, \sigma_{41}),\ \text{and } (\sigma_{36}, \sigma_{46}).$$

The geometric methods can be applied to particular numerical solutions and in some cases result in new solutions which do not involve overly large integers. Examples are

$$(A, B, C, D) = (15265, 6101, 13085, 12743),\ (27407, 758, 27374, 7217),$$

and

$$(31731, 5468, 27661, 25596)$$

derived from σ_{25}, σ_{38}, and σ_{15}, respectively.

2.5. Another solution. T. Hayashi **[16]** showed that every solution of the Diophantine equation

$$3u^4 + v^4 = w^2 \tag{16}$$

leads to a solution of (2). We can express his result by stating that if u, v, w satisfy (16), then

$$p = 2u^3(2u^4 + v^4) \qquad r = 2u^6 v$$
$$q = uv^4 w \qquad s = vw(2u^4 + v^4)$$

satisfy (4).

Several systems are known which produce from one solution u_1, v_1, w_1, to (16) a new solution u_2, v_2, w_2; for example:

$$u_2 = 2u_1v_1w_1 \qquad v_2 = 2v_1^4 - w_1^2 \qquad w_2 = w_1^4 + 12u_1^4v_1^4.$$

The least solutions to (16) are $(u, v, w)=(1, 1, 2)$, $(2, 1, 7)$ and $(3, 11, 122)$ which yield, respectively $(A, B, C, D)=(2, 1, 1, 2)$, $(542, 103, 514, 359)=\sigma_5$, and $(4970416, 1139811, 4962397, 1539492)$.

3.0. Equal sums of four fifth powers. We shall next derive three parametric solutions of the Diophantine equation

$$A_1^5 + A_2^5 + A_3^5 + A_4^5 = B_1^5 + B_2^5 + B_3^5 + B_4^5 \tag{17}$$

for which we also use the notation $(A_1, A_2, A_3, A_4)^5=(B_1, B_2, B_3, B_4)^5$. In examin-

ing numerical solutions found by computer search it was observed that a large proportion of these solutions satisfy the additional conditions

$$A_1 + A_2 = B_1 + B_2, \qquad A_3 + A_4 = B_3 + B_4. \tag{18}$$

For example, the solution of (17) in least integers is $(5, 6, 6, 8)^5 = (4, 7, 7, 7)^5$ which also satisfies (18) since $5+6=4+7$, $6+8=7+7$.

If we set

$$\begin{aligned} A_1 &= u_1 + v_1 + w_1 & B_1 &= u_1 + v_1 - w_1 \\ A_2 &= u_1 - v_1 - w_1 & B_2 &= u_1 - v_1 + w_1 \\ A_3 &= u_2 + v_2 - w_2 & B_3 &= u_2 + v_2 + w_2 \\ A_4 &= u_2 - v_2 + w_2 & B_4 &= u_2 - v_2 - w_2 \end{aligned} \tag{19}$$

it follows at once that (19) is a solution of (18). Equation (17) will also be satisfied provided that

$$u_1v_1w_1(u_1^2 + v_1^2 + w_1^2) = u_2v_2w_2(u_2^2 + v_2^2 + w_2^2). \tag{20}$$

For the least solution, $(u_1, v_1, w_1) = (11, 2, 1)$, $(u_2, v_2, w_2) = (14, 1, 1)$ and $11\cdot 2 \cdot 1(11^2+2^2+1^2) = 22\cdot 126 = 2\cdot 11\cdot 9\cdot 14 = 14\cdot 198 = 14\cdot 1\cdot 1(14^2+1^2+1^2)$.

3.1. Solution derived from an asymptote. Consider the Cartesian curve Γ with the equation

$$xab(x^2 + a^2 + b^2) = yac(y^2 + a^2 + c^2), \qquad abc \neq 0,\ b \neq c \tag{21}$$

which has an asymptote $bx^3 = cy^3$ of rational slope m provided $b = cm^3$. Then $x = c$, $y = b$ is trivially a rational point P on Γ. The line through P with slope m intersects Γ again in the point with coordinates

$$\begin{aligned} x &= \left[a^2 + c^2 + c^2m^2(m^2 - 1)^2\right]/3m^2c \\ y &= mx + mc(m^2 - 1). \end{aligned}$$

After clearing of fractions this gives the following parametric solution to (20):

$$\begin{aligned} u_1 &= a^2 + (m^6 - 2m^4 + m^2 + 1)c^2 & u_2 &= ma^2 + (m^7 + m^5 - 2m^3 + m)c^2 \\ v_1 &= 3m^5c^2 & v_2 &= 3m^2c^2 \\ w_1 &= 3m^2ac & w_2 &= 3m^2ac. \end{aligned} \tag{22}$$

For each rational m there results a solution to (17) in homogeneous polynomials of the second degree in two variables. The least primitive solutions to (17) produced by (22) with integer arguments have $(m, a, c) = (2, 1, 1)$ and $(2, 1, 3)$ which give $(23, 73, 74, 74)^5 = (35, 61, 62, 86)^5$ and $(7, 70, 89, 94)^5 = (43, 53, 58, 106)^5$, respectively.

3.2. Solution derived from a tangent. The tangent to Γ at P has the equation

$$y = k_1x + k_2, \tag{23}$$

where $k_1=[b(a^2+b^2+3c^2)]/[c(a^2+3b^2+c^2)]$, $k_2=b-k_1c$ and intersects Γ again in a point with abscissa

$$x=\frac{3ck_1^2k_2}{b-ck_1^3}-2c. \tag{24}$$

Equations (23) and (24) together with

$$\begin{array}{lll} u_1=x & v_1=b & w_1=a \\ u_2=y & v_2=c & w_2=a \end{array} \tag{25}$$

provide another solution to (20) in the three parameters a, b, c. The two least primitive solutions to (17) produced by (25) have $(a, b, c)=(2, 1, 5)$ and $(2, 1, 3)$ giving respectively $(11, 17, 22, 28)^5=(7, 21, 24, 26)^5$ and $(28, 30, 39, 45)^5=(24, 34, 41, 43)^5$.

3.3. An auxiliary condition suggested by numerical solutions. A third solution is derived by taking (20) together with the additional condition

$$u_1v_1w_1=u_2v_2w_2 \tag{26}$$

which implies (for nontrivial solutions)

$$u_1^2+v_1^2+w_1^2=u_2^2+v_2^2+w_2^2. \tag{27}$$

When (19), (26) and (27) are satisfied, $(A_1, A_2, A_3, A_4)^n=(B_1, B_2, B_3, B_4)^n$ for $n=1$, 3 and 5.

A solution of (27) can be written

$$\begin{array}{lll} u_1=X+A & v_1=Y-B & w_1=Z-C \\ u_2=X-A & v_2=Y+B & w_2=Z+C \end{array} \tag{28}$$

where A, B, C, X, Y, Z are any rational numbers satisfying the single condition

$$AX=BY+CZ. \tag{29}$$

Substituting (28) into (26) and using (29) to eliminate X gives

$$(YZ)B^2+C(Y^2+Z^2-A^2)B+YZ(C^2-A^2)=0. \tag{30}$$

If B is rational, the discriminant of this quadratic equation must be a square, say K^2. This condition may be written

$$\begin{aligned} C^2(Y+A+Z)(Y+A-Z)(Y-A+Z)(Y-A-Z) \\ =(K+2AYZ)(K-2AYZ). \end{aligned} \tag{31}$$

Assuming that

$$\begin{aligned} K+2AYZ&=C(Y+A+Z)(Y-A+Z) \\ K-2AYZ&=C(Y+A-Z)(Y-A-Z) \end{aligned} \tag{32}$$

we find using (30) and (29) that

$$A = C, \qquad K = C(Y^2 + Z^2 - C^2),$$
$$(33) \qquad B = -\frac{C}{YZ}(Y^2 + Z^2 - C^2), \qquad X = \frac{C^2 - Y^2}{Z}.$$

After clearing of fractions and setting $Y=b$, $Z=a$, $C=c$, the resulting solution to (20) is

$$(34) \qquad \begin{aligned} u_1 &= b(c^2 - b^2 + ac) & u_2 &= b(b^2 - c^2 + ac) \\ v_1 &= (a + c)(b^2 - c^2 + ac) & v_2 &= (a - c)(c^2 - b^2 + ac) \\ w_1 &= ab(a - c) & w_2 &= ab(a + c). \end{aligned}$$

The primitive solutions to (17) in integers not exceeding 50 produced by (34) are given in Table I. In preparing this table terms were transposed so as to be written positively, and common factors were removed.

TABLE I

Primitive Solutions to $A_1^5 + A_2^5 + A_3^5 + A_4^5 = B_1^5 + B_2^5 + B_3^5 + B_4^5$

	a	b	c	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4
1	1	6	3	1	13	17	23	3	9	21	21
2	1	6	2	6	16	18	24	7	13	21	23
3	1	1	3	7	21	23	33	11	13	29	31
4	3	14	10	5	20	22	34	9	12	27	33
5	1	2	4	1	21	27	39	7	11	33	37
6	1	2	5	9	21	31	39	13	15	35	37
7	3	20	7	11	28	34	44	13	23	39	42
8	1	12	5	12	23	39	45	14	20	42	43
9	1	12	3	21	30	39	45	22	28	41	44

3.4. Related work. The following two-parameter solution to (17) was given by G. Xeroudakes and A. Moessner [17]:

$$(35) \qquad \begin{aligned} A_1 &= -p^2 + 4pq + 9q^2 & A_2 &= 3p^2 + 12pq + 21q^2 \\ A_3 &= p^2 + 8pq + 3q^2 & A_4 &= 3p^2 + 12pq + 21q^2 \\ B_1 &= 3p^2 + 16pq + 17q^2 & B_2 &= -p^2 + 13q^2 \\ B_3 &= p^2 + 12pq + 23q^2 & B_4 &= 3p^2 + 8pq + q^2. \end{aligned}$$

This can be obtained from (34) by setting $a=p+2q$, $b=2q$, $c=q$. The first solution in Table I corresponds to this case for $p=-5$, $q=3$.

Equations (34) produce solutions to (17) which can be written with at most one term nonpositive. Two primitive solutions involving a zero term are obtained by setting $(a, b, c)=(1, 10, 14)$ and $(9, 20, 5)$ which give $(38, 105, 123)^5 = (13, 23, 110, 120)^5$ and $(30, 40, 55)^5 = (6, 19, 49, 51)^5$, respectively.

The general solution of (20) is yet to be obtained; none of the parametric solutions contained here produce, for example, the least solution

$$11\cdot 2\cdot 1(11^2 + 2^2 + 1^2) = 14\cdot 1\cdot 1(14^2 + 1^2 + 1^2)$$

mentioned previously. The least solution of (17) without the restriction imposed by (18) was found on the computer to be $(1, 1, 3, 24)^5 = (6, 15, 15, 23)^5$.

4.0. Equal sums of three fifth powers. The Diophantine equation

$$A_1^5 + A_2^5 + A_3^5 = B_1^5 + B_2^5 + B_3^5 \tag{36}$$

was solved in positive integers first by A. Moessner [**18**] who gave $49^5+75^5+107^5 = 39^5+92^5+100^5$. A two-parameter solution of the seventh degree was found by Moessner [**19**] and two parametric solutions were found by H. Swinnerton-Dyer [**20**] which result in polynomial solutions of rather high degree that were not explicitly given. (Dyer did give a single numerical result which appears to be incorrect.) Both Moessner's and Dyer's solutions satisfy (36) together with the additional condition

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3. \tag{37}$$

In a computer study [**1**] the solution of (36) in least integers was found to be $24^5+28^5+67^5=3^5+54^5+62^5$ which also satisfies (37), as do a great many of the other solutions obtained by search. The least solution of (36) which does not also satisfy (37) is $26^5+85^5+118^5=53^5+90^5+116^5$ and Moessner [**19**] exhibited a parametric solution of the 36th degree having this property. We shall derive a 3-parameter solution to the system (36), (37) which yields Moessner's first numbers as well as other solutions in moderately small integers. A particular version of this solution is given explicitly as a set of polynomials of the ninth degree in a single variable.

4.1. Three-parameter solution. Suppose that the Diophantine system (36), (37) has the known solution $A_i=a_i$, $B_i=b_i$, $i=1, 2, 3$. If $u=b_3-a_3$, $v=a_2-b_2$, $x_0=a_1-u$, $y_0=b_2$, $z_0=a_3$, then

$$(x + u)^5 + (y + v)^5 + z^5 = (x + v)^5 + y^5 + (z + u)^5 \tag{38}$$

is the Cartesian equation of a surface S which passes through the point $P_0(x_0, y_0, z_0)$ and contains the parallel lines

$$L_1\colon x = y = z, \qquad L_2\colon x = y - u = z - v.$$

Each rational point $P(x, y, z)$ of S gives a rational solution of (36), (37) through the equations

$$A_1 = x + u,\quad A_2 = y + v,\quad A_3 = z,\quad B_1 = x + v,\quad B_2 = y,\quad B_3 = z + u. \tag{39}$$

A solution in integers may be obtained by multiplying $A_1, A_2, \cdots, B_3$ so as to clear of fractions.

In expanding (38) the terms in x^5, y^5, z^5 vanish and therefore S is a surface

of the fourth degree. Let T be the plane tangent to S at P_0 and P_j the intersection of T with the line L_j, $j=1$ or 2.

The equation of T is rational, P_j is rational and thus P_0P_j is a rational line intersecting S in three rational points; that is, twice at P_0 and once at P_j. Hence P_0P_j intersects S in a fourth rational point Q_j which in some cases gives a new solution to (36). If P_0 falls on $L_1(L_2)$, the solution degenerates because then T contains $L_1(L_2)$ and is parallel to $L_2(L_1)$. It may also happen that the new solution is trivial. However, for certain initial solutions a_i, b_i new nontrivial solutions are in fact obtained.

The normal to S at P_0 has directions $d_1:d_2:d_3$ where $d_i=a_i^4-b_i^4$ and the coordinates of $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ may be written

$$x_1 = y_1 = z_1 = (x_0d_1 + y_0d_2 + z_0d_3)/(d_1 + d_2 + d_3)$$
$$x_2 = y_2 - u = z_2 - v = x_1 - (d_2u + d_3v)/(d_1 + d_2 + d_3).$$

The equation of P_0P_j can be written parametrically as

$$x = x_0 + h_1t \qquad y = y_0 + h_2t \qquad z = z_0 + h_3t \tag{40}$$

for $j=1$ or 2, where $h_1=x_j-x_0$, $h_2=y_j-y_0$, $h_3=z_j-z_0$.

On substituting (40) into (38) there results an equation of the form $c_4t^4+c_3t^3+c_2t^2+c_1t+c_0=0$ in which

$$c_4 = 5\sum_{i=1}^{3} h_i^4(a_i - b_i), \qquad c_3 = 10\sum_{i=1}^{3} h_i^3(a_i^2 - b_i^2). \tag{41}$$

Three of the roots of this equation are $t=0$, 0, 1 and so the fourth root is $t=-(c_3/c_4)-1$ which together with (40) gives the coordinates of Q_j. If the numerically minimal solution of (36) is used as an initial solution, the two resulting new solutions (actually there are several sets corresponding to permutations of the a_i and b_i) involve integers of 17 digits. If instead we attempt to use a trivial initial solution in which the b_i are simply a permutation of the a_i, the new solution is either degenerate or trivial.

TABLE II

Primitive Solutions of $A_1^5 + A_2^5 + A_3^5 = B_1^5 + B_2^5 + B_3^5$

p	q	r	A_1	A_2	A_3	B_1	B_2	B_3
1	−3	2	−907	−549	1378	−1087	−414	1423
1	−2	3	−49	−107	−75	−100	−39	−92
1	2	3	803	289	561	808	309	536
1	3	2	293	−501	754	959	498	−911
1	3	5	1157	543	1135	1271	885	679
2	−4	1	346	−1162	641	−1724	−127	1676
2	3	1	1084	−252	−433	1249	243	−1093
3	4	1	1783	−763	−699	1974	1	−1654

However, a trivial initial solution of the type

$$a_1 = p, \quad a_2 = -p, \quad a_3 = r, \quad b_1 = q, \quad b_2 = r, \quad b_3 = -q \tag{42}$$

does produce new solutions. It is sufficient to use Q_2, to take $r>0$, $p>0$, $|q|>p$ and have p, q, r all distinct in magnitude. Some results for small integer values of the arguments p, q, r are given in Table II; it will be noted that Moessner's solution is the smallest obtained.

4.2. Polynomial solution. The foregoing provides an implicit parametric solution of (36), (37) in three integer variables (or two rational variables) but does not give the results explicitly in a form which facilitates the calculation of numerical solutions. Accordingly, a specialized solution was computed for (42) with $p=-2$, $q=1$, using the case $j=2$ of (40). After clearing of fractions the result is

$$\begin{aligned}
A_1 &= -2r^8+10r^7+20r^6+20r^5+34r^4-10r^3-270r^2-20r-682\\
A_2 &= 2r^8+10r^7-20r^6+20r^5-34r^4-10r^3+270r^2-20r+682\\
A_3 &= r^9-22r^5-125r^3-79r\\
B_1 &= r^8+10r^7-10r^6+20r^5-92r^4-160r^3-15r^2-320r+341\\
B_2 &= r^9-22r^5+175r^3+521r\\
B_3 &= -r^8+10r^7+10r^6+20r^5+92r^4-160r^3+15r^2-320r-341
\end{aligned} \tag{43}$$

The values of these polynomials for several small rational arguments (after clearing of fractions and removing any common factor) are presented in Table III.

TABLE III

Primitive Solutions of $A_1^5+A_2^5+A_3^5=B_1^5+B_2^5+B_3^5$

Derived from a Polynomial Solution

r	A_1	A_2	A_3	B_1	B_2	B_3
3	100	92	39	49	75	107
4	614	1018	1028	773	1124	763
1/2	−1724	1676	−127	346	641	−1162
3/2	−13964	24572	−13701	−19334	25467	−9226
5/2	44684	36516	−2505	2558	29495	46642
8	−2654	20126	53912	14271	53976	3137
2/3	−57738	55866	−6626	4137	28366	−41001
1/4	−74552	73464	−2301	27244	14083	−44716

In conclusion we note that the solutions obtained can be put in one of the two forms $A^5+B^5+C^5=D^5+E^5+F^5$ or $A^5+B^5=C^5+D^5+E^5+F^5$ where A, B, C, D, E, F are positive integers. The last three entries in Table II are examples of the latter type.

References

1. L. Lander, T. Parkin and J. Selfridge, A survey of equal sums of like powers, Math. Comp., 21 (1967) 446–459.

2. L. Lander and T. Parkin, A counterexample to Euler's conjecture on like powers, Bull. Amer. Math. Soc., 72 (1966) 173.

3. E. Lucas, Nouv. Ann. Math. (2), 17 (1878) 507–509. Cited in L. E. Dickson, History of the Theory of Numbers, v. 2, p. 590, Carnegie, Washington D.C., 1920.

4. B. Segre, Arithmetic upon an algebraic surface, Bull. Amer. Math. Soc., 51 (1945) 152–161.

5. ———, On arithmetical properties of quartic surfaces, Proc. London Math. Soc., 2, 49 (1947) 353–395.

6. L. Euler, Novi Commentarii Acad. Petropol., v. 17, 1772, p. 64; Commentationes Arithmeticae, I, pp. 473–6; Opera Omnia, v. 1, III, p. 211. Cited in Dickson, Ibid., p. 644.

7. ———, Nova Acta Acad. Petrop, v. 13, ad annos 1795–6, 1802 (1778), 45; Comment. Arith., II, p. 281. Cited in Dickson, ibid., pp. 645–646.

8. L. Lander and T. Parkin, Equal sums of biquadrates, Math. Comp., 20 (1966) 450–451.

9. L. Euler, Mémoires de l'Acad. Imp. de St. Petersbourg, v. 11, 1830 (1780), p. 49; Comment. Arith., II, pp. 450–456. Cited in Dickson, ibid., p. 645.

10. A. Gérardin, L'Intermédiaire des Mathématiciens, 24 (1917) 51.

11. P. S. Dyer, A solution of $A^4+B^4=C^4+D^4$, J. London Math. Soc., 18 (1943) 2–4.

12. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., p. 201, Oxford University Press, London, 1960.

13. E. Fauquembergue, L'Intermédiaire des Math., 21 (1914) 17.

14. A. S. Werebrusow, L'Intermédiaire des Math., 20 (1913) 197.

15. ———, ibid., 21 (1914) 18.

16. T. Hayashi, On the Diophantine Equation $x^4+y^4=z^4+t^4$, Tôhoku Math. J., 1 (1912) 143–145.

17. G. Xeroudakes and A. Moessner, On equal sums of like powers, Proc. Indian Acad. Sci. Sect. A, 48 (1958) 245–255.

18. A. Moessner, Einige numerische Identitäten, Proc. Indian Acad. Sci., Section A, 10 (1939) 296–306.

19. ———, Due Sistemi Diofantei, Boll. Un. Mat. Ital., (3), 6 (1951) 117–118.

20. H. P. F. Swinnerton-Dyer, A solution of $A^5+B^5+C^5=D^5+E^5+F^5$, Proc. Cambridge Philos. Soc., 48 (1952) 516–518.

PARTITIONING A SET INTO MUTUALLY HOMEOMORPHIC SUBSETS

BENJAMIN HALPERN, University of California, Berkeley

Let S be a subset of Euclidean n-dimensional space, R^n. We shall consider the problem of partitioning S into a fixed number N of mutually homeomorphic subsets. That is, we shall investigate when it is or is not possible to find sets $E_0, E_1, \cdots, E_{N-1}$ such that $E_i \cap E_j = \emptyset$ if $i \neq j$, E_i is homeomorphic to E_j for any i and j, and $\bigcup_{i=0}^{N-1} E_i = S$. This is not always possible. Any finite set S with m points can be so partitioned into N parts if and only if N divides m. If $S = \{p, p_1, p_2 \cdots\}$ where all the p's are distinct and $p_i \to p$ as $i \to \infty$ then S cannot be so partitioned into two or more homeomorphic parts. But as soon as S has a

nonempty interior it is possible for any $N=1, 2, \cdots$. This is the content of Theorem 1 below.

The problem of partitioning two sets A and B into subsets $A_1, A_2, \cdots A_n$ and $B_1, B_2, \cdots, B_N$ respectively such that $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^N A_i = A$, $\bigcup_{i=1}^N B_i = B$, and A_i is congruent to B_i for $i=1, 2, \cdots, N$ has been investigated (Sierpinski [1]).

We shall consider a related problem: to partition A and B into subsets $A_1, A_2, \cdots, A_N$ and $B_1, B_2, \cdots, B_N$ respectively such that $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$ if $i=j$, $\bigcup_{i=1}^N A_i = A$, $\bigcup_{i=1}^N B_i = B$, and A_i is homeomorphic to B_i for $i=1, 2, \cdots, N$. Theorem 2 below states that if A and B are both subsets of $\boldsymbol{R}^n$ and both have nonempty interiors then it is possible with $N=2$.

The author is indebted to Professor Theodore Motzkin for his help in generalizing the author's original results.

Definitions and notations. Let $\boldsymbol{R}^n$ denote n-dimensional Euclidean space and $0=(0, 0, \cdots, 0)$ its origin. Set

$$\|(x_1, x_2, \cdots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_2^2)^{1/2}.$$

If $A, B, \subset \boldsymbol{R}^n$ and $p \in \boldsymbol{R}^n$ then set

$$A \sim B = \{z \mid z \in A \text{ and } z \notin B\} \quad \text{and} \quad A + p = \{z + p \mid z \in A\}.$$

Define $[(x_1, x_2, \cdots, x_n)] = \max(|x_1|, |x_2|, \cdots, |x_n|)$. Note that $[\quad]$ satisfies $[\alpha z] = |\alpha|[z]$, $[z+w] \leqq [z]+[w]$ and $1/n\|z\| \leqq [z] \leqq \|z\|$ for all $z, w \in \boldsymbol{R}^n$ and $\alpha \in \boldsymbol{R}$ (the reals). Thus $[z]$ is continuous in z. Also note that

$$\{(x_1, x_2, \cdots, x_n) \mid [x_1, x_2, \cdots, x_n)] \leqq r\} = \{(x_1, x_2, \cdots, x_n) \mid |x_i| \leqq r$$

for $i=1, 2, \cdots, n\}$ for any real r.

Define $\phi(z) = [z]^{-2}z$ for $z \neq 0$. Note that ϕ is 1-1 and continuous for $z \neq 0$ with $\phi^{-1} = \phi$. Also $\phi(\{z \mid [z]=r\}) = \{z \mid [z] = 1/r\}$ for any $r > 0$.

Results. Theorem 1. *Let T be a subset of $\boldsymbol{R}^n$ with $n \geqq 1$. If T has a nonempty interior then for each integer $N \geqq 2$ there exists sets T_i, $i=0, 1, \cdots, N-1$ such that $T_i \cap T_j = \emptyset$ if $i \neq j$, $\bigcup_{i=0}^{N-1} = T$ and T_i is homeomorphic to T_j for all $i, j = 0, 1, \cdots, N-1$.*

Proof. Since T has a nonempty interior and $z \to kz+w$ is a homeomorphism of $\boldsymbol{R}^n$ onto $\boldsymbol{R}^n$ for $k \neq 0$ and $w \in \boldsymbol{R}^n$ we may assume without loss of generality that $R \subset T$, where

$$R = \{(x_1, x_2, \cdots, x_n) \mid -1 < x_1 < 2N-1 \text{ and }$$
$$-1 < x_i < +1 \text{ for } i = 2, 3, \cdots, n\}.$$

It is easy to construct a homeomorphism η from $\boldsymbol{R}^n$ onto $\subset \{z \in \boldsymbol{R}^n \mid [z] \leqq 2N\}$ such that $\eta(z) = z$ for all $z \in R$. Thus it is sufficient to consider the case where $R \subset T \subset \{z \in \boldsymbol{R}^n \mid [z] \leqq 2N\}$.

Set

$$S = T - R$$
$$p_j = (2j, 0, \cdots, 0)$$
$$r = 2N + 1$$
$$\phi_{i,j}(z) = \phi(r^{6i+2}z) + p_j$$
$$* \qquad A_i = \{z \mid r^{-3i} \leqq [z] \leqq r^{-3i+1}\} \qquad A_{i,j} = A_i + p_j$$
$$** \qquad B_i = \{z \mid r^{-3i-2} < [z] < r^{-3i}\} \qquad B_{i,j} = B_i + p_j$$

where $z \neq 0$, $i = 0, 1, 2, \cdots$ and $j = 0, 1, \cdots, N-1$. Define S_{2i} for $i = 0, 1, \cdots$ inductively by

$$S_0 = S, \qquad S_{2i+2} = \overset{-1}{\phi}_{2i+2,j} \circ \phi_{2i+1,j}(S_{2i}),$$

where j is any integer $0, 1, \cdots, N-1$. Set $S_{2i+1,j} = \phi_{2i+1,j}(S_{2i})$, $i = 0, 1, \cdots$ and $j = 0, 1, \cdots, N-1$. Define S'_{2i} for $i = 0, 1, \cdots$ inductively by

$$S'_0 = A_0 \sim S$$
$$S'_{2i+2} = \overset{-1}{\phi}_{2i+2,j} \circ \phi_{2i+1,j}(S'_{2i}) = A_{2i+2} \sim S_{2i+2}$$

where j is any integer $0, 1, \cdots, N-1$.
Set $S'_{2i+1,j} = \phi'_{2i+1,j}(S'_{2i})$ for $i = 0, 1, \cdots; j = 0, 1, \cdots, n-1$. Now for $i = 0, 1, \cdots$ let

$$C_{2i,0} = S_{2i} \cup B_{2i}$$
$$C_{2i,j} = S_{2i+1,j} \cup B_{2i,j} \qquad \text{for } j = 1, 2, \cdots, N-1$$
$$D_{2i+1,j} = S'_{2i+1,j+1} \cup B_{2i+1,j+1} \cup A_{2i+2,j+1} \qquad \text{for } j = 0, 1, \cdots, N-2$$
$$D_{2i+1,N-1} = A_{2i+1} \cup B_{2i+1} \cup S'_{2i+2}$$
$$T_j = \{p_j\} \cup \bigcup_{i=0}^{\infty} (C_{2i,j} \cup D_{2i+1,j}).$$

Finally, define $\Psi_j \colon T_0 \to T_j$, $j = 1, 2, \cdots, N-2$ by

$$\psi_j(z) \quad \begin{cases} p_j & \text{if } z = p_0 \\ \phi_{2i,j}(z) & \text{if } z \in C_{2i} \quad i = 0, 1, \cdots \\ z + p_j & \text{if } z \in D_{2i+1} \quad i = 0, 1, \cdots \end{cases}$$

and define Ψ_{N-1} by

$$\psi_{N-1}(z) = \begin{cases} p_{N-1} & \text{if } z = p_0 \\ \phi_{2i,N-1}(z) & \text{if } z \in C_i \quad i = 0, 1, \cdots \\ \overset{-1}{\phi}_{2i+1,1}(z) & \text{if } z \in D_i \quad i = 0, 1, \cdots \end{cases}$$

Ψ_j for $j=1, 2, \cdots, N-1$ is obviously 1-1 and onto. To see that Ψ_j is continuous observe that each of the sets $C_{2i,0}$ and $D_{2i+,0}$ is separated from the rest of T_0 by a positive distance. Thus continuity of Ψ_j at some point $z_0 \in C_{2i}$ or $z_0 \in D_{2i+1}$ $i=0, 1, \cdots$ reduces to the continuity of $\phi_{2i,j}$, $\phi_{2i+1,1}$ or $h(z)=z+p_j$ at z_0. But these latter maps are continuous. Now for the continuity of Ψ_j at p_0. If $z_m \to p_0$ as $m \to \infty$ then $z_m \in C_{2i_m}$ where $i_m \to \infty$ as $m \to \infty$. Thus $\Psi_j(z_m) \in C_{2i_m,j}$ and thus $\Psi_0(z_m) \to p_0$ as $m \to \infty$. The continuity of Ψ_j follows in the same way. This establishes that the T_j are mutually homeomorphic. The other required properties of the T_j are obvious from the construction. This completes the proof.

The subsets T_j constructed in the above proof are wildly disconnected. This is not always necessary. For a nontrivial example consider the method for dividing a closed square into two disjoint homeomorphic subsets illustrated in Figure 1. The homeomorphism between the dark and light parts is accomplished by rotating the dark through 180° and then pirouetting each of the dark rockets about its tip through 180° and finally distorting the fins a little.

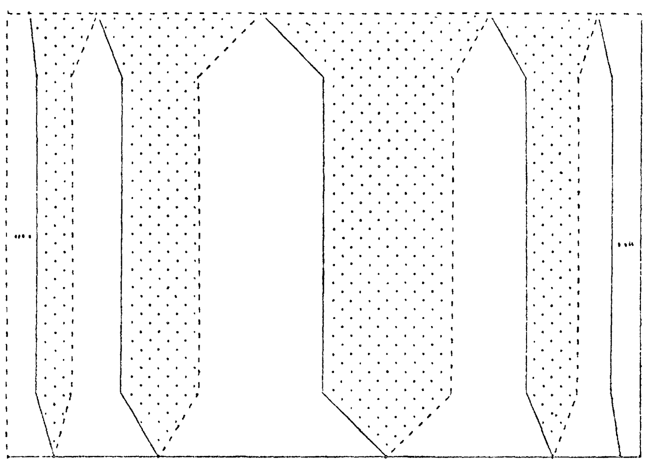

FIG. 1

To partition a subset of $\boldsymbol{R}^n$ with a nonempty interior into a countable number of disjoint mutually homeomorphic subsets is an easy matter. We will illustrate the method on the unit interval $[0, 1] \subset \boldsymbol{R}^1$. Set $A_i=(2^{-3i-1}, 2^{-3i}]$ for $i=0, 1, \cdots$, $B_0=\{0\}$, $B_i=\{2^{-3i+1}\}$ for $i=1, 2, \cdots$, and $C_i=(2^{-3i-3}, 2^{-3i-2}) \cup (2^{-3i-2}, 2^{-3i-1}]$ for $i=0, 1, \cdots$. Now let $T_l=A_{i_l} \cup B_{j_l} \cup C_{k_l}$ where i_l, j_l, and k_l are permutations of the integers $0, 1, 2, \cdots$ with the property that for each l the three sets A_{i_l}, B_{i_l} and C_{i_l} are pairwise separated by a positive distance.

THEOREM 2. *Let* $F, G \subset \boldsymbol{R}^n$, $n \geq 1$, *interior* $F \neq \emptyset$, *and interior* $G \neq 0$. *Then there exist sets* F_1, F_2, G_1 *and* G_2 *such that* $F_1 \cap F_2=\emptyset$, $G_1 \cap G_2=\emptyset$, $F_1 \cup F_2=F$, $G_1 \cup G_2=G$ *and* F_i *is homeomorphic to* G_i *for* $i=1, 2$.

Proof. As in Theorem 1 we may assume without loss of generality that

$$\{z \mid [z] < 1\} \subset F, G \subset \{z \mid [z] \leqq 2\}.$$

Let $r=2$ and define A_i and B_i by * and ** above. Set $\phi_i(z)=\phi(r^{6i+2}z)$. Define Q_i and T_i inductively by

$$Q_0 = A_0 \cap F, \qquad T = A \cap G$$
$$Q_{i+1} = \phi_{i+1}(Q_i), \qquad T_{i+1} = \phi_{i+1}(T_i)$$

Set $Q_i' = A_i \sim Q_i$ and $I_i' = A_i \sim T_i$ Then let

$$F_1 = \{0\} \cup \bigcup_{i=0}^{\infty} (Q_{2i} \cup B_{2i} \cup T_{2i+1})$$

$$F_2 = \bigcup_{i=0}^{\infty} (T'_{2i+1} \cup B_{2i+1} \cup Q'_{2i+2})$$

$$G_1 = \{0\} \cup \bigcup_{i=0}^{\infty} (T_{2i} \cup B_{2i} \cup Q_{2i+1})$$

$$G_2 = \bigcup_{i=0}^{\infty} (Q'_{2i+1} \cup B_{2i+1} \cup T'_{2i+2})$$

Define ψ_1: $F_1 \to G_1$ and ψ_2: $F_2 \to G_2$ by

$$\psi_1(z) = \begin{cases} \phi_{2i}(z) & \text{if } z \in Q_{2i} \cup B_{2i} \cup T_{2i+1} \qquad i = 0, 1, \cdots \\ 0 & \text{if } z = 0 \end{cases}$$

$\psi_2(z) = \phi_{2i+1}(z)$ if $z \in T'_{2i+1} \cup B_{2i+1} \cup Q'_{2i+2}$ $i=0, 1, \cdots$.

It is apparent from the construction that ψ_1 and ψ_2 are 1-1 and onto. By the same considerations as in Theorem 1 we see that ψ_1, ψ_1^{-1}, ψ_2 and ψ_2^{-1} are all continuous. This proves the theorem.

Reference

1. Waclaw Sierpínski, On the congruence of sets and their equivalence by finite decomposition, The Lucknow University, 1954.

A THIN SET OF CIRCLES

J. R. KINNEY, Michigan State University

At a recent symposium on combinatorial geometry held at Michigan State University the following question was posed: Is there a set K with plane Lebesgue measure zero, such that, for every positive $\alpha \leqq 1$, the set K contains a circle of diameter α? We construct K as follows: on $\{0 \leqq x \leqq 1, y=0\}$ we construct a Cantor set by the process of removing middle thirds. It is well known that the distance set of this Cantor set is the interval $0 \leqq \alpha \leqq 1$. For each α we

select the left most pair, $x_1(\alpha)$, $x_2(\alpha)$ of points from the Cantor set for which $x_2(\alpha)-x_1(\alpha)=\alpha$. We let $C(\alpha)$ be the circle with

$$\{(x,y)\,|\,x\in[x_1(\alpha),\,x_2(\alpha)],\ y=0\}$$

as diameter, and define $K=\bigcup_{0\leqq\alpha\leqq 1}C(\alpha)$. It is clear that for each α, $C(\alpha)$ is a subset of K, so the question will be answered in the affirmative if we can show that

THEOREM. *$m_2(K)=0$ where $m_2(\ )$ is two-dimensional Lebesgue measure.*

We first decompose K into a set of similar configurations $\{K_i\}$ and show it to be sufficient to prove the largest K_1 to have $m_2(K_1)=0$. Let $L(c)$ denote the points on $y=c$. We show that $K_1\cap L(c)$ is a set of linear measure zero for each c, and complete the proof by applying Fubini's theorem.

Let

$$K_1=\bigcup_{\frac{1}{3}\leqq\alpha\leqq 1}C(\alpha),\qquad K_i=\bigcup_{3^{-i}\leqq\alpha\leqq 3^{-i+1}}C(\alpha).$$

For $1/3<\alpha<1$, $0\leqq x_1(\alpha)\leqq 1/3$, $2/3\leqq x_2(\alpha)\leqq 1$. Since either $x_1(\alpha)\geqq 2/3$, or $x_2(\alpha)\leqq 1/3$, would imply $x_2(\alpha)-x_1(\alpha)\leqq 1/3$. Hence

$$C_\alpha\subset D_0=\left\{(x,y)\,\middle|\,\frac{1}{36}\leqq\left(x-\tfrac{1}{2}\right)^2+y^2\leqq\tfrac{1}{4}\right\}. \tag{1}$$

For $3^{-i}\leqq\alpha\leqq 3^{-i+1}$, $x_1(\alpha)=3^{-i+1}x_1(3^{i-1}\alpha)$, $x_2(\alpha)=3^{-i+1}x_2(3^{i-1}\alpha)$. This is seen from the property of the similarity of subsections of the Cantor set to its whole and the construction property of choosing always the left most pair to realize the distance α. Thus we see that for $3^{-i}\leqq\alpha\leqq 3^{-i+1}$,

$$C(\alpha)=\{(x,y)\,|\,(3^{i-1}x,\,3^{i-1}y)\in C(3^{i-1}\alpha)\}$$

and so

$$K_i=\{(x,y)\,|\,(3^{i-1}x,\,3^{i-1}y)\in K_1\}.$$

Since $K=\bigcup_{i\geqq 1}K_i$, it is clear that

$$m_2(K)\leqq\sum_{i\geqq 1}m_2(K_i)=\sum_{i\geqq 1}3^{-2(i-1)}m_2(K_1)=9m_2(K_1)/8.$$

Hence it is sufficient to show $m_2(K_1)=0$.

To complete the proof we construct K_1 in a different way. We decompose D_0, see (1), into $D_0=D_1\cup D_2\cup D_3$ where

$$D_1=\{(x,y)\,|\,(7/18)^2<(x-1/2)^2+y^2\leqq 1/4\}$$
$$D_2=\{(x,y)\,|\,(5/18)^2<(x-1/2)^2+y^2\leqq(7/18)^2\}$$
$$D_3=\{(x,y)\,|\,(1/16)^2<(x-1/2)^2+y^2\leqq(5/18)^2\}.$$

We define

$$D_2^*=\{(x,y)\,|\,(x+\tfrac{1}{9},\,y)\in D_2\}.$$

An argument slightly more complicated than that which showed that for $1/3 < \alpha < 5/9$, we have $C(\alpha) \subset D_0$ will show that for $1/3 < \alpha < 5/9$ we have $C(\alpha) \subset D_3$ and that for $7/9 < \alpha \leqq 1$ we have $C(\alpha) \subset D_1$. The same argument, plus the use of the "left most" property of the construction, will show that for $5/9 < \alpha \leqq 7/9$, we have $C(\alpha) \subset D_2^*$. Hence $K_1 \subset D_1 \cup D_2^* \cup D_3 \subset D_0$.

Each of the annuli D_1, D_2^*, and D_3 we treat as we treated D_0, that is, we divide it into three annuli of equal width, and slide the middle annulus to the left so that the left hand intersection with the x-axis coincides with that of the largest annulus and the right hand intersection coincides with that of the smallest annulus. Also as before, we will have

$$D_1 \cap K \subset D_{11} \cup D_{12}^{*} \cup D_{13} \subset D_1$$

in the obvious notation. If performing this operation on each annulus of the configuration be denoted by T, we have $K \subset T^n D_0 \subset T^{n-1} D_0$ and $K = \bigcap_n T^n D_0$. Figure 1 shows the effect of T on the annulus D_0.

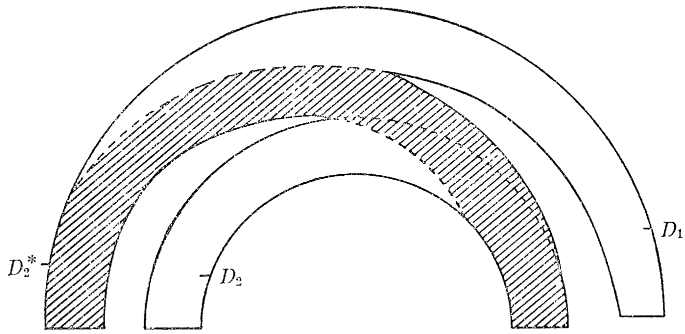

FIG. 1

Since T works the same way on each annulus, it will be sufficient to show that $m_2(T^n D) \to 0$ for any of the annuli. An appeal to Fubini's theorem shows that proving

$$\underset{n \to \infty}{m_1} (T^n D \cap L(k)) \to 0$$

for each $L(k)$ intersecting one of the annuli, where $m_1(\ )$ is linear Lebesgue measure, is sufficient to prove the theorem.

We look at the measure F constructed on $D \cap L(k)$ as follows:

First we normalize to $[0, 1]$, and let $F_1(x)$ be uniform measure on $[0, 1]$. After the first shift of the middle annulus, we let $I_{10}(x)$, $I_1^*(x)$, $I_{12}(x)$ be the characteristic functions of the intersections of the annuli with $L(k)$ and let

$$F_{11}(x) = \int_0^x \sum I_{1i}(y) dy.$$

We work from left to right on the annuli at each stage, letting

$$F_{ni}(x) = \int_0^x \left\{ \sum_{j=0}^{j=3^n} I_{n+1j}(y) + \sum_{j=i+1}^{3^n-1} I_{nj}(y) \right\} dy.$$

Since at each stage we always shift the annuli to the left, $F_{ni}(x)$ is monotonically increasing in n and i, and since $F_{ni}(x) \leqq 1$, increases to some monotone function $F(x)$. It is clear from the construction process that if $F'_{ni}(x) = 0$ on some interval (a, b) then $F'_{mj}(x) = 0$ on that interval for $j > i$, $m \geqq n$ and in fact that $F_{mj}(a) = F_{ni}(a)$, $F_{mj}(b) = F_{ni}(b)$ and hence that $F(a) = F_{ni}(a) = F_{ni}(b) = F(b)$. Also by the definition of F_{ni}, it is clear that $F'_{ni}(x) = 0$ or $F'_{ni}(x) \geqq 1$. Hence it will be sufficient to show that with probability 1, $F'_{ni}(x) \neq 0$ for but finitely many steps of the procedure.

We consider the stage $(n\ i)$. For $x \in I_{n+1\ 3i}(x) \cap c(I^*_{n+1\ 3i+1}(x))$, $F'_{n+1\ 3i}(x) = F'_{ni}(x)$; for $x \in I_{n+1\ 3i}(x) \cap I^*_{n+1\ 3i+1}(x)$, $F'_{n+1\ 3i}(x) = F'_{ni}(x) + 1$; for $x \in I_{n+1\ 3i+1}(x) \cap I^*_{n+1\ 3i+1}(x)$, $F'_{n+1\ 3i+1}(x) = F'_{ni}(x)$; for $x \in I_{n+1\ 3i+1}(x) \cap cI^*_{n+1\ 3i+1}(x)$, $F'_{n+1\ 3i+1}(x) = F'_{ni}(x) - 1$; for $x \in I_{n+1\ 3i+2}(x)$, $F'_{n+1\ 3i+2}(x) = F'_{ni}(x)$. Hence if

$$a_{ni} = \left| I_{n+1\ 3i}(x) \cap I_{n+1\ 3i+1}(x) \right| / \left| I_{ni}(x) \right|$$

we have $F'_{n+1\ 3i}(x) = F'_{ni}(x) + \delta_{ni}$
where

$$\delta_{ni} = \begin{array}{rl} 1 & \text{prob } a_{ni} \\ 0 & \text{prob } 1 - 2a_{ni} \\ -1 & \text{prob } a_{ni}. \end{array}$$

The basic argument is that since

$$\left| I_{n+1\ 3i+1}(x) \cap cI^*{}_{n+1\ 3i+1}(x) \right| = \left| I_{n+1\ 3i}(x) \cap I^*{}_{n+1\ 3i+1}(x) \right|,$$

as much of the line is covered less as is covered more. Hence $F'_{ni}(x)$ is a random walk, up or down with equal probability, having an absorbing barrier at 0. Such a random walk, with equal probability of going up or down, is as likely eventually to hit zero, as one for which the probabilities of going up or down are 1/2, 1/2, as long as the a_{ni} are bounded away from zero, which is true in our case. (First passage times of this sort are discussed in W. Feller's book, An Introduction to Probability Theory and its Applications, Vol. 1, second edition, Chapter XIV.) The conclusion is that for almost all x, (*almost all* here would refer to Lebesgue measure on the line), $F'_{ni}(x) = 0$ for n sufficiently large. Hence $F'(x) = 0$ for almost all x. Hence $m_1(K_1 \cap L(k)) = 0$ and hence, by Fubini's theorem, $m_2(K) = 0$.

The problem was also posed of finding configurations of plane measure zero containing the α-similitudes of the boundaries of other convex sets. It is not difficult to see that a construction similar to the above using the left most differerences of the Cantor set as the diagonals for the similar rectangles will yield a set of plane measure zero, but one using the base of the rectangle will yield a

figure with positive two dimensional measure.

We denote by T the Cantor middle third set, and let

$$Z = \{(x, y) \mid x \in T, 0 \leqq y \leqq 1\} \cup \{(x, y) \mid 0 \leqq x \leqq 1, y \in T\}.$$

Then, choosing $\{x_1(\alpha), x_2(\alpha)\}$, $\{y_1(\beta), y_2(\beta)\}$ as the choice was made above, we see that for α, β with $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, we can find a rectangle $R(\alpha, \beta)$ contained in Z, with sides of length α, β. Also it is easily seen that Z is of plane measure zero.

The author is indebted to B. Grunbaum, L. Danzer, and especially to L. M. Kelly for introduction to the problem and helpful discussion.

Since this paper was submitted, the author has been informed that Professor A. S. Besicovitch has another construction of a set with properties similar to those of K.

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REDUCIBILITY OF POLYNOMIALS OF ODD DEGREE

D. B. LLOYD, District of Columbia College

In a previous article in this journal [3] the author established a necessary condition for reducibility of the general polynomial over the field of rational numbers. This paper offers a condition which is *sufficient* for reducibility over the rationals for polynomials of odd degree.

No absolute criterion, both necessary and sufficient, for reducibility of the general polynomial is available in the literature. Although there are a number of criteria which apply to certain restricted classes, for most polynomials the tests now available entail an excessive amount of computation. Among the writers who have found criteria sufficient for *irreducibility* of certain types of polynomials are the following: Bauer, Dumas, Netto, Ore, Eisenstein, Perron, Schoenemann, Stackel, Polya and Szego, and Koenigsberger. Their work is reviewed in a historical survey made by Dorwart [1] in 1935.

The complete problem of determining the factors of a numerical polynomial, including a reducibility test, was first solved by Kronecker [6], but his calculations are prohibitive in length. He used the principle that $f(x)$ is divisible by $g(x)$ only if $f(n)$ is divisible by $g(n)$ for any integer n. His attack consisted in arranging in a vertical column the expressions

$$\cdots f(-3), f(-2), f(-1), f(0), f(1), f(2), \cdots$$

and placing after each of these numbers all its factors written with both plus and minus signs. If $r_i^{(s)}$ be one of these factors, lying in the horizontal row of $f(s)$, add s^2 to it. Then the numbers $r_i^{(s)} + s^2$ thus formed are kept on the line of $f(s)$, and constitute our actual working table. If now an arithmetical sequence can be discovered running vertically through this table, the difference of this sequence is a provisional value of n, to be tried out by synthetic division.

Refinements of his method have been devised by Runge, Mandl, Glenn, Frumveller (references to these authors are listed in [3]) and more recently by J. B. Kelly [2]. Frumveller, for instance, set up coefficient equations; using this system, by tedious elimination, he found upper limits of roots, constructed tables of possible values, and after a series of conjectures, with certain cross-elimination, the coefficients of the polynomial factors would be forthcoming. His method is about the simplest yet proposed, but is still circuitous.

Herein below the author will extend the work in this field by establishing a sufficient condition for reducibility which is applicable to all polynomials of odd degree. We shall exclude as trivial the case of a linear factor and confine attention to the factorization of the monic polynomial $f(x)=x^n+\sum_{i=1}^{n} c_i x^{n-i}$, $n\geqq 5$, and odd, into irreducible polynomials of quadratic or higher degree. We shall use the n-ic equation $f(x)=0$ and its related equation $S_N(x)=\Pi[x-(\theta_i+\theta_j)]=0$, of degree $N=\binom{n}{2}$ whose roots are $\theta_i+\theta_j$, sums of all possible pairs of the n roots of $f(n)=0$. We present and prove the key

THEOREM. *If $S_N(x)=0$ has a rational root, then $f(x)$ is reducible.*

Proof: Assume $\theta_1+\theta_2=r$, a rational integer; for if r were a fraction, θ_1 or θ_2, or both, would be fractional. But without affecting reducibility we could transform $f(x)$ to make any fractional root integral. Assume also that $f(x)$ is irreducible. We shall show that this leads to a contradiction.

(1) $$\text{As } r-\theta_1 \text{ is a root of } f,\ f(r-\theta_1)=0.$$

As (1) is an n-ic in θ_1, it must reduce identically to $f(\theta_1)=0$, as θ_1 satisfies a unique n-ic, [5, page 91]. As the other θ_i are conjugates of θ_1, they satisfy the same principal equation. That is, $f(r-\theta_i)=0$; and $r-\theta_i$ are the n roots of f. Then

(2) $$c_1=-\sum(r-\theta_i)=-nr-c_1 \quad\text{or}\quad r=(-2/n)\cdot c_1.$$

Since r is an integer, a contradiction is evident in (2) unless c_1 is a multiple of n. In the latter case we may attain a contradiction as follows:

Suppose $c_1=nk$, k an integer. Now a linear transformation does not affect reducibility; so increase the roots of f by k. This annuls the x^{n-1}-coefficient. Then, using primes to denote the new values, $c_1'=0$.

And from (2) $r'=(2/n)\cdot 0=0$. Hence, $\theta_1'+\theta_2'=0$.

Then the roots of f are both θ_i' and $-\theta_i'$, so $f(\theta')=f(-\theta')$. The negative of each root is also a root, and since f is of odd degree it must therefore have a zero root and hence be reducible.

An alternate variation of the above proof is the following. Assume as before that $f(x)$ is a monic irreducible polynomial of odd degree, and that the sum of any two of its roots is a rational integer r. The following then leads to a contradiction:

Since the roots of $f(r-x)$ are identical with those of $f(x)$ then

$$f(r-x)=kf(x).$$

Thence clearly $k=(-1)^n$. But selecting a value $x=r/2$ gives $1=(-1)^n$, since $f(r/2)\neq 0$. It follows that n is even, a contradiction.

The converse theorem is necessarily true only if f has a quadratic factor. Thus the condition is both necessary and sufficient only if f is a quintic.

Since the coefficients of the auxiliary function S_N are symmetric functions of the roots of f, they may be expressed in terms of the coefficients of f. S_N may thus be computed by machine methods for various values of $\binom{n}{2}$. For the case of the quintic, $N=\binom{5}{2}=10$. This decimic equation is:

$$\begin{aligned} S_{10}(x) = {} & x^{10} + 4c_1x^9 + (6c_1^2 + 3c_2)x^8 + (4c_1^3 + 9c_1c_2 + c_3)x^7 \\ & + (c_1^4 + 9c_1^2c_2 + 3c_2^2 + 4c_1c_3 - 3c_4)x^6 + (3c_1^3c_2 + 5c_1^2c_3 + 6c_1c_2^2 \\ & - 5c_1c_4 + 2c_2c_3 - 11c_5)x^5 + (2c_1^3c_3 + 3c_1^2c_2^2 - 2c_1^2c_4 - 22c_1c_5 \\ & + 6c_1c_2c_3 + c_2^3 - 2c_2c_4 - c_3^2)x^4 + (4c_1^2c_2c_3 - 16c_1^2c_5 + c_1c_2^3 + c_2^2c_3 \\ & - 4c_2c_5 - 4c_3c_4)x^3 + (c_1^2c_3^2 + c_1^2c_2c_4 - 4c_1^3c_5 + 2c_1c_2^2c_3 - 9c_1c_2c_5 \\ & - 3c_1c_3c_4 + c_2^2c_4 - c_2c_3^2 + 7c_3c_5 - 4c_4^2)x^2 + (c_1c_2c_3^2 + c_1c^2c_4 \\ & + 4c_1c_3c_5 - 4c_1c_4^2 - 4c_1^2c_2c_5 - c_2^2c_5 - c_3^3 + 4c_4c_5)x + (c_1c_2c_3c_4 \\ & - c_1c_2^2c_5 + 2c_1c_4c_5 - c_1^2c_4^2 + c_2c_3c_5 - c_3^2c_4 - c_5^2) = 0. \end{aligned}$$

Illustration: Given $f=x^5+4x^4+x^3+x^2+9x-6$. Compute

$$\begin{aligned} S_{10}(x) = {} & x^{10} + 16x^9 + 99x^8 + 293x^7 + 392x^6 + 184x^5 \\ & + 422x^4 + 1593x^3 + 1454x^2 - 1179x - 1719 = 0. \end{aligned}$$

The only possible integral roots are ± 3, or ± 1, or ± 9, factors of 1719. The wiser guess is -3 due to the size and signs of the coefficients. This checks, and as S_{10} has a rational root, then f is reducible, in this case into quadratic and cubic factors.

To find the factors, set $f=(x^2+mx+n).\ (x^3+px^2+qx+r)$. From the above, $\theta_1+\theta_2=-3=-m$. Multiplying, and equating like powers of coefficients, we find the unknown values. Substituting, and solving simultaneously, we have at once $p=1$. Thence by successive elimination, we get $q=(r^2-3r)/2$. Solving for r,

$$r = \frac{3 \pm \sqrt{9+8q}}{2}.$$

The only suitable values here are $q=0$, $r=3$ (We know r is a factor of 6). So, finally

$$f = (x^2 + 3x - 2)(x^3 + x^2 + 3).$$

The reader might wish to try the polynomial

$$x^5 + 2x^4 - 3x^3 - 5x^2 - 3.$$

It is a shorter example, since here S_{10} has a zero root, as evidenced by the vanishing of its constant term. In such cases the entire S_{10} function need not be computed.

The above reducibility criterion is not restricted in its application. If f has some zero coefficients the process is shortened but not invalidated. S_N is materially shortened by such zero coefficients, particularly those of higher degree.

The author tenders cordial acknowledgement to Dr. Morris Newman for reading this paper and offering kindly suggestions thereon.

References

1. H. L. Dorwart, Irreducibility of polynomials, this MONTHLY, 42 (1935) 369.
2. J. B. Kelly, On factorization of polynomials, this MONTHLY, 60 (1953) 375.
3. D. B. Lloyd, Factorization of the general polynomial by means of its homomorphic congruential functions, this MONTHLY, 71 (1964) 863–870.
4. ———, J. Res. Nat. Bur. Standards, 69B (July 1965) 189–212.
5. C. C. MacDuffie, An Introduction to Abstract Algebra, Wiley, New York, 1956.
6. L. J. Kronecker, J. Reine Agnew. Math., 92 (1882) 111–114.
7. M. V. Yakovkin, On reducibility of polynomials, Dokl. Akad. Nauk SSSR, 93 (1953) 629–631, 783–785.

MATHEMATICAL NOTES

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

THE PROBABILITY OF DUPLICATION IN SAMPLING WITH REPLACEMENT

R. S. KLEBER, St. Olaf College and Iowa State University

Three related problems in combinatorial probability could be stated as follows:

I. A random sample of size K is taken with replacement from a population of size N. What is the probability P that at least one item is chosen more than once?

II. Given a population of size N and a probability P, what is the size K of the smallest random sample that can be taken with replacement such that the probability is at least P that at least one item is chosen more than once?

III. Given a random sample size K and a probability P, what is the size N of the largest population from which the sample can be taken with replacement such that the probability is at least P that at least one item is chosen more than once?

The answer to the first problem is well known and is given by

$$P = 1 - N!/[N^K(N-K)!], \qquad K \leqq N.$$

The second and third problems are more difficult to answer since there are no direct formulas for these values. Heuer [1] has shown that K is given asymptot-

ically by $\sqrt{DN}$, where $D=-2\log(1-P)$. This gives reasonably accurate results for K, but the inverse formula for N is not very accurate. It is the purpose of this paper to provide formulas for both K and N which give highly accurate results.

To solve problem II one needs to find the smallest integer K such that $1-N!/[N^K(N-K)!\geqq P$, or equivalently such that $N!/[N^K(N-K)!]\leqq 1-P$, for fixed values of N and P. Similarly, to solve problem III one needs to find the largest integer N such that $N!/[N^K(N-K)!]\leqq 1-P$ for fixed values of K and P.

In order to obtain estimates of these values of K and N we shall consider approximations to the solutions of the following equation, obtained by using Stirling's formula in both the numerator and denominator of $N!/[N^K(N-K)!]$:

$$1/e^K(1-K/N)^{N-K+1/2}=1-P. \tag{1}$$

Inverting and taking natural logarithms in equation (1) one obtains:

$$K+(N-K+1/2)\log(1-K/N)=-\log(1-P). \tag{2}$$

Expanding the logarithm on the left in equation (2) and simplifying one obtains:

$$\begin{aligned}&(K^2/1\cdot 2N+K^3/2\cdot 3N^2+K^4/3\cdot 4N^3+\cdots)\\ &\qquad -(K/2N+K^2/4N^2+K^3/6N^3+\cdots)=-\log(1-P).\end{aligned} \tag{3}$$

We consider first a simple extension of Heuer's result as it applies to problem II. Let $K=A\sqrt{N}+B$ and substitute this into equation (3). Expanding and grouping terms in descending powers of N one obtains:

$$A^2/2+(6AB+A^3-3A)/6\sqrt{N}+O(1/N)=-\log(1-P). \tag{4}$$

Letting $A^2/2=-\log(1-P)$ and choosing B so that $6AB+A^3-3A=0$, one obtains the approximation:

$$K_1=\sqrt{DN}+(3-D)/6, \tag{5}$$

where $D=-2\log(1-P)$. For any particular choice of N and P one would then choose the smallest integer greater than or equal to K_1, $-[-K_1]$, as the solution to problem II. Table 1 gives a partial listing of the error in this approximation for certain values of N and P.

TABLE 1

Predicted values minus true values for prediction formula (5)

N	$P=0.01$	$P=0.05$	$P=0.25$	$P=0.50$	$P=0.75$	$P=0.95$	$P=0.99$
5^2	-------	−0.08−	−0.032	−0.016	−0.001	0.04−	0.09−
10^2	−0.086	−0.04−	−0.016	−0.008	−0.001	0.017	0.038
25^2	−0.035	−0.016	−0.006	−0.003	−0.000	0.006	0.014
50^2	−0.018	−0.008	−0.003	−0.002	−0.000	0.003	0.007
100^2	−0.009	−0.004	−0.002	−0.001	−0.000	0.002	0.004

Tables 1, 2 and 3 were obtained by finding the actual error at integer values of N and K for values of P above and below the given value and then by using linear interpolation, retaining as many significant digits as were reasonable in the interpolation. In each table a negative error indicates an error which could lead to a result with a probability less than P. A positive error indicates an error which does lead to a result with probability of at least P, but not necessarily the correct solution to problem II or III. The computations were done on Iowa State University's IBM 360/50.

To obtain a more precise estimate we let $K=A\sqrt{N}+B+C/\sqrt{N}$ and substitute this into equation (3). In this case expanding and grouping terms in descending powers of N one obtains:

$$\begin{aligned}(6)\qquad & A^2/2+(6AB+A^3-3A)/6\sqrt{N}+(12AC+6B^2\\ &\qquad -6B+6A^2B-3A^2+A^4)/12N+O(1/N^{3/2}) = -\log(1-P).\end{aligned}$$

Letting $A^2/2=-\log(1-P)$ and choosing B and C so that $6AB+A^3-3A=0$ and $12AC+6B^2-6B+6A^2B-3A^2+A^4=0$ one obtains the approximation:

$$K_2=\sqrt{DN}+(3-D)/6+(9-D^2)/72\sqrt{DN}, \tag{7}$$

where again $D=-2\log(1-P)$. As before one would choose $-[-K_2]$ as the solution to problem II. Table 2 gives a partial listing of the error in this approximation.

TABLE 2

Predicted values minus true values for prediction formula (7)

N	$P=0.01$	$P=0.05$	$P=0.25$	$P=0.50$	$P=0.75$	$P=0.95$	$P=0.99$
5^2	-------	0.002--	0.00010	0.0002-	0.0011-	0.006--	0.016--
10^2	0.00258	0.0003-	0.00002	0.00006	0.00028	0.0014-	0.0035-
25^2	0.00018	0.00002	0.00000	0.00001	0.00005	0.00022	0.00052
50^2	0.00002	0.00000	0.00000	0.00000	0.00001	0.00005	0.00013
100^2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00003

One might consider continuing this procedure to get better approximations to the roots of equation (1). However it is not our purpose to estimate these roots but rather to solve problem II. Using a root finding program on the IBM 360/50 the true roots of equation (1) were approximated for various values of N and P. These computations showed that prediction formula (7) gives more exact answers to problem II than the true roots of equation (1) except for values of P greater than approximately 0.90. If one lets $K=K_2$ in equation (6) the coefficient of $1/N^{3/2}$ becomes $\sqrt{D}(2D^2-45)/540$. This coefficient is zero when $P\doteq 0.907$ and as one might expect the approximation to the root is best near this value, which supports the observation above.

Turning now to problem III we let $N=AK^2+BK+C$ and substitute this

into equation (3). Expanding each term in powers of $1/K$ and grouping terms in descending powers of K one obtains:

$$(8)\quad \begin{aligned} &1/2A + (1 - 3B - 3A)/6A^2K + (1 - 4B + 6B^2 \\ &\qquad - 6AC - 3A + 6AB)/12A^3K^2 + O(1/K^3) = -\log(1 - P). \end{aligned}$$

Letting $1/2A = -\log(1-P)$ and choosing B and C so that $1-3B-3A=0$ and $1-4B+6B^2-6AC-3A+6AB=0$ one obtains the approximation:

$$(9)\qquad N_1 = (K^2 - K)/D + K/3 + (D - 3)/18,$$

where again $D = -2\log(1-P)$. For any particular choice of K and P one would in this case choose the greatest integer less than or equal to N_1, $[N_1]$, as the solution to problem III. Table 3 gives a partial listing of the error in this approximation.

TABLE 3

True values minus predicted values for prediction formula (9)

K	$P=0.01$	$P=0.05$	$P=0.25$	$P=0.50$	$P=0.75$	$P=0.95$	$P=0.99$
2	−0.00028	−0.00143	−0.00802	−0.01971	-------	-------	-------
5	−0.00003	−0.00013	−0.00029	0.0011-	0.009--	0.06---	-------
10	−0.00001	−0.00002	0.00008	0.00108	0.0053-	0.029--	0.08---
25	−0.00000	−0.00000	0.00007	0.00052	0.00223	0.0110-	0.0269-
50	−0.00000	0.00000	0.00004	0.00027	0.00113	0.00541	0.01299
100	0.00000	0.00000	0.00002	0.00014	0.00057	0.00268	0.00639

If one lets $N=N_1$ in equation (8) the coefficient of $1/K^3$ becomes $D^2(2D^2-45)/540$. Similar to the previous case computations showed that prediction formula (9) gives more exact answers to problem III than the true roots of equation (1) except for values of P greater than approximately 0.90.

Reference

1. Gerald A. Heuer, Estimation in a certain probability problem, this MONTHLY, 66 (1959) 704–706.

UNDER-EXACT SEQUENCES

F. RHODES and R. F. TURNER-SMITH, Wesleyan University, Connecticut and University of Southampton, England

Sequences of groups and homomorphisms

$$\longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow 0$$

with the property that at each stage every element in the image of α_{n+1} is mapped by α_n to the identity element of A_{n-1} have been studied extensively. Those elements of A_n which are mapped by α_n to the identity element of A_{n-1}

form a subgroup of A_n which is called the *kernel* of α_n, and the property just described can be expressed in symbols by $\operatorname{Im} \alpha_{n+1} \subseteq \operatorname{Ker} \alpha_n$.

If $\operatorname{Im} \alpha_{n+1} = \operatorname{Ker} \alpha_n$ for every integer n, then the sequence is said to be *exact*. The occurrence in an exact sequence of the trivial group 0 (the group with only one element) imposes conditions on some of the homomorphisms. For example, if

$$0 \rightarrow A \xrightarrow{\theta} B \xrightarrow{\phi} C \rightarrow 0$$

is an exact sequence, then θ is a monomorphism (a homomorphism which is 1-1) and ϕ is an epimorphism (a homomorphism such that each element of C is the image of an element of B). In this case B is said to be an *extension* of A by C. If, in addition, there is a homomorphism $\psi: C \rightarrow B$ such that $\phi\psi$ is the identity map of C, then both A and C can be regarded as subgroups of B, and each element of B can be expressed uniquely as a product of an element of A and an element of C. In this case B is said to be a *split extension* of A by C.

On the other hand, if $\operatorname{Im} \alpha_{n+1}$ is a proper normal subgroup of $\operatorname{Ker} \alpha_n$, then the factor group $\operatorname{Ker} \alpha_n / \operatorname{Im} \alpha_{n+1}$ can be regarded as a measure of the deviation of the sequence from exactness at A_n. The application of these ideas to the homology theory of topological spaces is expounded by E. H. Spanier in his book, *On Algebraic Topology* [3]. The development of the algebraic ideas is found, for example, in the book on homological algebra by D. G. Northcott [1].

In contrast, sequences in which the reverse inclusion holds ($\operatorname{Ker} \subseteq \operatorname{Im}$) have not yet been studied. However, such sequences arose recently in connection with work on the homotopy groups of a transformation group [2], and they have proved to have some interesting properties. We change Northcott's name for sequences such that $\operatorname{Im} \subseteq \operatorname{Ker}$ to *over-exact* sequences, and use the name *under-exact* sequences for sequences such that $\operatorname{Ker} \subseteq \operatorname{Im}$. We prove that under certain conditions an under-exact sequence must arise as a sequence of split extensions of an exact sequence, and we state a form of the 5-lemma for under-exact sequences from which the 5-lemma for exact sequences follows as a corollary.

DEFINITION. *A sequence of homomorphisms*

$$A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1}$$

will be said to be under-exact at A_n if $\operatorname{Ker} \alpha_n \subseteq \operatorname{Im} \alpha_{n+1}$. *If G_n is a subgroup of A_n such that G_n* $\operatorname{Ker} \alpha_n = \operatorname{Im} \alpha_{n+1}$, *then the sequence will be said to be under-exact relative to G_n at A_n.*

Clearly, if the sequence is under-exact at A_n, it is under-exact relative to $\operatorname{Im} \alpha_{n+1}$ at A_n. Moreover, if G_n is the subgroup of $\operatorname{Im} \alpha_{n+1}$ generated by a transversal of $\operatorname{Ker} \alpha_n$ in $\operatorname{Im} \alpha_{n+1}$, then the sequence is under-exact relative to G_n at A_n.

THEOREM 1. *Suppose that, for every integer n, the sequence*

$$(1) \qquad \cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is under-exact at A_n, *and that* $K_n = \text{Ker } \alpha_{n-1}\alpha_n$. *Then* α_n *induces a homomorphism of* K_n *into* K_{n-1}, *and the sequence*

$$(2) \qquad \cdots \longrightarrow K_{n+1} \xrightarrow{\alpha_{n+1}} K_n \xrightarrow{\alpha_n} K_{n-1} \longrightarrow \cdots$$

is exact at K_n. *Moreover, if* (1) *is under-exact relative to* G_n *at* A_n, *and for each integer* n, α_n *induces an isomorphism of* G_n *to* G_{n-1}, *then* A_n *is a split extension of* K_n *by* G_n.

Proof: By definition, $\alpha_{n-1}\alpha_n K_n = 1$, and so $\alpha_n K_n \subseteq K_{n-1}$ and $\alpha_{n+1}K_{n+1} \subseteq \text{Ker } \alpha_n \cap K_n$. On the other hand if $k \in \text{Ker } \alpha_n \cap K_n$ then, since (1) is under-exact at A_n, there is an element $a \in A_{n+1}$ such that $\alpha_{n+1}a = k$. But $\alpha_n\alpha_{n+1}a = \alpha_n k = 1$, so $a \in K_{n+1}$, and $\alpha_{n+1}K_{n+1} \supseteq \text{Ker } \alpha_n \cap K_n$. Hence (2) is exact at K_n.

For the second part of the theorem note that since $\alpha_{n-1}\alpha_n$ induces a monomorphism on G_n, $K_n \cap G_n = 1$. On the other hand for each $a \in A_n$, $\alpha_{n-1}\alpha_n a \in \alpha_{n-1}G_{n-1}$. Consequently there exists $g \in G_n$ such that $\alpha_{n-1}\alpha_n a = \alpha_{n-1}\alpha_n g$ and $ag^{-1} \in K_n$. Thus $A_n = K_n G_n$.

On the other hand, if it is known that each group A_n arises as a split extension, then the following result holds:

THEOREM 2. *Suppose that, for every integer* n, A_n *is a split extension of* K_n *by* G_n *and that* $\alpha_n: A_n \to A_{n-1}$ *is a homomorphism with the properties*:

$$\text{(a)} \quad \alpha_n K_n \subseteq K_{n-1}, \qquad \text{(b)} \quad \alpha_n G_n = G_{n-1}.$$

Then the sequence (1) *is under-exact relative to* G_n *at* A_n *if and only if the sequence* (2) *is exact at* K_n.

Proof: Suppose that in the sequence (1), $\text{Im } \alpha_{n+1} = G_n \text{Ker } \alpha_n$. Then $\alpha_n\alpha_{n+1}K_{n+1} = \alpha_n G_n = G_{n-1}$, hence $\alpha_n\alpha_{n+1}K_{n+1} \subseteq K_{n-1} \cap G_{n-1} = 1$. Moreover, if $k_n \in K_n$ is such that $\alpha_n k_n = 1$, then there is an element $a_{n+1} = g_{n+1}k_{n+1} \in A_{n+1}$ such that $\alpha_{n+1}a_{n+1} = k_n$. Thus $\alpha_{n+1}g_{n+1} \in K_n \cap G_n = 1$, and hence $\alpha_{n+1}k_{n+1} = k_n$. This proves the necessity of the condition.

Suppose that, in the sequence (2), $\text{Im } \alpha_{n+1} = \text{Ker } \alpha_n$. Then

$$\alpha_n\alpha_{n+1}A_{n+1} = (\alpha_n\alpha_{n+1}K_{n+1})G_{n-1} = G_{n-1} = \alpha_n G_n,$$

and $\alpha_{n+1}A_{n+1} \subseteq G_n \text{ Ker } \alpha_n$. Now if $g_n k_n \in G_n \text{ Ker } \alpha_n$, then $\alpha_n g_n k_n \in G_{n-1}$. Thus $\alpha_n k_n \in K_{n-1} \cap G_{n-1} = 1$, and there is an element $k_{n+1} \in K_{n+1}$ such that $\alpha_{n+1}k_{n+1} = k_n$. Moreover, by condition (b), there is an element $g_{n+1} \in G_{n+1}$ such that $\alpha_{n+1}g_{n+1} = g_n$. Now $\alpha_{n+1}g_{n+1}k_{n+1} = g_n k_n$. This shows that $\alpha_{n+1}A_{n+1} \supseteq G_n \text{ Ker } \alpha_n$, and so proves the sufficiency of the condition.

It is not possible to sharpen Theorem 1 to an exact converse of Theorem 2. This is demonstrated by the following example. Let A_n be the cyclic group of order p^n generated by a_n, where p is a given prime. The mapping $\alpha_n: a_n \to a_{n-1}^p$ induces a homomorphism of A_n onto the subgroup of A_{n-1} of order p^{n-2}. Now in the sequence

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

$\operatorname{Im} \alpha_{n+1} = gp\{a_n^p\} \supset gp\{a_n^{p^{n-1}}\} = \operatorname{Ker} \alpha_n$, for $n \geqq 2$. Notice the inclusion is strict, except when $n=3$, and that the sequence is under-exact at A_n relative to $\operatorname{Im} \alpha_{n+1}$. (In fact this is the only subgroup of A_n relative to which the sequence is under-exact at A_n.) We now have a counter example since A_n is not a split extension of any of its nontrivial subgroups.

In the application of exact sequences to homology theory the following result is often used and is referred to as the 5-lemma. If

$$\begin{tikzcd} A_4 \arrow[r, "\alpha_4"] \arrow[d, "\phi_4"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\phi_3"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "\phi_1"] & A_0 \arrow[d, "\phi_0"] \\ B_4 \arrow[r, "\beta_4"] & B_3 \arrow[r, "\beta_3"] & B_2 \arrow[r, "\beta_2"] & B_1 \arrow[r, "\beta_1"] & B_0 \end{tikzcd}$$

is a commutative diagram in which each row is exact and ϕ_4, ϕ_3, ϕ_1 and ϕ_0 are isomorphisms, then ϕ_2 is an isomorphism. That this result depends not so much on exactness of rows as on under-exactness is shown by the next theorem, from which the result just quoted follows as a corollary. That there is no result in this direction for over-exact sequences is shown by examples in which A_2 and B_2 are non-isomorphic groups while the other eight groups are all trivial.

Note first that without any condition on the rows ϕ_4 induces a homomorphism $\operatorname{Ker} \alpha_3\alpha_4 \to \operatorname{Ker} \beta_3\beta_4$, and ϕ_0 a homomorphism $\operatorname{Im} \alpha_1\alpha_2 \to \operatorname{Im} \beta_1\beta_2$.

THEOREM 3. *Suppose that the diagram is commutative with under-exact rows.*

(1) *If* ϕ_3, ϕ_1 *are epimorphic and* $\phi_0 \operatorname{Im} \alpha_1\alpha_2 = \operatorname{Im} \beta_1\beta_2$, *and* $\operatorname{Ker} \phi_0 \subseteq \operatorname{Im} \alpha_1\alpha_2$, *then* ϕ_2 *is epimorphic.*

(2) *If* ϕ_1, ϕ_3 *are monomorphic and* $\phi_4 \operatorname{Ker} \alpha_3\alpha_4 = \operatorname{Ker} \beta_3\beta_4$, *then* ϕ_2 *is monomorphic.*

The proof follows the same lines as that of 4.5.11 of [3] and is left to the reader.

References

1. D. G. Northcott, An introduction to homological algebra, C.U.P., 1960.

2. F. Rhodes, Homotopy groups of transformation groups, Notices Amer. Math. Soc., 14 (1967) 544.

3. E. H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

A NOTE ON GROUP COMMUTATORS OF 2×2 MATRICES

C. J. ELIEZER, University of Malaya

1. In a recent paper, I. Sinha [1] has stated and proved the following:

THEOREM I. *If A, B are nonsingular 2×2 matrices, and if the group commutator $ABA^{-1}B^{-1}$ commutes with both A and B, then A and B either commute or anticommute.*

The following simple proof of this result should be of interest. We use

THEOREM II. *If A is a 2×2 matrix ($\neq pI$) then any matrix $X(\neq qI)$ which commutes with A is of the form $X=\lambda A+\mu I$, where p, q, λ, μ are scalars and I is the 2×2 unit matrix.*

The proof is elementary and not reproduced here. [Taking

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we examine the implications of $AX=XA$.]

If $Y=ABA^{-1}B^{-1}$ commutes with A and B, then by Theorem II,

$$Y = kA + lI \quad \text{and} \quad Y = mB + nI$$

where k, l, m, n are scalars. If $k\neq 0$, we obtain $A=(m/k)B+(n-l)I/k$ which satisfies $AB=BA$. If $k=0$ but $m\neq 0$, then $B=(l-n)I/m$, and $AB=BA$. If $k=0$ and $m=0$, then $l=n$ and $Y=ABA^{-1}B^{-1}=lI$. Taking determinants, $|Y|=l^2$. But $|Y|=1$ and hence $l^2=1$, giving $l=1$ or -1. Hence $ABA^{-1}B^{-1}=I$ or $-I$, and $AB=BA$ or $-BA$.

2. Theorem II may also be used to prove another result quoted by I. Sinha, viz: If 2×2 nonsingular matrices A, B are such that $AB-BA$ commutes with A and B, then $AB=BA$. As before $AB-BA=pA+qI=rB+sI$. If p or r is not zero, then it is easy to see that A and B commute. If $p=r=0$, then $q=s$, $AB-BA=qI$. Taking the trace of each side, we obtain $q=0$.

3. It is of interest to note the analogue of Theorem II for anticommuting matrices:

THEOREM III. *If a 2×2 matrix X anticommutes with a given 2×2 matrix A, then for $X\neq 0$ exist:*

$$\textit{either (i)} \ |A| = 0 \quad \textit{or} \quad \text{(ii)} \ \textit{Trace}\ A = 0.$$

Further, if $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then in case (i), if A is expressed, using the summation convention, in the form $A=a_0I+a_r\sigma_r$ where $r=1, 2, 3$ and $a_0^2=a_1^2+a_2^2+a_3^2$, then X may be written $\lambda(a_0I-a_r\sigma_r)$ and $AX=XA=0$. In case (ii), A is of the form $a_r\sigma_r$ and $X=x_r\sigma_r$ where $a_rx_r=0$, i.e., the vectors (a_1, a_2, a_3) and (x_1, x_2, x_3) are orthogonal.

Reference

1. I. Sinha, On group-commutators of 2×2 matrices, this MONTHLY, 72 (1965) 525.

$$\sum_{t=1}^{r-1}(x_t^p - x_{t+1}\cdots x_{t+p}) \geqq \sum_{t=r}^{n}(x_{t+1}\cdots x_{t+p} - x_t^p), \tag{5}$$

where all the terms are nonnegative, r being the least positive integer such that $x_r^p < x_{r+1}\cdots x_{r+p}$.

The proof of the theorem now follows by induction on p. Clearly (4) holds with equality, for $p=1$. Let (4) hold for some p, $1 \leqq p < n$, and write (4) in the form (5). Since all the terms in (5) are nonnegative, by multiplying by x_r and noting that

$$x_t \geqq x_r \quad \text{for } t < r, \qquad x_t \leqq x_r \quad \text{for } t > r,$$

we see that

$$\sum_{t=1}^{r-1} x_t(x_t^p - x_{t+1}\cdots x_{t+p}) \geqq \sum_{t=r}^{n} x_t(x_{t+1}\cdots x_{t+p} - x_t^p)$$

which is (4) with p replaced by $p+1$. This proves the theorem.

NOTE. A slight adjustment in the proof enables one to show that under the hypotheses of the theorem strict inequality holds in (4) when $p>1$.

COROLLARY. *If we define $x_{sn+t} = x_t$ for all positive integers s and t, $1 \leqq t \leqq n$, then* (4) *holds for all p.*

Proof. By the arithmetic geometric mean inequality

$$\sum_{j=1}^{p} x_{t+j}^p \geqq p x_{t+1}\cdots x_{t+p},$$

for $1 \leqq t \leqq n$. Summation over t, $1 \leqq t \leqq n$, and division by p gives the required result.

I wish to thank the referee for his comments.

A NOTE ON SCHWARZ DIFFERENTIABILITY

JOHN SWETITS, Lafayette College

Let $f(x)$ be a real valued function defined on an open interval I and let $[a, b]$ be a closed subinterval of I. Assume $f(x)$ is Schwarz differentiable at each point in I, i.e.,

$$\lim_{h\to 0}\frac{f(x+h) - f(x-h)}{2h}$$

exists $\forall x \in I$. Suppose that

$$\psi(x, h) = \frac{f(x+h) - f(x-h)}{2h} - f^{(1)}(x),$$

where $f^{(1)}(x)$ denotes the Schwarz derivative of $f(x)$. If $\lim_{h\to 0} \psi(x, h) = 0$ uniformly on $[a, b]$, then $f(x)$ is said to be uniformly Schwarz differentiable on $[a, b]$ [2], where uniformly Schwarz differentiable in I means uniformly in each closed subinterval of I.

In [2] Mukhapadhyay posed the question: Does the uniform Schwarz differentiability of $f(x)$ imply the continuity of $f(x)$? It is the purpose of this note to prove a stronger result.

Throughout the paper we assume that if $\exists x_0$ such that $\lim_{h\to 0}|f(x_0+h)| = \infty$, then $\exists x_1$ such that $f(x)$ is locally bounded in $[x_1, x_0)$.

THEOREM 1. *If* $\exists x_0 \in I$ *such that* $\lim_{h\to 0} |f(x_0+h)| = \infty$, *then* $f(x)$ *is not uniformly Schwarz differentiable in* I.

Proof. Let $[a, b]$ be a closed subinterval of I such that $x_0 \in (a, b)$. Let δ be such that $0<\delta<1$. Let x_1 be such that $f(x)$ is locally bounded in $[x_1, x_0)$ and $x_0 - x_1 < \delta$. Let x_2 be the midpoint of $[x_1, x_0)$. Then $|f(x)| \leqq M < \infty$ $\forall x \in [x_1, x_2]$. Fix h, $0<h<\delta/2$, so that

$$|f(x_2 + h)| > 1 + M + |f^{(1)}(x_2)|.$$

Then

$$\begin{aligned} |\psi(x_2, h)| &\geqq \left|\frac{f(x_2+h) - f(x_2-h)}{2h}\right| - |f^{(1)}(x_2)| \\ &\geqq |f(x_2+h)| - |f(x_2-h)| - |f^{(1)}(x_2)| \\ &\geqq |f(x_2+h)| - M - |f^{(1)}(x_2)| \\ &> 1. \end{aligned}$$

Thus $f(x)$ is not uniformly Schwarz differentiable in I.

THEOREM 2. $f(x)$ *is uniformly Schwarz differentiable in I if and only if* $f(x)$ *is continuously differentiable in* I.

Proof. To prove necessity, let $x_0 \in I$ and let $[a, b]$ be a closed subinterval of I such that $x_0 \in (a, b)$. Theorem 1 implies that $f(x)$ is locally bounded in I. Let $\delta_1 > 0$ be such that $(x_0 - \delta_1, x_0 + \delta_1) \subset (a, b)$ and

$$|f(x)| \leqq M < \infty \ \forall x \in (x_0 - \delta_1, x_0 + \delta_1).$$

Let $I_1 = (x_0 - \delta_1, x_0 + \delta_1)$ and let $I_2 = (x_0 - (\delta_1/2), x_0 + (\delta_1/2))$. Let $\delta > 0$ be such that $|\psi(x, h)| < 1$ if $|h| < \delta$ and $x \in [a, b]$. Fix h_1 so that $|h_1| < \delta$ and $x \pm h_1 \in I_1$ if $x \in I_2$. Then, $\forall x \in I_2$

$$|f^{(1)}(x)| < 1 + \left|\frac{f(x+h_1) - f(x-h_1)}{2h_1}\right|$$
$$< 1 + \frac{2M}{2|h_1|} = 1 + \frac{M}{|h_1|}\cdot$$

Thus $f^{(1)}(x)$ is locally bounded in I. By Theorem 1 and Theorem A of [**2**], $f(x)$ and $f^{(1)}(x)$ are continuous in I. It follows from a result of Aull, [**1**, p. 709, Theorem 3], that $f(x)$ is differentiable in I. Since $f^{(1)}(x) = f'(x)$ and $f^{(1)}(x)$ is continuous in I, $f'(x)$ is continuous in I.

Sufficiency follows from Theorem 2 of [2].

The author is indebted to the referee for some helpful suggestions.

References

1. C. E. Aull, The first symmetric derivative, this MONTHLY, 74(1967) 708.
2. S. N. Mukhapadhyay, On Schwarz differentiability—V, this MONTHLY, 74(1967) 542.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

A NOTE ON THE THEOREMS OF GREEN AND STOKES

P. J. LEDDEN, University of California, San Diego

1. Introduction. The theorems to which I have reference are these:

THEOREM 1. (Green) *Let S^* be a region in the plane with boundary ∂S^* and let $F^*(x, y) = (\eta(x, y), \zeta(x, y))$ be a vector field on S^*. Under suitable assumptions concerning the regularity of S^* and the differentiability of F^*, the following equality holds:*

$$\int_{\partial S^*} F^* = \iint_{S^*}\left(\frac{\partial\zeta}{\partial x} - \frac{\partial\eta}{\partial y}\right)dxdy. \tag{1}$$

THEOREM 2. (Stokes) *Let S be the graph of the function $z = f(x, y)$, where $f(x, y)$ is defined on some region S^* for which Theorem 1 is true. Further, let $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ be a vector field on S. Then, under suitable differentiability and orientation assumptions, the following holds:*

$$\int_{\partial S} F = \iint_S [(R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma]dA. \tag{2}$$

(Here $n=(\cos\alpha, \cos\beta, \cos\gamma)$ is the unit normal to the surface S. The subscripts indicate partial differentiation, and the integral on the right is the ordinary surface integral.)

In the standard advanced calculus texts there are two procedures for proving Theorem 2. One depends upon repeated applications of Theorem 1 [cf. **1**]; the other starts from first principles and follows closely the technique involved in proving Theorem 1 itself [cf. **2**]. One can, however, by a slight modification of the first of these obtain Theorem 2 as a simple extension of Theorem 1. The relation between them is thus seen to be precisely that of Rolle's Theorem and the mean value theorem. This is surely valuable from the pedagogical point of view. It perhaps is also a good preparation for the modern approach in which there is just one "Stokes' Theorem." I refer to the basic theorem which relates the integral of a form ω over a boundary $\partial\sigma$ to the integral of the exterior derivative $d\omega$ over the chain σ:

$$\int_{\partial\sigma}\omega = \int_{\sigma} d\omega.$$

(cf. [3], [4].)

2. Proof of Theorem 2. According to the hypotheses, the surface S is given by $z=f(x, y)$, where $f(x, y)$ is defined on some region S^* in which Theorem 1 is true. Let $F^*(x, y)=(\eta(x, y), \zeta(x, y))$ be the vector field on S^* given by:

$$\begin{aligned} \eta(x, y) &= P(x, y, f(x, y)) + R(x, y, f(x, y))f_1(x, y) \\ \zeta(x, y) &= Q(x, y, f(x, y)) + R(x, y, f(x, y))f_2(x, y). \end{aligned} \tag{3}$$

For future reference, I note that:

$$\begin{aligned} \frac{\partial\eta}{\partial y} &= P_2 + P_3f_2 + R_2f_1 + R_3f_2f_1 + Rf_{12} \\ \frac{\partial\zeta}{\partial x} &= Q_1 + Q_3f_1 + R_1f_2 + R_3f_1f_2 + Rf_{21}. \end{aligned} \tag{4}$$

I claim that

$$\int_{\partial S} F = \int_{\partial S^*} F^*. \tag{5}$$

This is simply a matter of computation. Let $X^*(t)=(\phi(t), \psi(t))$, $a\leqq t\leqq b$, be a parametric representation of ∂S^*; then $X(t)=(\phi(t), \psi(t), f(\phi(t), \psi(t)))$, $a\leqq t\leqq b$, is a representation for ∂S. Since

$$\frac{dX^*}{dt} = (\dot\phi(t), \dot\psi(t)), \quad \text{and} \quad \frac{dX}{dt} = (\dot\phi(t), \dot\psi(t), f_1\dot\phi(t) + f_2\dot\psi(t)),$$

we have

$$\int_{\partial S^*} F^* = \int_a^b \left[\left(F^* \cdot \frac{dX^*}{dt}\right)\right] dt = \int_a^b [(P + Rf_1, Q + Rf_2) \cdot (\dot\phi, \dot\psi)] dt$$

$$= \int_a^b [P\dot\phi + Rf_1\dot\phi + Q\dot\psi + Rf_2\dot\psi] dt,$$

and

$$\int_{\partial S} F = \int_a^b \left[F \cdot \frac{dX}{dt}\right] dt = \int_a^b [(P, Q, R) \cdot (\dot\phi, \dot\psi, f_1\dot\phi + f_2\dot\psi)] dt$$

$$= \int_a^b [P\dot\phi + Rf_1\dot\phi + Q\dot\psi + Rf_2\dot\psi] dt.$$

F^* was chosen, of course, so that this equality would hold; it also has a physical interpretation: the work done by F in moving a particle around ∂S is precisely that done by F^* in moving the particle around ∂S^*.

Now apply Theorem 1 to the right side of (5):

$$\int_{\partial S^*} F^* = \iint_{S^*} \left[\frac{\partial \zeta}{\partial x} - \frac{\partial \eta}{\partial y}\right] dxdy \tag{6}$$

$$= \iint_{S^*} [(Q_3 - R_2)f_1 + (R_1 - P_3)f_2 + (Q_1 - P_2)] dxdy.$$

This follows from (4) and the equality of cross partials. To lift this back to S, I require the following remarks on surface integrals. (cf. [1]). The unit normal, n, of S is determined by the gradient of the function $g(x, y, z) = z - f(x, y)$. That is,

$$n = \frac{\nabla g}{\|\nabla g\|} = \frac{(-f_1, -f_2, 1)}{\sqrt{1 + f_1^2 + f_2^2}} = (\cos \alpha, \cos \beta, \cos \gamma).$$

Thus $-f_1 \cos \gamma = \cos \alpha$ and $-f_2 \cos \gamma = \cos \beta$. If $h(x, y, z)$ is any function defined on S, then $\iint_S h(x, y, z) dA = \iint_{S^*} h(x, y, z) \sec \gamma dxdy$; equivalently,

$$\iint_S h(x, y, z) \cos \gamma dA = \iint_{S^*} h(x, y, z) dxdy.$$

Applying these facts to the right side of (6) gives:

$$\iint_{S^*} [(Q_3 - R_2)f_1 + (R_1 - P_3)f_2 + (Q_1 - P_2)] dxdy \tag{7}$$

$$= \iint [(R_2 - Q_3) \cos \alpha + (P_3 - R_1) \cos \beta + (Q_1 - P_2) \cos \gamma] dA.$$

The theorem is thus contained in equations (5), (6), and (7).

3. Conclusion. I wish to add here a remark and a question. The remark is this: one can carry through the same argument for surfaces in higher dimensional spaces. For example, let S be the (two dimensional!) surface in R^4 given by $x_3=f(x_1, x_2)$, $x_4=g(x_1, x_2)$, where f and g are defined on some suitable region S^* in the (x_1, x_2) plane. If $F(X)=(P(X), Q(X), R(X), S(X))$ is a vector field on S, (here I have let $X=(x_1, x_2, x_3, x_4)$), then one can show in the same fashion as before:

$$\int_{\partial S} F = \iint_{S^*} (dF)\,dx_1dx_2. \tag{8}$$

By dF I mean the "exterior derivative":

$$(Q_1-P_2)+(Q_3-R_2)f_1+(R_1-P_3)f_2+(Q_4-S_2)g_1 \\ +(S_1-P_4)g_2+(S_3-R_4)(f_1g_2-f_2g_1).$$

The question is: what is the correct formula for the element of surface area, the analogue to the formula sec $\gamma dxdy=dA$? This would make it possible to convert the right side of (8) to a surface integral. Presumably it would involve the direction cosines of the normals n_1 and n_2 determined by the functions $x_3=f(x_1, x_2)$ and $x_4=g(x_1, x_2)$.

References

1. D. V. Widder, Advanced Calculus, Prentice-Hall, Englewood Cliffs, New Jersey, 1947.
2. E. Kreyszig, Advanced Engineering Mathematics, Wiley, New York, 1962.
3. N. J. Hicks, Notes on Differential Geometry, Van Nostrand, Princeton, 1965.
4. M. Spivak, Calculus on Manifolds, W. A. Benjamin, New York, 1965.

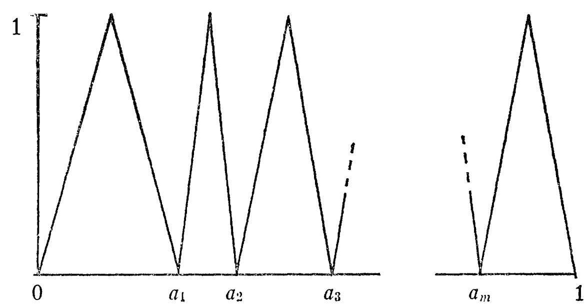
NETS AND SEQUENCES, AN EXAMPLE

W. M. Priestley, The University of the South

When introduced to the notion of a net and to the idea that nets are adequate in general for what sequences can do only in first countable topological spaces, the student naturally asks for relevant examples. Especially desired is an illustration of the fact that a subnet of a sequence does not have to be a subsequence, or—what is virtually the same thing—an example of a countable set S with a limit point to which converges no sequence of elements of S. To be sure, there are such examples, and interesting ones, such as occur in some topological spaces of ordinal numbers and in the space of planar lattice points of pairs of nonnegative integers with a certain topology [see 2, pp. 76–7]. But, on seeing these, one may be left with the erroneous impression that this seemingly anomalous situation occurs only in topological spaces which are themselves rather anomalous. The following example of such a countable set S in a more natural setting (the topological product of closed intervals) may be helpful in this connection.

Let X denote the topological space of all functions $x=x(t)$ from $[0, 1]$ into itself with the product topology, i.e., a neighborhood $N(x', \epsilon; t_1, t_2, \cdots, t_n)$ of x' is of the form $\{x: |x(t_i)-x'(t_i)|<\epsilon \text{ for } i=1, 2, \cdots, n\}$ where $\epsilon>0$ and $t_i\in[0, 1]$, $i=1, 2, \cdots, n$. Note that a sequence x_n converges to x in the product topology if and only if $\lim_{n\to\infty}x_n(t)=x(t)$ for each $t\in[0, 1]$. By Tychonoff's theorem [2, p. 143] X is compact.

To each finite subset $A=\{a_i\}$ of rational numbers such that $0<a_1<a_2 \cdots a_m<1$ associate the function x_A whose graph is indicated below:



Let $S\subseteq X$ be the set of all such x_A. S is countable because the collection of finite subsets of a countable set is countable. It is easy to see (by induction on n) that for each neighborhood $N=N(x_0, \epsilon; t_1, t_2, \cdots, t_n)$ of the function x_0 defined by $x_0(t)\equiv 0$, we have $N\cap S\neq\emptyset$, and thus x_0 is a limit point of the countable set S. However, there is no sequence x_n in S converging to x_0, for if there were, we should have $\int_0^1 x_n(t)dt\to\int_0^1 x_0(t)dt=0$ by the bounded convergence theorem [1, p. 288], an impossibility since $\int_0^1 x(t)dt=\frac{1}{2}$ for any $x\in S$.

S is "almost" dense in X (its closure is the set of functions vanishing at 0 and 1) but the collection of sequential limits of S is rather small, containing no function which is not a Baire function with integral $\frac{1}{2}$. Does X have a countable dense subset with no sequential limits other than its own elements?

References

1. R. G. Bartle, The Elements of Real Analysis, Wiley, New York, 1964.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, 1955.

CONSTRUCTION OF CHEBYSHEV QUADRATURE FORMULAE

K. SALKAUSKAS, University of Calgary, Canada

Current methods of constructing Chebyshev quadrature formulae do not make use of the fact that, like the Newton-Cotes and Gauss formulae, they are also interpolatory. We shall construct the polynomial whose zeros, when they are real, distinct and in the interval of integration, are the nodes of a Chebyshev quadrature formula by using Lagrange interpolation as a starting point.

Let

$$I(f) = \int_a^b w(x)f(x)dx,$$

where (a, b) and w are fixed, while f belongs to a class of functions for which $I(f)$ exists, and which contains all polynomials. We require w to be nonnegative on (a, b) and define

$$\alpha_j = I(f_j), \qquad f_j(x) = x^j \quad (j = 0, 1, \cdots).$$

An interpolatory n-point formula approximates $I(f)$ by

$$Q_n(f) \equiv \sum_{i=1}^{n} h_i f(x_i), \qquad a \leqq x_1 < x_2 < \cdots < x_n \leqq b,$$

where

$$h_i = \frac{1}{P'(x_i)} \int_a^b \frac{w(x)P(x)}{x - x_i}\, dx, \qquad P(x) = \prod_{i=1}^{n} (x - x_i). \tag{1}$$

In the Chebyshev case $h_i = h$ (a constant), and

$$Q_n(f_j) = I(f_j) \qquad (j = 0, 1, \cdots, n). \tag{2}$$

In fact, the first n conditions given in (2) imply that the formula is interpolatory. All conditions in (2) imply that $I(P) = 0$. This follows from the fact that I and Q_n are linear and the x_i's are zeros of P. We determine the common value h of the weights by noting that $\alpha_0 = I(f_0) = Q_n(f_0) = nh$, so that $h = \alpha_0/n$.

The polynomial P can now be determined from (1) and $I(P) = 0$ by the simple device of constructing a rational function R such that $R(x_i) = h_i$, and then requiring that

$$R(x_i) \equiv \frac{\alpha_0}{n} \qquad (i = 1, 2, \cdots, n).$$

Let

$$R(x) = \frac{1}{P'(x)} \int_a^b w(t)\, \frac{P(t) - P(x)}{t - x}\, dt.$$

Clearly, $R(x_i) = h_i$ since $P(x_i) = 0$. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

The integral is easily evaluated and yields

$$R(x) = \frac{1}{P'(x)} \sum_{k=0}^{n-1} x^k \sum_{j=0}^{n-1-k} \alpha_j a_{k+1+j}.$$

Dividing once by P' we have

$$R(x) = \frac{\alpha_0}{n} + \frac{Q(x)}{P'(x)},$$

where

$$Q(x) = \sum_{k=0}^{n-2} x^k \left\{ \alpha_0 a_{k+1} \frac{n-k-1}{n} + \sum_{j=1}^{n-1-k} \alpha_j a_{k+1+j} \right\}.$$

Now $R(x_i) = \alpha_0/n$ if, and only if, $Q(x_i) = 0$ $(i = 1, \cdots, n)$. But Q is of degree $n-2$, hence $Q \equiv 0$. It follows that the coefficients of P must satisfy the equations

$$\text{(3a)} \qquad \alpha_0 a_{k+1} \frac{n-k-1}{n} + \sum_{j=1}^{n-1-k} \alpha_j a_{k+1+j} = 0 \qquad (k = 0, 1, \cdots, n-2)$$

together with the condition $I(P) = 0$, which is

$$\text{(3b)} \qquad \sum_{j=0}^{n} a_j \alpha_j = 0 \qquad (a_n = 1).$$

These equations are identical with those obtainable from (2) by the use of Newton's identities which relate the sums of powers of the zeros of a polynomial with the coefficients. However, such an approach presupposes familiarity with these identities, and obscures the interpolatory origin of the formula. Of course, the existence of P does not guarantee the existence of Q_n since we can say (almost) nothing about the zeros x_i except in special cases.

References

1. Z. Kopal, Numerical Analysis, Wiley, New York, 1955.
2. V. I. Krylov, Approximate Calculation of Integrals, trans. A. H. Stroud, Macmillan, New York, 1962.
3. P. L. Tchebichef, Sur les quadratures, J. de Math., 2, 19 (1874) 19.

A NOTE ON THE ALGEBRAIC CLOSURE OF A FIELD

ROBERT GILMER, Florida State University

In order to prove that a given field F has an algebraic closure, it is sufficient to prove the existence of an extension field K of F such that each nonconstant polynomial f over F splits into linear factors over K, for if such a field K exists, then the set of elements of K algebraic over F is an algebraic closure of F [3, p. 194]. To establish the existence of such a field K, Lang in his text *Algebra* [2] proceeds as follows. He first proves that if E is any field, then there is an extension field E_1 of E such that each nonconstant polynomial f over E has a root in E_1. Applying this result successively to $E_1, E_2, \cdots$, Lang obtains an ascending chain $E_1 \subset E_2 \subset \cdots$ of extension fields of E such that for each i, every non-

MOTIVATION FOR DEFINING THE CONDITIONAL

G. A. KRAUS, Bucknell University and Southern Illinois University

It is relatively easy to motivate the definitions of the truth functional operations "$\sim$," "$\wedge$," "$\vee$" and "$\Leftrightarrow$" which are used, respectively, to construct the negation, conjunction, disjunction and biconditional of statements. Often, however, students suspect that "$\Rightarrow$" is defined too arbitrarily to express the usual meaning of the English fragment "if . . . then . . ." in conditional statements. A motivational argument using degenerate conditionals such as "$p\Rightarrow p$" is liable to be unsatisfactory to anyone who is not in the habit of speaking degenerately. The student who is familiar with the truth tables for negation, conjunction and the biconditional should appreciate the following argument. Frequent reference will be made to Table I which originally has entries only in the first and second columns and headings on columns 1–3. There are abundant examples which suggest that "F" should be entered in the second row of column three; the problem consists in completing this column.

TABLE I

p	q	$p\Rightarrow q$	$p\Leftrightarrow q$	$(p\Rightarrow q)\wedge(\sim p\Rightarrow\sim q)$	$(p\Rightarrow q)\wedge(\sim p\Rightarrow q)$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	T	F	F	T
F	F	T	T	T	F

Consider a statement of the form

$$(p \Rightarrow q) \wedge (\sim p \Rightarrow \sim q). \tag{1}$$

For example:

If they settle the steel strike the economy will benefit; but if they don't the economy will suffer.

This seems to mean that "They will settle the steel strike," is materially equivalent to "The economy will benefit." Such examples suggest that (1) and "$p\Leftrightarrow q$" are logically equivalent; with this assumption columns four and five may be added to Table I. To be consistent with the definition of "$\wedge$" the letter "F" may not be inserted in row one nor in the last row of column three; so "T" is inserted.

Now consider a statement of the form

$$(p \Rightarrow q) \wedge (\sim p \Rightarrow q). \tag{2}$$

For example:

We will have a happy Christmas if the war is over and if the war goes on we will still enjoy Christmas.

Apparently we will enjoy Christmas regardless of the state of hostilities. Ex-

amples of this type lead to the conclusion that (2) is logically equivalent to "q," whence the last column is added to the table. Clearly "F" may not be placed in the third row of column three. Thus "T" is entered to complete the table for "$\Rightarrow$."

REMARK. This argument is indirect; it rests on the assumption that column three should be completed by insertion of either "F" or "T" in each space; *i.e.*, "$\Rightarrow$" is assumed to be truth functional. One who is troubled by the philosophical question, "Should 'if . . . then . . .' be regarded as truth functional?" must at least agree that if the answer is affirmative then the usual definition of "$\Rightarrow$" is appropriate.

Reference

1. W. V. Quine, Mathematical Logic, rev. ed., Harvard University Press, Cambridge, 1951.

ON THE RATIONAL POSITIVE SOLUTIONS OF $m^n=n^m$ WITH $m>n$

SIMEON REICH, Technion, Haifa, Israel

In the first part of [1], the equation $m^n=n^m$, with $m>n$, is solved for positive rational numbers. In this note we show a shorter way to do this.

Denoting m/n by $k>1$, we get from $m^n=n^m$ that $(nk)^n=n^{nk}$, or $n=k^{1/(k-1)}$. Let the rational positive number $1/(k-1)$ be denoted by p/q where p, q are natural numbers such that $(p, q)=1$. Then we have $n=(1+q/p)^{p/q}$ and $m=(1+q/p)^{p/q+1}$. For m, n to be rational, p, as well as $p+q$, should be of the form a^q where a is a natural number (since $1+q/p=(p+q)/p$ and $(p+q, p)=1$). Noting that for $q\geqq 2$ and $p=a^q$ we have $a^q<p+q\leqq a^q+qa^{q-1}<(a+1)^q$, we arrive at the conclusion that there are no solutions with $q>1$. If $q=1$ and p is any natural number we verify that we have a solution. Hence, all positive rational solutions of $m^n=n^m$ with $m>n$ are $n=(1+1/p)^p$ and $m=(1+1/p)^{p+1}$ where p is a natural number.

I would like to thank the referee for his remarks.

Reference

1. Solomon Hurwitz, On the rational solutions of $m^n=n^m$ with $m\neq n$, this MONTHLY, 74(1967) 298–300.

AN UNCOMPLETABLE FUNCTION

CHARLES DUNHAM, University of Western Ontario

In this note we obtain an interesting class of uncomputable functions related to Turing machines started with a zero (blank) tape and hence related to the Busy Beaver [1] problem. A partially-computable function ω exists such that when given the Gödel number of a Turing machine as argument, the function

takes as value the result of the computation started at zero and if there is no result of the computation (due to looping) the function is undefined: the function is computed by a universal Turing machine whose arguments are the Gödel number and the contents of the initial tape (zero in this case). We now consider replacing ω by a computable function ϕ taking the same values as ω whenever it is defined. More precisely, consider a function ϕ such that if the Turing machine M_N with Gödel number N eventually stops when started with zero on the tape, $\phi(N)=M_N(0)$ and if the machine does not stop, $\phi(N)$ takes some (unspecified and variable) value. Since there is nothing about the values of ϕ to indicate whether the corresponding arguments represent stopping or nonstopping machines, it would seem that some such function ϕ should be computable, but such is not the case. Suppose such a function ϕ is computable. For given n we can generate the Gödel numbers of all machines with n statements in their program. There are only a finite number of such machines. We use each of these Gödel numbers as arguments of ϕ and define $\psi(n)$ to be the largest value of ϕ over Gödel numbers of n-statement machines. It is easily seen that ψ is computable if ϕ is computable. Now let $\Sigma(n)$ be the largest number written by an n-statement machine which when started with a zero tape eventually stops. By definition of ϕ, ψ, Σ, we have $\Sigma(n)\leqq\psi(n)$ for all n. We have a contradiction as Rado [1] has shown that for any computable function ψ, $\Sigma(n)>\psi(n)$ for all n sufficiently large. It follows that ψ is not computable and hence neither is ϕ. We have found a partially computable function ω such that no computable completion ϕ can exist.

Reference

1. T. Rado, On noncomputable functions, Bell Telephone System Technical Journal, 41 (1962) 877–884.

CHARACTERISTIC ROOTS OF RANK 1 MATRICES

LARRY CUMMINGS, University of Waterloo, Canada

The purpose of this note is to prove a result which generalizes popular elementary Problem E 1859 (this MONTHLY, 74(1967) 598, 724).

THEOREM. *Let A denote an $n\times n$ rank 1 matrix over any field. When Trace (A) is nonzero, it is a simple characteristic root of A and 0 is a root of multiplicity $n-1$. Otherwise, the only characteristic root of A is 0.*

Proof. When $n=1$ the result is trivial. We first show directly that Trace (A) is a characteristic root of A. Suppose the row vectors of A are $A_1, \cdots, A_n$. Since A is not zero, A has a nonzero row, say A_k. Then there are scalars $\lambda_1, \cdots, \lambda_n$ with $\lambda_k=1$ for which

$$A_i = \lambda_i A_k \qquad (i = 1, \cdots, n).$$

Let $A_k=(a_1, \cdots, a_n)$. Then A may be written as $[\lambda_i a_j]$. For any n-vector

$x=(x_1, \cdots, x_n)$, (A_k, x) will denote the usual inner product, $\sum_{i=1}^n a_i x_i$. In particular, $(A_k, \omega)=\text{Trace }(A)$, where ω is the nonzero vector $(\lambda_1, \cdots, \lambda_n)$. A simple calculation shows

$$Ax^T = (A_k, x)\omega^T.$$

Thus, ω is a characteristic vector of A corresponding to $(A_k, \omega)=\text{Trace (A)}$ and $Ax=0$ only if $(A_k, x)=0$.

To determine the multiplicities of 0 and Trace (A), we note that A must be similar to a matrix all of whose rows except the kth are zero. For, if A is not already in this form, $\lambda_j\neq 0$ for some $j\neq k$ and $C=B_{jk}(-\lambda_j)AB_{jk}(\lambda_j)$ is a matrix similar to A with jth row zero. $B_{ij}(\lambda)$ here denotes the elementary matrix $I+\lambda E_{jk}$, where E_{ij} is the matrix with entry (i, j) equal to 1 and zeros elsewhere. Since C is also a rank 1 matrix we may repeat the argument as necessary with a possibly different set of scalars $\lambda_1, \cdots, \lambda_n$ to obtain a matrix $C=[c_{ij}]$ which is similar to A with only $C_k\neq 0$. Since the characteristic equation of C is $\lambda^{n-1}(\lambda-c_{kk})=0$, 0 is a root of A with multiplicity $\geqq n-1$. Since one root of the characteristic equation is Trace (A), the multiplicity of zero is $n-1$ if Trace (A) is nonzero and otherwise is n. This completes the proof.

The fact that Trace (A) is a characteristic root is also seen by noting that the trace is a similarity invariant and Trace (C) $=c_{kk}$ is a root of C with characteristic vector $e_k=(\delta_{1k}, \cdots, \delta_{nk})$.

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MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

Material for this department should be sent to John G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706

A REPORT ON THE CUPM RECOMMENDATIONS IN THE STATE OF TEXAS

P. E. THOMPSON and R. L. POE, Texas Technological College

The Mathematical Association of America appointed its Committee on the Undergraduate Programs in Mathematics (CUPM) in January, 1959. CUPM is charged with the responsibilities of recommending and influencing undergraduate mathematics curriculum changes. While carrying out its responsibilities, CUPM has exerted a dominant effect within a national mathematics curriculum reform movement. Many colleges and university mathematics departments have used CUPM recommendations as guidelines in structuring their present undergraduate mathematics course offerings to provide an efficient program of studies for the students of today. In view of this national mathematics curriculum reform movement, the question of strengths and weaknesses in the mathematics programs of colleges and universities of a given geographical region arises. That is to ask, how well do the mathematics programs of the colleges and universities in an area compare with the CUPM recommendations and with each other? The question is asked with respect to the schools of a geographical area rather than the entire nation or a specific level of institutions over the nation since the problem in its entirety is impractical and perhaps not feasible in the sense of sampling techniques. Also, it is not unreasonable to believe that the mathematics staffs of colleges and universities in a specific area are interested in comparing their mathematics programs, especially if each institution's programs are compared to recognized national standards such as the CUPM recommendations.

Comparisons as asked for above can be made by examining the mathematics course descriptions listed in each institution's catalog. Catalog course descriptions are the means by which an institution advertises its offerings in an academic field and therefore form a practical estimation of a valid basis for comparison—in essence, colleges and universities are theoretically obligated to teach the contents of each catalog course description. The course listings and descriptions, however, as published in the current catalog of an institution may be out of date. The time lapse since the last catalog printing or the administrative procedures necessary to add or delete catalog course offerings can cause the present catalog of an institution to be out of phase with the actual classes and course content being presented. If the mathematics department of each institution compared has taken the opportunity to modify (correct) its catalog mathematics course listing and descriptions the comparisons may be considered as reliable.

This report concerns the mathematics programs of the twenty-two state supported four-year colleges and universities of Texas. (The state supported institutions were chosen since they enroll the majority of the students in Texas.) The mathematics curriculum of each of the institutions was examined, as listed in their 1967–68 catalogs, and compared with the CUPM recommended courses for each of five areas. A letter was sent to the department of mathematics chairman at each institution requesting modifications. The table which follows represents the comparison of each institution's mathematics program with the CUPM recommendations for the area of Mathematics for General Curriculum. Similar tables were constructed for the areas of (1) Mathematics for Engineers and Physicists, (2) Mathematics for Undergraduate Research and Ph.D. Training, (3) Mathematics for Training of Teachers, and (4) Mathematics for Biological, Management and Social Sciences, but are not included in this report since Table 1 illustrates the technique used for the comparison. The columns of each table are labeled according to the general classifications of mathematics courses recom-

TABLE 1. MATHEMATICS FOR GENERAL CURRICULUM

Institution	Precalculus	Calculus	Linear Algebra	Probability and Statistics	Advanced Calculus	Geometry	Abstract Algebra	Applied Math and Numerical Analysis	Real Analysis-	Complex Analysis
CUPM	1 (3)	3(9–12)	1 (3)	3 (9)	1 (3)	1 (3)	1 (3)	2 (6)	2 (6)	1 (3)
1101	2 (6)	3 (9)	2 (6)	5 (15)	1 (3)	1 (3)	2 (6)	4 (12)	2 (6)	2 (6)
1102	6 (19)	5 (17)	0 (0)	1 (3)	2 (6)	0 (0)	2 (6)	1 (3)	2 (6)	1 (3)
1103	3 (9)	3 (13)	2 (6)	4 (12)	2 (6)	3 (9)	1 (3)	6 (18)	4 (12)	1 (3)
1104	7 (21)	6 (23)	2 (6)	5 (15)	4 (12)	5 (15)	2 (6)	9 (27)	4 (12)	1 (3)
1105	4 (12)	3 (12)	1 (3)	1 (3)	2 (6)	1 (3)	2 (6)	0 (0)	1 (3)	0 (0)
1106	4 (12)	3 (9)	1 (3)	3 (9)	2 (6)	3 (9)	1 (3)	2 (6)	2 (6)	1 (3)
1107	3 (9)	2 (6)	0 (0)	1 (3)	2 (6)	3 (9)	1 (3)	1 (3)	1 (3)	1 (3)
1108	3 (9)	3 (9)	1 (3)	0 (0)	1 (3)	1 (3)	1 (3)	1 (3)	1 (3)	0 (0)
1109	3 (9)	3 (9)	2 (6)	2 (6)	2 (6)	1 (3)	1 (3)	1 (3)	1 (3)	1 (3)
1110	3 (9)	3 (9)	0 (0)	0 (0)	1 (3)	1 (3)	2 (6)	1 (3)	0 (0)	0 (0)
1111	7 (21)	5 (15)	0 (0)	1 (3)	1 (3)	1 (3)	1 (3)	2 (6)	2 (6)	1 (3)
1112	2 (5)	3 (12)	1 (3)	1 (3)	0 (0)	1 (3)	2 (6)	3 (9)	3 (9)	1 (3)
1113	4 (14)	3 (12)	1 (3)	2 (6)	2 (6)	2 (6)	1 (3)	2 (6)	1 (3)	0 (0)
1114	3 (9)	3 (9)	1 (3)	1 (3)	2 (6)	1 (3)	3 (9)	2 (6)	1 (3)	0 (0)
1115	4 (12)	4 (12)	1 (3)	3 (9)	2 (6)	2 (6)	2 (6)	2 (6)	2 (6)	2 (6)
1116	3 (9)	3 (9)	1 (3)	0 (0)	2 (6)	4 (12)	1 (3)	1 (3)	1 (3)	0 (0)
1117	3 (9)	4 (12)	1 (3)	2 (6)	1 (3)	0 (0)	1 (3)	0 (0)	0 (0)	0 (0)
1118	4 (12)	4 (12)	1 (3)	1 (3)	C (0)	2 (6)	1 (3)	0 (0)	0 (0)	0 (0)
1119	4 (12)	3 (9)	1 (3)	0 (0)	1 (3)	1 (3)	2 (6)	1 (3)	1 (3)	0 (0)
1120	3 (9)	3 (9)	0 (0)	2 (6)	1 (3)	0 (0)	1 (3)	1 (3)	0 (0)	0 (0)
1121	5 (15)	3 (9)	1 (3)	3 (9)	2 (6)	2 (6)	2 (6)	4 (12)	1 (3)	1 (3)
1122	3 (9)	5 (21)	1 (3)	5 (15)	1 (3)	1 (3)	1 (3)	1 (3)	0 (0)	0 (0)

mended by the CUPM. Row one of each table lists in the columns first the number of courses recommended by CUPM and then in parenthesis the number of credit hours recommended for each general classification. The Texas institutions are not identified by name; the numbers 1101 through 1122 were assigned them at random. (The names of the schools and copies of the other four tables may be obtained upon request.) The rows of each table designated by an institution number presents, in each column, first the number of equivalent courses the school offers and in parenthesis the number of credit hours the school gives for each general classification.

To summarize the tables a "Table of Comparison" was constructed arbi-

TABLE 2. TABLE OF COMPARISONS

School No.	Area 1 General Curriculum $N(*)$	Area 2 Engineering and Physics $N(*)$	Area 3 Research Mathematics $N(*)$	Area 4 Training of Teachers of Mathematics $N(*)$	Area 5 Biological, Management and Social Sciences $N(*)$	$\Sigma N(*)$
CUPM	1000.0	1000.00	1000.00	1000.00	1000.00	5000.00
1101	1000.00 (1)	933.33 (1)	888.80 (1)	750.00 (3)	800.00 (2)	4372.13 (2)
1102	687.50 (5)	666.67 (5)	611.05 (5)	625.00 (5)	600.00 (3)	3190.22 (11)
1103	1000.00 (1)	866.67 (2)	833.25 (2)	875.00 (1)	600.00 (3)	4174.92 (3)
1104	1000.00 (1)	933.33 (1)	888.80 (1)	875.00 (1)	800.00 (2)	4497.13 (1)
1105	625.00 (6)	666.67 (5)	611.05 (5)	750.00 (3)	400.00 (4)	3115.21 (13)
1106	1000.00 (1)	933.33 (1)	722.15 (3)	812.50 (2)	000.00 (6)	3467.98 (8)
1107	687.50 (5)	533.33 (7)	500.00 (7)	687.50 (4)	200.00 (5)	2608.33 (18)
1108	625.00 (6)	533.33 (7)	555.55 (6)	500.00 (6)	600.00 (3)	2813.88 (14)
1109	812.50 (3)	800.00 (3)	666.60 (4)	687.50 (4)	600.00 (3)	3566.60 (7)
1110	500.00 (8)	400.00 (9)	500.00 (7)	687.50 (4)	000.00 (5)	2087.50 (22)
1111	812.50 (3)	733.33 (4)	611.05 (5)	750.00 (3)	400.00 (4)	3306.88 (10)
1112	812.50 (3)	866.07 (2)	722.15 (3)	625.00 (5)	400.00 (4)	3426.32 (9)
1113	812.50 (3)	800.00 (3)	666.60 (4)	687.50 (4)	1000.00 (1)	3966.60 (5)
1114	750.00 (4)	600.00 (6)	611.05 (5)	750.00 (3)	000.00 (6)	2711.05 (16)
1115	1000.00 (1)	933.33 (1)	888.80 (1)	812.50 (2)	000.00 (6)	3634.63 (6)
1116	625.00 (6)	600.00 (6)	500.00 (7)	500.00 (6)	400.00 (4)	2625.00 (17)
1117	562.50 (7)	600.00 (6)	500.00 (7)	687.50 (4)	400.00 (4)	2750.00 (15)
1118	500.00 (8)	466.67 (8)	500.00 (7)	625.00 (5)	400.00 (4)	2491.67 (19)
1119	625.00 (6)	600.00 (6)	555.55 (6)	625.00 (5)	000.00 (6)	2405.55 (20)
1120	562.50 (7)	600.00 (6)	388.85 (9)	625.00 (5)	200.00 (5)	2376.35 (21)
1121	937.50 (2)	866.67 (2)	833.25 (2)	875.00 (1)	600.00 (3)	4112.42 (4)
1122	750.00 (4)	666.67 (5)	444.40 (8)	687.50 (4)	600.00 (3)	3148.57 (12)
	$\frac{1000}{16}=62.50$	$\frac{1000}{15}=66.67$	$\frac{1000}{18}=55.55$	$\frac{1000}{16}=62.50$	$\frac{1000}{5}=200.00$	

$N=\frac{1000}{C}\times n$ (* position rank among schools)

trarily by determining a numerical value, N, for each school in each of the five undergraduate mathematics curriculum areas considered; i.e., $N=1000n/C$, where n is the number of courses the institution offers comparable to the CUPM number of recommended courses, C; and 1000 is the maximum value an institution may have in one area. The numbers in parenthesis show how the schools rank with each other in relation to the CUPM recommendations. Additional value was not given a school for offering more courses comparable to the CUPM recommended courses than the number CUPM actually recommends. The N scores of each school for each area are summed in column six, ΣN. A school meeting all the CUPM recommendations for all five areas would have a perfect summed score of 5000 points.

The mean score of the Texas schools is 65.08% of the CUPM perfect score; the median is 66.16%; the mode lies between 55% and 60%. Of the four university type institutions in Texas, one has 91.19% of the CUPM perfect score, one has 87.44%, another has 84.75% and the other 67.4%. Of the thirteen public senior colleges granting the master's degree as the highest degree in mathematics, 81.69% is the highest score and 41.75% the lowest. Of the five public senior colleges that do not grant advanced degrees in mathematics, as of the spring session of 1967, 83% is the highest score and 47.53% is the lowest.

No institution of Texas offers all the CUPM recommended courses for more than one area. Only six of the twenty-two schools meet the CUPM recommendations for the General Curriculum, Area 1. No school meets the CUPM recommendations for the Engineers and Physicists Curriculum, Area 2, the Research Curriculum, Area 3, and the Training of Teachers of Mathematics, Area 4. Only one school meets the CUPM recommendations for the Biological, Management and Social Sciences Curriculum, Area 5.

The CUPM recommended courses for each of the five areas of interest are acknowledged to be a minimal program. However, it is also acknowledgeable that, in terms of efficiency, the undergraduate programs of mathematics should consist only of those courses necessary for a complete curriculum.

Relating the mathematics courses offered by a school to the CUPM recommended courses, an efficiency index, for each of the five areas can be obtained; i.e., efficiency, E, of a school is the quotient of the number of CUPM courses offered for that area divided by the number of CUPM courses recommended plus the number of courses the school offers in excess of the number of courses CUPM recommends for that area plus the number of courses the school does not offer of the CUPM recommended courses. Summing the efficiency E of all five areas results in a total efficiency, Total E, for each school. The resulting efficiencies are shown in Table 3.

The efficiency index determined by the CUPM recommendations indicate that the Texas public institutions of higher learning taken as an entirety are most efficient in Area 4, the Training of Teachers of Mathematics. The second highest efficiency is in Area 1, the General Curriculum, and the least efficient in Area 5, Mathematics for the Biological, Management and Social Sciences. Eight of the twenty-two Texas schools have a Total E greater than .500. Eleven

TABLE 3. EFFICIENCY INDEX OF COURSE OFFERINGS

School	Area I	Area II	Area III	Area IV	Area V	Total E
1101	.667	.518	.571	.545	.571	.574
1102	.367	.400	.367	.417	.429	.388
1103	.552	.542	.500	.737	.375	.555
1104	.355	.400	.457	.609	.571	.441
1105	.370	.476	.423	.600	.250	.441
1106	.727	.700	.538	.684	.000	.586
1107	.407	.348	.310	.524	.111	.367
1108	.417	.320	.384	.333	.429	.368
1109	.565	.545	.444	.524	.429	.510
1110	.296	.240	.321	.524	.000	.306
1111	.481	.500	.379	.545	.250	.454
1112	.565	.619	.520	.435	.250	.510
1113	.542	.600	.444	.524	.833	.541
1114	.480	.375	.379	.578	.000	.396
1115	.667	.583	.615	.619	.000	.562
1116	.357	.409	.300	.296	.250	.330
1117	.346	.409	.310	.500	.250	.374
1118	.276	.292	.310	.435	.250	.319
1119	.385	.409	.370	.345	.000	.361
1120	.360	.428	.233	.454	.111	.336
1121	.577	.650	.600	.667	.429	.606
1122	.462	.370	.265	.500	.429	.388

$$E = \frac{\text{CUPM courses given}}{\text{CUPM courses recommended} + \text{overage} + \text{deficiency}}.$$

schools have $.300 < \text{Total } E < .400$. The greatest Total E of any school is .606; the lowest, .306. Of the four university type schools, two have a Total $E > .500$ and two have a Total $E < .500$.

COMMENTS ON RANKING MATHEMATICS DEPARTMENTS

Prompted by Cartter's recent assessment of quality in graduate education, Siebring [this MONTHLY, 74 (1967) 1126–30] attempted another type of evaluation of graduate instruction in mathematics. Siebring's study is concerned with mathematicians who received their Ph.D. degrees in 1950–59; who are listed in the Combined Membership List (AMS-MAA-SIAM), 1963–64; and whose biographies are found in the 9th or 10th editions of *American Men of Science*. Departments of Mathematics (by institutions) are ranked in four tables. The numbers assigned to each department for purposes of ranking in all of the tables are dependent in some way upon the listing of the Ph.D. graduates of an institution in *American Men of Science*.

A critic of the Siebring report, who checked the numbers reported by Siebring

for his university, found that only approximately 50 percent of the Ph.D. graduates in mathematics of his university, 1950–59, are listed in *American Men of Science.* Furthermore, in checking publications of the actual graduates, rather than only those listed in *American Men of Science,* he finds the record for publication of all graduates in mathematics of his university is much better than that shown in the Siebring survey in which a limited population was used.

In his report Siebring calls attention to the possibility that the *American Men of Science* source may vary in completeness from one university to another. Those who read his report should recognize that the use of the complete population of mathematics Ph.D. graduates would affect the rankings.

Persons who are interested in quality factors of graduate departments of mathematics will want to consider the Albert report (*A Survey of Research Potential and Training in the Mathematical Sciences,* University of Chicago, 1957); Volume III of the Survey of Mathematics, a study sponsored by CBMS; and the COSRIMS report, *The Mathematical Sciences,* to be published by the National Academy of Sciences. The Survey and COSRIMS reports are scheduled for publication at an early date.

JOHN R. MAYOR

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before April 30, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2133. *Proposed by H. D. Ruderman, Hunter College High School, N. Y.*

Let function f_n be defined on $W \times W$ to W, where $W = \{0, 1, 2, \cdots\}$, the set of nonnegative integers with $n \in W$, as follows:

For $n \geqq (x+y)$: $f_n(x, y) = n - (x+y)$;

For $n < (x+y)$: $f_n(x, y) = \operatorname{Min}(\{f_n(0, y), f_n(1, y), f_n(2, y), \cdots, f_n(x-1, y), f_n(x, 0), f_n(x, 1), \cdots, f_n(x, y-1)\}')$,

where $A' = W - A$.

Find an efficient way of computing $f_n(x, y)$ and, in particular, $f_2(30, 40)$.

NOTE: If one considers $n=1$, one obtains the set of losing triples for the game of NIM and $f_1(x, y)$ is obtained by considering the numbers expressed in binary numeration and assigning that number which yields an even number of ones in each column, a well known strategy.

E 2134. *Proposed by Harley Flanders, Purdue University*

Find the rank of the system of linear equations over a field of characteristic zero:

$$\begin{cases} x_{ijkl} + x_{jkil} + x_{ijlk} + x_{jlik} = 0 \\ x_{ijkl} + x_{jkil} + x_{jikl} + x_{ikjl} = 0. \end{cases}$$

The indices run independently from 1 to n.

E 2135. *Proposed by F. A. Butter, Jr., California State College at Long Beach.*

Two radii (distinct or not) of lengths r_1, r_2 with respective slopes m_1, m_2 are drawn from the center of the ellipse $b^2x^2+a^2y^2=a^2b^2$, $0<b<a$, to the open quadrant for which $x>0$, $y>0$. Prove that (r_1r_2-ab) and $(ba^{-1}-m_1m_2)$ are alike in sign.

E 2136. *Proposed by A. Inselberg and B. Dimsdale, IBM Los Angeles Scientific Center.*

Let

$$S_r = \sum_{k=1}^{n} k^r.$$

It is well known that $S_3 = S_1^2$. Are there other values of p, q, u, v such that $S_p^u = S_q^v$, for all n?

E 2137. *Proposed by R. A. Christiansen, University of Victoria, Canada*

Show that each integer q has a unique decomposition

$$q = \sum_{j=0}^{n} d_j 2^j, \quad \text{where} \quad d_j \in \{0, 1, -1\}$$

and no two consecutive d_j are nonzero. Show further that if f is any nonincreasing function from the nonnegative integers into the nonnegative reals, and if

$$q = \sum_{j=0}^{m} c_j 2^j,$$

then

$$\sum_{j=0}^{n} |d_j| f(j) \leqq \sum_{j=0}^{m} |c_j| f(j).$$

E 2138. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha*

Show that for all positive integers n, the integer

$$\left[\frac{(n-1)!}{n(n+1)}\right]$$

is even.

E 2139. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Consider the following four points of the triangle: the circumcenter, the incenter, the orthocenter, and the nine point center. Show that no three of these points can be the vertices of a nondegenerate equilateral triangle.

E 2140. *Proposed by H. Guggenheimer, Polytechnic Institute of Brooklyn, N. Y.*

Let A, B, C be three concurrent circles that do not form a pencil, and let D be the circle through the other points of intersection of the three circles taken in pairs. Denoting inversion in a circle by the letter representing the circle, show that the product of the inversions

$ADABDBCDCADABDBCDCBDBCDCADABDBCDCADACDCBDBADACDCBDBCDCBDBADACDCBDB$,

where the inversions are performed from right to left, is the identity mapping.

SOLUTIONS OF ELEMENTARY PROBLEMS

Property of the Isosceles Tetrahedron

E 2017 [1967, 1005]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let h be the length of an altitude of an isosceles tetrahedron and suppose the orthocenter of a face divides an altitude of that face into segments of lengths h_1 and h_2. Prove that $h^2 = 4h_1h_2$.

Solution by William S. Wynne Willson, Cheltenham Grammar School, England. Let $ABCD$ be the tetrahedron (with $AB = CD$, $AC = BD$, $AD = BC$). Rotate the faces DBC, DCA, DAB about the edges BC, CA, AB into the plane of ABC to form triangles D_1BC, D_2CA, D_3AB, respectively. Then since the tetrahedron is isosceles, $D_1D_2D_3$ is the triangle with A, B, C as the midpoints of its edges. Let H be the orthocenter of triangle $D_1D_2D_3$, and D_1P be an altitude meeting D_2D_3 at P and BC at X. D lies on the circle with center X and radius XD_1 whose plane is perpendicular to BC. Also DH is an altitude of the tetrahedron. Thus by intersecting chords, $DH^2 = D_1H \cdot HP$.

Since $D_1D_2D_3$ is similar to ABC with enlargement factor 2, this gives $h^2=(2h_1)(2h_2)=4h_1h_2$.

Also solved by Ragnar Dybvik (Norway), Michael Goldberg, Harry Ploss, Simeon Reich (Israel), B. M. Shah (India), P. D. Thomas, and the proposer.

Conjugate Matrices

E 2018 [1967, 1005]. *Proposed by W. A. McWorter, Ohio State University*

Let M_n be the group of all nonsingular $n\times n$ matrices over a field F of characteristic 0. Let S_n be the subgroup of M_n consisting of all permutation matrices. Prove that if two elements of S_n are conjugate in M_n, then they are already conjugate in S_n.

I. *Solution by Simeon Reich, Israel Institute of Technology, Israel.* Suppose P_1, $P_2\in S_n$ and $P_1=M^{-1}P_2M$ with $M\in M_n$. Then P_1, P_2 have the same characteristic polynomial. In general, let P be associated with the permutation π and let π have type $(t_1, t_2, \cdots, t_r)$. It is known that P has characteristic polynomial $(-1)^{n-r}(1-\lambda^{t_1})(1-\lambda^{t_2})\cdots(1-\lambda^{t_r})$. Since, in our case, P_1, P_2 have the same characteristic polynomial, it follows that π_1, π_2 have the same type. Hence π_1, π_2 are conjugate elements of the symmetric group of degree n, S_n'. But S_n' is isomorphic with S_n. Therefore P_1, P_2 are conjugate in S_n.

II. *Solution by M. G. Greening, University of New South Wales, Australia.* S_n is isomorphic to the symmetric group of degree n, each of whose automorphisms is inner. As MS_nM^{-1} is isomorphic to S_n and MAM^{-1} is in S_n for each A in S_n, we have $MS_nM^{-1}=S_n$. It follows that M induces an automorphism, hence an inner automorphism, of S_n.

Also solved by F. Ž. Djoković, J. R. Purdy, and the proposer.

Four Points and Circles Through Three of Them

E 2019 [1967, 1005]. *Proposed by Norman Miller, Queen's University, Kingston, Ontario*

Given four noncollinear, nonconcyclic points in a plane, show that (a) at most three of the points can each be outside the circle that passes through the other three, and (b) at most two of the points can each be inside the circle that passes through the other three.

Solution by D. Ž. Djoković, University of Waterloo, Ontario. Let us denote the points by A, B, C, D. If one of these points, say A, lies outside (resp. inside) the circle which passes through the remaining three points, we shall say that A has property O (resp. property I). Two cases are possible:

I. The convex closure of the points A, B, C, D is a triangle, say ABC. Then the point D has property I, while at least two of the points A, B, C have property O. (It is possible that D lies on some side of ABC.) Hence, both assertions (a) and (b) are true.

II. The convex closure of A, B, C, D is a quadrilateral $ABCD$. We can assume that A and C are opposite vertices. We have

$$\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB = 2\pi. \tag{1}$$

Since A, B, C, D are not concyclic the sums $\measuredangle ABC + \measuredangle CDA$ and $\measuredangle BCD + \measuredangle DAB$ are distinct. Let us take, for instance, that the first sum is greater. Then (1) implies that

$$\measuredangle ABC + \measuredangle CDA > \pi > \measuredangle BCD + \measuredangle DAB. \tag{2}$$

From (2) we conclude that A and D have property I, while B and C have property O. Hence (a) and (b) are again true.

Also solved by Anders Bager (Denmark), Neal Felsinger, Michael Goldberg, M. G. Greening (Australia), Peter Hajdu, Ned Harrell, Peter Kornya, Bohuslav Míšek (Czechoslovakia), C. B. A. Peck, Harry Ploss, Simeon Reich (Israel), Robin Robinson, N. Sivaramakrishnan (India), Dan Sunday, Necdet Üçoluk, and the proposer.

Circle and a Set of n points

E 2020 [1967, 1006]. *Proposed by G. F. Schumm, University of Chicago*

Given n points ($n>3$) such that no three are collinear and no four concyclic. Must there exist a circle which passes through three of them and contains the remaining points in its interior?

Solution by Michael Goldberg, Washington, D. C. Yes. The smallest circle that encloses all the points has three of the points on its circumference, unless it is a circle through two points which are the ends of a diameter. See Problem E 1866, [1967, 726]. In the latter case, if the circle is enlarged, while still passing through these two points, until it touches a third point, it will include all but three of the points unless all the points lie along a line. Hence, the conditions of the problem can be made stronger. Three or more of the points may be collinear, although they may not all be in one line.

Also solved by Anders Bager (Denmark), Walter Bluger, D. Ž. Djoković, Neal Felsinger, T. Fujinawa (Japan), J. W. Grossman, Peter Kornya, Bohuslav Míšek (Czechoslovakia), Hugh Noland, C. C. Oursler, C. B. A. Peck, Harry Ploss, Simeon Reich (Israel), Robin Robinson, David Sibley, Dan Sunday, Necdet Üçoluk, and the proposer.

Root of a Complex Quadratic

E 2021 [1967, 1006]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

If $|\bar{a}b-\bar{b}c|=|a\bar{a}-c\bar{c}|$, determine whether the equation $az^2+bz+c=0$ (a, b, c complex numbers, $a\neq 0$, $|a|\neq|c|$) has a root z_1 with the property $|z_1|=1$. The converse is known.

Solution by W. D. Bouwsma, Southern Illinois University. We may assume without loss of generality that $b\geqq 0$. Then $b=|a\bar{a}-c\bar{c}|/|\bar{a}-c|$, and one obtains

$$b^2-4ac=(|a|^2+|c|^2-2ac)^2/(\bar{a}-c)(a-\bar{c}).$$

Then the equation has a root

$$z_1 = \frac{1}{2a}\left[\frac{-|a|^2+|c|^2}{\sqrt{(\bar{a}-c)(a-\bar{c})}} - \frac{|a|^2+|c|^2-2ac}{\sqrt{(\bar{a}-c)(a-\bar{c})}}\right] \quad \text{if } |a| > |c|,$$

or

$$z_1 = \frac{1}{2a}\left[\frac{|a|^2-|c|^2}{\sqrt{(\bar{a}-c)(a-\bar{c})}} - \frac{|a|^2+|c|^2-2ac}{\sqrt{(\bar{a}-c)(a-\bar{c})}}\right] \quad \text{if } |c| > |a|.$$

In either case, $|z_1| = 1$.

Also solved by D. A. Brannan, T. Fujinawa (Japan), Nancy E. Keyton, Peter Kornya, L. Kuipers, Joseph Lehner, M. J. Merscher, and J. P. Ruebsamen.

Brannan obtains the result as a corollary of the theorem: *If* $f(x) = c_0 + c_1x + \cdots + c_nx^n$, $|c_0| \neq |c_n|$, *then* $f_1(x) = \sum_{k=0}^{n-1} (\bar{c}_0c_k - c_n\bar{c}_{n-k})x^k$ *has the same number of roots on* $|x| = 1$ *as has* $f(x)$. See M. Marden, *Geometry of the Zeros of a Polynomial in a Complex Variable*, AMS, New York (1949), Ch. 10.

An Application of the Fibonacci Numbers

E 2022 [1967, 1006]. *Proposed by Richard Parris, Tufts University*

Ordered k-tuples are formed using the two symbols A, B. In how many ways can this be done, if it is required that "A" does not occur in adjacent positions? I.e., for $k = 3$, the triples (A, A, A), (A, A, B), (B, A, A) are not counted.

I. *Solution by N. Sivaramakrishnan, Sri Palaniandavar Arts College, Palni, India.* If we take $(k-r)$ B's, then a single A can be placed in any r of the $(k-r+1)$ interspaces, including the two ends. Hence there are

$$\binom{k-r+1}{r}$$

ways in this arrangement. The total number of ways is therefore

$$\sum_{r=0}^{[\frac{1}{2}(k+1)]} \binom{k-r+1}{r},$$

and this is the $(k-2)$th Fibonacci number.

II. *Solution by Robert Prener, Long Island University.* Let s_k be the number of satisfactory k-tuples. To count s_{k+1} note that each $(k+1)$-tuple begins with either B or A. If it begins with B, any of the s_k k-tuples may fill the remaining k places. If it begins with A, then B must follow and any of the s_{k-1} $(k-1)$-tuples may fill the next $k-1$ places. Therefore $s_{k+1} = s_k + s_{k-1}$; and as $s_1 = 2$ and $s_2 = 3$, $s_k = F_{k-2}$ where F_k is the kth Fibonacci number. $(F_1 = F_2 = 1)$.

Also solved by seventy-six other readers. The result is known. Douglas Lind refers to J. L. Brown, Jr., *Zeckendorf's theorem and some applications*, Fibonacci Quarterly, 2 (1964), 163–168. Other references are J. Riordan, *An Introduction to Combinatorial Analysis*, Problem 1(b), p. 14; also H. J. Ryser, *Combinatorial Mathematics*, No. 14 Carus Mathematical Monographs, p. 30.

Fields in which $a^2+b^2=1$, $ab\neq 0$

E 2023 [1967, 1006]. *Proposed by L. F. Meyers, The Ohio State University*

Which fields have nonzero elements a and b such that $a^2+b^2=1$? Generalize.

I. *Solution by the proposer.* If a field of characteristic different from 2 has more than 5 elements, then there is an element x such that

$$2x(x^2+1)(x^2-1)\neq 0.$$

For any such element x, let

$$a=\frac{x^2-1}{x^2+1}\quad\text{and}\quad b=\frac{2x}{x^2+1}\cdot$$

If a field of characteristic 2 has more than 2 elements, then let a be an element different from 0 and 1, and let $b=a+1$. In either case a and b have the required properties. Direct computation shows that no other fields have such elements a, b.

II. *Generalization by Jonathan Ryshpan, University of Wisconsin.* Let $\mathfrak{F}=GF(q)$, q odd. For $a\in\mathfrak{F}$, $a\neq 0$, put

$$\sigma(a)=\begin{cases}1 & \text{if } a \text{ is square}\\ -1 & \text{if } a \text{ is nonsquare.}\end{cases}$$

We prove that the number of solutions of $x^2+y^2=a$ with x, $y\in\mathfrak{F}$; x, $y\neq 0$ is $q-2-2\sigma(a)-\sigma(-1)$.

Consider first the quadratic character of pairs of successive field elements, i.e., let $A=$ the number of pairs $(x, x+1)$ of nonzero elements of $\mathfrak{F}$ with x and $x+1$ both squares. Similarly, let $B=$ the number of pairs with x square, $x+1$ nonsquare; $C=$ the number of pairs with x nonsquare, $x+1$ square; $D=$ the number of pairs with x and $x+1$ both nonsquare.

There are just as many squares as nonsquares among the nonzero elements of $\mathfrak{F}$, thus each set has just $T=\frac{1}{2}(q-1)$ elements.

$$\begin{aligned}A+B &= \text{number of pairs with the first element square}\\ &= \text{number of squares in } \mathfrak{F}-\{0,-1\} = T-\epsilon,\end{aligned}$$

where ϵ equals 1 or 0 according as -1 is or is not a square.

Similarly: $C+D=T-1+\epsilon$, $A+C=T-1$, $B+D=T$. It is clear that

$$\sigma(x)\sigma(x+1)=\begin{cases}1 & \text{if both } x \text{ and } x+1 \text{ are squares or nonsquares}\\ -1 & \text{if one is square and the other is not.}\end{cases}$$

Thus

$$(A+D)-(B+C)=\sum_{x\neq 0,-1}\sigma(x)\sigma(x+1)$$

$$= \sum \sigma(x(x+1)) = \sum \sigma\left(x^2\left(1+\frac{1}{x}\right)\right)$$
$$= \sum_{x\neq 0,-1} \sigma\left(1+\frac{1}{x}\right) = \sum_{y\neq 0,-1} \sigma(1+y) = \sum_{z\neq 1,0} \sigma(z)$$
$$= \text{number of squares in } \mathfrak{F} - \{0, 1\}$$
$$- \text{number of nonsquares in } \mathfrak{F} - \{0, 1\}$$
$$= -1.$$

Solving the equations we get

$$A = \tfrac{1}{2}(T-1-\epsilon), \quad B = \tfrac{1}{2}(T+1-\epsilon), \quad C = \tfrac{1}{2}(T-1+\epsilon), \quad D = \tfrac{1}{2}(T-1+\epsilon).$$

We can now turn our attention to $a=x^2+y^2$. If $a=\alpha^2$, we divide by α^2 to get

$$(*) \qquad 1 = \xi^2+\eta^2, \qquad 1-\xi^2=\eta^2.$$

Each solution of (*) corresponds to a solution of $a=x^2+y^2$.

If -1 is a square, $-\xi^2$ is a square, and (*) has four times as many solutions as $1+s_1=s_2$, where s_1 and s_2 are squares. But this number is A, so the number of solutions to (*) is $4A=q-5$.

If -1 is nonsquare, $-\xi^2$ is nonsquare, and (*) has four times as many solutions as $1+n=s$, where n is nonsquare and s is square. The number of solutions of (*) is $4C=q-3$.

If a is nonsquare, $a=x^2+y^2$ has the same number of solutions as $1=x^2/a+y^2/a$, which has four times as many as $1=n_1+n_2$, where n_1 and n_2 are nonsquare. Then, if -1 is square the number of solutions of (*) is $4D=q-1$. If -1 is nonsquare the number of solutions is $4B=q+1$. This completes the proof.

We note that if we allow $x=0$ or $y=0$ we find that in the case $a=\alpha^2$ we have omitted the four solutions $x=\pm 1$, $y=0$ and $x=0$, $y=\pm 1$; and in the other case we have omitted nothing. Therefore the total number of solutions is $q-\sigma(-1)$.

Also solved by Z. Z. Uoiea.

A Super-number

E 2024 [1967, 1006]. *Proposed by R. B. Eggleton, Avondale College, Cooranbong, Australia*

In discussing the number $F_{73}=2^{2^{73}}+1$, W. W. Rouse Ball states: "Its digits are so numerous that, if it were printed in full with the type and number of pages used in this book, many more volumes would be required than are contained in all the public libraries in the world." (*Mathematical Recreations and Essays*, Eleventh (revised) edition, 1963.) (a) Is this an exaggeration? (b) What are the last three digits of F_{73}?

Solution by Harry Ploss, Cooper Union, New York City. $2^{10}>10^3$ implies $2^{73}>8\cdot 10^{21}$, whence $2^{2^{73}}>10^{24\cdot 10^{20}}$. Therefore F_{73} has more than $24\cdot 10^{20}$ digits.

Can this number of digits be stored in the world's libraries? We make an over-generous estimate:

3000 digits per page,	700 pages per volume,
1,400,000 volumes per library,	1,000,000 libraries in the world.

(There are less than 10,000 public libraries in the U.S.A.) This accounts for less than $3\cdot 10^{18}$ digits and falls short by a factor of at least 800. So Ball is guilty of understatement rather than exaggeration.

(b) It is easily shown that for every n,

$$n^{22} \equiv n^2 \pmod{100}, \qquad n^{103} \equiv n^3 \pmod{1000}.$$

Therefore $2^{73} \equiv 2^{13} \equiv 8192 \pmod{100}$, and

$$2^{2^{73}} \equiv 2^{8192} \equiv 2^{92} \equiv (2^{23})^4 \equiv (608)^4 \equiv 896 \pmod{1000}.$$

Hence $F_{73} \equiv 897 \pmod{1000}$.

Also solved by Leon Bankoff, Walter Bluger, W. D. Bouwsma, Rosalie Farrand, Neal Felsinger, Bengt Fornberg (Sweden), Octavio Garcia (Mexico), Jerry Goodman, Michael Goldberg, J. W. Grossman, John Kieffer, Sidney Kravitz, E. S. Langford, M. J. Merscher, Hans Penkuhn (Italy), Ann M. Penton, J. R. Purdy, Simeon Reich (Israel), S. L. Robinson, Eric Rosenthal, H. J. de St. Germain & G. E. Steen, Jr., B. P. Sarkar (India), Z. Z. Uoiea, Bob Walcott, J. H. Weintraub, and Aleksandras Zujus.

Ball's statement is discussed in detail in A. H. Beiler, *Recreations in the Theory of Numbers*, p. 174. Dimitrios Vathis notes that the size of F_{73} is also discussed in P. Maghiras, *An Introduction to Number Theory* (in Greek), Athens, Greece, 1964.

Several contributors compared F_{73} with Eddington's Cosmical Number N, representing the total number of electrons and protons in the universe. Eddington's number requires 90 digits, while F_{73} requires more than 2.4 billion trillion digits. de St. Germain and Steen used a computer to obtain the last 40 digits of F_{73}, i.e. 8947301518995672165296243935786246864897.

Many readers were so carried away with the enormity of F_{73} that they tried to accommodate a number having F_{73} digits instead of only the number F_{73}.

A Polynomial and a Related Sequence

E 2025 [1967, 1133]. *Proposed by Stephen Spindler, Purdue University*

Let $P(x)$ denote a polynomial with integer coefficients such that $P(0)=P(1)=1$. If the sequence $\{a_n\}$ is defined by

$$a_{n+1} = P(a_n), \qquad (a_0,\text{ arbitrary integer}),$$

show that the terms of the sequence are pairwise relatively prime.

Solution by Anders Bager, Hjørring, Denmark. We know that

$$P(x) = x(x-1)Q(x) + 1,$$

where $Q(x)$ has integer coefficients. From this it follows easily by induction that

$$a_{n+1} \equiv 1 \pmod{a_0 a_1 a_2 \cdots a_n}.$$

Hence the conclusion.

Also solved by sixty-three other readers.

Postulates for a Group

E 2026 [1967, 1133]. *Proposed by Qazi Zameeruddin and Surjeet Singh, K. M. College, Delhi, India*

Let G be a set with a binary composition $(\cdot)$ such that
(i) G is closed with respect to $(\cdot)$,
(ii) $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ for all a, b, $c\in G$,
(iii) There exists $e\in G$ such that $ae=ea=a$ for all $a\in G$,
(iv) For each $a\in G$ there exists $a'\in G$ such that either $aa'=e$ or $a'a=e$.
Show that G is a group.

Solution by the Echols Mathematics Club, University of Virginia. It is sufficient to show that every element has a two-sided inverse. Assume that $a\cdot a'=e$ (the case $a'\cdot a=e$ is identical) and let $a'\cdot a=f$. Then

$$f\cdot f = (a'\cdot a)\cdot(a'\cdot a) = a'\cdot(a\cdot a')\cdot a = a'\cdot a = f.$$

Multiplying by f' on either the right or the left, we obtain $e=f$.

Also solved by one hundred thirty-nine other readers.

An Iterative Sequence of Polynomials

E 2027 [1967, 1133]. *Proposed by A. Zachariou, Eylenja-Nicosia, Cyprus*

The polynomials $P_n(x)$, $n=0, 1, 2, \cdots$, are given by

$$P_0(x) = x, \qquad P_{n+1}(x) = P_n^2(x) + \alpha^{2^n}P_n(x),$$

where α is constant and the additions are carried mod 2. For instance:

$$P_1(x) = x^2+\alpha x, \qquad P_2(x) = x^4+\alpha^3 x, \qquad P_3(x) = x^8+\alpha^4x^4+\alpha^6x^2+\alpha^7x.$$

Determine some properties of $P_n(x)$ and, if possible, find an explicit expression for $P_n(x)$.

Solution by D. C. B. Marsh, Colorado School of Mines. Some obvious properties of these polynomials are: each is a multiple of its predecessor, each is homogeneous in x and α, each has 0 as a zero and, for positive n and real α, has precisely one other real zero and only one real critical value which is a regional minimum. It is easy to demonstrate by mathematical induction that

$$P_n(x) = \sum_{j=0}^{n} C(n,j)_2 x^{2^{n-j}}\alpha^{2n-2^{n-j}}$$

where $C(n,j)_2$ represents the corresponding binomial coefficient, reduced modulo 2.

Also solved by Arnold Adelberg, W. D. Bouwsma, L. Carlitz, D. Ž. Djoković, R. B. Eggleton (Australia), M. G. Greening (Australia), C. B. A. Peck, Michael Stolnicki, Martin Tangora, Gregory Wulczyn, and the proposer.

An Impossible Representation of the Function $f(x, y) = xy$

E 2028 [1967, 1133]. *Proposed by Wayne Roberts, Macalester College*

It is known that any function of two variables $f(x, y)$ may be represented in the form $f(x, y) = g\{\phi(x) + \psi(y)\}$. Show that the function $f(x, y) = xy$ cannot be so represented if ϕ and ψ are required to be continuous.

Solution by Sidney Spital, California State College at Haywood. The variation of $xy = g\{\phi(x) + \psi(y)\}$ for fixed $y \neq 0$ shows that ϕ is not constant. If ϕ is assumed continuous its image is some interval I. Therefore, from $y = 0$,

$$0 = g\{w + \psi(0)\} \quad \text{for all } w \text{ in } I. \tag{1}$$

An $x' \neq 0$ can now be chosen such that $\phi(x')$ is interior to I. If, in addition, ψ is assumed continuous (at $y = 0$), a sufficiently small $y' \neq 0$ can be chosen such that

$$\phi(x') + (\psi(y') - \psi(0)) \text{ remains in } I, \tag{2}$$

while

$$0 \neq x'y' = g\{(\phi(x') + \psi(y') - \psi(0)) + \psi(0)\}.$$

The contradiction of (2) with (1) shows that ϕ or ψ (or both) must be discontinuous.

Also solved by D. D. Adamović (Yugoslavia), Arnold Adelberg, W. D. Bouwsma, C. V. Heuer & G. A. Heuer, E. S. Langford, and the proposer.

Apologies are offered for the misprint in the original statement which read $f(xy) = xy$ instead of the correct $f(x, y) = xy$.

A Consequence of the Erdös-Mordell Inequality

E 2029 [1967, 1133]. *Proposed by J. Garfunkel, Forest Hills High School, N. Y.*

If A, B, C are the angles of a triangle, show that

$$3 \sum \cos A \geqq 2 \sum \sin A \sin B.$$

Solution by M. G. Greening, University of New South Wales, Australia. The Erdös-Mordell inequality states that $PA + PB + PC \geqq 2(p_a + p_b + p_c)$, where P is any point inside or on the boundary of triangle ABC and p_a is the length of the perpendicular from P on BC, etc. Applied to P as orthocenter of ABC, this yields (where R is the circumradius of ABC)

$$\sum 2R \cos A \geqq 2 \sum 2R \cos A \cos B,$$

or

$$\sum \cos A - 2 \sum \cos A \cos B \geqq 0.$$

But

$$\sum \cos A - 2 \sum \cos A \cos B = 3 \sum \cos A - 2 \sum \sin A \sin B.$$

Also solved by A. N. Aheart, Anders Bager (Denmark), Leon Bankoff, M. T. L. Bizley (England), M. A. Ettrick, Robert Heller, J. M. Quoniam (France), Simeon Reich (Israel), Marlow Sholander, and the proposer.

The Fermat Relation as a Matrix Equation

E 2030 [1967, 1133]. *Proposed by J. L. Brenner, Stanford Research Institute, and Bernard Jacobson, Franklin and Marshall College, independently.*

In the article, Solutions of $x^4 + y^4 = z^4$ in 2×2 integral matrices (this MONTHLY, 1966, p. 631) R. Z. Domaity gives examples. Show further (1) that $x^n + y^n = z^n$ is solvable in nonzero 2×2 integral matrices with nonnegative elements; (2) that $x^r + y^r = z^r$ is solvable in $r \times r$ nonsingular matrices with nonnegative elements.

Solution by Stephen Spindler, Purdue University. (1) is solved by letting a, b be any positive integers and noting that

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}^n + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n.$$

For (2), let $A = B = I$, the $k \times k$ identity matrix, and let $C = wI$, where w is a positive real rth root of 2. Then A, B, C are nonsingular with nonnegative elements, and $A^r + B^r = C^r$ for all k.

Also solved by D. C. B. Marsh and the proposers.

The problem is solved also in the paper by Ethan Bolker, *Solutions of $A^k + B^k = C^k$ in $n \times n$ Integral Matrices*, this MONTHLY, 75 (1968) 759–760.

Shades of Cardano

E 2031 [1967, 1134]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

a is any real number. Prove the inequality

$$\left| \sqrt[3]{-\frac{a}{2} + \sqrt{\frac{a^2}{4} + \frac{1}{27}}} + \sqrt[3]{-\frac{a}{2} - \sqrt{\frac{a^2}{4} + \frac{1}{27}}} \right| < \sqrt[3]{|a|}.$$

Solution by E. J. F. Primrose, University of Leicester, England. If we call the left-hand side $|x|$, it is not difficult to show that $x^3 + x + a = 0$. Since x^3 and x have the same sign, $|x^3| < |a|$, and the result follows provided $a \neq 0$.

Also solved by forty-two other readers.

Zujus points out that the given cubic radicals are assumed to be real. If arbitrary cube roots are taken, the inequality does not necessarily hold true.

An Inequality

E 2032 [1967, 1134]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove the following inequality

$$\min[(b-c)^2, (c-a)^2, (a-b)^2] \leqq \tfrac{1}{2}(a^2+b^2+c^2),$$

with a, b, c real numbers.

Study the analogous problem for $\min[(a_k-a_i)^2]$, $k<i$; $k, i=1, 2, \cdots, n$.

Solution by Joseph Lehner, University of Maryland. The general inequality is

$$\min_{i\neq j, 1\leq i,j\leq n} (a_i-a_j)^2 \leqq \frac{12}{n(n^2-1)}(a_1^2+\cdots+a_n^2), \qquad n=2, 3, 4, \cdots.$$

To prove it, observe that multiplying each a_i by $\lambda>0$ does not change the inequality. Hence we may assume $\sum_{i=1}^n a_i^2=1$. Choose the numbering so that $a=a_1\leqq a_2\leqq \cdots \leqq a_n$, and set $\mu^2=12/n(n^2-1)$. If the inequality is false, $a_{i+1}-a_i>\mu$ for $i=1, \cdots, n-1$ and so $a_i>a+(i-1)\mu$, $i=2, \cdots, n$. Hence

$$\sum_{i=1}^n a_i^2 > \sum_{i=0}^{n-1}(a+i\mu)^2.$$

Denote the right member by $f(a)$. Since $f\to+\infty$ as $a\to\pm\infty$ and f is quadratic in a, it has a unique minimum, which occurs at $a=-(n-1)\mu/2$. Calculation shows that $f(-(n-1)\mu/2)=1$. Hence

$$\sum_{i=1}^n a_i^2>1,$$

a contradiction to our normalization.

Also solved by D. D. Adamović (Yugoslavia), Anders Bager (Denmark), R. F. Dembek, N. J. Fine, T. Fujinawa (Japan), Leon Gerber, Michael Goldberg, M. G. Greening (Australia), C. V. Heuer & G. A. Heuer, Donald Jeffords, Peter Kornya, E. S. Langford, D. C. B. Marsh, G. V. McWilliams, Bohuslav Míšek (Czechoslovakia), S. Ćuk & J. Polajnar (Yugoslavia), E. J. F. Primrose (England), G. J. Rieger, M. A. Roondog, Steven Russ, Sidney Spital, and Gregory Wulczyn.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before June 30, 1969. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5636. *Proposed by E. C. Jones, University of Michigan*

Which dihedral groups can be given a presentation with two generators and two defining relations?

5637. *Proposed by P. M. Cohn, University of London, England*

Let ω be a primitive nth root of unity. Given two linear operators A, B satisfying the commutation rule $BA=\omega AB$, show that

$$(A+B)^n = A^n + B^n.$$

5638. *Proposed by Oswald Wyler, Carnegie-Mellon University*

Show that the free group G with two generators a, b and the three relations $a^4=b^4=abab=e$ is an extension of the free abelian group $Z\times Z$ (where Z is the additive group of integers) by the cyclic group Z_4.

5639. *Proposed by Oswald Wyler, Carnegie-Mellon University*

Show that the free group G with two generators a, b and the three relations $a^4=b^4=aba^2b^2=e$ is an extension of the cyclic group Z_5 by the cyclic group Z_4.

5640. *Proposed by R. E. Shafer, University of California at Livermore*

Show that

$$\gamma = \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\frac{1}{2}\log\left(n^2+n+\frac{1}{3}\right)$$
$$+\frac{1}{9}\sum_{k=n+1}^{\infty}(k-n)\int_0^1 \frac{dx}{(k+x)^2[(k+x)^4-\frac{1}{3}(k+x)^2+\frac{1}{9}]}.$$

5641. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

From the set $\{1, 2, 3, \cdots, n^2\}$ how many arrangements of the n^2 elements are there such that there is no subsequence of $n+1$ elements either monotone increasing or monotone decreasing?

5642. *Proposed by Raymond Redheffer, University of California at Los Angeles*

If x is real, show that

$$\sin(\pi x/\pi x) \geqq (1-x^2)/(1+x^2).$$

5643. *Proposed by S. Zaidman, Nyon, Switzerland*

Is there a positive function $a(x)$, $x\in R^n$, $a(x)>\alpha>0$, such that

$$a(x) = (2\pi)^{-n/2}\int e^{ix\lambda}A(\lambda)d\lambda,$$

with $A(\lambda)\in L^1$, however $\sqrt{a(x)}$ does not admit any such representation?

SOLUTIONS OF ADVANCED PROBLEMS

Distribution of a Product

5535 [1967, 1144]. *Proposed by C. J. Mozzochi, University of Connecticut*

Suppose $0<(\delta-\frac{1}{2})<\epsilon<\frac{1}{2}$; m and n positive integers such that $n\nmid m$; G

$=\bigcup_{k=-\infty}^{+\infty} (k,\ k+\delta)$; $H=\sim G$; $g(\nu)=m(x+\nu/n)$, $\nu=0,\ 1,\ \cdots,\ (n-1)$; $mx\in H$. Then for arbitrarily small $\epsilon>0$: (a) If n is even, then (number of $g(\nu)\in G$) $\geqq\frac{1}{2}(n-2)$. (b) If n is odd, then (number of $g(\nu)\in G)\geqq\frac{1}{2}(n-3)$.

Solution by Oswald Wyler, Carnegie-Mellon University. Let

$$G_0 = \bigcup_{k=-\infty}^{+\infty} (k,\ k+\tfrac{1}{2}).$$

Then $G_0\subset G$ for every admissible choice of ϵ and δ. Let $(m,\ n)=d$, let $n=dn'$, and let $mx=(p+y)/n'$, for an integer p and $0<y\leqq 1$. Then the numbers $g(\nu)$ are congruent modulo 1 to the numbers $h(\mu)=(y+\mu)/n'$, $\mu=0,\ 1,\ \cdots,\ n'-1$, each number $h(\mu)$ being counted d times. Since G_0 is invariant under congruence modulo 1, and $h(\mu)\in(0,\ 1]$ for $\mu=0,\ 1,\ \cdots,\ (n'-1)$, the number of $g(\nu)$'s in G_0 is d times the number of $h(\mu)$'s in $(0,\ \frac{1}{2}]$.

If n' is even, then $h(\mu)$ is in $(0,\ \frac{1}{2}]$ for $\mu=0,\ 1,\ \cdots,\ \frac{1}{2}n'-1$, but not otherwise. Thus there are exactly $\frac{1}{2}n'$ numbers $h(\mu)$ in $(0,\ \frac{1}{2}]$, and hence exactly $\frac{1}{2}n$ numbers $g(\nu)$ in G_0.

If n' is odd and $\frac{1}{2}(n'-1)=n''$, then $h(\mu)$ is in $(0,\ \frac{1}{2}]$ for $\mu=0,\ 1,\ \cdots,\ n''-1$, and $h(n'')\in(0,\ \frac{1}{2}]$ if (and only if) $y\leqq\frac{1}{2}$. No other numbers $h(\mu)$ are in $(0,\ \frac{1}{2}]$. Thus there are $n''+1$ numbers $h(\mu)$ in $(0,\ \frac{1}{2}]$ if $0<y\leqq\frac{1}{2}$, and n'' numbers $h(\mu)$ in $(0,\ \frac{1}{2}]$ if $\frac{1}{2}<y\leqq 1$. The number of $g(\nu)$'s in G_0 is $d(n''+1)=\frac{1}{2}(n+d)$ or $\frac{1}{2}(n-d)$ accordingly.

Comparing this solution with the statement of the problem, we find: If n' is even or $(m,\ n)=1$, then there is at least one $g(\nu)$ more in G than claimed. If n' is odd and $(m,\ n)=2$ or $(m,\ n)=3$, then the statement of the problem is a "best possible" estimate. If n' is odd, $(m,\ n)>3$, and $\frac{1}{2}<y\leqq 1$, then the statement of the problem is false. Whether or not $mx\in H$ seems to have no bearing on the solution.

Also solved by the proposer, who states that the problem was suggested by results in Walter Rudin, *An Arithmetic Property of Riemann Sums*, Proceedings of the Amer. Math. Soc., 15 (1964) 322.

An Incomplete Ordered Field

5551 [1968, 84]. *Proposed by G. A. Heuer, University of California at Berkeley and Concordia College*

Find an ordered field in which every sequence is bounded, and every convergent sequence is ultimately constant.

Solution by L. F. Meyers, Ohio State University. Let Q be any ordered field, and let ω be the set of all positive integers. Let (t_α: α is a countable ordinal) be a transfinite independent sequence of indeterminates over Q. The required field F is the result of adjoining all the indeterminates to Q. Positivity in F is defined recursively:

1. If $x\in Q$, then $x>_F 0 \leftrightarrow x>_Q 0$.

2. If $k\in\omega$, then a polynomial using exactly k indeterminates nontrivially is positive in F just when the coefficient of the highest power of its "highest" in-

determinate is positive in F (as a polynomial in at most $k-1$ of these indeterminates, or as an element of Q).

3. $x_1/x_2 >_F 0 \leftrightarrow x_1 \cdot x_2 >_F 0$.

It is easily verified that, with these definitions, F becomes an ordered field.

Now let $(x_n)_{n\in\omega}$ be a sequence of elements of F, and for each n in ω let α_n be the index of the "highest" indeterminate used nontrivially in some fixed representation of x_n as a quotient of polynomials (or -1, if no indeterminates are used). Let $\alpha = \sup\{\alpha_n : n\in\omega\}$. Then $\alpha+1$ is a countable ordinal, so that $t_{\alpha+1}$ is a bound for $(x_n)_{n\in\omega}$.

The second condition (that every convergent sequence is eventually constant) is equivalent to the first. In fact, if $(x_n)_{n\in\omega}$ is a not ultimately constant sequence with limit x, then it has either a strictly increasing or a strictly decreasing subsequence $(y_k)_{k\in\omega}$ with the same limit. But then the sequence $((x-y_k)^{-1})_{k\in\omega}$ is unbounded. Conversely, if $(x_n)_{n\in\omega}$ is an unbounded sequence, then the sequence $(x_n^{-1})_{n\in\omega}$ has limit 0, but is not ultimately constant.

Also solved by P. R. Chernoff, D. H. Frank, D. A. Hejhal, Tassos Nakossis (Greece), J. J. Uhl, Oswald Wyler, and the proposer.

Uhl uses the non-standard real line (*Non-standard Analysis*, Cal. Tech Lecture Notes, W. A. J. Luxemburg, 1964) for his example.

The Characteristic Vector of the Adjoint Matrix

5552 [1968, 84]. *Proposed by D. E. Crabtree, Amherst College*

It is well known that each characteristic vector for a nonsingular $n\times n$ matrix A is also a characteristic vector for the matrix adj A of (transposed) cofactors of elements of A. Prove that this is true for singular A also.

I. *Solution by W. G. Dotson, Jr., North Carolina State University.* We have $A\cdot\text{adj } A = \text{adj } A\cdot A = |A|\cdot I$. Suppose $Ax=\lambda x$ $(x\neq 0)$. If $\lambda\neq 0$, then $(\text{adj } A)x = (1/\lambda)(\text{adj } A)Ax = (|A|/\lambda)x$. Suppose $\lambda=0$. Then $Ax=0$ $(x\neq 0)$, and so $|A|=0$. If rank $A\leqq n-2$, then adj $A=0$, and so $(\text{adj } A)x=(0)x=0\cdot x$. The remaining case is rank $A=n-1$. Let $S=\{y: Ay=0\}$; then $\dim S = n-(n-1)=1$. Since $x\in S$ and $x\neq 0$, $\{x\}$ is a basis for S. Since

$$A((\text{adj } A)x) = |A|\cdot x = 0\cdot x = 0, \qquad (\text{adj } A)x \in S.$$

Hence there exists μ such that $(\text{adj } A)x = \mu\cdot x$.

II. *Solution by J. V. Michalowicz, Catholic University of America.* We give a proof valid for an arbitrary $n\times n$ matrix $(n>1)$ A over a field F. Let the characteristic polynomial for A be given by

$$f = \det(xI - A) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0.$$

In Gantmacher, *The Theory of Matrices*, Vol. 1, p. 85, an expression is obtained for $\text{adj}(xI-A)$. By evaluating this expression at 0, we get

$$\text{adj } A = (-1)^{n-1}\,\text{adj}(-A) = (-1)^{n-1}[A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_2A + c_1I].$$

Consequently, if $AX=\lambda X$ where $\lambda\in F$ and $X\neq 0$ is an $n\times 1$ matrix over F, then

$$\begin{aligned}(\text{adj } A)X &= (-1)^{n-1}[A^{n-1}+c_{n-1}A^{n-2}+\cdots+c_1 I]X\\ &= (-1)^{n-1}(\lambda^{n-1}+c_{n-1}\lambda^{n-2}+\cdots+c_1)X,\end{aligned}$$

so X is a characteristic vector for adj A.

Also solved by P. R. Chernoff, C. G. Cullen, Crist Dixon, D. Ž. Djoković, Ralph Frese, Wallace Givens, D. Z. Hearon, D. A. Hejhal, R. A. Horn, A. S. Householder, R. A. Howland, Rudolf Kochendörffer (Germany), T. L. Markham, C. D. Meyer, P. J. Nikolai, S. J. Pierce, C. M. Price, Simeon Reich (Israel), R. F. Rinehart, P. V. Subba Rao & B. Ramachandra Rao (India), Michael Skalsky, Antonio Villanueva, W. C. Waterhouse, Oswald Wyler, and the proposer.

Givens notes that the property of common eigenvectors for the matrices A, adj A may be interpreted for matrices C, D which commute, as follows: If x is an eigenvector of C then so is Dx with immediate consequences when the eigenvalues are simple.

Topologies of Separate Continuity in a Cross Space

5553 [1968, 84]. *Proposed by P. R. Meyer, Herbert H. Lehman College, New York*

It is well known that the topology of joint continuity is the product topology, but is there a topology of separate continuity? More precisely, if X, Y, and Z are topological spaces and $f: X\times Y\to Z$, is there a topology T for $X\times Y$ such that f is T-continuous if and only if f is separately continuous in each variable?

Solution by Red Cougar, University of Houston. Suppose X and Y are topological spaces, $a\in X$, $b\in Y$, u is an open set in X and v is an open set in Y such that $a\in u$ and $b\in v$. The set $a\times v\cup b\times u$ in $X\times Y$ will be called a "plus at (a, b)." A set N in $X\times Y$ will be called a "plus neighborhood" if and only if for each (x, y) in N, there is a plus at (x, y) which is a subset of N. This system of plus neighborhoods induces a topology on the space $X\times Y$ in which a function $f: X\times Y\to Z$ is continuous if and only if it is separately continuous in each variable.

To see this, suppose $f: X\times Y\to Z$ is separately continuous in each variable. Let V be an open set in Z and (a, b) be a point in $f^{-1}(V)$. Since $f(a, b)\in V$ and f is separately continuous there exist neighborhoods u of a in X and v of b in Y such that $f[a\times v]\subseteq V$ and $f[u\times b]\subseteq V$. Thus $f^{-1}(V)$ is a plus neighborhood in $X\times Y$ and the function f is continuous. If f is continuous in $X\times Y$ it follows immediately that f is continuous in each variable separately.

Also solved by D. R. Anderson, P. R. Chernoff, H. D. Keesing, C. J. Knight (England), L. F. Meyers, Charles Riley, Linda E. Wells, Oswald Wyler, and the proposer.

The proposer notes that for the real line, the problem was studied by J. Novák, *Induktion partiell stetiger Funktionen*, Math. Annalen, 118 (1942) 449–461.

A Class of Infinite Matrices

5554 [1968, 84]. *Proposed by Surjeet Singh, Kirori Mal College, Delhi, India*

Consider matrices with denumerably many rows and columns over a division ring D. Let R be the ring generated by all scalar matrices together with

matrices having only a finite number of nonzero columns. Show that R is regular in the sense of von Neumann, and that R, considered as a right R-module, is not injective.

Solution by Hans H. Storrer, Eidgen. Technische Hochschule, Zürich, Switzerland. We first prove that R is regular. R is generated by the ring S of scalar matrices and the ring F of matrices with only finitely many nonzero columns. Since F is a two-sided ideal of R, we have $R=S+F$. F is isomorphic to the ring of all linear transformations of finite rank of a vector space of countably infinite dimension over D (the transformations being written on the right), and hence regular by a known result (see, for example, N. H. McCoy, *The Theory of Rings*, Theorem 7.4).

Since S is clearly regular, for every $A\in R$ there exists $B\in R$ such that $A-ABA\in F$. Then there is $C\in F$ such that

$$A - ABA = (A - ABA)C(A - ABA),$$

i.e.

$$A = A[B + (1 - BA)C(1 - AB)]A.$$

We now show that R_R is not injective. F is a right ideal of R. We define a right R-homomorphism ϕ: $F\rightarrow R$, which is not a left multiplication by an element of R. Let $f=(f_{ij})$ be a matrix in F and put

$$(\phi f)_{ij} = \begin{cases} f_{i-1,j} & i \geqq 2 \\ 0 & i = 1. \end{cases}$$

A direct calculation shows that ϕ is in fact an R-homomorphism. However, there exists no matrix $a=(a_{ij})$ in R with $\phi f=af$ for all $f\in F$, since such an a would have $a_{k+1,k}\neq 0$ for $k=1, 2, \cdots$, as one sees by letting f in turn be equal to the "matrix units" $e_{[k1]}$, $k=1, 2, \cdots$, i.e., the matrices having 1 at the position k, 1 and zeros otherwise.

Also solved by Oswald Wyler and by the proposer.

Functional Inequality for a Probability Integral

5555 [1968, 84]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove or disprove the inequality $f(x)f(y)\geqq f(x)+f(y)-f(x+y)$, $(x, y\geqq 0)$, where

$$f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Solution by R. J. Weinacht, University of Delaware. We prove the inequality is correct for $x\geqq 0$, $y\geqq 0$ with equality if and only if x or y is an end point of the closed interval $[0, +\infty]$.

First note that $f(0)=0$ and $f(+\infty)=1$. Then for any fixed $y>0$, the function defined for $x\geqq 0$ by

$$F(x) = f(x)f(y) - f(x) - f(y) + f(x+y)$$

satisfies $F(0)=F(+\infty)=0$. A "Rolle's theorem" for $[0, +\infty]$ insures the existence of a z in $(0, +\infty)$ where $F'(z)=0$. Now

$$F'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}[f(y) - 1 + e^{-2xy-y^2}],$$

which is a monotonically decreasing product for $x>0$. It follows that such a z is unique and $F'(x)$ has the same sign as $z-x$. Thus 0 and $+\infty$ are the only zeros of F and F is positive elsewhere.

Also solved by I. N. Baker (England), Bruce Berndt, D. Borwein, N. A. Bowen (Malta), F. A. Butter, Jr., L. Carlitz, P. R. Chernoff & W. C. Waterhouse, L. E. Clarke (England), T. W. Daniel, D. Ž. Djoković, D. A. Hejhal, Paul Jurovietzky (Venezuela), M. S. Kaplan, Eric Langford, Charles McCracken, A. Meir, George Mitchell, G. W. Petrie III, Simeon Reich (Israel), L. W. Ringenbach, J. J. Schäffer (Uruguay), Marlow Sholander, Michael Skalsky, S. H. Smith, Chang Sung-sheng (Taiwan), O. Wolowyna (Argentina), and Oswald Wyler.

Chernoff and Waterhouse generalize and replace $f(x)$ with $\int_0^x \rho(t)dt$ where $\log \rho(t)$ is concave and $\int_0^1 \rho(t)dt=1$.

Normal Polynomials

5556 [1968, 84]. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Every polynomial $f(x)$ irreducible over a perfect field R, and with an abelian Galois group, is normal. (An irreducible equation $f(x)=0$, $f(x)\in R[x]$, is called normal if $f(x)$ splits in the field obtained by the adjunction of an arbitrary root of $f(x)$ to R.)

Solution by Robert Gilmer, Florida State University. We prove: *If $f(x)$ is an irreducible separable polynomial over a field R, and if the Galois group of $f(x)$ over R is Hamiltonian, then $f(x)$ is normal over R,* which includes the result of the problem.

Let G be the Galois group of the splitting field K of $f(x)$ over R. Since G is Hamiltonian, each subgroup of G is normal in G, so that each intermediate field F between R and K is normal over R. In particular, if θ is any root of $f(x)$ in K, then $R(\theta)$ is normal over R. Therefore $R(\theta)=K$ and $f(x)$ is normal over R.

Also solved by D. Ž. Djoković, Harley Flanders, D. A. Hejhal, J. M. Katz, M. G. Greening (Australia), Kenneth Kramer, Brian Parshall & Pierre Burr, S. E. Payne, A. K. Pizer, Surjeet Singh & Kamlesh Wasan (India), W. C. Waterhouse, Oswald Wyler, and the proposer.

The Derivative of Area

5558 [1968, 84]. *Proposed by A. W. Goodman, University of South Florida*

The derivative of area is arc length. Let $\overline{R}(s)$ be the vector equation of a

smooth simple closed curve C that encloses a plane region of area A. Let $\overline{N}(s)$ be the unit outward normal. Define a parallel curve $C(r)$ at distance r from C by the vector equation $\overline{R}_r(s)=\overline{R}(s)+r\overline{N}(s)$. If $0\leqq r$ and r is sufficiently small then $C(r)$ will be a smooth simple closed curve that encloses a region with area $A(r)$. Prove that at $r=0$, $dA(r)/dr=L$, where L is the length of the curve C.

Solution by D. A. Hejhal, University of Chicago. Let

$$\{x = x(s),\, y = y(s),\, s \nearrow,\, 0 \leqq s \leqq L\}$$

be the arc length parameterization of a smooth simple closed positively oriented path $\vec{\Gamma}$ in the (x, y)-plane. Now the outward unit normal to $\vec{\Gamma}$ at $(x(s), y(s))$ has components $(y'(s), -x'(s))$. Define $\vec{\Gamma}(r)$ to be the path

$$\{x = x(s) + ry'(s),\, y = y(s) - rx'(s),\, s \nearrow,\, 0 \leqq s \leqq L\}, \qquad r \geqq 0.$$

Let $A(r)$ be the area enclosed by $\vec{\Gamma}(r)$.

By means of Green's theorem we have $\oint_C xdy$ equals the area enclosed by C. For small $r\geqq 0$,

$$\begin{aligned} A(r) &= \int_{\vec{\Gamma}(r)} xdy = \int_0^L [x(s) + ry'(s)][y'(s) - rx''(s)]ds \\ &= \int_0^L x(s)y'(s)ds + r\left\{\int_0^L [y'(s)]^2 ds - \int_0^L x(s)x''(s)ds\right\} \\ &\quad - r^2\int_0^L y'(s)x''(s)ds. \end{aligned}$$

Thus,

$$A'(r) = \int_0^L [y'(s)]^2ds - \int_0^L x(s)x''(s)ds - 2r\int_0^L y'(s)x''(s)ds.$$

Now, $0=\int_0^L d[x(s)x'(s)]=\int_0^L x(s)x''(s)ds+\int_0^L [x'(s)]^2ds$. Hence,

$$-\int_0^L x(s)x''(s)ds = \int_0^L [x'(s)]^2ds.$$

Therefore,

$$\begin{aligned} A'(r) &= \int_0^L [x'(s)^2 + y'(s)^2]ds - 2r\int_0^L y'(s)x''(s)ds \\ &= L - 2r\int_0^L y'(s)x''(s)ds. \end{aligned}$$

Hence, $A'(0+)=L$.

Also solved by A. S. Adikesavan (India), D. Ž. Djoković, P. R. Chernoff & W. C. Waterhouse, and Marlow Sholander.

Rings and Integral Domains

5559 [1968, 85]. *Proposed by Kwangil Koh, North Carolina State University, Raleigh*

A ring is said to be prime if and only if $aRb=0$, for a, b in R, implies that either $a=0$ or $b=0$. It is said to be semi-prime if and only if $aRa=0$ implies that $a=0$. Prove that a prime ring R with no nilpotent element except 0 is an integral domain (not necessarily commutative) and that there exists a semi-prime ring with no nilpotent element except 0, but with zero divisors.

Solution by H. L. Penn, Southeastern State College, Durant, Oklahoma. Suppose a, b are in R, and $ab=0$. Then for $x \in R$, $bxabxa=0$ and, thus, $bxa=0$, because of the unique nilpotent element. Therefore $bRa=0$ which implies either $a=0$ or $b=0$.

The integers mod 6 is an example of a semiprime ring with zero divisors and no nonzero nilpotent elements.

Also solved by D. T. Adams, J. C. Adams, Christine Ayoub, Kanchan Basho (India), C. B. Baytop & J. E. Joseph, Prabir Bhattacharya (India), W. E. Bodden, J. J. Bode & J. E. Cicero, P. R. Chernoff & W. C. Waterhouse, W. J. Clover, J. W. Duke, William Fox, D. H. Frank, Octavio Garcia R. (Mexico), R. R. Hallett, M. E. Harris, D. A. Hejhal, R. A. Howland, T. P. Kezlan, John Koelzer, Kenneth Kramer, Eric Langford, Charles Lanski, W. G. Leavitt, Jiang Luh, Wanda J. Mourant, Brian Parshall, S. J. Pierce, A. K. Pizer, Jonathan Ryshpan, R. L. Snider, H. H. Storrer (Switzerland), D. P. Sumner, W. A. Thrash, Jr., Annapurna Upadhyayula (India), Kamlesh Wasan (India), D. R. Weidman, J. C. Wenger, Oswald Wyler, Kenneth Yanosko, Qazi Zameeruddin (India), and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. All unsigned material is by the editor. Correspondence about reviews will be welcome.

Beginning with the January 1969 issue, film reviews will be edited by Seymour Schuster, Carleton College, Northfield, MN 55057. All correspondence concerning films should be sent to him.

Conformal Mapping on Riemann Surfaces. By Harvey Cohn. McGraw-Hill, New York, 1967. xiv+325 pp. $12.95. (Telegraphic Review, Feb. 1968.)

This book, like some other recent ones (e.g., Spivak's "Calculus on Manifolds," Rudin's "Real and Complex Analysis," etc.) is written with a highly personal flavour. One either loves it or hates it. The style is colloquial throughout. Few teachers will quarrel with the author's introductory statements that " . . . (the) subject should be made available to the senior undergraduate or beginning graduate student . . . ," or that it " . . . can be made available . . . at the desired level only in the spirit of compromise . . . ," so that it is " . . . necessary to compromise unwillingly with rigor and completeness and, more sadly,

with historic motivation and elegance . . . ." However, when it comes to decide just where to draw the line, each one will have, presumably, his own ideas. Whatever one's personal opinion, the author never leaves one in doubt about his own and, at the beginning of each chapter (and even of many sections) states quite precisely his immediate aims, such as (p. 151): "We shall never pursue the matter beyond the grossly intuitive level . . . ," or (concerning the proof of the Riemann mapping theorem, p. 203) " . . . we shall follow a somewhat rambling path of least resistance in order to show the interplay of the field . . . ," or again (p. 198), " . . . what we shall do is to give a rough (indeed an incorrect) proof . . . ."

The focal point of the book is Riemann's theorem about the correspondence between algebraic functions and compact Riemann surfaces. The theorem is stated early in Chapter 6 and much of the preceding material is there in order to build up the necessary concepts and provide the motivation and historic perspective. Quoting again: "The most remarkable feature of Riemann's theorem is not even the depth of the proof, which will occupy practically the remainder of the text! . . . (it) is that at one time it represented almost all of mathematics in microcosm . . . ." While no textbook writer can afford to follow exactly the vagaries of the evolution of mathematical thinking, the author never loses sight of the significance of this evolution—an attitude which makes this text particularly appealing, at least to this reviewer. The quoted sentences (inevitably taken somewhat out of their context) may give the impression of lack of rigor (or even of correctness) or that the author is of an old-fashioned outlook. Nothing could be further from the truth! In fact, except for peripheral topics, the proofs are painstakingly complete. Also, the most "modern" minded mathematician will be happy to know that homology (without the term) is introduced on page 5, and formally on page 17, while the Cauchy-Riemann equations (assumed known and alluded to in a footnote on page 4) do not appear until Chapter 8 (p. 171) and that the book contains the most lucid (to an analyst) presentation of the Riemann-Roch theorem known to the reviewer.

The detailed content of the text (aptly presented in the telegraphic review) will not be repeated, nor is this a place to list the (relatively few) printing errors, but the generally excellent selection of exercises deserves mention. This reviewer was less happy about the presentation of Dirichlet's problem and about the fact that the proofs of a large number of far from trivial theorems, stated and used in the text, are left as exercises (although, fortunately, often with copious hints). On the other hand, the presentation of modular functions, abelian differentials and the whole Chapter 14 (Riemann-Roch theorem, Weierstrass points, Schwarz's theorem on self-mappings of manifolds, Abel's theorem on prescribing zeros and poles for a meromorphic function, application to cubics, divisor classes—and even the Jacobi variety of a manifold, all in a scant 20 pages) is masterly. In conclusion: a remarkable text, both for the classroom (if the instructor likes it) and for self-study.

EMIL GROSSWALD, Temple University

A Deductive Theory of Space and Time. By Saul A. Basri. North Holland, Amsterdam, 1966. xi+163 pp. $7.00.

This monograph sets out to construct a completely deductive theory of space and time, and as such, is along the lines of Robb's classical work of 1914, and that of Reichenbach et al. It uses, however, exclusively, the language of symbolic logic.

As in all deductive theories, this work starts with a finite number (eight to be exact) of primitive concepts, axioms, postulates (to be differentiated from axioms) and the end products are the logical consequences of the axioms. What is novel about the present theory is the thorough analysis of the primitive concepts. Thus the author starts with the class of observers H, which is "the class of living humans who have adequately functioning sense organs, can communicate with each other, and do so honestly and without bias." Then, he proceeds to define carefully a subjective entity of an observer H, and finally establishes a criterion for objectivity of an entity, which is essentially a measure of agreement between at least two observers from the class H. The author argues that the concept of objectivity is possible because the probability of agreement among the observers having the above mentioned qualifications should converge rapidly to unity.

Next, a precise definition of a macroscopic particle, as perceived by an observer H, is given. At this point one must be prepared for an anti-climax—one learns that "the concept of a particle is not objective . . . (and) there is no way out of this, either in the macroscopic or microscopic domain. Since events are happenings to particles, events are not objective either, which explains why the subscript 'H' referring to the observer H, persists throughout the theory." If it is not possible to establish the objectivity of the concept of a particle, then any Physics based on the present (i.e. author's) theory of space and time would necessarily be subjective and one might as well have started with a class of observers who, albeit possessing adequately functioning sense organs, can only communicate with each other dishonestly, and with bias!

Must all concepts of a particle be subjective? I think a way out is possible along the lines of thought expressed by Max Born (in "Natural Philosophy of Cause and Chance," Oxford, 1949, p. 104). The concept of an electron, for example, acquires objectivity if by an electron we mean not a macroscopic particle having a definite position in space and time but as the sum total of its "observational invariants" (e.g. charge, mass, spin, etc.) The idea is to apply some sort of invariance principle in Physics similar to Klein's Erlanger Program in mathematics.

To many physicists all this may seem metaphysical hair-splitting, but to those who are philosophically minded and/or are interested in the foundations of their subject, this book should serve as an excellent application of logic to problems which can appear to be deceptively simple but are in reality exceedingly complex.

For those (and that includes most bread and butter physicists) not versed in the language of symbolic logic, there is an excellent appendix on symbolic

logic and set theory. Finally, there is a comprehensive list of logical theorems, references, symbols, conventions and abbreviations. The production of the book is beyond reproach.

D. K. SEN, University of Toronto

Calculus of variations and optimal control theory. By Magnus F. Hestenes (Univ. of California, Los Angeles). Wiley, New York, 1966. xii+405 pp. $12.95.

Optimal control, which is here treated as part of the calculus of variations, is also partly an engineering subject. Although the book has been to some extent addressed to a fairly advanced level, its tone is therefore set rather lower than this. There are few hard to digest, abstract concepts or dry existence theorems, but many examples of a rather introductory nature. Proofs involve complicated analytical manipulation, but they are generally put off until some illustrations have been given. At this level it is unusual to treat much more than necessary conditions. In the calculus of variations this means setting the clock back. Therefore the author, who is an authority in his field, avoids this basic limitation, or makes up for it, by providing, in the classical case of no constraints, a brief account of sufficiency theorems and also of an alternative indirect method, which he devised some 20 years ago and which can be extended to cover certain problems with constraints. Further the book does manage to be virtually self-contained, which is quite an achievement at this level, and it contains many ingenious and interesting side-remarks, which are certainly not generally known, for instance the fact, noted on p. 43, that, for a positively homogeneous Lagrangian of degree $p>1$ subject to the Legendre conditions, the Hamiltonian transformation maps n-space into itself. Thus the book is a very valuable addition to the existing literature.

L. C. YOUNG, University of Wisconsin

A FILM SERIES FOR GRADUATE STUDENTS

During the 1968 winter and spring quarters at the University of Minnesota, the Graduate Math Committee presented a series of mathematics films. This program was motivated by a desire to determine whether mathematics films could be used effectively on the graduate level. The purpose of this note is to describe the organization, content, and success of this program.

The films were shown on a weekly basis at a regular time and place, scheduled so that they might fit into the program of seminars and colloquia in the School of Mathematics. Publicity was handled in the form of a note in the weekly departmental bulletin and by means of flyers patterned after commercial movie advertisements (*e.g.*, in advertising the film on the life of John von Neumann: "An Ergodic Masterpiece"). Our principal source for films was the CEM Individual Lecture Series whose films were rented from Modern Learning Aids. We also made use of the University's Audio-Visual Aids Department in obtaining films. Orders with outside firms had to be placed well ahead of schedule in order to insure prompt arrival. We found it most convenient to order films in monthly groups, each film being rented for a three-day period. The average

cost per film was around $10.00 and projectors and screens were rented from the University AVA Department. Most films ran about one hour and consisted of more than one reel, so an intermission was usually required to change reels. We often used this interruption as a discussion period. It was found very helpful to have a faculty "expert" in the area with which the film at hand dealt. Popcorn was occasionally served.

We were seeking in this program films that presented advanced material in a straight-forward, comprehensible manner and films that might present new ideas concerning the teaching of elementary mathematical concepts (a large number of the graduate mathematics students are also engaged in teaching). The former were in ample supply; the latter were hard to find. With one exception, we showed only expository films. That one exception was a film about R. L. Moore, in which his teaching method was a central concern. The other films which were shown could be roughly classified as falling into one of three categories: biographical, "content," and historical. As might be expected, these areas of classification are not mutually exclusive.

The series was led off by a film on the life and work of John von Neumann. This was one of the best received of the series. A few weeks later a similar film about Richard Courant (*Göttingen and New York*) was shown which included among its highlights live shots of David Hilbert shoveling snow. Both films are highly recommended.

Among the "content" films were a three-part series by John Milnor on Differential Topology (in these films, manifolds and their classification are studied from a categorical viewpoint); three short films on geometry narrated by Daniel Pedoe; a two-part series with Marston Morse on extremum points in differentiable manifolds (*Pits, Peaks, and Passes*); and Andrew Gleason speaking on oriented graph games (this is one of the "classics" of mathematics films). All the above films, we might mention, were entirely comprehensible to most of the graduate students attending. We found that these films provide an effective method for giving the average graduate student the feel of the nature of previously unfamiliar branches of mathematics.

The one historical film was a lecture by Kenneth May (*Who Killed Determinants?*) presenting a survey of the history of research in the field of determinants. This was an especially interesting film describing, among other things, the relationship between the volume and originality of work published in that field at a given time.

In all respects, the program was well-received. Attendance varied greatly from a low of fifteen to a high of seventy-five and included many senior faculty members. The biographical films drew best, while attendance at the later installments of the two- and three-part series on a given topic tended to fall off, reflecting the difficulty of maintaining interest in a single film over the course of weeks. An informal poll indicated that the participants found the program of value and that there was considerable interest in continuing it during the next academic year.

PHILIP SILLER AND DANIEL SUNDAY, University of Minnesota

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)-18 (second graduate year). A boldface star (★) marks a notable book that might be overlooked.

Calculus

Analytic Geometry and Calculus. 2nd ed. By L. J. Adams (Santa Monica City College) and Paul A. White (Univ. of Southern California). Oxford Univ. Press, New York, 1968. xiii+975 pp. $11.75. Final chapters on partial derivatives, double and triple integrals, differential equations.

Calculus. By Robert G. Bartle (Univ. of Illinois) and C. Ionescu Tulcea (Northwestern Univ.). Scott, Foresman, Glenview, Ill. 1968. xi+718 pp. $11.50. After a short chapter on sets and numbers, analytic geometry, functions and graphs, calculus is introduced via limits of sequences, and the usual topics relating to functions of one variable are covered in just under 500 pages. There follow two chapters on series and two on partial differentiation and multiple integration.

Calculus for Engineering Technology. By Walter R. Blakeley (Ryerson Pol. Inst. Toronto). Wiley, New York, 1968. xi+441 pp. $8.95. Cook book.

Calculus I: Differential Calculus. By Albert A. Blank (New York Univ.). With the assistance of Florence L. Elder (West Hempstead High School), and Clarence W. Leeds III (SMSG). Houghton Mifflin, Boston, Mass., 1968. xx+436 pp. $7.50. The first of four volumes (the next three being entitled Integral Calculus, Applications of Single Variable Calculus, Linear Algebra, Multivariate Calculus and its Applications). Based on the text prepared by the editorial group and summer writing teams of the School Mathematics Study Group from 1964 to 1966. The first two volumes are designed for advanced placement courses in the high school as well as for the first year of college work. The preface states many good goals such as "depth of understanding rather than superficial coverage" and "to reverse Gresham's Law in calculus texts. . . ." Clearly not just another calculus!

A Programed Course in Calculus. I. Functions, Limits and the Derivative. II. The Definite Integral. III. Transcendental Functions. IV. Applications and Techniques of Integration. V. Infinite Sequences and Series. Prepared by the Committee on Educational Media of the Mathematical Association of America with the support of the National Science Foundation. Benjamin, New York, 1968. I: xii+295 pp., II: xiii+237 pp., III: xiii+129 pp., IV: xii+213 pp., V: xiii+121 pp. $2.95 each (paper). These materials were prepared as part of the programed learning project of the CEM during writing sessions at Stanford University, 1964–1966. Preliminary versions were tested and used as a basis for the final product. Eighteen writers were involved. If students write answers on separate sheets, the books can be used again. This is probably the best programed calculus course now available and should be considered for use as part of a system containing other components such as lectures and books explaining concepts (for example Toeplitz Calculus, A Genetic Approach and W. W. Sawyer, What is Calculus About?). T, S.

Calculus with Analytic Geometry. By R. H. Crowell and W. E. Slesnick (both of Dartmouth College). Norton, New York, 1968. x+727 pp. $9.95. For a one year course ending with a taste of differential equations. Some numerical approximation suitable for computer use.

Calcul Infinitésimal. By Jean Dieudonné. Hermann, Paris, 1968. 479 pp. 42 F (paper). A textbook for students who have already passed through a year of the "first cycle," which includes some analysis, this book focuses on calculation. The author sums up his emphasis in the slogan "majorer, minorer, approcher." As one might expect, there are many interesting comments among which I cannot resist quoting the following: ". . . there is no 'modern mathematics' in opposition to 'classic mathematics,' but simply a single mathematics of today which continues that of yesterday without any profound break, and which, above all, tries to solve the great problems that have been willed to us by our predecessors. If, in order to do this, mathematicians have been led to develop new abstract ideas in substantial numbers, it is that these ideas have often made it possible, by so to speak concentrating light on the heart of the problem and by eliminating troublesome details, to progress in giant steps in areas that were considered inaccessible only fifty years ago. Mathematicians who abstract for the love of abstraction are usually mediocrities." P.

Vector Calculus and Differential Equations. By Albert G. Fadell (SUNY at Buffalo). Van Nostrand, Princeton, N. J., 1968. xii+558 pp. $11.95. To follow a first year course in calculus. Ends with chapters on the Laplace transform, series solutions, and Fourier series.

Basic Calculus. By Nathaniel A. Friedman (SUNY at Albany). Scott, Foresman, Glenview, Ill. 1968. 295 pp. $8.75. Intended for a one semester or two quarter introduction to calculus stressing ideas, this text uses a geometric approach. T (13).

Modern Calculus with Analytic Geometry. Volume II. By A. W. Goodman (Univ. of South Florida). Macmillan, New York, 1968. x+454 pp. $10.95. Linear algebra and determinants (introduced axiomatically), partial differentiation and multiple integration, line and surface integrals, differential equations and an appendix on the classical theory of determinants.

Introduction to Calculus. By Donald Greenspan (Univ. of Wisconsin). Harper & Row, New York, 1968. xi+439 pp. $9.95. Offbeat! (See ch. 8 on interval arithmetic, p. 13 on "packets of time" and p. x on "using the notation $|\Delta x|$ when Δx is to be only positive."

A Second Course in Calculus. 2nd ed. By Serge Lang (Columbia Univ.). Addison-Wesley, Reading, Mass., 1968. xii+305 pp. $7.95. Intended to follow the author's *A First Course in Calculus* and available also as part of a single volume containing both books. A treatment of calculus of several variables tightly integrated with linear algebra. There is no indication of what changes, if any, have been made in the 2nd edition.

Calculus of Several Variables. By E. K. McLachlan (Oklahoma State Univ.). Brooks /Cole, Belmont, Calif., 1968. xii+384 pp. $11.95. Follows the CUPM recommendations for a course in functions of several variables as part of the curriculum for engineers and physicists, departing from it only by adding a chapter on topology in Euclidean n-space. It appears to be suitable for any second year course.

A Calculus Notebook. By C. Stanley Ogilvy (Hamilton College) Prindle, Weber & Schmidt, Boston, Mass., 1968. viii+100 pp. $2.95 (paper). Not a textbook but a collection of problems in which to browse. S, P, L.

Differential and Integral Calculus with Problems, Hints for Solutions, and Solutions. By A. Ostrowski (Univ. of Basel). Translation Editor, Donald W. Crowe (Univ. of Wisconsin). Scott, Foresman, Glenview, Ill. 1968. xii+627 pp. $13.50. A classic for the first year course. T (13), L.

★*Calculus in the First Three Dimensions.* By Sherman K. Stein (Univ. of California, Davis). McGraw-Hill, New York, 1967. xiv+613 pp. $9.95. Those who are familiar with the author's *Mathematics, The Man-Made Universe* will expect high quality and novelty, and it appears that they will not be disappointed in this book. Part I treats three main topics, the definite integral, the derivative, and the fundamental theorem of calculus simultaneously in the first three dimensions. Part 2 covers expected topics, including series, vector functions and Green's theorem. Part 3 deals with applications to growth, business management and economics, psychology, traffic, rockets, and gravity. There are appendices on precalculus topics and on epsilontics.

Calculus and Analytic Geometry. 4th edition. By George B. Thomas, Jr. (M.I.T.) Addison-Wesley, Reading, Mass., 1968. xii+818 pp. $13.50. The most obvious change from previous editions is a larger page size with two column format and two color printing. There is a new chapter on limits and two on linear algebra and vector analysis (but these are not fully exploited in the following calculus of several variables).

Calculus and Analytic Geometry. By John A. Tierney (U. S. Naval Acad.). Allyn and Bacon, Boston, 1968. xii+641 pp. $10.95. Usual topics for a two year course with "valuable applications."

A Short Course in Differential Equations. By W. R. Utz (Univ. of Missouri). McGraw-Hill, New York, 1967. 168 pp. $5.95. Traditional elementary material available at lower cost in many more comprehensive paperbacks.

Calculus with Analytic Geometry. By Bevan K. Youse (Emory Univ.). International Textbook, Scranton, Penn., 1968. ix+416 pp. $8.50. For a one year course.

History

Edmond Halley. By Angus Armitage. Nelson, London, 1966. xii+220 pp. 42/-. Halley (1656–1742) was primarily an astronomer but, as was usual in that period, played a role in the development of mathematics. P.

Collected Works of Hidehiko Yamabe. Edited by Ralph P. Boas (Northwestern Univ.). Gordon and Breach, New York, 1967. xii+142 pp. $7.50. Biographical notes, a list of papers, and reprints of his twenty papers. P, L.

Dictionary of Inventions and Discoveries. Edited by E. F. Carter. Muller, London, 1966. 193 pp. $3.00. Mathematics is included, but the coverage is thin and uneven. For example quaternions and zero are included, but complex numbers and non-euclidean geometry are not. Brief biographical information is given on many scientists and inventors, but not on Gauss, Fermat, or Lobachevsky.

Mathematical Papers. By William Kingdon Clifford. Edited by Robert Tucker. Chelsea, New York, 1968. lxx+658 pp. $15.00. A reprint with correction of errata of the first edition of 1862, including a facsimile sample of Clifford's handwriting. L.

The Society of Arcueil. A view of French science at the time of Napoleon I. By Maurice Crosland. Harvard Univ. Press, Cambridge, Mass., 1967. xx+514 pp. $15.00. Among the mathematicians discussed are Laplace, Biot, Arago, Cauchy, and Poisson. (Reviewed by Schofield in *Science*, 29 Dec. 1967.) P.

Britannica Book of the Year. 1967. Irving Kaplansky summarizes the major developments in "the principal divisions of modern mathematics: algebra, analysis and topology," and warns that continued growth of the literature at current rates would yield 20,000 pages of abstracts per year by the end of the century. The increased use of mathematical methods in economic planning in the Soviet Union, the infiltration of new mathematical tools in geology, and the philosophical importance of mathematical publications (including the translation of *Disquisitiones Arithmeticae* and the *Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability*) are mentioned in other articles.

Encyclopædia Britannica Book of the Year. 1968. Irving Kaplansky in his report of events for 1967 concentrates on number theory but mentions significant results also on rings of operators, finite groups, topology, and Fourier series.

Émile Borel. Philosophe et Homme d'action. Pages choisies et présentées par Maurice Fréchet. Gauthier-Villars, Paris, 1967. 406 pp. $8.00. A representative selection of the writings of Borel on teaching and mathematical education, biography, the foundations of physics, economics, philosophy of science and mathematics, and the relations of science and society. Only his political writings are not included. Borel has to be considered by those who enjoy identifying another "last man who was familiar with all aspects of human culture." P, L.

Dialogue Concerning the Two Chief World Systems—Ptolemaic and Copernican. By Galileo Galilei. Translated by Stillman Drake. Foreword by Albert Einstein. Univ. of Calif. Press, Berkeley, 1967. xxvii+505 pp. $2.95 (paper). A reprint of the finest translation of this great work, with Drake's notes and a good index. Although Galileo is rightly considered a physicist rather than a mathematician, much of what he has to say is of interest to mathematicians. P, L.

The Golden Age of Science. Thirty Portraits of the Giants of 19th Century Science by their Scientific Contemporaries. Edited by B. Z. Jones, with an Introduction by E. Mendelsohn. Simon and Schuster (In cooperation with the Smithsonian Inst.), New York, 1966. xxxiii+659 pp. $12.00. Of special interest to mathematicians are the articles on Laplace, Legendre, Quetelet, and Poincaré. L.

★*Où vont les mathématiques? Réflexions sur l'enseignement et la recherche.* By Jean Kuntzmann (Grenoble Univ.). Illustrations by Avoine. Hermann, Paris, 1967. 166 pp. 15 F. (paper). This is an extraordinary attempt to discuss the major issues of mathematics: its present nature, its likely future, teaching, and organization of research. The author has published in the field of applied analysis, numerical methods, and computer science, but the presentation is nontechnical and accessible to any educated person without university training in mathematics. Mathematicians may disagree with the author's predictions for 1980, his views of the future of the mathematical education, his opinions on the role of computers, and his ideas on proper organization of mathematical research, but they cannot help finding the book interesting and

challenging. The whimsical drawings add spice to an already tasty dish. This volume is a good start for the new series to be called *Science Publique.* P, L.

Histoire des Sciences Mathématiques en Italie, depuis la renaissance des lettres jusqu'à la fin du dix-septième siècle. By Guillaume Libri. Four volumes. Paris, 1838–1841. Reprinted by Johnson Reprint, New York, 1967. $67.50. L.

The Pythagorean Proposition. By Elisha Scott Loomis. National Council of Teachers of Mathematics, Washington, D. C. 1968. xvi+284 pp. $3.00. The first in a new series called *Classics in Mathematics Education,* this is a facsimile reproduction of the second edition of 1940. It contains 256 proofs, many historical notes (some quite amateurish), a bibliography (often so incomplete as to be useless), and portraits of mathematicians. The publisher could have corrected many weaknesses without spoiling the original quaint flavor. S, L.

Guide to Russian Reference Books Vol. 5. Science Technology and Medicine. By Karol Maichel (Hoover Institution, Stanford Univ.), 1967. 384 pp. $22.50. A carefully annotated and classified bibliography, including a section on mathematics and references to mathematical literature in other sections also. The mathematics section begins with the following statement "The usefulness of mathematics is not limited to pragmatic calculations: mathematics, like music, is a universal language in which complex concepts can be expressed more precisely than is possible in ordinary word-symbols. Without recourse to mathematical expression, scientists would obviously often be at a great loss to describe many of their discoveries. Furthermore, mathematics has come increasingly to be recognized as a philosophy in its own right." L.

Probleme des Unendlichen. Werk und Leben Georg Cantors. By Herbert Meschkowski. Friedr. Vieweg, Braunschweig, 1967. 288 pp. 38 DM. This extraordinarily interesting looking book is devoted mostly to a detailed scientific biography of Cantor, including many pictures of him and his family, twenty three letters to various mathematicians, a list of publications by and about Cantor and a fine facsimile of one of his letters. In addition there is discussion of the historical and mathematical context in which Cantor worked and of his heritage. A prime candidate for translation! P, L.

Galileo. Man of Science. Edited by Ernan McMullin. Basic, New York, 1967. xiv+455 +cii pp. $15.00. Twenty-three papers by distinguished authorities including Stillman Drake, Paul Henry, Carl B. Boyer, I. Bernard Cohen, and Ernst Cassirer. The editor provides a Galileo bibliography from 1940 to 1964 and an addendum to previous bibliographies. There is also an annotated list of works by Galileo and his contemporaries prepared by Michael J. Crowe. P, L.

The Construction of the Wonderful Canon of Logarithms. By John Napier. Translated from Latin into English with Notes, and a Catalogue of the various editions of Napier's works, by William Rae Macdonald, F. F. A. Dawsons of Pall Mall, London, 1966. xix+169 pp. £3. 10s. This is a fine reprint of the original publication of 1889 and ought to be in every college mathematics library. P, L.

The Mathematical Papers of Isaac Newton. Volume II, 1667–1670. Edited by D. T. Whiteside. Cambridge Univ. Press, New York, 1968. xxii+520 pp. $35.00. Another magnificent volume in the sequence of eight projected, this contains Newton's work on analytic geometry, calculus, and algebra from the years indicated. Every mathe-

matical library should contain the set, which for the first time, will make easily accessible the work of this mathematical giant. T, L (!).

Theodore Von Karman. 1881–1963. In Memoriam. SIAM, Philadelphia, 1965. 178 pp. $5.50 (SIAM members $4.85). This collection of articles from the SIAM journal contains a portrait, a nine page biographical sketch by W. R. Sears, but no bibliography. P, L.

H. P. Robertson. January 27, 1903–August 26, 1961. In Memoriam. SIAM, Philadelphia, 1963. 63 pp. $3.50 (SIAM members $2.85). Contains a nine page biography by A. H. Taub, bibliography, a portrait, and several papers, all reprinted from the SIAM journal. P, L.

Studies in Approximation and Analysis. SIAM, Philadelphia, 1966. 195 pp. $5.75 (SIAM members $4.85). This is a collection of papers presented to Professor J. L. Walsh on the occasion of his seventieth birthday. Such collections ought to at least contain an adequate biography of the man being honored, including a list of his publications. This contains a portrait, less than two pages of biography and no bibliography. The material is reprinted from the SIAM Journal on Numerical Analysis. P.

Cumulative Index 1901–1960. School Science and Mathematics. (Vol. LXVI, Num. 9, Pt. II of II.). Central Association of Science and Mathematics Teachers, Bloomington, Ind., 1968. xi+204 pp. $2.50. Although the mathematical material in this journal has been of uneven quality, much of it is valuable and worthy of retrieval. Unfortunately, this index will be practically useless for this or any other purpose. Anyone attempting to find material thinks of authors or detailed subjects. The entries in this index under mathematics are arranged alphabetically by title under very general headings such as curriculum or teaching techniques. There is no author or subject listing. Because of the vagaries of word order in titles, this amounts to a practically random arrangement. Many items, such as book reviews and problems, are "indexed" by simply listing in order the volume numbers and pages in each volume where such material appears. Such information is completely useless since no page references are needed to locate book reviews or other special departments in a given volume. This index is only one of many that testify to the low state of this art in the mathematical community.

Most Significant New Books on Mathematics. 1966. Edited by D. E. Seabrook. Robert Maxwell, Oxford, England, 1968. 39 pp. $3.50. Appears to be a random sample rather than a selection. There are quotations from reviews in largely nonmathematical journals. The Mathematical Gazette, this MONTHLY and Mathematical Reviews are not quoted.

History of Science. Edited with a general preface by Rene Taton. Translated by A. J. Pomerans. Four volumes. I. Ancient and Medieval Science from the Beginning to 1450. II. The Beginning of Modern Science from 1450 to 1800. III. Science in the Nineteenth Century. IV. Science in the Twentieth Century. Basic Books, New York, 1963–1966. xx+551 pp. xx+665 pp. xxi+623 pp. xxiv+638 pp. $17.50 per volume. Under the editorship of this distinguished historian of science, these four volumes are the most ambitious recent attempt to write a comprehensive survey of the entire subject. Whether or not such a unified history is possible at all and to what extent this work succeeds in approaching it, there is no doubt that these volumes contain much interesting material on the history of mathematics and that the set should be

in mathematics libraries. Authors include J. Needham, J. Itard, A. Koyre, P. Montel, J. Dieudonné, A. Denjoy, M. Janet, M. Fréchet, L. Godeaux, G. Darmois, D. Dugue, F. Le Lionnais. P, L.

Precalculus

Elementary Algebra for College Students. 2nd ed. By Irving Drooyan and William Wooton (both of Pierce College, Los Angeles). Wiley, New York, 1968. x+302 pp. $6.95. Eighteenth century algebra through quadratics printed in two colors.

College Algebra and Trigonometry. By Daniel E. Dupree and Frank L. Harmon (both of Northeast Louisiana State College). Prentice-Hall, Englewood Cliffs, N. J. 1968. ix+288 pp. $7.95. Includes "complete axiomatic treatment of real numbers" (in nine pages) and "a thorough presentation of matrices and determinants" (in thirty five pages), as well as touching on logic, sets, functions, inequalities, and "hyperbolic trigonometric functions."

Contemporary College Algebra and Trigonometry. By William A. Gager (Univ. of Florida). Macmillan, New York, 1968. xvii+476 pp. $8.95. The author states that he has taken account of the recommendations of C.M.C.E.B., C.U.P.M. and S.M.S.G. Topics include linear systems by determinants, sequences and series, polynomials, complex numbers and probability.

Introduction to College Mathematics. By Vincent H. Haag and Donald W. Western (Franklin and Marshall College). Holt, Rinehart and Winston, New York, 1968. x+676 pp. $9.95. In the tradition of recent unified freshman texts, this begins with logic, sets and relations and touches on the real number system, matrices, probability, graphs, elementary functions, vectors, and elementary calculus.

Trigonometry. A Study of Certain Real Functions. By Donald R. Horner (Eastern Washington S. C.). Holt, Rinehart and Winston, New York, 1968. xiii+303 pp. $6.95. Includes complex numbers and ends with a twenty five page introduction to vector spaces. T.

Elementary Functions. By Florence D. Jacobson (Albertus Magnus College, New Haven) and William G. Chinn (San Francisco Unified School District). Silver Burdett, Morristown, N. J., 1968. 274 pp. $4.60. Starts with real numbers, relations and functions and ends with circular functions and a few pages on limits. T (12–13).

College Algebra. By Adele Leonhardy. 2nd ed. Wiley, New York, 1968. xiii+468 pp. $7.50. Largely traditional, with some attention to modern ideas. From the integers through theory of equations to combinatorial probability.

Basic College Algebra. Revised Edition. By Julian D. Mancill and Mario O. Gonzalez (both of Univ. of Alabama). Allyn and Bacon, Boston, Mass., 1968. x+470 pp. $8.95. The first edition of 1963 has been revised by some rearrangements and a new chapter on combinatorics and probability. There remains a mix of traditional and modern topics with more weight on the former.

New College Algebra. By Marvin Marcus and Henryk Minc (Univ. of Calif. at Santa Barbara). Houghton Mifflin, Boston, Mass., 1968. $6.50. Chapter headings are Sets, Numbers and Functions; Number Systems, and Polynomials; Linear Equations, Matrices and Inequalities; Combinatorics (including mathematical induction);

Exponential and Logarithmic Functions. Each section is followed by a true false quiz in addition to exercises.

Modern Trigonometry. Eugene D. Nichols and E. Henry Garland (both of Florida State Univ.). Holt, Rinehart and Winston, New York, 1968. viii+342 pp. $6.95. This does contain modern features including the wrapping function, vectors and two-color printing, but even more modern would be the abolition of separate courses and textbooks on trigonometry and the incorporation of this material in the mathematics course as appropriate during the first twelve grades.

Sequences. By Katharine E. O'Brien (Univ. of Maine, Portland). Houghton Mifflin, Boston, Mass., 1966. iii+90 pp. $1.20. Apparently the first in the Houghton Mifflin Mathematics Enrichment Series, this book may be useful as a supplement in precalculus or calculus courses. S.

Elements of Trigonometry. By Tullio J. Pignani and Paul W. Haggard (both of East Carolina Univ.). Harcourt, Brace & World, New York, 1968. xv+284 pp. $7.50. Includes solution of triangles, vectors and complex numbers. T.

Introduction to Algebra for College Students. 2nd ed. By William A. Rutledge (Old Dominion College) and Simon Green (California State Polytechnic College). Prentice-Hall, Englewood Cliffs, N.J., 1968. x+310 pp. $7.50. Begins with sets and numbers and ends with quadratic equations and inequalities.

NOTABLE PAPERS

In *Science* for May 10, 1968 (Vol. 160, pp. 661–663) A. P. Dempster makes interesting comments on basic issues of mathematical statistics while reviewing three recent collections of papers by J. Neyman and E. S. Pearson.

Acknowledgement. The following have generously helped in evaluating books: Edward G. Begle, Truman Botts, Jean M. Calloway, Trevor Evans, Aaron Fialkov, Florence D. Jacobson, Paul J. Kelly, Joseph P. LaSalle, Howard Levi, Harry Levy, Bert Mendelson, Kunio Murasugi, Karl K. Norton, H. O. Pollak, Steve Regoczei, Andrew Sterrett.

NEWS AND NOTICES

Edited by Raoul Hailpern, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dr. L. N. H. Bunt, University of Utrecht, has been appointed Professor at Arizona State University.

Professor Ralph Crouch, Drexel Institute of Technology, has been appointed Dean of the College of Science.

Assistant Professor J. J. Fisher, Virginia Polytechnic Institute, has been appointed Associate Professor at Southern Colorado State College.

Dr. Raj N. Kaul, Desh Bandhu College, has been appointed Reader in Mathematics at the University of Delhi.

Dr. W. B. Laffer II, Armstrong State College, has been appointed Associate Professor at Chicago State College.

Professor K. D. Magill, Jr., SUNY at Buffalo, has been appointed Visiting Professor at the University of Leeds, England, for the Fall term 1968.

Dr. Chull Park, University of Minnesota, has been appointed Assistant Professor at Miami University.

Professor H. C. Saar, Bloomington Public School System, has been appointed Dean of Shawnee Community College, Karnak.

Professor Emeritus J. Rogers Musselman, Case Western Reserve University, died on August 8, 1968. He was a Charter Member of the Association.

Assistant Professor W. V. Nevins III, Alfred University, died on October 18, 1967. He was a member of the Association for twenty-four years.

Professor Emeritus Oystein Ore, Yale University, died on August 13, 1968. He was a member of the Association for thirty-eight years.

Erratum: Contrary to the information we received some months ago and which was published in the October issue of this MONTHLY, we have just been advised that Professor Emeritus G. H. Hunt is alive and well: our deepest apologies and our very best wishes to him for a long life ahead.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The spring meeting of the Allegheny Mountain Section of the MAA was held at Indiana University of Pennsylvania on April 27, 1968, with over one hundred and thirty in attendance.

During the business session, an amendment to the by-laws of the section was proposed and adopted.

Dr. Charles Cunkle, Slippery Rock State College, was elected to the Executive Committee; Dr. H. L. Krall, Pennsylvania State University, was continued as Chairman and Dr. W. A. Hallam, West Virginia Wesleyan College, was continued as Secretary-Treasurer.

Professor I. D. Peters reported for the High School Contest in West Virginia, and Dr. Frank Kocher reported for Pennsylvania.

The section voted to give membership in the Association to the top six scorers of the section area in the Putnam Contest.

The following program was presented:

1. *Integrating polynomial functions with base 'b' numeration systems*, by Edwin Bailey, Indiana University of Pennsylvania.
2. *An extension of phase plane analysis*, by G. DiAntonio, Indiana University of Pennsylvania.
3. *Matrix representations of certain semigroups*, by J. B. Kim, West Virginia University.
4. *Some remarks on inverse series relations*, by H. W. Gould, West Virginia University.
5. *A remark on the divergence theorem in algebraic geometry*, by Mario Benedicty, University of Pittsburgh.
6. *On extensions of certain integral results of Ramanujan*, by M. K. Jain.

7. *Periodic sequences*, by J. P. Hoyt, Indiana University of Pennsylvania.
8. *Mappings*, by Gail Young, Tulane University (by invitation).
9. *Some contrary ideas which led and lead to good results*, by J. C. Eaves, West Virginia University (by invitation).

W. A. HALLAM, *Secretary-Treasurer*

MAY MEETING OF THE MINNESOTA SECTION

The spring meeting of the Minnesota Section of the MAA was held at the College of Saint Teresa in Winona, Minnesota, on May 4, 1968. There were 97 persons in attendance of whom 58 were members of the Association. Sister Thomas a Kempis of the College of Saint Teresa presided over the morning session; Professor Roy Dowling of the University of Manitoba presided over the afternoon session.

At the business meeting, Professor Wayne Roberts reported on the mathematical High School Contest. The nominating committee, composed of Murray Braden of Macalester College, Roger Kirchner of Carleton College, and Warren Loud of the University of Minnesota, submitted a list of candidates for the coming school year, and the following were elected: Chairman: E. J. Camp, Macalester College; Secretary-Treasurer: Warren Thomsen, Moorhead State College; Executive Committee: Roy Dowling, University of Manitoba, and Robert Cameron, University of Minnesota. The Secretary-Treasurer made a short report.

A welcome address was given by Sister M. Camille, President of the College of Saint Teresa, after which the following program was presented:

1. *Infinite compositions of functions*, by J. A. Seebach, Jr., St. Olaf College.
2. *Teach graphs!* by K. W. Wegner, Carleton College.
3. *Permutable analytic functions with a common fixed point*, by R. H. Cameron, University of Minnesota.
4. *Some examples of radial limits of lacunary power series*, by W. D. Serbyn, University of Minnesota.
5. *Newton-Raphson by-products of two constructive proofs of the theorem on implicit functions*, by W. L. Hart, University of Minnesota.
6. *A converse to Sard's theorem and applications*, by D. E. Varberg, Hamline University.
7. *Identification of fields*, by L. Gaal, University of Minnesota.
8. *Roles of computers in scientific experiments*, by Paul Homeyer, C-E-I-R, Inc., Beverly Hills, California (by invitation).

WALBERT KALINOWSKI, *Secretary-Treasurer*

MAY MEETING OF THE WISCONSIN SECTION

The annual meeting of the Wisconsin Section of the MAA was held at Wisconsin State University—La Crosse on May 4, 1968. Chairman E. F. Wilde, Beloit College, presided. Approximately 100 persons attended. The morning session was devoted to a short business meeting and the presentation of several short papers.

During the business meeting, Dr. J. A. Raab, Wisconsin State University—Oshkosh, was elected Chairman; Dr. Marshall Wick, Wisconsin State University—Eau Claire, was elected Vice-Chairman; and Dr. R. W. Christensen, Wisconsin State University—La Crosse, was elected Secretary-Treasurer.

The remainder of the morning session was devoted to the presentation of the following papers:

1. *Least square curve fitting, theory versus practice*, by W. W. Wallace, University of Wisconsin, Madison.
2. *Horocyclic cluster sets of functions defined in the unit disk*, by S. V. Dragosh, Wisconsin State University—Oshkosh.

3. *Pairwise compactness and bitopological function space*, by Yong-Woon Kim, Wisconsin State University—Eau Claire.

4. *Modular forms and analytic number theory*, by J. R. Smart, University of Wisconsin, Madison.

5. *On shuffling cards*, by E. F. Wilde, Beloit College, Wisconsin.

The afternoon session began with a lecture on the qualifications for college teachers of mathematics by a representative from the MAA. A brief question and answer period followed the lecture. This session concluded with a viewing of the MAA films 'John von Neumann' and 'The Kakeya Problem.'

R W. CHRISTENSEN, *Secretary-Treasurer*

ACKNOWLEDGEMENT

The Editorial Board acknowledges with thanks the services of the following mathematicians, not members of the Board, who have kindly assisted by evaluating papers submitted for publication in the MONTHLY.

Janos Aczel, Claire Adler, Max Agoston, Felix Albrecht, Henry L. Alder, Carl B. Allendoerfer, W. A. Al-Salam, Douglas Anderson, K. W. Anderson, Richard D. Anderson, Richard V. Andree, Tom M. Apostol, Steve Armentrout, Harvey J. Arnold, Joseph Auslander, William Bade, J. Barlaz, Robert G. Bartle, Paul T. Bateman, Ross A. Beaumont, Edwin F. Beckenbach, Alexander Bednarek, Richard E. Bellman, Theodore J. Benac, Kent Bendall, S. K. Berberian, John Berglund, Salvatore Bernardi, Herman Biesterfeldt, R. H. Bing, Garrett Birkhoff, Donald W. Blackett, David Blackwell, A. A. Blank, Leonard Blumenthal, Ralph Boas, Edward Bolger, Bruno Bosbach, Truman Botts, D. G. Bourgin, Julia Bower, John Brace, Louis Bragg, Louis Brand, Alfred Brauer, Fred Brauer, J. L. Brenner, Robert H. Breusch, William E. Briggs, H. W. Brinkmann, Arthur B. Brown, Richard A. Brualdi, Andrew Bruckner, Hugh Brunk, R. Creighton Buck, C. Edmund Burgess, Robert G. Buschman, Herbert Busemann, Kenneth A. Bush, Donald W. Bushaw, Stewart S. Cairns, Jean M. Calloway, Leonard Carlitz, Stephen Chase, Yuh-Ching Chen, Theodore S. Chihara, Richard L. Chittim, Hubert Chrestenson, D. E. Christie, William E. Christilles, Alford H. Clifford, E. Cohen, Haskell Cohen, Marion Cohen, Harvey Cohn, Richard Cohn, A. J. Coleman, Courtney Coleman, Wistar Comfort, P. F. Conrad, H. S. Coxeter, Donald W. Crowe, Richard Crowell, Charles W. Curtis, Arthur Danese, Martin Davis, Philip Davis, M. M. Day, Frank DeSua, J. B. Diaz, Alice Dickinson, David Dickinson, Stephen Diliberto, R. P. Dilworth, Nathan Divinsky, Joseph L. Doob, Harold L. Dorwart, Burton S. Dreben, James Dugundji, Peter Duren, Michael Edelstein, Louis Ehrlich, Samuel Eilenberg, Murray Eisenberg, Howard Eves, Ky Fan, Albert Farnell, Marilynn L. Faulkner, Walter Feit, William Feller, Thomas Ferguson, F. A. Ficken, Daniel Finkbeiner, M. J. Fisher, Harley Flanders, Aviezri Fraenkel, Thomas Frayne, Peter Freyd, Avner Friedman, Orrin Frink, L. Fuchs, Watson Fulks, Wilson Gately, Harry M. Gehman, Bernard R. Gelbaum, Murray Gerstenhaber, Leonard Gillman, Robert Gilmer, Samuel Goldberg, Alan J. Goldman, Solomon Golomb, R. L. Goodstein, Basil Gordon, Walter Gottschalk, H. W. Gould, William Gragg, Ronald L. Graham, George Gratzer, Thomas Greville, Leonard Gross, George Grossman, Emil Grosswald, Branko Grunbaum, V. H. Haag, Franklin Haimo, H. R. Halkin, Dick Wick Hall, Douglas W. Hall, Marshall Hall, Paul Halmos, Frank Harary, William Harris, P. Hartman, G. A. Hedlund, D. W. Henderson, Peter Henrici, Israel N. Herstein, Edwin Hewitt, F. B. Hildebrand, Einar Hille, Peter J. Hilton, I. I. Hirschman, Gerhard P. Hochschild, J. G. Hocking, John Hodges, Alan Hoffman, Kenneth M. Hoffman, Stephen Hoffman, Franz Hohn, John Horvath, Charles Houghton, S. T. Hu, Philip J. Huneke, B. Hunt, Ronald Huston, W. R. Hutcherson, William T. Ingram, Rufus Isaacs, John Isbell, R. C. James, Ludvik Janos, J. P. Jans, James A. Jenkins, Fritz John, G. P. Johnson, P. B. Johnson, Richard E. Johnson, Burton

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CALENDAR OF FUTURE MEETINGS

Fifty-Second Annual Meeting, New Orleans, Louisiana, January 25–27, 1969.

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 26, 1969.

FLORIDA, Florida Atlantic University, Boca Raton, March 21–22, 1969.

ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.

INDIANA

IOWA, University of Northern Iowa, Cedar Falls, April 18, 1969.

KANSAS, Wichita State University, Wichita, March 29, 1969.

KENTUCKY, Morehead State University, Morehead, Spring 1969.

LOUISIANA-MISSISSIPPI, New Orleans, January 25–27, 1969.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 29, 1969.

MINNESOTA, College of St. Catherine, St. Paul, April 26, 1969.

MISSOURI, St. Louis University, St. Louis, April 26, 1969.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 25–26, 1969.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of Santa Clara, Santa Clara, February 8, 1969.

OHIO

OKLAHOMA-ARKANSAS, Arkansas State University, Jonesboro, March 21–22, 1969.

PACIFIC NORTHWEST, University of Oregon, Eugene, August 1969.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, May 9–10, 1969.

SOUTHEASTERN, Winthrop College, Rock Hill, South Carolina, March 28–29, 1969.

SOUTHERN CALIFORNIA, California State College at Fullerton, March 15, 1969.

SOUTHWESTERN, Northern Arizona University, Flagstaff, Spring 1969.

TEXAS, Texarkana College, Texarkana, April 18–19, 1969.

UPPER NEW YORK STATE

WISCONSIN, Oshkosh, Wisconsin, May 2–3, 1969.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, New Orleans, Louisiana, January 23–26, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Statler-Hilton Hotel, Washington, D. C., May 7–9, 1969.

ASSOCIATION FOR SYMBOLIC LOGIC, New Orleans, Louisiana, January 22–23, 1969.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS

FIBONACCI ASSOCIATION, University of San Francisco, January 18, 1969.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, New Orleans, Louisiana, January 25–26, 1969.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Brown Palace Hotel, Denver, Colorado, June 17–20, 1969.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 10–12, 1969.